Convergence Analysis for Operator Splitting Methods to Conservation Laws with Stiff Source Terms

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Abstract

We analyze the order of convergence for operator splitting methods applied to conservation laws with stiff source terms. We suppose that the source term $q(u)$ is dissipative. It is proved that the L^1 error introduced by the time-splitting can be bounded by $O(\Delta t\|q(u_0)\|_{L^1(\mathbf{R})})$, which is an improvement of the $O(Q\Delta t)$ upper bound, where Δt is the splitting time step, Q is the Lipschitz constant of q or $Q = \max_u |q'(u)|$ in case q is smooth. We also propose a non-uniform temporal mesh which can eliminate the effect of the initial layer introduced by the stiff source term.

1 Introduction

The purpose of this paper is to study operator splitting procedures for multidimensional conservation laws with stiff source terms. Without loss of generality, we consider Cauchy problems for the two-dimensional conservation law:

$$u_t + f_1(u)_x + f_2(u)_y = q(u), \qquad (x,t) \in \mathbf{R} \times (0,T], \tag{1.1}$$

$$u(x,y,0) = u_0(x,y), \tag{1.2}$$

where f_1, f_2 and q are smooth functions of u. The operator splitting procedures are defined from local solutions of the corresponding homogeneous equation

$$u_t + f_1(u)_x + f_2(u)_y = 0 \tag{1.3}$$

and the ordinary differential equation

$$u_t = q(u). \tag{1.4}$$

Operator splitting methods have been used frequently as part of a computational strategy for solving (1.1)-(1.2). One of the advantages of using the splitting methods is that we can use some successful numerical methods, e.g. TVD, ENO schemes [8, 9, 27], to solve the homogeneous equation (1.3). In particular, LeVeque's CLAWPACK [17], a software package for solving multi-dimensional conservation

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laws, make time-splitting more attractive. If $q \equiv 0$, first-order dimensional splitting method was introduced by Godunov [7], which was modified by Strang [29]. Numerical and theoretical investigations of their methods were carried out by many authors (see, e.g. [4, 10, 28, 32]). If q is a nonstiff source term, splitting methods for (1.1)-(1.2) have been studied recently by Tang and Teng [30] and Langseth et al. [12].

In this research, we wish to analyze splitting methods to (1.1) with stiff source terms. There have been extensive research for the stiff problems in recent years (see, e.g. [1, 2, 11, 18] and references therein). For stiff problems, most of the numerical methods are semi-implicit, which are related to operator splitting (explicit method for the homogeneous conservation laws and implicit method for the stiff ODEs). Theoretical analysis for the semi-implicit methods have been given by Chalabi [1], Schroll et al. [25], and Schroll and Winther [26]. In this work we will concentrate on time-splitting and assume that the solution of (1.3) are found exactly. The orders of convergence obtained in this paper can be easily extended to operator splitting methods with (i) time splitting plus dimensional splitting for (1.1), or (ii) fully discretization (say, monotone schemes) for the homogeneous problem (1.3) and backward Euler in time for (1.4). These can be done following Section 6.2 of [30] and Section 3.2 of [12], respectively. Since the extension is quite straightforward, we will omit the detail.

For ease of exposition we shall consider the one-dimensional problem

$$u_t + f(u)_x = q(u), \qquad (x,t) \in \mathbf{R} \times (0,T], \tag{1.5}$$

$$u(x,0) = u_0(x), \tag{1.6}$$

in the subsequent analysis. The extension to the multi-dimensional equation will be discussed in Section 6. According to Pember [22, 23], the conservation law (1.1) is stiff if the time scale introduced by the source term q is small compared with the characteristic speed f' and some appropriate length scale. In particular,

$$Q := \max_{|u| \le \|u_0\|_\infty} |q'(u)| \tag{1.7}$$

may be related to a very small parameter $\delta > 0$. Previous error analysis, e.g. [12], shows that the L^1 error between the entropy solution and the operator splitting solution can be bounded by

$$C\Delta t \left(1 + \max_{|u| \le \|u_0\|_\infty} |q'(u)|\right), \tag{1.8}$$

provided that $u_0 \in BV \cap L^\infty \cap L^1$, where C is a constant independent of Δt and q. That is, the error bound is of the order $O(Q\Delta t)$. A typical relationship between Q and δ is $Q = O(\delta^{-1})$. If this is the case, the splitting error obtained by [12] is $O(\delta^{-1}\Delta t)$. This error bound is unsatisfactory since convergence requires that $\Delta t \sim \delta \ll 1$.

It is observed that a realistic assumption on the source term is $q'(u) \le 0$ for all $u \in \mathbf{R}$. It is indeed the case for many practical problems, e.g. in the models of combustion [5, 21], water waves in presence of the friction force in the bottom [35]. This assumption is also used in many theoretical papers, for example Chalabi [1], Chen et al. [2], Liu [19], and Schroll and Winther [26]. In the sense of Chen et al. [2], $q' \le 0$ means the dissipativity of the source term. Furthermore, as usual, we assume that $u = 0$ is an equilibrium. Hence, throughout this paper, we assume that

$$q(0) = 0, \qquad q'(u) \le 0. \tag{1.9}$$

With the above assumptions, we will show in this paper that the error for the operator splitting method based on backward Euler for (1.4) can be bounded by

$$C\Delta t\left(1+\int_{\mathbf{R}}|q(u_0(x))|dx\right), \tag{1.10}$$

provided that $u_0 \in BV \cap L^\infty \cap L^1$, where C is a constant independent of Δt and q. It follows from the assumption $q(0)=0$ that

$$\begin{aligned} & \Delta t\left(1+\int_{\mathbf{R}}|q(u_0(x))|dx\right) \\ \leq\ & \Delta t\left(1+\max_{|u|\leq\|u_0\|_\infty}|q'(u)|\bullet\|u_0\|_{L^1(\mathbf{R})}\right)\leq C\Delta t\left(1+\max_{|u|\leq\|u_0\|_\infty}|q'(u)|\right). \end{aligned} \tag{1.11}$$

In other words, (1.10) is an improvement of the error bound (1.8). Moreover, it can be shown that even if $Q=O(\delta^{-1})$, the error bound (1.10) may be δ-free for some classes of source terms q, or initial data u_0. This can be regarded as a substantial improvement of the $O(\delta^{-1}\Delta t)$ bound given by (1.8). We will give some examples of such q and u_0 in Sections 3 and 5.

We will also discuss the initial layer problem which is quite common for stiff conservation laws. We propose to divide $[0,T]$ into two subintervals and then use two different meshes in the two subintervals. This will eliminate the effect of the initial layer and leads to uniform first order convergence rate.

The error estimates for dimension splitting methods and time splitting methods obtained before have all been based on the work of Kuznetsov [15] who was the first to provide error bounds for numerical schemes for (1.3). Inspired by the work of Kruzkov [14], he established a key approximation inequality. His approach has been extended to obtain error estimates for several numerical schemes to scalar conservation laws, see e.g. [3, 20, 24]. In this paper, we will not use the theory of Kuznetsov. Our error estimates will be established by introducing parabolic regularizations. This approach can also be applied to deal with nonstiff problem so that the results in [12, 30] can be recovered.

The paper is organized as follows: In Section 2 we introduce some notation and prove some stability properties for the solutions to (1.5)-(1.6) with nonpositive q'. These results play important roles in the error analysis. Time-splitting methods based on backward Euler for time is studied in Section 3. We prove that the L^1 error introduced by the splitting is bounded by (1.10). Precise value for the constant C in (1.10) is also obtained. In Section 4, we give an analysis for time-splitting methods based on forward Euler in time. It is shown that if $Q\Delta t\leq 1$, then the error introduced by the splitting is bounded by (1.10). Unlike the backward Euler method, there is a restriction for the time step for the forward Euler approach. A generic model investigated by Schroll and Winther [26] is reconsidered in Section 5. For this model, the splitting error is small only when u_0 is close to equilibrium. This is due to the presence of the initial layer. We propose in Section 5 a non-uniform mesh which can resolve the initial layer. In Section 6 we discuss the extension of our one-dimensional analysis to several space dimensions.

2 Preliminaries

We begin by introducing some definitions and summarizing some properties of solutions to the nonhomogeneous scalar conservation law (1.5)-(1.6). A bounded measurable function, u, is a weak solution of (1.5)-(1.6) if for all $\phi \in C_0^1(\mathbf{R}\times[0,T))$,

$$\int_{\mathbf{R}}\int_0^T \Big(u\phi_t + f(u)\phi_x\Big)dxdt + \int_{\mathbf{R}} u_0(x)\phi(x,0)dx = -\int_{\mathbf{R}}\int_0^T q\phi dxdt. \tag{2.1}$$

The bounded-variation seminorm of a function $u \in \mathrm{BV}$ is defined by

$$|u(\cdot,t)|_{\mathrm{BV}} = \sup_{0<|h|<\infty}\int_{\mathbf{R}}\frac{|u(x+h,t)-u(x,t)|}{|h|}dx.$$

In this paper, we also deal with composite functions which belong to BV space. A composite function $p(u(x))$ belongs to BV if

$$|p(u)|_{BV} = \sup_{0<|h|<\infty}\int_{\mathbf{R}}\frac{|p(u(x+h,t))-p(u(x,t))|}{|h|}dx < \infty.$$

The norm on $L^p(\mathbf{R})$ is denoted by $\|\bullet\|_p$. The L^∞ and L^1 norms for $p(u(x))$ is defined by

$$\|p(u)\|_\infty = \sup_{x\in\mathbf{R}}|p(u(x))|, \qquad \|p(u)\|_1 = \int_{\mathbf{R}}|p(u(x))|dx.$$

The following property of the source term is a consequence of the monotonicity of q, i.e. $q' \le 0$, and is fundamental in the analysis below. For all $u, v \in \mathbf{R}$, we have

$$\mathrm{sgn}(u-v)\Big(q(u)-q(v)\Big) = -|q(u)-q(v)|. \tag{2.2}$$

Weak solutions are not uniquely determined by their initial data and additional principles, entropy conditions, are needed to select the appropriate physical solution. A weak solution, u, of (1.5)-(1.6) is an entropy solution if for all $\phi \in C_0^1(\mathbf{R}\times(0,T))$ with $\phi(x,t)\ge 0$ and any $c \in \mathbf{R}$,

$$\int_{\mathbf{R}}\int_0^T \Big(|u-c|\phi_t + \mathrm{sgn}(u-c)(f(u)-f(c))\phi_x + \mathrm{sgn}(u-c)q\phi\Big)dxdt \ge 0. \tag{2.3}$$

There are several different (and equivalent) forms of entropy conditions. It can be shown that entropy solution of (1.5)-(1.6) can be constructed, as the limit of solutions u^ϵ of the parabolic equation

$$u_t^\epsilon + f(u^\epsilon)_x = q(u^\epsilon) + \epsilon u_{xx}^\epsilon, \tag{2.4}$$

$$u^\epsilon(x,0) = u_0(x), \tag{2.5}$$

where $\epsilon > 0$. It follows easily from the maximum principle that (2.4)-(2.5) has at most one solution. It is also true that a solution u^ϵ exists for all t, and that as $\epsilon \to 0+$ these solutions converge in the L^1 sense to u, the entropy solution of (1.5)-(1.6).

