The Energy Transport and the Drift Diffusion Equations as Relaxation Limits of the Hydrodynamic Model for Semiconductors

Ingenuin Gasser
FB Mathematik MA 6-2, TU–Berlin
Straße des 17. Juni 136, D–10623 Berlin, Germany
E–mail: gasser@math.tu-berlin.de

Roberto Natalini
Istituto per le Applicazioni del Calcolo "M. Picone"
Viale del Policlinico 137, I–00161 Roma, Italy
E–mail: natalini@asterix.iac.rm.cnr.it

Abstract

Two relaxation limits of the hydrodynamic model for semiconductors are investigated. Using the compensated compactness tools we show the convergence of (scaled) entropy solutions of the hydrodynamic model to the solutions of the energy transport and the drift–diffusion equations, according respectively to different time scales.

1 Introduction

We consider an Euler–Poisson hydrodynamic model for semiconductors (HD) in one space dimension. Denoting by n, j, p and E the charge density, current density, pressure and electric field ($n,\ p \geq 0$), the (non–dimensional) equations of the HD model for semiconductors are given by a hydrodynamic part

$$
\begin{aligned}
&n_t + j_x = 0 \ , \\
&j_t + \Big(\frac{j^2}{n} + p\Big)_x = nE - \frac{j}{\tau_p} \ , \\
&\Big(\frac{j^2}{2n} + \frac{1}{\gamma-1}p\Big)_t + \Big(\frac{j^3}{2n^2} + \frac{\gamma}{\gamma-1}\frac{j}{n}p\Big)_x = jE - \frac{1}{\tau_w}\Big(\frac{j^2}{2n} + \frac{1}{\gamma-1}(p - nT_l)\Big) \ ,
\end{aligned}
\tag{1}
$$

supplemented by the Poisson equation for the electric field

$$E_x = n - b(x). \tag{2}$$

The positive constants τ_p, τ_w, γ ($\gamma > 1$) denote respectively the momentum relaxation time, the energy relaxation time and the ratio of specific heats of the system. The functions $T_l(x)$ and $b(x)$ are the lattice temperature and the doping profile of the semiconductor. Here we are assuming the equation of state of polytropic ideal gas given by

$$p = (\gamma - 1) n\, e,$$

where e is the specifc internal energy of the electron fluid.

As well known, see [MRS], the time evolution of a distribution of electrons in a semiconductor device is well described by the semiclassical Boltzmann-Poisson Transport Equations. Unfortunately, dealing with this kinetic model remains for the moment too expensive from a computational point of view. Nevertheless, it is possible to derive from transport equations simpler fluid dynamical equations for macroscopic (integral) quantities like particle, current or energy densities, which could represent a compromise between physical accuracy and reduction of computational cost. A standard approach for this derivation is the classical moments method. According to different ansatzes for the phase space densities, we recover different limit models and in particular the popular drift–diffusion equations (DD) and the hydrodynamic Euler–Poisson system (HD) (see [An] or again the reference book [MRS] and the literature quoted therein).

The (rigorous) derivation of the DD model from the Boltzmann-Poisson equations can be found in [P] and [GP]. The dominant collision mechanism in the derivation is the electron–phonon scattering. This model works very well under the assumptions of low carrier densities and small electric fields. By contrast, hydrodynamic models are usually considered to describe high field phenomena or submicronic devices, because they take into account the transport of energy in the semiconductor. The particular hydrodynamic model (1)-(2) under consideration in this paper, was introduced in [B] and [BW] (see again [MRS]) and intensively studied in recent years.

However, it turns out that also the qualitative study and the numerical approximation of the hydrodynamic models is far to be trivial. Since the equations form a quasilinear hyperbolic–elliptic system of balance laws (to be supplemented by some suitable initial–boundary conditions), the solutions can became discontinuous in finite time, see for example [L] for general references on various shock waves phenomena.

A possible way to overcame this difficulty is given by the energy transport models (ET). Actually, under a suitable relaxation limit, the hydrodynamic models can be well–approximated by a simpler class of diffusive evolution equations which still describe with some accuracy the basic energy transport mechanism.

In the present work we assume that the relevant scattering mechanisms can be described by the relaxation time approximation on the macroscopic HD level. Observe that Monte Carlo simulations of the Boltzmann–Poisson equations show that the momentum relaxation time τ_p is much smaller then the energy relaxation time τ_w [An]. Therefore we define the small parameter

$$\tau^2 := \frac{\tau_p}{\tau_w}. \tag{3}$$

In order to perform the relaxation limits we introduce the new time scale $s = \tau_p\, t$, $\tau_p = \tau^m$ $(m = 1, 2)$ and denote the scaled quantities by

$$\begin{aligned}
\mathcal{N}^\tau(x,s) &= n\Big(x, \tfrac{s}{\tau^m}\Big), \quad \mathcal{J}^\tau(x,s) = \frac{1}{\tau^m} j\Big(x, \frac{s}{\tau^m}\Big), \\
\mathcal{P}^\tau(x,s) &= p\Big(x, \tfrac{s}{\tau^m}\Big), \quad \mathcal{E}^\tau(x,s) = E\Big(x, \frac{s}{\tau^m}\Big).
\end{aligned} \tag{4}$$

