On the Front Tracking Algorithm

Paolo Baiti * Helge Kristian Jenssen †

Abstract

In this paper we present an improved version of the front-tracking algorithm for systems of conservation laws. The formulation and the theoretical analysis are here somewhat simpler than in previous algorithms. At the same time, our version leads to a more efficient numerical scheme.

1991 *Mathematical Subject Classification:* 35L65
Key Words: Systems of conservation laws, wave front-tracking.

1 Introduction

We are concerned with the construction of a global weak solution to the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws

$$\begin{aligned} &u_t + (f(u))_x = 0, \\ &u(0, \cdot) = \bar{u}. \end{aligned} \tag{1}$$

The basic idea of front-tracking for systems of conservation laws is to construct approximate solutions within a class of piecewise constant functions. One approximates the initial data by a piecewise constant function and solves the resulting Riemann problems. Rarefactions are replaced by many small discontinuities. One tracks the outgoing fronts until the first time two waves interact. This defines a new Riemann problem, etc. One of the main problems in this construction is to keep the number of wave-fronts finite for all times $t > 0$. For this purpose there are presently three types of front-tracking algorithms available [5, 1, 8, 2].

In [1, 2] one defines the notion of generation order which tells how many interactions were needed to produce a wave-front. In order to keep the number of waves finite, one solves in an accurate way the Riemann problems arising from interactions between waves of low order, and in a less accurate way those arising from interactions between high order waves. This simplified solution is constructed by letting the incoming waves pass through each other, slightly changing their speeds, and by collecting all the remaining waves into a so-called "non-physical" front. All non-physical waves propagate with a constant speed greater than all characteristic speeds.

*S.I.S.S.A., Via Beirut 4, Trieste 34014,Italy
†Department of Mathematical Sciences, NTNU, N-7034 Trondheim, Norway

In [8] one does not use the concept of generation order. Instead, for each time where two waves interact one considers a functional depending on the future interactions. If this functional is small enough, some of the small waves are removed and the algorithm is restarted. This guarantees that the approximate solution can be prolonged for a positive time. One then shows that it is only necessary to apply this restarting procedure a finite number of times. In the special case of 2×2 systems [5], the number of fronts remains automatically finite, and no restarting procedure is needed.

By the algorithms defined in these papers piecewise constant approximate solutions $u_\nu(t, x)$ are defined for all $t \geq 0$. By a compactness argument one then shows that a subsequence converges to a global weak entropic solution of (1). Indeed, by the results in [3, 4] the entire sequence will converge to the unique semigroup solution.

From a theoretical point of view these methods yield the same existence result. However from a numerical point of view they are not efficient. The method presented in [8] demands knowledge of the future history of the approximate solution, whereas in [1, 2] one has to keep track of the past history by counting the generation order of each wave.

In this paper we present a new algorithm which avoids these problems. More precisely, we introduce a threshold parameter ε which dictates how to solve the Riemann problems. If the product of the strengths of the colliding waves is greater than ε then we use an accurate approximation to the Riemann problem, whereas if the product is smaller than ε we use a crude approximation. We prove that if ε tends to zero sufficiently fast, compared with other approximation parameters (the number of initial jumps and the maximal size of rarefaction fronts), then our sequence of approximations u_ν converges to an entropy weak solution of the Cauchy Problem (1).

2 Preliminaries

We consider a strictly hyperbolic $n \times n$ system of conservation laws (1) in which each characteristic family is either genuinely nonlinear or linearly degenerate, and where the flux f is C^2 on a set $\Omega \subset \mathbb{R}^n$. The function $\bar{u}$ is assumed to be of sufficiently small total variation. We recall that, given two states u^-, u^+ sufficiently close, the corresponding Riemann problem admits a self-similar solution given by at most $n+1$ constant states separated by shocks or rarefaction waves [7]. More precisely, there exist C^2 curves $\sigma \mapsto \psi_i(\sigma)(u^-)$, $i = 1, \dots, n$, parametrized by arclength, such that

$$u^+ = \psi_n(\sigma_n) \circ \cdots \circ \psi_1(\sigma_1)(u^-), \tag{2}$$

for some $\sigma_1, \dots, \sigma_n$. We define $u_0 \doteq u^-$ and $u_i \doteq \psi_i(\sigma_i) \circ \cdots \circ \psi_1(\sigma_1)(u_0)$. When σ_i is positive (negative) and the i-th characteristic family is genuinely nonlinear, the states u_{i-1} and u_i are separated by an i-rarefaction (i-shock) wave. If the i-th characteristic family is linearly degenerate the states u_{i-1} and u_i are separated by a contact discontinuity. The *strength* of the i-wave is defined as $|\sigma_i|$.