The error estimates for dimension splitting [4, 32] and time splitting methods [12, 30] obtained before have been based on the entropy condition (2.3). Our estimate in this paper will be based on the parabolic regularization (2.4)-(2.5). Moreover, we need the following propositions.

Proposition 2.1 *If $u_0 \in L^\infty(\mathbf{R})\cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$ and $q \in C^1(\mathbf{R})$ such that $q(0)=0, q'\le 0$, then the nonhomogeneous conservation law (1.5)-(1.6) possesses a unique entropy solution $u(x,t) = S(t)u_0$ satisfying*

$$\|S(t)u_0\|_\infty \le \|u_0\|_\infty. \tag{2.6}$$

Moreover, for any $v_0 \in L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, the following inequalities hold:

$$\|S(t)u_0 - S(t)v_0\|_1 \le \|u_0 - v_0\|_1, \tag{2.7}$$

$$\int_0^T \left\| q\Big(S(\tau)u_0\Big) - q\Big(S(\tau)v_0\Big) \right\|_1 d\tau \le \|u_0 - v_0\|_1. \tag{2.8}$$

Proof. The estimates (2.6) and (2.7) have been obtained by Kruzkov [14]. Next we follow Lax [16] and Tang and Teng [31] to prove (2.8). Consider the following parabolic regularizations of (1.5)-(1.6),

$$u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx} + q(u^\epsilon), \qquad u^\epsilon(x,0) = u_0(x),$$
$$v^\epsilon_t + f(v^\epsilon)_x = \epsilon v^\epsilon_{xx} + q(v^\epsilon), \qquad v^\epsilon(x,0) = v_0(x).$$

Let $w := u^\epsilon - v^\epsilon$. It follows from the above two equations that

$$w_t + (f'(\zeta)w)_x = \epsilon w_{xx} + q(u^\epsilon) - q(v^\epsilon), \tag{2.9}$$

where ζ is some intermediate value between u^ϵ and v^ϵ. Assume that w change signs at $p_1(t)$ and $p_2(t)$ and w does not change sign for $x \in (p_1, p_2)$. Since u^ϵ and v^ϵ are continuous, we have $w = 0$ at $x = p_1$ and $x = p_2$. Multiplying (2.9) by $\mathrm{sgn}(w)$ and integrating the resulting equation over (p_1, p_2) give

$$\begin{aligned} \int_{p_1}^{p_2} \mathrm{sgn}(w) w_t dx \;&=\; \epsilon\Big(w_x(p_2 - 0, t) - w_x(p_1 + 0, t)\Big)\,\mathrm{sgn}(w) \\ &\quad + \int_{p_1}^{p_2} \mathrm{sgn}(u^\epsilon - v^\epsilon)(q(u^\epsilon) - q(v^\epsilon))dx. \end{aligned} \tag{2.10}$$

Since $w(p_1,t) = w(p_2,t) = 0$, we have

$$\frac{d}{dt}\int_{p_1}^{p_2} |w| dx = \int_{p_1}^{p_2} \mathrm{sgn}(w) w_t dx.$$

Since w changes sign at p_1 and p_2, it is easy to check that

$$w_x(p_2 - 0, t)\,\mathrm{sgn}(w) \le 0, \qquad w_x(p_1 + 0, t)\,\mathrm{sgn}(w) \ge 0.$$

Moreover, it follows from (2.2) that

$$\int_{p_1}^{p_2} \mathrm{sgn}(u^\epsilon - v^\epsilon)(q(u^\epsilon) - q(v^\epsilon))dx = -\int_{p_1}^{p_2} |q(u^\epsilon) - q(v^\epsilon)| dx.$$

The inequality (2.10), together with the above three results, yield

$$\frac{d}{dt}\int_{p_1}^{p_2} |u^\epsilon - v^\epsilon| dx \le -\int_{p_1}^{p_2} |q(u^\epsilon) - q(v^\epsilon)| dx. \tag{2.11}$$

Similar inequality holds for any interval in which w does not change signs. Following the same arguments used in Lax [16], we conclude that

$$\frac{d}{dt}\|u^\epsilon - v^\epsilon\|_1 \le -\|q(u^\epsilon) - q(v^\epsilon)\|_1. \tag{2.12}$$

Integrating (2.12) with respect to t over $[0,T]$ and letting $\epsilon \to 0$ yield (2.8). □

Remark. There is an alternative way to prove (2.8). It is to use splitting method with forward Euler in time, which has been studied in [12, 30]. This technique will be used to prove (2.18) below. □

Proposition 2.2 *If $u_0 \in \mathrm{BV} \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, $q \in C^1(\mathbf{R})$ such that $q(0) = 0, q' \leq 0$, then the problem (1.5)-(1.6) possesses a unique entropy solution $u(x,t) = S(t)u_0$ satisfying*

$$|S(t)u_0|_{\mathrm{BV}} \leq |u_0|_{\mathrm{BV}}, \tag{2.13}$$

$$\|S(t)u_0\|_1 \leq \|u_0\|_1, \tag{2.14}$$

$$\int_0^T |q(u)|_{BV} d\tau \leq |u_0|_{BV}, \tag{2.15}$$

$$\int_0^T \|q(u)\|_1 d\tau \leq \|u_0\|_1. \tag{2.16}$$

Moreover, for any $\tau_2 > \tau_1 \geq 0$, we have

$$\|u(\cdot,\tau_2) - u(\cdot,\tau_1)\|_1 \leq \Big(C_0|u_0|_{BV} + Q\|u_0\|_1\Big)(\tau_2 - \tau_1), \tag{2.17}$$

$$\|u(\cdot,\tau_2) - u(\cdot,\tau_1)\|_1 \leq C_0|u_0|_{BV}(\tau_2 - \tau_1) + \int_{\tau_1}^{\tau_2} \|q(u(\cdot,\tau))\|_1 d\tau, \tag{2.18}$$

where the constants Q is defined by (1.7) and C_0 is defined by

$$C_0 = \max_{|u| \leq \|u_0\|_\infty} |f'(u)|. \tag{2.19}$$

Proof. Setting $v_0(x) = u_0(x+h)$ in (2.7) and (2.8) will lead to (2.13) and (2.15), respectively. Setting $v_0 \equiv 0$ in (2.7) and (2.8) will lead to (2.14) and (2.16), respectively. It remains to check (2.17) and (2.18). For any given $\tau_2 > \tau_1 \geq 0$, let $k = (\tau_2 - \tau_1)/M$ be the mesh size in the time direction, where M is fixed integer such that

$$kQ \leq 1. \tag{2.20}$$

Let $t_m = \tau_1 + mk$ and $v(\cdot,\tau_1) = u(\cdot,\tau_1)$. Given $v(x,t_m)$, we compute $v(x,t_{m+1})$ by the following forward Euler splitting method. We first compute the intermediate value $\eta(\cdot,t_{m+1})$ by the following equation:

$$\eta(x,t_{m+1}) = v(x,t_m) + kq(v(x,t_m)). \tag{2.21}$$

Then compute $v(x,t_{m+1})$ by solving the following homogeneous hyperbolic equation with initial data $\eta(\cdot,t_{m+1})$:

$$v_t + f(v)_x = 0, \quad t_m < t \leq t_{m+1}; \quad v(x,t_m) = \eta(x,t_{m+1}). \tag{2.22}$$

Starting from $m = 0$, the above procedure will give $\{v(x,t_m)\}_{m=0}^M$. It is proved in [12, 30] that as $k \to 0$ (i.e. $M \to \infty$), we have the following result:

$$\sup_{0 \leq m \leq M} \|v(\cdot,t_m) - u(\cdot,t_m)\|_1 \to 0, \qquad \text{as} \quad k \to 0. \tag{2.23}$$

It follows from (2.21) that

$$\|\eta(\cdot,t_{m+1}) - v(\cdot,t_m)\|_1 \leq k\|q(v(\cdot,t_m))\|_1. \tag{2.24}$$

Using the standard theory of homogeneous conservation laws for (2.22) gives

$$\|v(\cdot,t_{m+1}) - \eta(\cdot,t_{m+1})\|_1 \leq C_0 k|v(\cdot,t_m)|_{BV}. \tag{2.25}$$

Since $q' \le 0$, it follows from (2.21) and (2.22) that

$$|v(\cdot,t_{m+1})|_{BV} \le |\eta(\cdot,t_{m+1})|_{BV} \le |v(\cdot,t_m)|_{BV}, \quad \text{for} \quad 0 \le m \le M-1,$$

provided that (2.20) holds. The above result leads to

$$|v(\cdot,t_m)|_{BV} \le |v(\cdot,t_0)|_{BV} = |v(\cdot,\tau_1)|_{BV} = |u(\cdot,\tau_1)|_{BV} \le |u_0|_{BV},$$

for $0 \le m \le M$. This, together with (2.24) and (2.25), yield

$$\|v(\cdot,t_{m+1}) - v(\cdot,t_m)\|_1 \le C_0 k |u_0|_{BV} + k\|q(v(\cdot,t_m))\|_1,$$

which implies that

$$\|v(\cdot,\tau_2) - v(\cdot,\tau_1)\|_1 \le C_0(\tau_2-\tau_1)|u_0|_{BV} + \sum_{m=0}^{M-1} k\|q(v(\cdot,t_m))\|_1. \tag{2.26}$$

The above result, together with (2.14) and (2.23), lead to (2.17). Using (2.17) and (2.23), it is readily shown that

$$\sum_{m=0}^{M-1} k\|q(v(\cdot,t_m))\|_1 \to \int_{\tau_1}^{\tau_2} \|q(u(\cdot,\tau))\|_1 d\tau, \qquad \text{as} \quad k \to 0.$$