In the case $m = 1$ the (formal) limits $\mathcal{N}$, $\mathcal{J}$, $\mathcal{P}$, $\mathcal{E}$ of $\mathcal{N}^\tau$, $\mathcal{J}^\tau$, $\mathcal{P}^\tau$, $\mathcal{E}^\tau$ as $\tau \to 0$ satisfy (formally) the following energy transport equations (ET)

$$\begin{aligned}
&\mathcal{N}_s = (\mathcal{P}_x - \mathcal{N}\mathcal{E})_x \\
&(\frac{1}{\gamma - 1}\mathcal{P})_s + \Big(\frac{\gamma}{\gamma - 1}\frac{\mathcal{N}\mathcal{E} - \mathcal{P}_x}{\mathcal{N}}\mathcal{P}\Big)_x = (\mathcal{N}\mathcal{E} - \mathcal{P}_x)\mathcal{E} - \frac{1}{\gamma - 1}(\mathcal{P} - \mathcal{N}T_l) \\
&\mathcal{E}_x = \mathcal{N} - b(x).
\end{aligned} \tag{5}$$

In the case $m = 2$ the limits $\mathcal{N}$, $\mathcal{J}$, $\mathcal{P}$, $\mathcal{E}$ satisfy the well known drift diffusion equations (DD)

$$\begin{aligned}
\mathcal{N}_s &= ((\mathcal{N}T_l)_x - \mathcal{N}\mathcal{E})_x \\
\mathcal{E}_x &= \mathcal{N} - b(x).
\end{aligned} \tag{6}$$

Both limiting systems describe asymptotic regions of the solutions of the HD equations. On the ET time scale, variations of the temperature (or pressure) are not neglected as in the DD model, where the temperature is approximated by the lattice temperature. The above (formal) limits from HD to ET and DD have first been obtained in [Al] (see also [AMN]), where more general hydrodynamic models, arising from the extended thermodynamic framework [An], were considered. A somewhat related idea, a vanishing effective mass limit, was mentioned in [BDG].

A different approach in order to derive general ET models can be found in [BD] (see also [De] and [BDG]). In this derivation they start directly from the Boltzmann equation. The dominant scattering mechanisms are assumed to be both electron–electron and elastic electron–phonon scattering.

We investigate the rigorous limits as $\tau \to 0$ in both cases $m = 1$ and $m = 2$. The resulting limit models are the ET model (5) and the standard DD model (6). The present ET model is just a particular case, due to the choice of the HD model, of the ET models already derived both in [BDG] and [Al].

Let us recall that the existence of the entropy solutions and the rigorous relaxation analysis in the case of the isentropic hydrodynamic equations was first done in [MN1], [MN2]. In that case the pressure depends only on the density and the hydrodynamic system consists of a continuity and a momentum equation. The limits of the density and the current satisfy the drift–diffusion equations with the pressure depending in a nonlinear way on the density. Analogous results, but in the bipolar case, can be found in [N].

In the case of the general non–isentropic model (1)-(2), a first open problem is to establish the global existence of entropy solutions, even for some special classes of initial data. Let us observe that a local (in time) existence theorem for system (1)-(2), for initial data having small total variation and in the class of weak entropy BV solutions, could be proved by a straightforward extension of the arguments in [DH], where a simple fractional step version of the Glimm scheme [G] was used (see also the recent semigroup version of this result in [CP]). Actually in [DH] the authors also obtained global solutions under some complete dissipative assumptions on the source term, which however are not verified in the present case, even in the isentropic case. By contrast the source terms in system (1)-(2) have a linear growth in their arguments. Then, in principle, it is not clear whether the solutions stay for all time in a fixed (small) bounded set of the BV space, as required by the interaction estimates of the Glimm proof.

On the other hand, we observe that the arguments used in [MN1] to establish the global existence of entropy solutions in the isentropic case seem do not be expedient in this case, since the essential tool in that analysis was given by the methods of compensated compactness and the analysis of the Young measure of approximating sequences [T], [Di], which do not work, in general, in dealing with $N \times N$ systems, for $N \geq 3$. Therefore in the following we are just *assuming* the existence of uniformly bounded global entropy solutions to the Cauchy problem for (1)-(2).

To give the relaxation limits, the main tool we use is the "div–curl" Lemma, again from the theory of the compensated compactness [T], which we combine with energy estimates, given by the entropy inequalities, to yield the required strong convergence of the pressure term. This kind of technique was first introduced in [MMS] and [MM] and then successfully adapted to investigate many other hyperbolic–parabolic relaxation limits, [MN2], [MR], [R1], [R2].

Finally, let us briefly outline the organization of the paper. In section 2 we present some preliminaries and discuss the entropy conditions. Section 3 is devoted to prove the rigorous relaxation limit from the HD model to the ET model. In Section 4 we shortly prove the convergence of the HD model to the DD model.