In order to estimate the change in the strength of waves under interactions we need some well-known results, basically due to Glimm [6].

Lemma 2.1 *For any compact set $K \subset \Omega$, there exist constants C_1, μ_1 such that*

(i) If $u^- \in K$, $|\sigma_i'|, |\sigma_j'| \leq \mu_1$, and $u^+ = \psi_j(\sigma_j') \circ \psi_i(\sigma_i')(u^-)$ with $i > j$, then the coefficients σ_i defined in (2) satisfy

$$|\sigma_i - \sigma_i'| + |\sigma_j - \sigma_j'| + \sum_{k \neq i,j} |\sigma_k| \leq C_1 |\sigma_i' \sigma_j'|. \tag{3}$$

Moreover,

$$|u^+ - \psi_i(\sigma_i') \circ \psi_j(\sigma_j')(u^-)| \leq C_1 |\sigma_i' \sigma_j'|. \tag{4}$$

(ii) If $u^- \in K$, $|\sigma_i'|, |\sigma_i''| \leq \mu_1$, and $u^+ = \psi_i(\sigma_i'') \circ \psi_i(\sigma_i')(u^-)$, then the coefficients σ_i defined in (2) satisfy

$$|\sigma_i - \sigma_i' - \sigma_i''| + \sum_{k \neq i} |\sigma_k| \leq C_1 |\sigma_i' \sigma_i''|. \tag{5}$$

Moreover,

$$|u^+ - \psi_i(\sigma_i' + \sigma_i'')(u^-)| \leq C_1 |\sigma_i' \sigma_i''|. \tag{6}$$

(iii) If $u^- \in K$, $|\sigma|, |u^- - v^-| \leq \mu_1$ and $u^+ = \psi_i(\sigma)(u^-)$, $v^+ = \psi_i(\sigma)(v^-)$, then

$$|v^+ - u^+| - |v^- - u^-| \leq C_1 |\sigma| |v^- - u^-|. \tag{7}$$

3 The Algorithm

For a given initial data $\bar{u}$ let $\bar{u}_\nu$ be a sequence of piecewise constant functions approximating $\bar{u}$ in the L^1-norm. Let N_ν be the number of discontinuities in the function $\bar{u}_\nu$, and choose a parameter $\delta_\nu > 0$ controlling the maximum strength of rarefaction fronts.

In order to construct a piecewise constant approximation for all positive times, we introduce various ways of solving the Riemann Problems generated by wave-front interactions. More precisely, we will use an accurate solver when the product of the strengths of the incoming fronts is "large", a simplified solver yielding non-physical waves when the product is "small", or when one of the incoming waves is non-physical.

3.1 The Riemann Solvers

Assume for a positive time $\bar{t}$ we have an interaction at $\bar{x}$ between two waves of families i_α, i_β and strengths σ_α', σ_β', respectively. Here σ_α' denotes the left incoming wave. Let u^-, u^+ be the Riemann problem generated by the interaction and let $\sigma_1, \ldots, \sigma_n$ and $u_0, \ldots, u_n$ be defined as in (2). We define the following approximate Riemann solvers.

(A) *Accurate Solver*: if the i-th wave belongs to a genuinely nonlinear family and $\sigma_i > 0$ then we let

$$p_i \doteq \lceil \sigma_i / \delta_\nu \rceil, \tag{8}$$

where $\lceil s \rceil$ denotes the smallest integer number greater than s. For $l = 1, \ldots, p_i$ we define

$$u_{i,l} \doteq \psi_i(l\sigma_i/p_i)(u_{i-1}), \qquad x_{i,l}(t) \doteq \bar{x} + (t - \bar{t})\lambda_i(u_{i,l}), \tag{9}$$

where $\lambda_i(\cdot)$ denotes the i-th characteristic speed. Otherwise, if the i-th characteristic family is linearly degenerate, or it is genuinely nonlinear and $\sigma_i \le 0$, we define $p_i \doteq 1$ and

$$u_{i,l} \doteq u_i \qquad x_{i,l}(t) \doteq \bar{x} + (t - \bar{t})\lambda_i(u_{i-1}, u_i). \tag{10}$$

Here $\lambda_i(u_{i-1}, u_i)$ is the speed given by the Rankine-Hugoniot conditions of the i-shock connecting the states u_{i-1}, u_i. In this case the approximate solution of the Riemann problem is defined in the following way

$$v_a(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < x_{1,1}(t), \\ u^+ & \text{if} \quad x > x_{n,p_n}(t), \\ u_i & \text{if} \quad x_{i,p_i}(t) < x < x_{i+1,1}(t), \\ u_{i,l} & \text{if} \quad x_{i,l}(t) < x < x_{i,l+1}(t), \quad (l = 1, \ldots, p_i - 1). \end{cases} \tag{11}$$