Letting $k \to 0$ in (2.26) yields (2.18). $\square$

Proposition 2.3 *Let $u_0 \in \mathrm{BV} \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, $f \in C^1(\mathbf{R})$, $q \in C^1(\mathbf{R})$ such that $q(0)=0, q' \le 0$. Consider the parabolic regularization of (1.5)-(1.6)*

$$u^\epsilon_t + f(u^\epsilon)_x = \epsilon u^\epsilon_{xx} + q(u^\epsilon), \qquad u^\epsilon(x,0) = u_0(x). \tag{2.27}$$

Then the following inequalities hold:

$$\|u^\epsilon(\cdot,t)\|_\infty \le \|u_0\|_\infty, \tag{2.28}$$

$$\|u^\epsilon_x(\cdot,t)\|_1 \le |u_0|_{BV}, \tag{2.29}$$

$$\int_0^T |q(u^\epsilon(\cdot,\tau))|_{BV} d\tau \le |u_0|_{BV}, \tag{2.30}$$

$$\int_0^T \|q(u^\epsilon(\cdot,\tau))\|_\infty d\tau \le 2\|u_0\|_\infty. \tag{2.31}$$

Proof. The proof for (2.28) and (2.29) is simple and will not be given here. The inequality (2.30) follows directly from (2.12). It remains to prove (2.31). By a standard regularization of Heaviside function (which we omit), (2.27) implies for $u^{\epsilon,+} := H(u^\epsilon)u^\epsilon$,

$$u^{\epsilon,+}_t + f'(u^\epsilon)u^{\epsilon,+}_x = q(u^{\epsilon,+}) + \epsilon u^{\epsilon,+}_{xx}.$$

It follows from the maximum principle that

$$\frac{d}{dt}\left[\sup_x u^{\epsilon,+}(x,t)\right] \le q\left(\sup_x u^{\epsilon,+}(x,t)\right),$$

which leads to

$$-\int_0^T q\left(\sup_x u^{\epsilon,+}(x,t)\right)dt \le \sup_x u^{\epsilon,+}(x,0) \le \|u_0\|_\infty. \tag{2.32}$$

Similarly, we can show that

$$\int_0^T q\left(\inf_x u^{\epsilon,-}(x,t)\right)dt \le \|u_0\|_\infty, \tag{2.33}$$

where $u^{\epsilon,-}(x,t) := (1-H(u^\epsilon))u^\epsilon$. Since $q(0)=0, q' \le 0$, it is observed that

$$\begin{aligned}
& \|q(u^\epsilon(\cdot,t))\|_\infty \\
= \ & \max\left[q\left(\inf_x u^{\epsilon,-}(x,t)\right), -q\left(\sup_x u^{\epsilon,+}(x,t)\right)\right] \\
\le \ & q\left(\inf_x u^{\epsilon,-}(x,t)\right) - q\left(\sup_x u^{\epsilon,+}(x,t)\right).
\end{aligned}$$

This, together with (2.32) and (2.33), yield (2.31). □

Proposition 2.4 *If u^ϵ and v^ϵ are continuously differentiable functions satisfying the following non-homogeneous parabolic equations:*

$$u_t^\epsilon + f(u^\epsilon)_x = \epsilon u_{xx}^\epsilon + p_1(x,t), \qquad \tau_1 < t < \tau_2, \tag{2.34}$$

$$v_t^\epsilon + f(v^\epsilon)_x = \epsilon v_{xx}^\epsilon + p_2(x,t), \qquad \tau_1 < t < \tau_2, \tag{2.35}$$

then the following inequality holds:

$$\|u^\epsilon(\cdot,\tau_2) - v^\epsilon(\cdot,\tau_2)\|_1 \le \|u^\epsilon(\cdot,\tau_1) - v^\epsilon(\cdot,\tau_1)\|_1 + \int_{\tau_1}^{\tau_2} \|p_1(\cdot,\tau) - p_2(\cdot,\tau)\|_1 d\tau. \tag{2.36}$$

The above result can be found in [31, 33]. Its proof can be found in [31].

3 Operator Splitting

We will consider an operator splitting procedure for the initial value problem (1.5)-(1.6), where we alternate between solving an homogeneous equation and an ordinary differential equation. More precisely, let S_1 be the solution operator of the homogeneous problem, i.e. $v(\cdot,t) = S_1(t)v_0$ is the entropy solution of the initial value problem

$$v_t + f(v)_x = 0, \qquad v(x,0) = v_0(x). \tag{3.1}$$

Let S_2 be the *backward* Euler operator for the ordinary differential problem, i.e. $v(\cdot,t) = S_2(t)v_0$ is given by

$$v(\cdot,t) = v_0 + \Delta t q(v(\cdot,t)). \tag{3.2}$$

The operator splitting solution $\{v(x,t_n)\}_{n=0}^N$ is defined by

$$v(\cdot,t_n) = [S_2(\Delta t)S_1(\Delta t)]^n v_0, \tag{3.3}$$

where $t_n = n\Delta t$ and $t_N = T$. Note that this approximate solution is only defined at discrete t-values. We require in this section to solve (3.1) and (3.2) exactly.

For $t \in (t_n, t_{n+1})$, we consider the following procedure: First solve

$$\eta_t + f(\eta)_x = 0, \qquad t \in (t_n, t_{n+1}]; \qquad \eta(x, t_n) = v(x, t_n), \tag{3.4}$$

where $v(x, t_n)$ is defined by (3.3). Then compute $v(x, t_{n+1})$ by the following equation

$$v(x, t_{n+1}) = \eta(x, t_{n+1}) + \Delta t q(v(x, t_{n+1})). \tag{3.5}$$

The scheme (3.4)-(3.5) defines

$$\eta(\cdot, t_{n+1}) = S_1(\Delta t)v(\cdot, t_n), \qquad v(\cdot, t_{n+1}) = S_2(\Delta t)\eta(\cdot, t_{n+1}).$$

The main purpose of this section is to derive an error bound for the splitting scheme (3.3). To this end, we require the following two lemmas.

Lemma 3.1 *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^2(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \leq 0$. Let u be the entropy solution of (1.5)-(1.6), η and v be the solutions of (3.4) and (3.5), respectively. The following estimate holds*

$$\begin{aligned} &\sum_{n=0}^{N-1} \left\| u(\cdot, t_{n+1}) - \int_{t_n}^{t_{n+1}} q(u)d\tau - \eta(\cdot, t_{n+1}) \right\|_1 \\ \leq \; &\sum_{n=0}^{N-1} \|u(\cdot, t_n) - v(\cdot, t_n)\|_1 + \Delta t |u_0|_{BV} \Big(C_1 + 4C_2 \|u_0\|_\infty \Big), \end{aligned} \tag{3.6}$$

where the constants C_1 and C_2 are defined by

$$C_j = \max_{|u| \leq 3\|u_0\|_\infty} |f^{(j)}(u)|. \tag{3.7}$$

Proof. Consider the following equations:

$$u^\epsilon_t + f(u^\epsilon)_x = q(u^\epsilon) + \epsilon u^\epsilon_{xx}, \quad t > 0; \qquad u^\epsilon(x, 0) = u_0(x), \tag{3.8}$$

$$\eta^\epsilon_t + f(\eta^\epsilon)_x = \epsilon \eta^\epsilon_{xx}, \qquad t_n < t \leq t_{n+1}; \qquad \eta^\epsilon(x, t_n) = v(x, t_n). \tag{3.9}$$

As $\epsilon \to 0$, the limits for u^ϵ and η^ϵ are u and η, respectively. Let $\omega_\lambda(x) = \frac{1}{\lambda}\omega(x/\lambda)$ be a standard positive C_0^∞-mollifier with unit mass. Let

$$\phi^{\lambda,\epsilon}(x, t) = q(u^\epsilon(\cdot, t) * \omega_\lambda = \int \omega_\lambda(x - y) q(u^\epsilon(y, t))dy.$$

It is easy to verify that, for any fixed time $t \in [t_n, t_{n+1}]$,

$$\|\phi^{\lambda,\epsilon}(\cdot, t)\|_\infty \leq \|q(u^\epsilon(\cdot, t))\|_\infty, \qquad \|\phi^{\lambda,\epsilon}_x(\cdot, t)\|_1 \leq |q(u^\epsilon(\cdot, t))|_{BV}, \tag{3.10}$$

Using (2.28) we obtain

$$\|\phi^{\lambda,\epsilon}_{xx}(\cdot, t)\|_1 \leq \tilde{c}(\lambda), \tag{3.11}$$

where $\tilde{c}(\lambda)$ is a constant independent of ϵ. Rewrite (3.8) as

$$\begin{aligned} &u^{\lambda,\epsilon}_t + f(u^{\lambda,\epsilon})_x \\ = \; &\epsilon u^{\lambda,\epsilon}_{xx} + f(u^{\lambda,\epsilon})_x - f(u^\epsilon)_x + q(u^\epsilon) - \phi^{\lambda,\epsilon}(x, t) + \epsilon \int_{t_n}^{t} \phi^{\lambda,\epsilon}_{xx}(x, \tau)d\tau, \end{aligned} \tag{3.12}$$

where the function $u^{\lambda,\epsilon}$ is defined by

$$u^{\lambda,\epsilon}(x,t) = u^{\epsilon}(x,t) - \int_{t_n}^{t} \phi^{\lambda,\epsilon}(x,\tau)d\tau. \tag{3.13}$$

Using Proposition 2.4 for (3.9) and (3.12) and noting that $u^{\lambda,\epsilon}(\cdot,t_n) = u^{\epsilon}(\cdot,t_n)$ and $\eta^{\epsilon}(\cdot,t_n) = v(\cdot,t_n)$, we obtain

$$\begin{aligned} & \|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^{\epsilon}(\cdot,t_{n+1})\|_1 \\ \leq\ & \|u^{\epsilon}(\cdot,t_n) - v(\cdot,t_n)\|_1 + \int_{t_n}^{t_{n+1}} \|f(u^{\lambda,\epsilon})_x - f(u^{\epsilon})_x\|_1 dt \\ & + \int_{t_n}^{t_{n+1}} \|q(u^{\epsilon}) - \phi^{\lambda,\epsilon}\|_1 dt + \epsilon \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \|\phi^{\lambda,\epsilon}_{xx}(\cdot,\tau)\|_1 d\tau dt. \end{aligned} \tag{3.14}$$