2 Preliminaries

Let us recall some basic notions which will be used in the following. We assume the (phenomenological) relation (3) between the relaxation times. The doping profile is assumed to be $b \in L^1(I\!R)$, $(b \geq 0)$ and the lattice temperature $T_l \in L^\infty(I\!R)$, $(T_l \geq 0)$. We assume locally bounded initial conditions

$$\begin{aligned} &n(x,0) = n_0(x), \quad j(x,0) = j_0(x), \quad p(x,0) = p_0(x), \\ &E_0(x) = \int_{-\infty}^{x} (n_0(y) - b(y))dy + E_-, \end{aligned} \tag{7}$$

where n_0, p_0 are nonnegative functions and E_- is a given constant. A locally bounded measurable function (n, j, p, E) is called a weak solution of the Cauchy problem (1)-(2)-(7), if and only if for every smooth test function Φ with a compact support in $I\!R \times [0, +\infty)$ we have

$$\int\int (n\Phi_t + j\Phi_x) dx dt + \int_{t=0} n_0 \Phi dx = 0,$$
$$\int\int \{ j\Phi_t + \Big(\frac{j^2}{n} + p\Big)\Phi_x + \Big(nE - \frac{j}{\tau}\Big)\Phi\} dx dt + \int_{t=0} j_0 \Phi dx = 0,$$
$$\int\int \Big\{ \Big(\frac{j^2}{2n} + \frac{1}{\gamma - 1}p\Big)\Phi_t + \Big(\frac{j^3}{2n^2} + \frac{\gamma}{\gamma - 1}\frac{j}{n}p\Big)\Phi_x$$
$$+ \Big[jE - \tau\Big(\frac{j^2}{2n} + \frac{1}{\gamma - 1}(p - nT_l)\Big)\Big]\Phi\Big\} dx dt + \int_{t=0} \Big(\frac{j_0^2}{2n_0} + \frac{1}{\gamma - 1}p_0\Big)\Phi dx = 0,$$
$$\int\int [E\Phi_x + (n - b)\Phi] dx dt = 0.$$

Let us give the definition of (weak) entropy solutions for problem (1)-(2). Set

$$U = \Big(n, j, (\frac{j^2}{2n} + \frac{1}{\gamma - 1}p) \Big),$$
$$F(U) = \Big(j, (\frac{j^2}{n} + p), (\frac{j^3}{2n^2} + \frac{\gamma}{\gamma - 1}\frac{j}{n}p) \Big),$$
$$G^\tau(U, E) = \Big(0, nE - \frac{j}{\tau}, jE - \tau\Big(\frac{j^2}{2n} + \frac{1}{\gamma - 1}(p - nT_l)\Big) \Big).$$

The system (1) reads now
$$\partial_t U + \partial_x F(U) = G^\tau(U, E). \tag{8}$$

Recall that in general quasilinear systems of conservation laws do not possess global smooth solutions, for the presence of discontinuities (shock waves) in the solutions. Moreover, the weak solutions defined above are in general not unique and further admissibility conditions are needed to select stable and physically significant solutions.

Following [L] we say that an entropy–entropy flux pair (η, q) for (1) is a couple of smooth functions of U such that

$$\nabla q = \nabla \eta \cdot \nabla F^T ,$$

where ∇ denotes the gradient with respect to U. A classical example of a strictly convex entropy pair for the homogeneous part of system (1) is given by:

$$\eta(U) = -n \ln \frac{p}{n^\gamma}, \quad q(U) = -j \ln \frac{p}{n^\gamma}. \tag{9}$$

It can be shown that the physical entropy in (9) is strictly convex (see for instance [GR] or [S]). Furthermore, the entropy pair (9) is the unique non trivial entropy pair of the system [S] (up to addition of conserved quantities and constants).

We say that a weak solution (U, E) to the Cauchy problem (1)-(2)-(7) is an entropy solution if and only if for any entropy pair (η, q), with η convex, one has

$$\partial_t \eta(U) + \partial_x q(U) \leq (\nabla\eta)^T \cdot G^\tau(U, E), \qquad \text{in } \mathcal{D}' \ .$$

Let us give a first result concerning the entropy solutions, which will be useful in the following.

Lemma 2.1 *Let (η, q) be the entropy pair for the HD system given by (9).Then, for any $t_0 > 0$ and any $\tau > 0$, the following inequality holds*

$$\begin{aligned}(\gamma-1)\Big(1-\tfrac{\tau^2}{2}\Big)\tfrac{1}{\tau^2} \quad \int_0^{t_0}\int \frac{\Big(j(x,\frac{s}{\tau})\Big)^2}{p(x,\frac{s}{\tau})}dxds + \int_0^{t_0}\int \frac{\Big(n(x,\frac{s}{\tau})\Big)^2}{p(x,\frac{s}{\tau})}T_l(x)dxds \\ \leq t_0 \int n_0(x)dx + \int \eta(U_0(x))dx - \int \eta(U(x,\frac{t_0}{\tau}))dx\end{aligned} \tag{10}$$

for any entropy solution (U, E) of the HD system, with U having a compact support.