(B) *Simplified Solver*: define

$$u' \doteq \begin{cases} \psi_{i_\alpha}(\sigma'_\alpha) \circ \psi_{i_\beta}(\sigma'_\beta)(u^-) & \text{if} \quad i_\alpha > i_\beta, \\ \psi_{i_\alpha}(\sigma'_\alpha + \sigma'_\beta)(u^-) & \text{if} \quad i_\alpha = i_\beta. \end{cases} \tag{12}$$

Let $v_a(t,x)$ be the approximate solution of the Riemann problem (u^-, u') given by (11). By construction this function contains exactly two wave-fronts of strengths $\sigma'_\beta, \sigma'_\alpha$, if $i_\alpha > i_\beta$, or a single wave-front of strength $\sigma'_\alpha + \sigma'_\beta$, if $i_\alpha = i_\beta$. Observe that in general $u' \neq u^+$ hence we introduce a *non-physical front* between these states (while we shall call all the other waves physical). The strength of this wave is defined as $|\sigma| \doteq |u' - u^+|$. This non-physical front is given a speed $\hat{\lambda} > 0$ strictly greater than all characteristic speeds in Ω. We thus define the simplified solution in the following way

$$v_s(t,x) \doteq \begin{cases} v_a(t,x) & \text{if} \quad x - \bar{x} < \hat{\lambda}(t - \bar{t}), \\ u^+ & \text{if} \quad x - \bar{x} > \hat{\lambda}(t - \bar{t}). \end{cases} \tag{13}$$

(C) *Crude Solver*: assume that the left incoming wave is a non-physical wave. Now let $u' \doteq \psi_{i_\beta}(\sigma'_\beta)(u^-)$ and let $v_a(t,x)$ be the approximate solution of the Riemann problem (u^-, u') given by (11). In this case v_a contains only one wave of the i_β-th family with strength σ'_β, connecting the states u^- and u'. Again $u' \neq u^+$ in general, hence also in this case we introduce a non-physical wave between these two states with the same speed $\hat{\lambda}$. Finally we define the crude solution v_c as in (13).

We notice that since all the non-physical waves travel with the same speed $\hat{\lambda}$, they cannot interact with each other.

3.2 Construction of the Approximate Solutions

Given ν we construct the approximate solution $u_\nu(t,x)$ as follows. At time $t=0$ all Riemann problems in $\bar{u}_\nu$ are solved accurately as in (A). By slightly perturbating the speed of a wave, we can assume that at any positive time we have at most one collision involving only two incoming fronts. Suppose that at some time $t>0$ there is a collision between two waves belonging to the i_α-th and i_β-th families. Let σ_α and σ_β be the strengths of the two waves. The Riemann problem generated by this interaction is solved as follows. Let ε_ν be a fixed small parameter that will be specified later.

- if $|\sigma_\alpha\sigma_\beta| > \varepsilon_\nu$ and the two waves are physical, then we use the accurate solver (A);
- if $|\sigma_\alpha\sigma_\beta| < \varepsilon_\nu$ and the two waves are physical, then we use the simplified solver (B);
- if one wave is non-physical we use the crude solver (C).

We claim that this algorithm yields a converging sequence of approximate solutions defined for all times $t>0$, for any ε_ν. We prove this by introducing the standard functionals V and Q measuring the total variation and interaction potential, respectively.

For a fixed ν and time $t>0$ at which no interaction occurs in $u_\nu(t,\cdot)$, let $x_1(t) < \cdots < x_m(t)$ be the discontinuities in $u_\nu(t,\cdot)$, and denote by $\sigma_1,\ldots,\sigma_m$ and $i_1,\ldots,i_m$ their strengths and families, respectively. Two waves $\sigma_\alpha,\sigma_\beta$ with $x_\alpha < x_\beta$ are said to be *approaching* if one of the following conditions holds

(1) if either $i_\alpha = i_\beta$ and one of them is a shock, or if $i_\alpha > i_\beta$;

(2) if σ_α is a non-physical wave and σ_β is a physical wave.

The *total strength* V and the *interaction potential* Q are defined as

$$V(t) \doteq \sum_\alpha |\sigma_\alpha|, \qquad Q(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha\sigma_\beta|, \tag{14}$$

where $\mathcal{A}$ denotes the set of all approaching waves at time t. The following lemma is standard.