We first estimate the third term in the above inequality. Observe that

$$\begin{aligned} \Big(f(u^{\lambda,\epsilon}) - f(u^{\epsilon})\Big)_x &= -\int_0^1 f'(\theta u^{\lambda,\epsilon} + (1-\theta)u^{\epsilon})d\theta \int_{t_n}^{t} \phi^{\lambda,\epsilon}_x d\tau \\ & - \int_0^1 f''\Big(\theta u^{\lambda,\epsilon} + (1-\theta)u^{\epsilon}\Big)\Big(\theta u^{\lambda,\epsilon}_x + (1-\theta)u^{\epsilon}_x\Big)d\theta \int_{t_n}^{t} \phi^{\lambda,\epsilon} d\tau. \end{aligned} \tag{3.15}$$

Since

$$\theta u^{\lambda,\epsilon} + (1-\theta)u^{\epsilon} = u^{\epsilon} - \theta \int_{t_n}^{t} \phi^{\lambda,\epsilon} d\tau, \qquad t \in [t_n, t_{n+1}],$$

we obtain from Proposition 2.3 and (3.10) that, for $t \in [t_n, t_{n+1}]$ and $\theta \in (0,1)$,

$$\|\theta u^{\lambda,\epsilon}(\cdot,t) + (1-\theta)u^{\epsilon}(\cdot,t)\|_\infty \leq \|u_0\|_\infty + \int_{t_n}^{t} \|q(u^{\epsilon}(\cdot,t))\|_\infty d\tau \leq 3\|u_0\|_\infty,$$

$$\|\theta u^{\lambda,\epsilon}_x(\cdot,t) + (1-\theta)u^{\epsilon}_x(\cdot,t)\|_1 \leq |u_0|_{BV} + \int_{t_n}^{t} |q(u^{\epsilon}(\cdot,t))|_{BV} d\tau \leq 2|u_0|_{BV}.$$

The above results, together with (3.10), yield

$$\begin{aligned} & \|f(u^{\lambda,\epsilon})_x - f(u^{\epsilon})_x\|_1 \\ \leq\ & C_1 \int_{t_n}^{t} |q(u^{\epsilon}(\cdot,\tau))|_{BV} d\tau + 2C_2|u_0|_{BV} \int_{t_n}^{t} \|q(u^{\epsilon}(\cdot,\tau))\|_\infty d\tau, \end{aligned} \tag{3.16}$$

for $t_n < t < t_{n+1}$, where the constants C_1 and C_2 are defined by (3.7). Combining (3.14) and (3.16) yields

$$\begin{aligned} & \sum_{n=0}^{N-1} \|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^{\epsilon}(\cdot,t_{n+1})\|_1 \\ \leq\ & \sum_{n=0}^{N-1} \|u^{\epsilon}(\cdot,t_n) - v(\cdot,t_n)\|_1 + C_1\Delta t \int_0^{t_N} |q(u^{\epsilon}(\cdot,\tau))|_{BV} d\tau \\ & + 2C_2\Delta t|u_0|_{BV} \int_0^{t_N} \|q(u^{\epsilon}(\cdot,\tau))\|_\infty d\tau + \int_0^{t_N} \|q(u^{\epsilon}) - \phi^{\lambda,\epsilon}\|_1 dt + \tilde{c}(\lambda)\epsilon, \end{aligned} \tag{3.17}$$

where $\tilde{c}(\lambda)$ is a constant independent of ϵ, and for the last term we have used (3.11). Using Proposition 2.3, (3.17) leads to

$$\begin{aligned} \sum_{n=0}^{N-1} \|u^{\lambda,\epsilon}(\cdot,t_{n+1}) - \eta^{\epsilon}(\cdot,t_{n+1})\|_1 &\leq \sum_{n=0}^{N-1} \|u^{\epsilon}(\cdot,t_n) - v(\cdot,t_n)\|_1 \\ + \Delta t|u_0|_{BV}\Big(C_1 + 4C_2\|u_0\|_\infty\Big) &+ \int_0^{t_N} \|q(u^{\epsilon}(\cdot,t)) - \phi^{\lambda,\epsilon}(\cdot,t)\|_1 dt + \tilde{c}(\lambda)\epsilon. \end{aligned} \tag{3.18}$$

Letting $\epsilon \to 0+$ yields

$$(3.19) \qquad \sum_{n=0}^{N-1} \|u^\lambda(\cdot,t_{n+1}) - \eta(\cdot,t_{n+1})\|_1 \le \sum_{n=0}^{N-1} \|u(\cdot,t_n) - v(\cdot,t_n)\|_1$$
$$+\Delta t|u_0|_{BV}\Big(C_1 + 4C_2\|u_0\|_\infty\Big) + \int_0^{t_N} \|q(u(\cdot,t)) - \phi^\lambda(\cdot,t)\|_1 dt,$$

where

$$u^\lambda(x,t) = u(x,t) - \int_{t_n}^t \phi^\lambda(x,\tau)d\tau, \qquad \phi^\lambda(x,t) = \int \omega_\lambda(x-y)q(u(y,t))dy.$$

Since $q(u) \in BV$, it is easy to show that $\phi^\lambda(\cdot,t)$ converges to $q(u(\cdot,t))$ strongly in L^1 when $\lambda \to 0+$. Letting $\lambda \to 0+$ in (3.19) yields (3.6). $\Box$

Remark. From (3.15), we require that $f \in C^2(\mathbf{R})$. This assumption is also used in [12], but not in [30].

Lemma 3.2 *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^1(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \le 0$. Let u be the entropy solution of (1.5)-(1.6). The following estimate holds:*

$$(3.20) \qquad \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} q(u(\cdot,\tau))d\tau - \Delta t\, q(u(\cdot,t_{n+1}))\right\|_1 \le \Delta t\Big(C_0|u_0|_{BV} + \|q(u_0)\|_1\Big),$$

where C_0 is defined by (2.19).

Proof. We first extend $u(\cdot,t)$ to $-\infty < t < \infty$ by letting

$$U(x,t) = \begin{cases} u(x,t) & \text{if} \quad t \ge 0, \\ u_0(x) & \text{if} \quad t < 0\ . \end{cases}$$

It follows from (2.18) that

$$(3.21) \qquad \|U(\cdot,\tau_2) - U(\cdot,\tau_1)\|_1 \le \Big(C_0|u_0|_{BV} + Q\|u_0\|_1\Big)(\tau_2 - \tau_1), \qquad \text{for any} \quad \tau_2 > \tau_1.$$

Let $\omega_\lambda(x) = \frac{1}{\lambda}\omega(x/\lambda)$ be a positive C_0^∞-mollifier satisfying

$$\int \omega_\lambda(x)dx = 1; \qquad \omega_\lambda(x) = 0 \quad \text{if} \quad |x| \ge \lambda.$$

Let $\psi^\lambda(x,t) = q(U(x,\cdot)) * \omega$, i.e.

$$(3.22) \qquad \psi^\lambda(x,t) = \int \omega_\lambda(t-y)q(U(x,y))dy = \int \omega_\lambda(y)q(U(x,t-y))dy.$$

Observe that

$$(3.23) \qquad \text{LHS of (3.20)} \le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|q(u(\cdot,\tau)) - \psi^\lambda(\cdot,\tau)\|_1 d\tau$$
$$+ \sum_{n=0}^{N-1} \Delta t\|q(u(\cdot,t_{n+1})) - \psi^\lambda(\cdot,t_{n+1})\|_1 + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\psi^\lambda(\cdot,\tau) - \psi^\lambda(\cdot,t_{n+1})\|_1 d\tau$$
$$= \int_0^T \|q(U(\cdot,\tau)) - \psi^\lambda(\cdot,\tau)\|_1 d\tau + \Delta t \sum_{n=0}^{N-1} \|q(U(\cdot,t_{n+1})) - \psi^\lambda(\cdot,t_{n+1})\|_1$$
$$+ \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\psi^\lambda(\cdot,\tau) - \psi^\lambda(\cdot,t_{n+1})\|_1 =: J_1 + J_2 + J_3,$$

where $T = N\Delta t$. Using (3.21) and (3.22) we can show that

$$J_1 \le \tilde{c}\lambda, \qquad J_2 \le \tilde{c}\lambda, \tag{3.24}$$

where $\tilde{c}$ is a constant independent of λ. It is easy to see that ψ^λ is differentiable with respect to x and t. In particular, we have

$$\|\psi_t^\lambda(\cdot,t)\|_1 = \left\| \lim_{h\to 0+} \frac{1}{h} \int \omega_\lambda(y) \Big[q(U(\cdot, t-y+h)) - q(U(\cdot,t-y)) \Big] dy \right\|_1 . \tag{3.25}$$

It follows from Fatou's lemma [6] and (3.21) that

$$\|\psi_t^\lambda(\cdot,t)\|_1 \le Q \lim_{h\to 0+} \frac{1}{h} \int \omega_\lambda(y) \|U(\cdot,t-y+h) - U(\cdot,t-y)\|_1 dy \le \tilde{c}, \tag{3.26}$$

where the constant $\tilde{c}$ is independent of λ. Observe that

$$\begin{aligned} J_3 &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left\| \int_\tau^{t_{n+1}} \psi_t^\lambda(\cdot,t)dt \right\|_1 d\tau \le \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t_{n+1}} \|\psi_t^\lambda(\cdot,t)\|_1 dt d\tau \\ &= \Delta t \int_0^T \|\psi_t^\lambda(\cdot,t)\|_1 dt =: \Delta t (J_{31} + J_{32}), \end{aligned} \tag{3.27}$$

where

$$J_{31} = \int_0^\lambda \|\psi_t^\lambda(\cdot,t)\|_1 dt\,, \qquad J_{32} = \int_\lambda^T \|\psi_t^\lambda(\cdot,t)\|_1 dt\,.$$

We first estimate J_{32}. For $t \ge \lambda, |y| \le \lambda$ and $h > 0$, we have $t-y \ge 0, t-y+h \ge 0$ which suggest that $U(\cdot,t-y) = u(\cdot,t-y)$ and $U(\cdot,t-y+h) = u(\cdot,t-y+h)$. Noting that $\omega_\lambda(y) = 0$ for all $|y| \ge \lambda$, we obtain from (3.25) that

$$\begin{aligned} J_{32} &\le \lim_{h\to 0+} \frac{1}{h} \int_\lambda^T \int_{-\lambda}^\lambda \omega_\lambda(y) \|q(U(\cdot,t-y+h)) - q(U(\cdot,t-y))\|_1 dy dt \\ &= \lim_{h\to 0+} \frac{1}{h} \int_{-\lambda}^\lambda \omega_\lambda(y) \int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt dy. \end{aligned} \tag{3.28}$$