Proof: Integration of the entropy inequality

$$\partial_t \eta(U) + \partial_x q(U) \leq -(\gamma-1)\frac{1}{\tau}\Big(1-\frac{\tau^2}{2}\Big)\frac{j^2}{p} + \tau(n - \frac{n^2}{p}T_l) \tag{11}$$

in the x–variable gives

$$\begin{aligned}\frac{d}{dt}(\int \eta(U(x,t))dx) \leq \quad - \quad & (\gamma-1)\frac{1}{\tau}\Big(1-\frac{\tau^2}{2}\Big)\int \frac{\Big(j(x,t)\Big)^2}{p(x,t)}dx \\ + \quad & \tau \int n(x,t)dx - \tau \int \frac{\Big(n(x,t)\Big)^2}{p(x,t)}T_l(x)dx.\end{aligned}$$

Denoting by

$$f(t) = \int \eta(U(x,t))dx, \quad \psi(t) = \int \frac{\Big(j(x,t)\Big)^2}{p(x,t)}dx, \quad \phi(t) = \int \frac{\Big(n(x,t)\Big)^2}{p(x,t)}T_l(x)dx$$

and changing the time scale $s = \tau t$ we obtain

$$\frac{d}{ds}(f(\frac{s}{\tau})) \leq -(\gamma-1)(1-\frac{\tau^2}{2})\frac{1}{\tau^2}\psi(\frac{s}{\tau}) + \int n_0(x)dx - \phi(\frac{s}{\tau}).$$

(the total density is a conserved quantity). Integration yields

$$f(\frac{t_0}{\tau}) - f(\frac{0}{\tau}) \leq -(\gamma-1)(1-\frac{\tau^2}{2})\frac{1}{\tau^2}\int_0^{t_0}\psi(\frac{s}{\tau})ds + t_0\int n_0(x)dx - \int_0^{t_0}\phi(\frac{s}{\tau})ds.$$

and the lemma follows. □

3 Relaxation to an Energy Transport Model

In this section we investigate the relaxation of the HD model to the ET model. Changing the time scale $s = \tau t$ and using the new quantities $\mathcal{N}^\tau, \mathcal{J}^\tau, \mathcal{P}^\tau, \mathcal{E}^\tau$ defined in (4) for $m = 1$ we obtain the equations

$$\begin{aligned}
\mathcal{N}^\tau{}_t + \mathcal{J}^\tau{}_x &= 0, \\
\tau^2 \mathcal{J}^\tau{}_t + \Big(\tau^2 \frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau} + \mathcal{P}^\tau\Big)_x &= \mathcal{N}^\tau \mathcal{E}^\tau - \mathcal{J}^\tau, \\
\Big(\tau^2 \frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} + \frac{1}{\gamma - 1}\mathcal{P}^\tau\Big)_t &+ \Big(\tau^2 \frac{\mathcal{J}^{\tau 3}}{2\mathcal{N}^{\tau 2}} + \frac{\gamma}{\gamma - 1}\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau\Big)_x \\
&= \mathcal{J}^\tau \mathcal{E}^\tau - \tau^2 \frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} - \frac{1}{\gamma - 1}(\mathcal{P}^\tau - \mathcal{N}^\tau T_l), \\
\mathcal{E}^\tau{}_x &= \mathcal{N}^\tau - b(x).
\end{aligned} \tag{12}$$

Formally, as $\tau \to 0$, the solutions of the above equations relax to the solutions of the ET model

$$\begin{aligned}
&\mathcal{N}_s + \mathcal{J}_x = 0, \\
&\mathcal{P}_x = \mathcal{N}\mathcal{E} - \mathcal{J}, \\
&(\frac{1}{\gamma - 1}\mathcal{P})_s + (\frac{\gamma}{\gamma - 1}\frac{\mathcal{J}}{\mathcal{N}}\mathcal{P})_x = \mathcal{J}\mathcal{E} - \frac{1}{\gamma - 1}(\mathcal{P} - \mathcal{N}T_l), \\
&\mathcal{E}_x = \mathcal{N} - b(x).
\end{aligned} \tag{13}$$

Using the second equation $\mathcal{J} = \mathcal{N}\mathcal{E} - \mathcal{P}_x$ we recover the equations (5). Let us state the main result of this section.

Theorem 3.1 *Assume $b \in L^1(I\!R)$, $T_l \in L^\infty(I\!R)$, $b \geq 0$, $T_l \geq 0$. For any fixed $\tau > 0$, let $(n^\tau, j^\tau, p^\tau, E^\tau) \in (L^\infty(I\!R \times I\!R_+))^4$ be a global entropy solution of the HD system (1)-(2) which satisfies the following uniform bound:*

$$\|n^\tau\|_{L^\infty(I\!R\times I\!R_+)} + \|\frac{j^\tau}{n^\tau}\|_{L^\infty(I\!R\times I\!R_+)} + \|\frac{p^\tau}{n^\tau}\|_{L^\infty(I\!R\times I\!R_+)} \leq C\,, \tag{14}$$

where C is a positive constant independent of τ.