Lemma 3.1 *Given a compact set $K\subset\Omega$ there exist constants $C_2,\mu_2>0$ such that the following holds. Let u be any piecewise constant approximate solution of (1) constructed as above defined in the strip $[0,T[\times\mathbb{R}$ and with $T.V.(u(0,\cdot)) \leq \mu_2$, $\lim_{x\to-\infty} u(0,x)\in K$. Then*

$$T.V.(u(t,\cdot)) \leq C_2 \cdot T.V.(u(0,\cdot)), \qquad \forall\, t>0. \tag{15}$$

Moreover, if at time t two waves of strengths σ_α and σ_β interact, then

$$\Delta Q(t) \leq -\frac{|\sigma_\alpha\sigma_\beta|}{2}, \qquad \Delta V(t) + 2C_1\Delta Q(t) \leq 0. \tag{16}$$

The proof of this lemma relies on the classical estimates by Glimm, [6, 1, 8]; for the changes needed due to the presence of non-physical waves we use lemma 2.1, see [1, 2].

We use this lemma to prove that the number of wave-fronts is finite. For each ν consider the set of collisions for which the interaction potential between the incoming waves is greater than ε_ν. By the first bound in (16), Q decreases by at least $\varepsilon_\nu/2$ in these interactions. Since new physical waves can only be generated by this kind of interactions, it follows that their number is finite. Furthermore, since non-physical waves are introduced only when physical waves interact, the number of non-physical waves is also finite. This implies that the approximate solutions are defined for all positive times, i.e. $T=\infty$ for each ν. We can now state the main result of this paper.

Theorem 3.2 *Let $\bar{u}$ be of small total variation, and let $\bar{u}_\nu$ converge to $\bar{u}$ in the L^1-norm. Let N_ν be the number of jumps in $\bar{u}_\nu$, δ_ν the parameter controlling the maximum strengths of rarefaction fronts, and ε_ν the threshold parameter. If*

$$\lim_{\nu\to\infty}\delta_\nu=0, \qquad \lim_{\nu\to\infty}\varepsilon_\nu\left(N_\nu+\frac{1}{\delta_\nu}\right)^k=0, \tag{17}$$

for every positive integer k, then the sequence of piecewise constant approximations u_ν constructed by the above algorithm converges to an entropy weak solution of the Cauchy Problem (1).

4 Proof of the theorem

The above algorithm does not involve the notion of generation order of wave-fronts. However, the proof of the theorem will make use of this concept. Fix a ν. First we assign order 1 to all the waves at time $t=0^+$. For waves generated at times $t>0$ we assign orders inductively depending on the kind of interaction giving rise to the waves. Let the incoming waves be of strengths $\sigma_\alpha,\sigma_\beta$, of families i_α,i_β, and orders k_α,k_β, respectively. Three cases are considered.

(i) $|\sigma_\alpha\sigma_\beta|\geq\varepsilon_\nu$ and both waves are physical.

To each outgoing j-wave we assign order

$$\begin{cases} \max\{k_\alpha,k_\beta\}+1 & \text{if} \quad j\neq i_\alpha,i_\beta, \\ \min\{k_\alpha,k_\beta\} & \text{if} \quad j=i_\alpha=i_\beta, \\ k_\alpha & \text{if} \quad j=i_\alpha\neq i_\beta, \\ k_\beta & \text{if} \quad j=i_\beta\neq i_\alpha. \end{cases} \tag{18}$$

(ii) $|\sigma_\alpha\sigma_\beta|<\varepsilon_\nu$ and both waves are physical.

The outgoing physical fronts are assigned orders according to (18), while the non-physical wave is assigned order $\max\{k_\alpha,k_\beta\}+1$.

(iii) One of the incoming fronts is non-physical.

In this case the colliding waves maintain their order.

Now in order to have convergence to a weak solution of (1), we need to prove that both the maximal size of rarefaction waves and the total amount of non-physical waves present in u_ν tends to zero as $\nu \to \infty$.

By (8) it is clear that the strength of any rarefaction wave in u_ν is bounded by δ_ν.