Using the same technique for the proof of (2.8) we can obtain

$$\int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt \le \|u(\cdot,\lambda-y+h) - u(\cdot,\lambda-y)\|_1,$$

for all $|y| \le \lambda$ and $h > 0$. This, together with (2.18), yield

$$\begin{aligned} &\int_\lambda^T \|q(u(\cdot,t-y+h)) - q(u(\cdot,t-y))\|_1 dt \\ &\le C_0|u_0|_{BV} h + \int_{\lambda-y}^{\lambda-y+h} \|q(u(\cdot,\tau))\|_1 d\tau \\ &\le C_0|u_0|_{BV} h + \|q(u_0)\|_1 h + \int_{\lambda-y}^{\lambda-y+h} \|q(u(\cdot,\tau)) - q(u_0)\|_1 d\tau \\ &\le C_0|u_0|_{BV} h + \|q(u_0)\|_1 h + \tilde{c}(2\lambda+h)h, \end{aligned} \tag{3.29}$$

for all $|y| \le \lambda$ and $h > 0$; where the constant $\tilde{c}$ is independent of λ and h, and in the last step we have used (2.17) and the fact that $\lambda - y + h \le 2\lambda + h$. Combining (3.28) and (3.29) gives

$$J_{32} \le C_0|u_0|_{BV} + \|q(u_0)\|_1 + 2\tilde{c}\lambda. \tag{3.30}$$

Moreover, it follows from (3.26) that $J_{31} \leq \tilde{c}\lambda$. Therefore, we have proved that

$$\text{LHS of (3.20)} \leq \Delta t\Big(C_0|u_0|_{BV} + \|q(u_0)\|_1\Big) + \tilde{c}\lambda,$$

where $\tilde{c}$ is a constant independent of λ. Letting $\lambda \to 0$ in the above inequality gives (3.20). □

We are now able to proceed with the construction of a bound on the splitting error using the above two lemmas.

Theorem 3.1 *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^2(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \leq 0$. Let u be the entropy solution of (1.5)-(1.6), S_1 be the solution operator of the homogeneous problem (3.1) and S_2 be the backward Euler operator (3.2). The following estimate holds:*

$$\begin{aligned} &\Big\| u(\cdot, N\Delta t) - [S_2(\Delta t)S_1(\Delta t)]^N v_0 \Big\|_1 \\ \leq\ & \|u_0 - v_0\|_1 + \Delta t\left(C_1|u_0|_{BV} + 4C_2|u_0|_{BV}\|u_0\|_\infty + \|q(u_0)\|_1 + C_0|u_0|_{BV}\right), \end{aligned} \tag{3.31}$$

where $T = N\Delta t$, the constants C_0, C_1 and C_2 depend on the function f and are defined by (2.19) and (3.7).

Proof. It follows from (3.5) that

$$\begin{aligned} &\|u(\cdot,t_{n+1}) - \Delta t\, q(u(\cdot,t_{n+1})) - v(\cdot,t_{n+1}) + \Delta t q(v(\cdot,t_{n+1}))\|_1 \\ \leq\ & \left\| u(\cdot,t_{n+1}) - \int_{t_n}^{t_{n+1}} q(u)d\tau - \eta(\cdot,t_{n+1})\right\|_1 + \left\| \int_{t_n}^{t_{n+1}} q(u)d\tau - \Delta t\, q(u(\cdot,t_{n+1}))\right\|_1 . \end{aligned} \tag{3.32}$$

Observe that for any $x \in \mathbf{R}$,

$$\begin{aligned} & \operatorname{sgn}\Big[u(x,t_{n+1}) - \Delta t\, q(u(x,t_{n+1})) - v(x,t_{n+1}) + \Delta t\, q(v(x,t_{n+1}))\Big] \\ =\ & \operatorname{sgn}\Big[(1-\Delta t q'(\bullet))(u(x,t_{n+1}) - v(x,t_{n+1}))\Big] \\ =\ & \operatorname{sgn}\Big[u(x,t_{n+1}) - v(x,t_{n+1})\Big], \end{aligned}$$

where $\bullet$ denotes an intermidiate value between u and v, and in the last step we have used the fact that $q' \leq 0$. It follows from the above result that

$$\begin{aligned} \text{LHS of (3.32)} &= \int_{\mathbf{R}} \operatorname{sgn}(u(x,t_{n+1}) - v(x,t_{n+1}))\Big[u(x,t_{n+1}) - \Delta t\, q(u(x,t_{n+1})) \\ &\qquad -v(x,t_{n+1}) + \Delta t\, q(v(x,t_{n+1}))\Big]dx \\ &= \|u(\cdot,t_{n+1}) - v(\cdot,t_{n+1})\|_1 + \Delta t\,\|q(u(\cdot,t_{n+1})) - q(v(\cdot,t_{n+1}))\|_1, \end{aligned} \tag{3.33}$$

where in the last step we have used (2.2). It follows from (3.32) and (3.33) that

$$\begin{aligned} &\sum_{n=0}^{N-1}\Big(\|u(\cdot,t_{n+1}) - v(\cdot,t_{n+1})\|_1 + \Delta t\,\|q(u(\cdot,t_{n+1})) - q(v(\cdot,t_{n+1}))\|_1\Big) \\ \leq\ & \sum_{n=0}^{N-1}\left\| u(\cdot,t_{n+1}) - \int_{t_n}^{t_{n+1}} q(u)d\tau - \eta(\cdot,t_{n+1})\right\|_1 + \sum_{n=0}^{N-1}\left\|\int_{t_n}^{t_{n+1}} q(u)d\tau - \Delta t\, q(u(\cdot,t_{n+1}))\right\|_1 . \end{aligned} \tag{3.34}$$

Using Lemmas 3.1 and 3.2 yields

$$\begin{aligned} (3.35) \qquad & \|u(\cdot,t_N)-v(\cdot,t_N)\|_1+\sum_{n=1}^{N}\Delta t\,\|q(u(\cdot,t_n))-q(v(\cdot,t_n))\|_1 \\ \leq\ & \|u(\cdot,0)-v(\cdot,0)\|_1+\Delta t\left(C_1|u_0|_{BV}+4C_2|u_0|_{BV}\|u_0\|_\infty+\|q(u_0)\|_1+C_0|u_0|_{BV}\right), \end{aligned}$$

which leads to (3.31). This completes the proof of this theorem. □

Theorem 3.1 implies that if $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$ then we have the following error bound:

$$\|u(\cdot,N\Delta t)-[S_2(\Delta t)S_1(\Delta t)]^N u_0\|_1 \leq C\Big(\|q(u_0)\|_1\Big)\Delta t,$$

where $C(\|q(u_0)\|_1)$ is a constant depending on $\|q(u_0)\|_1$, but not on $Q=\max_{|u|\leq\|u_0\|_\infty}|q'(u)|$. Since $q(0)=0$, it is clear that

$$\|q(u_0)\|_1 \leq Q\|u_0\|_1.$$

This suggests that if Q is uniformly bounded in the sense that it is independent of some small parameter, then the splitting error is $O(\Delta t)$. The main point of Theorem 3.1 is that even if Q is not uniformly bounded, the splitting error may also be free of some small parameter. This can be seen from the following examples.

Example 3.1

$$q(u)=e^{-u/\delta}-1, \qquad \delta \ll 1.$$

Example 3.2

$$q(u)=\frac{-u/\delta}{\sqrt{1+(u/\delta)^2}}, \qquad \delta \ll 1.$$

Example 3.3

$$q(u)=-\tan^{-1}(u/\delta), \qquad \delta \ll 1.$$

In all the above examples, we can easily verify that $Q=O(\delta^{-1})$. We can also verify that in these examples, $q(0)=0, q'\leq 0$. If u_0 has a compact support, then it is readily shown that $\|q(u_0)\|_1=O(1)$. Theorem 3.1 suggests that with the above given q, the L^1 convergence rate for the operator splitting method (3.3) is $O(\Delta t)$, provided u_0 has a compact support.

Remark. From the proof of Theorem 3.1, we also obtain the following result:

$$(3.36) \qquad \sum_{n=1}^{N}\Delta t\,\|q(u(\cdot,t_n))-q(v(\cdot,t_n))\|_1 \leq \|u_0-v_0\|_1+C\left(\|q(u_0)\|_1\right)\Delta t.$$

Remark. Similar to the proof of Theorem 3.1, we can show that

$$\Big\|u(\cdot,N\Delta t)-[S_1(\Delta t)S_2(\Delta t)]^N v_0\Big\|_1 \leq \|u_0-v_0\|_1+C\Big(\|q(u_0)\|_1\Big)\Delta t,$$

provided that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$. In other words, the order of splitting procedure (3.4)-(3.5) can be changed by starting with the backward Euler method and then solving the homogeneous equation. □

By simple modifications for Lemmas 3.1, 3.2 and Theorem 3.1, we can obtain the following estimate.

Theorem 3.2 *Let the functions u_0, f and q in (1.5)-(1.6) be subject to the conditions stated in Theorem 3.1. Let u be the entropy solution of (1.5)-(1.6), S_1 be the solution operator of the homogeneous problem (3.1) and S_2 be the backward Euler operator (3.2). The following estimate holds:*

$$\begin{aligned}(3.37)\qquad & \max_{0\le n\le N}\left\|u(\cdot,t_n)-[S_2(\Delta t)S_1(\Delta t)]^n v_0\right\|_1 \\ \le\ & \|u_0-v_0\|_1+\Delta t\left(C_1|u_0|_{BV}+4C_2|u_0|_{BV}\|u_0\|_\infty+\|q(u_0)\|_1+C_0|u_0|_{BV}\right),\end{aligned}$$

where $t_n = n\Delta t$, the constants C_0, C_1 and C_2 are defined by (2.19) and (3.7).

The above result is useful if an initial layer is presented. It implies that the initial layer can be resolved if $\|q(u_0)\|_1$ is not large. See Section 5 for more details about resolving initial layers.

4 Forward Euler Method

In this section, we will briefly discuss splitting method with forward Euler operator. This approach has been used for conservation laws with nonstiff source terms ([12, 28, 30]). Again, let S_1 be the solution operator of the homogeneous problem, as defined by (3.1). Let E be the forward Euler operator for the ODE problem $v_t = q(v)$, i.e. $v(\cdot,t) = E(t)v_0$ is given by

$$(4.1)\qquad v(\cdot,t)=v_0(\cdot)+\Delta t\, q(v_0(\cdot)).$$

The operator splitting solution is defined by

$$(4.2)\qquad v(\cdot,t_n)=[S_1(\Delta t)E(\Delta t)]^n v_0.$$

Given a starting value $v(x,t_n)$, we compute $v(x,t_{n+1})$ using the following steps. First find the intermediate value $\eta(x,t_{n+1})$ by using a forward Euler method:

$$(4.3)\qquad \eta(x,t_{n+1})=v(x,t_n)+\Delta t q(v(x,t_n)).$$

Then solving the following homogeneous equation to find $v(x,t_{n+1})$:

$$(4.4)\qquad v_t+f(v)_x=0,\quad t\in(t_n,t_{n+1}];\quad v(x,t_n)=\eta(x,t_{n+1}).$$

Similar to the proof for Theorem 3.1, the following two lemmas are needed. The detail proof will be omitted.