Then there exist some functions $\mathcal{N}, \mathcal{P} \in L^\infty(I\!R \times I\!R_+)$, $\mathcal{J} \in L^2_{loc}(I\!R \times I\!R_+)$, $\mathcal{E} \in Lip_{loc}(I\!R \times I\!R_+)$ which are a (weak) solution of the ET equations (13) and such that, as $\tau \to 0$,

$$\begin{aligned}
&\mathcal{N}^\tau \rightharpoonup \mathcal{N} \quad \text{in} \quad L^\infty weak - *, \\
&\mathcal{J}^\tau \rightharpoonup \mathcal{J} \quad \text{weakly in} \quad L^2_{loc}, \\
&\mathcal{P}^\tau \to \mathcal{P} \quad \text{in} \quad L^p_{loc}\,,\ \forall\, 1 \leq p < \infty, \\
&\mathcal{E}^\tau \to \mathcal{E} \quad \text{uniformly in } C.
\end{aligned} \tag{15}$$

In the proof of theorem 3.1 the following results are needed.

Lemma 3.1 *For any (scaled) uniformly bounded solution $\mathcal{N}^\tau, \mathcal{J}^\tau, \mathcal{P}^\tau, \mathcal{E}^\tau$ of (12) with compact support we have*

$$\int_0^{t_0}\int \frac{\left(\mathcal{J}^\tau(x,t)\right)^2}{\mathcal{P}^\tau(x,t)}dxdt \le C_0 \tag{16}$$

for all $t_0 > 0$ (and $\tau \ll 1$), where the constant C_0 depends only on t_0.

Proof: The proof of the lemma is based on the entropy considerations of the previous section. We use the entropy pair given by (9) in order to obtain

$$\begin{aligned} -\int \eta(U^\tau(x,\frac{t_0}{\tau}))dx &= \int \mathcal{N}^\tau(x,t_0)\ln\mathcal{P}^\tau(x,t_0)dx - \gamma\int \mathcal{N}^\tau(x,t_0)\ln\mathcal{N}^\tau(x,t_0)dx \\ &\le \max\{0, \ln\|\mathcal{P}^\tau\|_{L^\infty(I\!R\times I\!R_t)}\}\int n_0(x)dx + \gamma\frac{1}{e}R_0, \end{aligned} \tag{17}$$

where R_0 is the measure of the support of the (scaled) solutions for $t \in [0, t_0)$. Since

$$\|\mathcal{P}^\tau\|_{L^\infty(I\!R\times I\!R_t)} = \|p^\tau\|_{L^\infty(I\!R\times I\!R_t)} \le \|\frac{p^\tau}{n^\tau}\|_{L^\infty(I\!R\times I\!R_t)}\|n^\tau\|_{L^\infty(I\!R\times I\!R_t)}, \tag{18}$$

the result follows easily from inequality (10). $\Box$

Let us recall now the second result we use in the proof of the main theorem, namely the "div-curl" lemma from the theory of compensated compactness [T].

Lemma 3.2 *Given two sequences $\{Z^\epsilon\}$ and $\{V^\epsilon\}$, uniformly bounded in L^2_{loc}, assume that $\{\mathrm{div}Z^\epsilon\}$ and $\{\mathrm{div}V^\epsilon\}$ belong to a bounded set of L^2_{loc}. Then $Z^\epsilon \cdot V^\epsilon \to Z\cdot V$ in $\mathcal{D}'$, where Z and V are the weak-$\star$ limits of $\{Z^\epsilon\}$ and $\{V^\epsilon\}$.*

Proof (of Theorem 3.1): By the assumed uniform bound (14) and the finite speed of propagation, we can consider initial data with compact support. Define the vectors

$$\begin{aligned} Z^\tau &= \left(\mathcal{N}^\tau, \mathcal{J}^\tau\right) \\ V^\tau &= \left(-\mathcal{P}^\tau - \tau^2\frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau}, \tau^2\mathcal{J}^\tau\right), \\ W^\tau &= \left(-\tau^2\frac{\mathcal{J}^{\tau 3}}{2\mathcal{N}^{\tau 2}} - \frac{\gamma}{\gamma-1}\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau, \tau^2\frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} + \frac{1}{\gamma-1}\mathcal{P}^\tau\right), \\ X^\tau &= \left(\tau^2\frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} + \frac{1}{\gamma-1}\mathcal{P}^\tau, \tau^2\frac{\mathcal{J}^{\tau 3}}{2\mathcal{N}^{\tau 2}} + \frac{\gamma}{\gamma-1}\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau\right). \end{aligned} \tag{19}$$

According to (14) the density $\mathcal{N}^\tau$ and the pressure $\mathcal{P}^\tau$ are bounded in $L^\infty(I\!R\times[0,t_0])$ uniformly in τ, for any fixed $t_0 > 0$. We denote their weak-$\star$ limits as $\tau \to 0$ by $\mathcal{N}$ and