Lemma 4.1 *Assume that $u_0\in BV\cap L^\infty(\mathbf{R})\cap L^1(\mathbf{R}), f\in C^2(\mathbf{R}), q\in C^1(\mathbf{R})$ satisfying $q(0)=0, q'\le 0$. Let u be the entropy solution of (1.5)-(1.6), η and v be the solutions of (4.3) and (4.4), respectively. The following estimate holds*

$$\begin{aligned}(4.5)\qquad & \sum_{n=0}^{N-1}\|u(\cdot,t_{n+1})-v(\cdot,t_{n+1})\|_1 \\ \le\ & \sum_{n=0}^{N-1}\left\|u(\cdot,t_n)+\int_{t_n}^{t_{n+1}}q(u)d\tau-\eta(\cdot,t_{n+1})\right\|_1+\Delta t|u_0|_{BV}\left(C_1+4C_2\|u_0\|_\infty\right),\end{aligned}$$

where the constants C_1 and C_2 are defined by (3.7).

Proof. The proof of this lemma is similar to that of Lemma 3.1. The most important change is to replace (3.12) and (3.13) by

$$
\begin{aligned}
(4.6)\qquad & u_t^{\lambda,\epsilon} + f(u^{\lambda,\epsilon})_x \\
= \;& \epsilon u_{xx}^{\lambda,\epsilon} + f(u^{\lambda,\epsilon})_x - f(u^\epsilon)_x + q(u^\epsilon) - \phi^{\lambda,\epsilon}(x,t) + \epsilon \int_{t_{n+1}}^{t} \phi_{xx}^{\lambda,\epsilon}(x,\tau)d\tau,
\end{aligned}
$$

where the function $u^{\lambda,\epsilon}$ is defined by

$$
(4.7)\qquad u^{\lambda,\epsilon}(x,t) = u^\epsilon(x,t) - \int_{t_{n+1}}^{t} \phi^{\lambda,\epsilon}(x,\tau)d\tau.
$$

It is noted that only the lower limit of the integral is changed. □

Lemma 4.2 *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^1(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \leq 0$. Let u be the entropy solution of (1.5)-(1.6). The following estimate holds:*

$$
(4.8)\qquad \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} q(u(\cdot,\tau))d\tau - \Delta t\, q(u(\cdot,t_n)) \right\|_1 \leq \Delta t\Big(\|q(u_0)\|_1 + C_0|u_0|_{BV} \Big),
$$

where C_0 is defined by (2.19). □

The proof of this lemma is similar to that of Lemma 3.2. Having the above two lemmas, we are ready to prove the following theorem.

Theorem 4.1 *Assume that $u_0 \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R}), f \in C^2(\mathbf{R}), q \in C^1(\mathbf{R})$ satisfying $q(0) = 0, q' \leq 0$. Let $u = u(x,t)$ be the entropy solution of (1.5)-(1.6), S_1 be the solution operator of the homogeneous problem (3.1) and E be the forward Euler operator (4.1). If*

$$
(4.9)\qquad \Delta t \max_{|u| \leq \|u_0\|_\infty} |q'(u)| \leq 1,
$$

then the following estimate holds:

$$
\begin{aligned}
(4.10)\qquad & \|u(\cdot, N\Delta t) - [S_1(\Delta t)E(\Delta t)]^N v_0\|_1 \\
\leq\; & \|u_0 - v_0\|_1 + \Delta t\left(C_1|u_0|_{BV} + 4C_2\|u_0\|_\infty + \|q(u_0)\|_1 + C_0|u_0|_{BV} \right),
\end{aligned}
$$

where $T = N\Delta t$, the constants C_0, C_1 and C_2 depend on the function f and are defined by (2.19) and (3.7).

Proof. It follows from (4.3) that

$$
\begin{aligned}
(4.11)\qquad & \|u(\cdot,t_n) + \Delta t\, q(u(\cdot,t_n)) - \eta(\cdot,t_{n+1})\|_1 \\
= \;& \|u(\cdot,t_n) - v(\cdot,t_n) + \Delta t\, q(u(\cdot,t_n)) - \Delta t\, q(v(\cdot,t_n))\|_1 \\
= \;& \int_{\mathbf{R}} \operatorname{sgn}\Big((1+\Delta t\, q'(\bullet))(u(x,t_n) - v(x,t_n))\Big) \Big[u(x,t_n) - v(x,t_n) \\
& \qquad + \Delta t\, q(u(x,t_n)) - \Delta t\, q(v(x,t_n))\Big] dx \\
= \;& \int_{\mathbf{R}} \operatorname{sgn}\Big(u(x,t_n) - v(x,t_n)\Big) \Big[u(x,t_n) - v(x,t_n) + \Delta t\, q(u(x,t_n)) - \Delta t\, q(v(x,t_n))\Big] dx,
\end{aligned}
$$

where $\bullet$ denote an intermediate value between $u(x,t_n)$ and $v(x,t_n)$, and in the last step we have used (4.9). It follows from (2.2) that

$$\begin{aligned}&\int_{\mathbf{R}} \operatorname{sgn}\Big(u(x,t_n)-v(x,t_n)\Big)\Big[u(x,t_n)-v(x,t_n)+\Delta t\, q(u(x,t_n))-\Delta t\, q(v(x,t_n))\Big]dx\\ =\ & \|u(\cdot,t_n)-v(\cdot,t_n)\|_1-\Delta t\,\|q(u(\cdot,t_n))-q(v(\cdot,t_n))\|_1.\end{aligned}$$

This, together with (4.11), yield

$$\|u(\cdot,t_n)+\Delta t q(u(\cdot,t_n))-\eta(\cdot,t_{n+1})\|_1\le\|u(\cdot,t_n)-v(\cdot,t_n)\|_1. \tag{4.12}$$

Combining (4.5), (4.8) and (4.12) gives

$$\begin{aligned}\sum_{n=0}^{N-1}\|u(\cdot,t_{n+1})-v(\cdot,t_{n+1})\|_1 &\le \sum_{n=0}^{N-1}\|u(\cdot,t_n)+\Delta t q(u(\cdot,t_n))-\eta(\cdot,t_{n+1})\|_1\\ &+\sum_{n=0}^{N-1}\Big\|\int_{t_n}^{t_{n+1}}q(u)d\tau-\Delta t q(u(\cdot,t_n))\Big\|_1+\Delta t|u_0|_{BV}\Big(C_1+4C_2\|u_0\|_\infty\Big)\\ \le\ & \sum_{n=0}^{N-1}\|u(\cdot,t_n)-v(\cdot,t_n)\|_1+\Delta t\Big(\|q(u_0)\|_1+C_0|u_0|_{BV}\Big)+\Delta t|u_0|_{BV}\Big(C_1+4C_2\|u_0\|_\infty\Big).\end{aligned} \tag{4.13}$$

This completes the proof of Theorem 4.1. □

It is clear that the main disadvantage of using the forward splitting method (4.3)-(4.4) is that the time step size has to depend on $\max|q'(u)|$, see (4.9). For conservation laws with stiff source terms, the maximum value for q' is usually very large. Therefore, the constraint (4.9) is in general unacceptable. This suggests that if a stiff source term is involved then the splitting method with forward Euler method should not be recommended.

5 A generic model and initial layer

According to Pember [22, 23], a conservation law is stiff if the time scale introduced by the source term is small compared to the characteristic speeds of the hyperbolic equation and some appropriate length scale. A generic form of a stiff conservation law is (1.5) with

$$q(u)=\frac{1}{\delta}g(u), \tag{5.1}$$

where $g\in C^1(\mathbf{R}), g(0)=0, g'\le 0$, g is independent of δ; the relaxation time $0<\delta<1$ is a very small parameter. In this section, we will consider this model problem:

$$u_t+f(u)_x=\frac{1}{\delta}g(u),\qquad (x,t)\in\mathbf{R}\times(0,T], \tag{5.2}$$

$$u(x,0)=u_0(x). \tag{5.3}$$

This model problem has been studied by several authors numerically and theoretically (e.g., [13, 18, 26]). The result in Theorem 3.2 suggests that

$$\begin{aligned}&\max_{0\le n\le N}\Big\|u(\cdot,t_n)-[S_2(\Delta t)S_1(\Delta t)]^n v_0\Big\|_1\\ \le\ & \|u_0-v_0\|_1+C\Delta t+\frac{C\Delta t}{\delta}\|u_0\|_1,\end{aligned} \tag{5.4}$$

provided that $u_0 \in BV \cap L^\infty$, where C is independent of Δt and δ. Therefore, the splitting error is in general related to δ. It is noted that if $\|u_0\| = O(\delta)$, then we can get an δ-free error bounds. In particular, we have the following result.

Theorem 5.1 *Assume that $f \in C^2(\mathbf{R})$, $g \in C^1(\mathbf{R})$ satisfying $g(0) = 0$, $g' \leq 0$. If $u_0 = U(x/\delta)$ with $U \in BV \cap L^\infty(\mathbf{R}) \cap L^1(\mathbf{R})$, then*

$$\max_{0 \leq n \leq N} \|u(\cdot, t_n) - [S_2(\Delta t) S_1(\Delta t)]^n v_0\|_1 \leq \|u_0 - v_0\|_1 + C\Delta t, \tag{5.5}$$

where C is independent of δ and Δt.

Proof. Observe that

$$\|u_0\|_1 = \int |U(x/\delta)|\, dx = \delta \|U\|_1 \leq C\delta. \tag{5.6}$$

This, together with (5.4), yield (5.5). □

We point out that the estimate (5.4) is the optimal one, in the sense that the $O(\Delta t \|u_0\|_1/\delta)$ term is not removable. To see this, we consider a simple example.

Example 5.1 *Consider the following conservation law with linear flux and linear source:*

$$u_t + u_x = -\frac{u}{\delta}, \qquad u(x,0) = u_0(x), \tag{5.7}$$

where the initial data $u_0 \in L^1(\mathbf{R})$.

The exact solution of (5.7) is $u(x,t) = e^{-t/\delta} u_0(x - t)$. It can be verified that

$$[S_2(\Delta t) S_1(\Delta t)]^n u_0 = \left(1 + \frac{\Delta t}{\delta}\right)^{-n} u_0(x - t_n).$$

The above solutions imply that

$$\begin{aligned} &\max_{0 \leq n \leq N} \left\| u(\cdot, t_n) - [S_2(\Delta t) S_1(\Delta t)]^n u_0 \right\|_1 \\ = &\max_{0 \leq n \leq N} \left(\left(1 + \frac{\Delta t}{\delta}\right)^{-n} - \exp\left(-\frac{n\Delta t}{\delta}\right) \right) \|u_0\|_1. \end{aligned}$$

Unless $\|u_0\|_1$ is proportional to δ, the right hand side of the above equation is small only when $\Delta t \ll \delta$. In particular, it can not be bounded by an δ-free upper bound if $\|u_0\| = O(1)$. This suggests that the last term in (5.4) can not be removed. The reason for this is that for $\delta \ll 1$, there is a very small initial layer which needs to be resolved accurately. To illustrate the idea clearly, we consider a rather simple model. That is, we follow Schroll and Winther (1996) by assuming that the function g in (5.1) decreases *strictly* with respect to u. In this case, zero is the only equilibrium. More precisely, the function g is assumed to satisfy

$$g(0) = 0, \qquad g'(u) < 0, \qquad \text{for} \quad u \in \mathbf{R}. \tag{5.8}$$

With the above assumptions, there exist two positive constants α, β independent of δ such that

$$\alpha = \min_{|u| \leq \|u_0\|_\infty} |g'(u)| > 0, \qquad \beta = \max_{|u| \leq \|u_0\|_\infty} |g'(u)| > 0. \tag{5.9}$$

The previous sections employed uniform meshes, and it is readily shown that the results are also valid for quasi-uniform meshes which satisfy

$$\max_{0\le n\le N-1}(t_{n+1}-t_n)\le C\min_{0\le n\le N-1}(t_{n+1}-t_n), \tag{5.10}$$

where C is independent of N. In the following, we consider an uneven mesh given by

$$t_n=\begin{cases} 4n\delta|\ln\delta|/\alpha N & \text{for } n\le N/2,\\ 2\delta|\ln\delta|/\alpha+(T-2\delta|\ln\delta|/\alpha)(2n/N-1) & \text{for } N/2<n\le N,\end{cases} \tag{5.11}$$

where N is chosen to be an even number. It is easy to verify that $\{t_n\}_{n=0}^N$ is a mesh sequence of $[0,T]$. Let $\Delta t_n=t_n-t_{n-1}$. It can be verified that

$$\Delta t_n=\begin{cases} 4\delta|\ln\delta|/\alpha N & \text{for } n\le N/2,\\ 2(T-2\delta|\ln\delta|/\alpha)/N & \text{for } N/2<n\le N.\end{cases} \tag{5.12}$$

In other owrds, we divide $[0,T]$ into two subintervals $[0,2\delta|\ln\delta|/\alpha]$ and $(2\delta|\ln\delta|/\alpha,T]$. Two different uniform meshes, with mesh sizes given by (5.12), are used in the subintervals. With the mesh (5.11), the backward Euler operator splitting procedure for (5.2)-(5.3) is given by

$$v(\cdot,t_m)=\prod_{n=1}^m S_2(\Delta t_n)S_1(\Delta t_n)v_0,\qquad 1\le m\le N. \tag{5.13}$$

In order to obtain sharper error bounds than (5.4), we need some additional properties for the solutions of (5.2)-(5.3). Using (5.9), we can modify Proposition 2.2 to get the following results.

Lemma 5.1 *If $u_0\in BV\cap L^1(\mathbf{R})\cap L^\infty(\mathbf{R})$, $f\in C^1(\mathbf{R})$ and $g\in C^1(\mathbf{R})$ satisfying (5.8), then the nonhomogeneous conservation law (5.2)-(5.3) possesses a unique entropy solution $u(x,t)$ satisfying*

$$|u(\cdot,t)|_{BV}\le e^{-\alpha t/\delta}|u_0|_{BV}, \tag{5.14}$$

$$\|u(\cdot,t)\|_1\le e^{-\alpha t/\delta}\|u_0\|_1, \tag{5.15}$$

$$\|u(\cdot,\tau_2)-u(\cdot,\tau_1)\|_1\le C_0\int_{\tau_1}^{\tau_2}|u(\cdot,\tau)|_{BV}d\tau+\frac{\beta}{\delta}\int_{\tau_1}^{\tau_2}\|u(\cdot,\tau)\|_1d\tau,\quad \tau_2\ge\tau_1, \tag{5.16}$$

where C_0 is defined by (2.19), α,β are given by (5.9).

In the proof of Theorem 3.1, the term containing $\|q(u_0)\|_1$ is introduced in the estimate (3.20). We will use Lemma 5.1 to modify (3.20). Observe

$$\begin{aligned} &\sum_{n=0}^{N-1}\left\|\int_{t_n}^{t_{n+1}}q(u(\cdot,\tau))d\tau-\Delta t_n q(u(\cdot,t_{n+1}))\right\|_1 \\ \le\ &\frac{\beta}{\delta}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\|u(\cdot,\tau)-u(\cdot,t_{n+1})\|_1d\tau \\ \le\ &\frac{\beta}{\delta}\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\left[C_0\int_\tau^{t_{n+1}}|u(\cdot,t)|_{BV}dt+\frac{\beta}{\delta}\int_\tau^{t_{n+1}}\|u(\cdot,t)\|_1dt\right]d\tau, \end{aligned} \tag{5.17}$$

where in the last step we have used (5.16). Note that $0 < \delta < 1$. Using (5.14) and (5.15), we obtain from (5.17) that

$$
\begin{aligned}
(5.18) \qquad & \sum_{n=0}^{N-1} \left\| \int_{t_n}^{t_{n+1}} q(u(\cdot,\tau))d\tau - \Delta t_n q(u(\cdot,t_{n+1})) \right\|_1 \\
\leq \; & \frac{C}{\delta^2} \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{\tau}^{t_{n+1}} e^{-\alpha t/\delta} dt d\tau \\
\leq \; & \frac{C}{\delta^2} \sum_{n<N/2} \Delta t_n \int_{t_n}^{t_{n+1}} e^{-\alpha t/\delta} dt + \frac{C}{\delta^2} \sum_{n \geq N/2} (\Delta t_n)^2 e^{-\alpha t_n/\delta} \\
\leq \; & \frac{C}{\delta^2} \frac{2\delta |\ln \delta|}{N} \int_0^{2\delta|\ln\delta|/\alpha} e^{-\alpha t/\delta} dt + \frac{C}{(N\delta)^2} \sum_{n \geq N/2} e^{-2\delta|\ln\delta|/\delta} \\
\leq \; & CN^{-1}|\ln\delta| + CN^{-1},
\end{aligned}
$$

where C is a constant independent of δ and N. Combining the above analysis, we obtain the following conclusions.

Theorem 5.2 *Assume that $u_0 \in BV \cap L^1(\mathbf{R}) \cap L^\infty(\mathbf{R}), f \in C^2(\mathbf{R})$, and $g \in C^1(\mathbf{R})$ satisfying $g(0) = 0, g'(0) < 0$. Let u be the entropy solution of (5.2)-(5.3) and the mesh $\{t_n\}_{n=0}^N$ be given by (5.11). Then the following estimate holds:*

$$
(5.19) \qquad \left\| u(\cdot, N\Delta t) - \prod_{n=1}^{N} S_2(\Delta t_n) S_1(\Delta t_n) v_0 \right\|_1 \leq \|u_0 - v_0\|_1 + CN^{-1}\Big(|\ln\delta| + 1\Big),
$$

where the constant C is independent of N and δ.

As mentioned at the end of Section 3, we can modify the above proof to get the following estimates.

Theorem 5.3 *Let the functions u_0, f and g in (5.2)-(5.3) be subject to the conditions stated in Theorem 5.2. Let u be the entropy solution of (5.2)-(5.3). Then the following estimates hold:*

$$
\begin{aligned}
(5.20) \qquad & \max_{1 \leq n \leq N} \left\| u(\cdot, t_n) - \prod_{m=1}^{n} S_2(\Delta t_m) S_1(\Delta t_m) u_0 \right\|_1 \\
\leq \; & \begin{cases} C(N\delta)^{-1} & \text{for uniform mesh,} \\ CN^{-1}\Big(|\ln\delta| + 1\Big) & \text{for mesh (5.11),} \end{cases}
\end{aligned}
$$

where C is independent of N and δ.

For numerical verification of the results stated in Theorem 5.3, we reconsider the problem in Example 5.1, with $\delta = 0.001$, $0 < t \leq 1$ and with an initial data such that $\|u_0\|_1 = 1$. The L^1 error for the operator splitting methods, given by the left hand side of (5.20), with uniform mesh and (5.11) are plotted in Figure 1. It is observed that the error obtained by using mesh (5.11) is linear with respect to N^{-1}. On the other hands, the error associated with uniform mesh is quite large since N^{-1} is not smaller than δ.

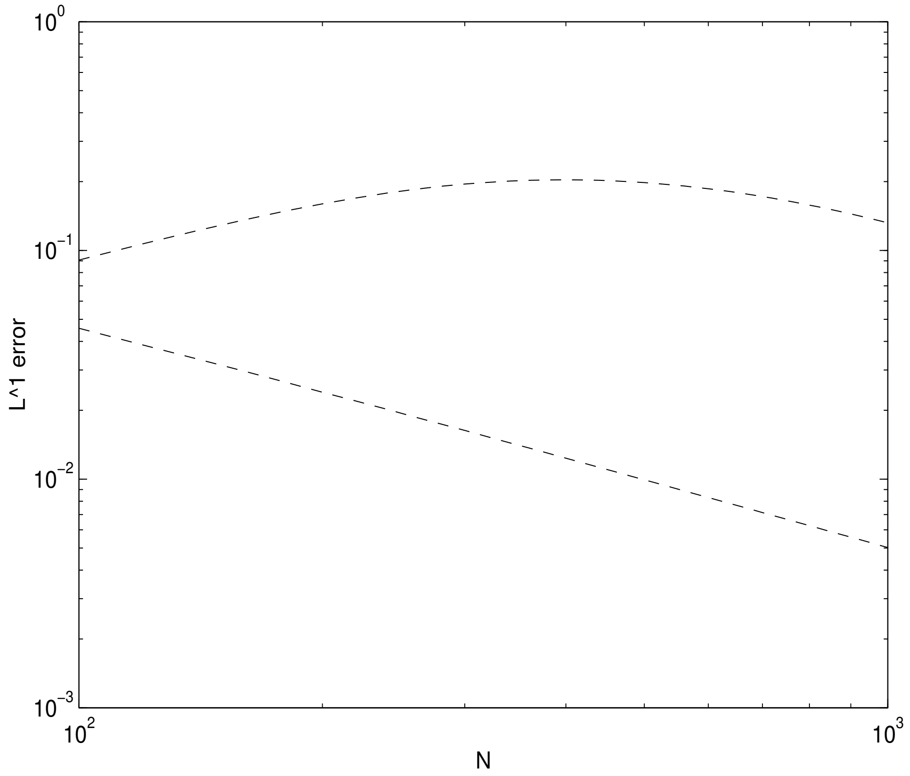

Figure 1: L^1-error for operator splitting method for Example 5.1: the upper curve is obtained by using uniform mesh and the lower one is given by mesh (5.11).