$\mathcal{P}$, respectively. In the following $\Omega \subseteq I\!R \times [0, t_0]$ denotes a relatively compact open region. The entropy inequality in Lemma 3.1 gives the following uniform estimates

$$
\begin{aligned}
&\|\mathcal{J}^\tau\|^2_{L^2(\Omega)} \le \|\frac{\mathcal{J}^{\tau 2}}{\mathcal{P}^\tau}\|_{L^1(\Omega)} \|\mathcal{P}^\tau\|_{L^\infty(I\!R\times[0,t_0])}, \\
&\|\frac{\mathcal{J}^\tau \mathcal{P}^\tau}{\mathcal{N}^\tau}\|_{L^2(\Omega)} \le \|\frac{\mathcal{P}^\tau}{\mathcal{N}^\tau}\|_{L^\infty(I\!R\times[0,t_0])} \|\mathcal{J}^\tau\|_{L^2(\Omega)}.
\end{aligned}
\tag{20}
$$

Therefore, as $\tau \to 0$, we have

$$
\begin{aligned}
\mathcal{J}^\tau \quad &\rightharpoonup \mathcal{J} \quad \text{weakly in} \quad L^2(\Omega), \\
\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau \quad &\rightharpoonup \mathcal{G} \quad \text{weakly in} \quad L^2(\Omega).
\end{aligned}
\tag{21}
$$

In addition, we have that, as $\tau \to 0$,

$$
\begin{aligned}
&\|\tau^2 \mathcal{J}^\tau\|_{L^2(\Omega)} = \tau^2 \|\mathcal{J}^\tau\|_{L^2(\Omega)} \to 0, \\
&\|\tau^2 \frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau}\|_{L^2(\Omega)} \le \tau \|\frac{\tau \mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^\infty(I\!R\times[0,t_0])} \|\mathcal{J}^\tau\|_{L^2(\Omega)} \to 0, \\
&\|\tau^2 \frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau}\mathcal{P}^\tau\|_{L^2(\Omega)} \le \tau \|\frac{\tau \mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^\infty(I\!R\times[0,t_0])} \|\mathcal{P}^\tau\|_{L^\infty(I\!R\times[0,t_0])} \|\mathcal{J}^\tau\|_{L^2(\Omega)} \to 0, \\
&\|\tau^2 \frac{\mathcal{J}^{\tau 3}}{\mathcal{N}^{\tau 2}}\|_{L^2(\Omega)} \to 0.
\end{aligned}
\tag{22}
$$

The last convergence readily follows from the uniform estimates

$$
\begin{aligned}
\|\tau^2 \tfrac{\mathcal{J}^{\tau 3}}{\mathcal{N}^{\tau 2}}\|_{L^2(\Omega)} \quad &\le \|\frac{\tau \mathcal{J}^\tau}{\mathcal{N}^\tau}\|^2_{L^\infty(I\!R\times[0,t_0])} \|\mathcal{J}^\tau\|_{L^2(\Omega)}, \\
\|\tau^2 \tfrac{\mathcal{J}^{\tau 3}}{\mathcal{N}^{\tau 2}}\|_{L^1(\Omega)} \quad &\le \tau \|\frac{\tau \mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^\infty(I\!R\times[0,t_0])} \|\frac{\mathcal{J}^{\tau 2}}{\mathcal{P}^\tau}\|_{L^1(\Omega)} \|\frac{\mathcal{P}^\tau}{\mathcal{N}^\tau}\|_{L^\infty(I\!R\times[0,t_0])}.
\end{aligned}
\tag{23}
$$

Therefore, as $\tau \to 0$, we have the following convergence in the weak topology of $(L^2)^2$

$$
\begin{aligned}
Z^\tau \quad &\rightharpoonup \quad Z = (\mathcal{N}, \mathcal{J}), \\
V^\tau \quad &\rightharpoonup \quad V = (-\mathcal{P}, 0), \\
W^\tau \quad &\rightharpoonup \quad W = (-\frac{\gamma}{\gamma - 1}\mathcal{G}, \frac{1}{\gamma - 1}\mathcal{P}), \\
X^\tau \quad &\rightharpoonup \quad X = (\frac{1}{\gamma - 1}\mathcal{P}, \frac{\gamma}{\gamma - 1}\mathcal{G}).
\end{aligned}
\tag{24}
$$

Then we use of vectors $Z^\tau, V^\tau, W^\tau, X^\tau$ to rewrite the HD equations (setting $\mathrm{d}iv = (\partial_s, \partial_x)$ and $\mathrm{r}ot = (\partial_x, -\partial_s)$) and conclude that the following functions

$$
\mathrm{d}iv Z^\tau \quad = 0,
$$

$$\begin{aligned}
\mathrm{r}ot V^\tau &= -(\mathcal{N}^\tau \mathcal{E}^\tau - \mathcal{J}^\tau),\\
\mathrm{r}ot W^\tau &= -(\mathcal{J}^\tau \mathcal{E}^\tau - \tau^2 \frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} - \frac{1}{\gamma-1}(\mathcal{P}^\tau - \mathcal{N}^\tau T_l)),\\
\mathrm{d}iv X^\tau &= -\mathrm{r}ot W^\tau,
\end{aligned} \tag{25}$$

belong to a bounded set of $L^2(\Omega)$. Now we can apply three times the "div-curl" Lemma 3.2 in order to obtain, as $\tau \to 0$, the following weak (distributional) convergence

$$\mathcal{N}^\tau \mathcal{P}^\tau = -Z^\tau \cdot V^\tau \quad \rightharpoonup \quad -Z \cdot V = \mathcal{N}\mathcal{P} \tag{26}$$

$$\mathcal{J}^\tau \mathcal{P}^\tau = -Z^\tau \cdot W^\tau \quad \rightharpoonup \quad -Z \cdot W = -\frac{1}{\gamma-1}\mathcal{J}\mathcal{P} + \frac{\gamma}{\gamma-1}\mathcal{G}\mathcal{N} \tag{27}$$

$$\tau^2 \frac{(\mathcal{J}^\tau)^2}{2\mathcal{N}^\tau}\mathcal{P}^\tau - \frac{1}{\gamma-1}(\mathcal{P}^\tau)^2 = X^\tau \cdot V^\tau \quad \rightharpoonup \quad X \cdot V = -\frac{1}{\gamma-1}\mathcal{P}^2. \tag{28}$$

The Poisson equation and the weak convergence of the density $\mathcal{N}^\tau$ imply strong convergence of the electric field

$$\mathcal{E}^\tau \to \mathcal{E} \quad \text{in} \quad L^2_{loc} \quad \text{as} \quad \tau \to 0 \tag{29}$$

and the following weak L^2 convergences, as $\tau \to 0$,

$$\begin{aligned}
\mathcal{J}^\tau \mathcal{E}^\tau &\rightharpoonup \mathcal{J}\mathcal{E},\\
\mathcal{N}^\tau \mathcal{E}^\tau &\rightharpoonup \mathcal{N}\mathcal{E}.
\end{aligned} \tag{30}$$

This means that , as $\tau \to 0$, we can pass to limit in all terms in the HD equations but the nonlinear term $\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau$, which we know only to converge (weakly) to $\mathcal{G}$. On the other hand the weak convergence of the pressure term $\mathcal{P}^\tau$ and the convergence of its square $\mathcal{P}^{\tau 2}$ to $\mathcal{P}^2$ in (28) give the strong convergence

$$\mathcal{P}^\tau \to \mathcal{P} \quad \text{in} \quad L^p_{loc}, \quad 1 \le p < \infty \quad \text{as} \quad \tau \to 0. \tag{31}$$

This yields

$$\mathcal{J}^\tau \mathcal{P}^\tau \rightharpoonup \mathcal{J}\mathcal{P} \quad \text{weakly in} \quad L^2_{loc} \quad \text{as} \quad \tau \to 0 \tag{32}$$

and from (27) we obtain

$$\mathcal{J}\mathcal{P} = -\frac{1}{\gamma-1}\mathcal{J}\mathcal{P} + \frac{\gamma}{\gamma-1}\mathcal{G}\mathcal{N}. \tag{33}$$

Therefore

$$\mathcal{G} = \frac{\mathcal{J}}{\mathcal{N}}\mathcal{P}, \tag{34}$$

and we can conclude that the limits $\mathcal{N}$, $\mathcal{J}$, $\mathcal{P}$ and $\mathcal{E}$ are a weak solution of the ET equations (13) and (5). □

Remember that for the (time dependent) ET model (13) no analytic results are known. In the stationary case existence and uniqueness results for general ET models can be found in [DGJ].

4 Relaxation to the Drift Diffusion Model

This section is devoted to the relaxation of the HD model to the DD model. On the long DD time scale $s = \tau^2 t$ we use the quantities $\mathcal{N}^\tau$, $\mathcal{J}^\tau$, $\mathcal{P}^\tau$ and $\mathcal{E}^\tau$ defined in (4) for $m = 2$ in order to obtain

$$\begin{aligned}
\mathcal{N}^\tau{}_t + \mathcal{J}^\tau{}_x &= 0,\\
\tau^4 \mathcal{J}^\tau{}_t + \left(\tau^4 \frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau} + \mathcal{P}^\tau\right)_x &= \mathcal{N}^\tau \mathcal{E}^\tau - \mathcal{J}^\tau,\\
\left(\tau^6 \frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} + \frac{1}{\gamma-1}\tau^2 \mathcal{P}^\tau\right)_t &+ \left(\tau^6 \frac{\mathcal{J}^{\tau 3}}{2\mathcal{N}^{\tau 2}} + \frac{\gamma}{\gamma-1}\tau^2 \frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\mathcal{P}^\tau\right)_x\\
&= \tau^2 \mathcal{J}^\tau \mathcal{E}^\tau - \tau^4 \frac{\mathcal{J}^{\tau 2}}{2\mathcal{N}^\tau} - \frac{1}{\gamma-1}(\mathcal{P}^\tau - \mathcal{N}^\tau T_l),\\
\mathcal{E}^\tau{}_x &= \mathcal{N}^\tau - b(x)
\end{aligned} \tag{35}$$