6 Multidimensional equation

We consider the two dimensional Cauchy problem:

$$u_t + f_1(u)_x + f_2(u)_y = q(u); \qquad (x, y) \in \mathbf{R}^2, \quad t \in (0, T], \tag{6.1}$$

$$u(x, y, 0) = u_0(x, y). \tag{6.2}$$

We point out that Propositions 2.1-2.4 can be extended to scalar conservation laws with several space dimensions. As an example, we will extend (2.9) to the two-dimensional problem (6.1)-(6.2). Assume that we have another equation:

$$v_t + f_1(v)_x + f_2(v)_y = q(v); \qquad (x, y) \in \mathbf{R}^2, \quad t \in (0, T], \tag{6.3}$$

$$v(x, y, 0) = v_0(x, y). \tag{6.4}$$

We will prove the following result, which is the extension of (2.9). The standard theory for *vanishing viscosity method*, which is based on the idea of passing to the limit as $\epsilon \to 0+$ in (6.9) or (6.12) below, can be found in Kruzkov [14] and Vol'pert [34].

Lemma 6.1 *If u_0 given by (6.2) and v_0 given by (6.4) belong to $BV(\mathbf{R}^2) \cap L^1(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2)$, then the following estimate holds:*

$$\int_0^T \|q(u(\cdot, \cdot, \tau)) - q(v(\cdot, \cdot, \tau))\|_1 d\tau \leq \|u_0 - v_0\|_1, \tag{6.5}$$

where u and v are solutions of (6.1)-(6.2) and (6.3-(6.4), respectively.

Proof. Again we consider the parabolic regularization of the following forms:

$$u_t^\epsilon + f_1(u^\epsilon)_x + f_2(u^\epsilon)_y = q(u^\epsilon) + \epsilon\left(u_{xx}^\epsilon + u_{yy}^\epsilon\right); \quad u^\epsilon(x,y,0) = u_0(x,y), \tag{6.6}$$

$$v_t^\epsilon + f_1(v^\epsilon)_x + f_2(v^\epsilon)_y = q(v^\epsilon) + \epsilon\left(v_{xx}^\epsilon + v_{yy}^\epsilon\right); \quad v^\epsilon(x,y,0) = v_0(x,y), \tag{6.7}$$

where $\epsilon > 0$. We will employ the techniques used in the proof of Proposition 2.2 to prove (6.5). Let $k = T/M$ and $t_m = mk$. Let $U^\epsilon(x,y,0) = u_0(x,y)$. For a given starting value $U^\epsilon(x,y,t_m)$, we will obtain $U^\epsilon(x,y,t_{m+1})$ by the following forward Euler splitting method. First find the intermediate value $\xi_m(x,y)$ by using the information of the given function $U^\epsilon(x,y,t_m)$:

$$\xi_m(x,y) = U^\epsilon(x,y,t_m) + kq(U^\epsilon(x,y,t_m)). \tag{6.8}$$

Then solve the following homogeneous parabolic equation to obtain $U^\epsilon(x,y,t_{m+1})$:

$$U_t^\epsilon + f_1(U^\epsilon)_x + f_2(U^\epsilon)_y = \epsilon\left(U_{xx}^\epsilon + U_{yy}^\epsilon\right), \qquad t \in (t_m, t_{m+1}), \tag{6.9}$$

$$U^\epsilon(x,y,t_m) = \xi_m(x,y). \tag{6.10}$$

This procedure will generate $U(x,y,t_{m+1})$ for $0 \le m \le M-1$. Similarly, we can obtain $V^\epsilon(x,y,t_{m+1})$ by first computing

$$\eta_m(x,y) = V^\epsilon(x,y,t_m) + kq(V^\epsilon(x,y,t_m)), \tag{6.11}$$

and then solving the homogeneous parabolic equations

$$V_t^\epsilon + f_1(V^\epsilon)_x + f_2(V^\epsilon)_y = \epsilon\left(V_{xx}^\epsilon + V_{yy}^\epsilon\right), \qquad t \in (t_m, t_{m+1}), \tag{6.12}$$

$$V^\epsilon(x,y,t_m) = \eta_m(x,y), \tag{6.13}$$

for $0 \le m \le M-1$. It can be shown that the above splitting procedures is convergent in the following sense:

$$\sup_{0\le m\le M} \|u^\epsilon(\cdot,\cdot,t_m) - U^\epsilon(\cdot,\cdot,t_m)\|_1 \to 0, \qquad \text{as} \quad k \to 0, \tag{6.14}$$

$$\sup_{0\le m\le M} \|v^\epsilon(\cdot,\cdot,t_m) - V^\epsilon(\cdot,\cdot,t_m)\|_1 \to 0, \qquad \text{as} \quad k \to 0. \tag{6.15}$$

Using the standard theory for homogeneous parabolic equations we obtain from (6.9)-(6.10) and (6.12)-(6.13) that

$$\|U^\epsilon(\cdot,\cdot,t_{m+1}) - V^\epsilon(\cdot,\cdot,t_{m+1})\|_1 \le \|\xi_m(\cdot,\cdot) - \eta_m(\cdot,\cdot)\|_1. \tag{6.16}$$

It follows from (6.8), (6.11) and the same technique used in the proof of Theorem 4.1 that

$$\begin{aligned} &\|\xi_m(\cdot,\cdot) - \eta_m(\cdot,\cdot)\|_1 \\ = \;& \|U^\epsilon(\cdot,\cdot,t_m) - V^\epsilon(\cdot,\cdot,t_m)\|_1 - k\|q(U^\epsilon(\cdot,\cdot,t_m)) - q(V^\epsilon(\cdot,\cdot,t_m))\|_1, \end{aligned} \tag{6.17}$$

provided that k is sufficiently small, where we have used the fact that $q' \le 0$. Combining (6.16) and (6.17) we obtain

$$\begin{aligned} &\|U^\epsilon(\cdot,\cdot,t_{m+1}) - V^\epsilon(\cdot,\cdot,t_{m+1})\|_1 \\ \le \;& \|U^\epsilon(\cdot,\cdot,t_m) - V^\epsilon(\cdot,\cdot,t_m)\|_1 - k\|q(U^\epsilon(\cdot,\cdot,t_m)) - q(V^\epsilon(\cdot,\cdot,t_m))\|_1, \end{aligned} \tag{6.18}$$

for $0 \leq m \leq M-1$, provided that k is sufficiently small. It is easy to obtain from the above inequality that

$$\sum_{m=0}^{M-1} k\|q(U^\epsilon(\cdot,\cdot,t_m)) - q(V^\epsilon(\cdot,\cdot,t_m))\|_1 \leq \|u_0 - v_0\|_1, \tag{6.19}$$

where we have used the facts that $U^\epsilon(x,y,0) = u_0(x,y)$ and $V^\epsilon(x,y,0) = v_0(x,y)$. Letting $k \to 0$, it follows from (6.14), (6.15) and (6.19) that

$$\int_0^T \|q(u^\epsilon(\cdot,\cdot,\tau)) - q(v^\epsilon(\cdot,\cdot,\tau))\|_1 d\tau \leq \|u_0 - v_0\|_1. \tag{6.20}$$

The inequality (6.5) is obtained by letting $\epsilon \to 0$ in (6.20). □

After extending Propositions 2.1-2.4, we can establish results similar to those in Sections 3 to 5. In particular, Theorems 3.1, 4.1 and 5.3 can be extended to the two-dimensional problem (6.1)-(6.2). We will omit the detail results and proofs here.

7 Concluding Remarks

In this work, we have analyzed time-splitting methods applied to hyperbolic conservation laws. The solution properties obtained in Section 2 are important in the error analysis. In a separate work, we will consider fully discretizations for the nonhomogeneous equation with stiff source terms. We will analyze the order of convergence of the fully discretized schemes. It will be shown that the schemes possess a discrete analogy of the stability properties given in Section 2.

This work concentrates on time-splitting based on Euler type methods. As expected, the error introduced by the splitting is at most linear with respect to the time step. A frequently used splitting method for inhomogeneous hyperbolic equation is Strang's splitting [29]. As before, if we call the stiff ODE operators as $S_2(t)$ and the homogeneous convection operator S_1 then the Strang splitting takes the form

$$u(\cdot,t_{n+1}) \approx S_2\left(\frac{\Delta t}{2}\right) S_1(\Delta t) S_2\left(\frac{\Delta t}{2}\right) u(\cdot,t_n).$$

Consider the simple model (5.2)-(5.3). It can be shown that the above scheme is a second-order splitting for $\delta = O(1)$, as long as the exact solution is sufficiently *smooth*. A natural quation is can Strang's splitting have second-order as $\delta \to 0$ while holding Δt fixed? In [11], Jin considered a prototypical relaxation model:

$$\begin{aligned} &h_t + w_x = 0, \\ &w_t + p(h)_x = -\frac{1}{\delta}(w - f(h)), \quad p'(h) > 0. \end{aligned}$$

By an asymptotic analysis, he claimed that in the regime $\delta \to 0$ and $\delta/\Delta t \to 0$ the Strang splitting is only a first order approach to the equilibrium equation. He also designed a second-order Runge-Kutta type splitting method that possesses the discrete analogy of the continuous asymptotic limit, which is able to capture the correct physical behaviors with high order accuracy, even if the initial layer is not numerically resolved. The convergence of Jin's second-order splitting scheme was proved by Chalabi

[1]. It is interesting to provide rigorous analysis for the convergence order of Strang's second-order splitting and Jin's second-order splitting.

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