(note that the quantities are not the same as in the previous section, since we are on a different time scale). Formally, the solutions of the above equations relax to the solutions of the DD model

$$\begin{aligned}
\mathcal{N}_s + \mathcal{J}_x &= 0,\\
\mathcal{P}_x &= \mathcal{N}\mathcal{E} - \mathcal{J},\\
\mathcal{P} &= \mathcal{N} T_l,\\
\mathcal{E}_x &= \mathcal{N} - b(x).
\end{aligned} \tag{36}$$

The equations (36) can be written as in (6) by substituting the current and the pressure given in the second and third equation of (36). Let us state our result in this case.

Theorem 4.1 *Assume $b \in L^1(\mathbb{R})$, $T_l \in L^\infty(\mathbb{R})$, $b \geq 0$, $T_l \geq 0$. Let $(n^\tau, j^\tau, p^\tau, E^\tau) \in (L^\infty(\mathbb{R} \times \mathbb{R}_+))^4$ be a global weak solution of the HD system (1)-(2) which satisfies the following uniform bound:*

$$\|n^\tau\|_{L^p(\Omega)} + \|\frac{j^\tau}{\tau^2 n^\tau}\|_{L^s(\Omega)} + \|p^\tau\|_{L^q(\Omega)} \leq C \tag{37}$$

where $p > 1$, $s \geq \frac{3p}{p-1}$, $q \geq \frac{3p}{2p+1}$, C is a positive constant independent of τ and $\Omega \in (\mathbb{R} \times [0, t_0])$ is a relatively compact open region.

Then the weak-$*$ limits $\mathcal{N}$, $\mathcal{J}$, $\mathcal{P}$ of the sequences $\{\mathcal{N}^\tau\}$, $\{\mathcal{J}^\tau\}$, $\{\mathcal{P}^\tau\}$, and the strong limit $\mathcal{E}$ of $\{\mathcal{E}^\tau\}$ satisfy the DD equations (36) in $\mathcal{D}'(\mathbb{R} \times [0, t_0])$ for every $t_0 > 0$.

Proof: The scaled current is uniformly bounded from the assumptions, since

$$\|\mathcal{J}^\tau\|_{L^{\frac{ps}{p+s}}(\Omega)} \leq \|\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^s(\Omega)} \|\mathcal{N}^\tau\|_{L^p(\Omega)}. \tag{38}$$

The uniform bounds of the electric field follows from the Poisson Equation (and the uniform bound on $\mathcal{N}^\tau$). Thus, as $\tau \to 0$, there holds

$$\begin{array}{rclcl} \mathcal{N}^\tau & \rightharpoonup & \mathcal{N} & \text{in} & L^p_{loc}, \\ \mathcal{J}^\tau & \rightharpoonup & \mathcal{J} & \text{in} & L^{\frac{ps}{p+s}}_{loc}, \\ \mathcal{P}^\tau & \rightharpoonup & \mathcal{P} & \text{in} & L^q_{loc}, \\ \mathcal{E}^\tau & \to & \mathcal{E} & \text{in} & L^2_{loc}. \end{array}$$

Also, using $\frac{ps}{2p+s} \geq 1$, $\frac{ps}{3p+s} \geq 1$, $\frac{qs}{q+s} \geq 1$ we obtain

$$\begin{aligned} \|\tau^4 \frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^\tau}\|_{L^{\frac{ps}{2p+s}}(\Omega)} &\leq \tau^4 \|\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^s(\Omega)} \|\mathcal{J}^\tau\|_{L^{\frac{ps}{p+s}}(\Omega)}, \\ \|\tau^6 \frac{\mathcal{J}^{\tau 3}}{\mathcal{N}^{\tau 2}}\|_{L^{\frac{ps}{3p+s}}(\Omega)} &\leq \tau^6 \|\frac{\mathcal{J}^{\tau 2}}{\mathcal{N}^{\tau 2}}\|_{L^{\frac{s}{2}}(\Omega)} \|\mathcal{J}^\tau\|_{L^{\frac{ps}{p+s}}(\Omega)}, \\ \|\tau^2 \frac{\mathcal{J}^\tau}{\mathcal{N}^\tau} \mathcal{P}^\tau\|_{L^{\frac{qs}{q+s}}(\Omega)} &\leq \tau^2 \|\frac{\mathcal{J}^\tau}{\mathcal{N}^\tau}\|_{L^s(\Omega)} \|\mathcal{P}^\tau\|_{L^q(\Omega)}. \end{aligned} \tag{39}$$

Thus, using the strong convergence of the electric field, we can easily pass to the limit in the scaled HD equations (35) . □

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