In order to estimate the total amount of non-physical waves we introduce the following functionals. Let $V_k(t)$ be the sum at time t of the strengths of all waves in $u_\nu(t,\cdot)$ of order $\geq k$. Also let $Q_k(t)$ be the interaction potential between all couples of approaching waves of orders k_α, k_β with $\max\{k_\alpha, k_\beta\} \geq k$. Let I_k be the set of times when two waves of orders k_α, k_β with $\max\{k_\alpha, k_\beta\} = k$ interact. Observe that, since the number of waves is finite, there is a maximal generation order for the waves in u_ν, call it s_ν. By using lemma 2.1 we now have the following estimates.

$$\begin{array}{rl} \Delta V_k(t) = 0 & t \in I_1 \cup \cdots \cup I_{k-2}, \\ \Delta V_k(t) + 2C_1 \Delta Q_{k-1}(t) \leq 0 & t \in I_{k-1} \cup \cdots \cup I_{s_\nu}, \\ \Delta Q_k(t) + 2C_1 \Delta Q(t) V_k(t-) \leq 0 & t \in I_1 \cup \cdots \cup I_{k-2}, \\ \Delta Q_k(t) + 2C_1 \Delta Q_{k-1}(t) V(t-) \leq 0 & t \in I_{k-1}, \\ \Delta Q_k(t) \leq 0 & t \in I_k \cup \cdots \cup I_{s_\nu}. \end{array} \tag{19}$$

As in [1, 2], it follows (if necessary by taking μ_2 smaller) that for all $k \geq 2$

$$V_k(t) \leq 4C_1 2^{-k}, \qquad \forall t \geq 0, \tag{20}$$

uniformly in ν.

Now we want to prove that the total amount of non-physical waves at a given time t tends to zero as $\nu \to \infty$. The bound (20) will be used to estimate the total strength of higher order non-physical fronts. For lower order non-physical waves we need another estimate. Let M_i denote the number of fronts of order i in the approximate solution u_ν. Since a k-th order wave can be generated only from an interaction between one of order $k-1$ and one of order $j \leq k-1$ and since only one non-physical wave can be generated from each interaction, we have (see also [2])

$$\begin{aligned} \#\left\{ \begin{array}{c} \text{non-physical waves} \\ \text{of order } k \end{array} \right\} &\leq (M_1 + \cdots + M_{k-1}) M_{k-1} \\ &\leq C(k) \left(N_\nu + \frac{1}{\delta_\nu} \right)^{p(k)}, \end{aligned} \tag{21}$$

for a suitable constant $C(k)$ and some positive integer $p(k)$. We note that the estimate (21) is useful only for each fixed k; since the constants $C(s_\nu)$ and $p(s_\nu)$ grow too fast as $\nu \to \infty$, we cannot use this to estimate the total amount of non-physical waves of all orders.

From (4) and (6) the strength of a non-physical wave generated by an interaction between two physical waves is bounded by $C_1 \varepsilon_\nu$. As a non-physical front interacts with physical ones, its strength can increase. However as in [2], lemma 2 pp. 115–116, there exists a constant C_3 such that for all times the strength of the wave remains bounded by $C_1 C_3 \varepsilon_\nu$.

Now we can estimate the total strength of the non-physical waves at time t in the following way. Denote by $\mathcal{N}(t)$ the set of indices α corresponding to non-physical fronts σ_α at time t. By (20) and (21) it follows that

$$\sum_{\alpha\in\mathcal{N}(t)} |\sigma_\alpha| \leq \sum_{k\leq k_0} C(k)\left(N_\nu + \frac{1}{\delta_\nu}\right)^{p(k)} \cdot C_1C_3\varepsilon_\nu + \sum_{k\geq k_0} 4C_1 2^{-k}, \tag{22}$$

for some integer number k_0.

Given $\rho > 0$, choose k_0 such that $\sum_{k\geq k_0} 4C_1 2^{-k} \leq \rho/2$. Next, by the second condition in (17), take ν so large that

$$\sum_{k\leq k_0} C(k)\left(N_\nu + \frac{1}{\delta_\nu}\right)^{p(k)} \cdot C_1C_3\varepsilon_\nu < \frac{\rho}{2}. \tag{23}$$

By (22) and (23) it follows

$$\sum_{\alpha\in\mathcal{N}(t)} |\sigma_\alpha| \leq \rho, \tag{24}$$

for large ν, uniformly in t.

Having established these estimates we can now show that the sequence $\{u_\nu\}$ yields a weak solution of (1) as ν tends to ∞. We recall here that a weak solution u of (1) is a distributional solution, i.e. for any fixed smooth function ϕ with compact support in $\mathbb{R}\times\mathbb{R}$ it satisfies

$$\int_{-\infty}^{+\infty} \bar{u}(x)\phi(0,x)\,dx + \int_0^{+\infty}\int_{-\infty}^{+\infty} \Big(u(t,x)\phi_t(t,x) + f(u(t,x))\phi_x(t,x)\Big)\,dxdt = 0. \tag{25}$$

We can choose the approximate initial data $\bar{u}_\nu$ so that $\text{T.V.}(\bar{u}_\nu) \leq \mu$, where $\mu \leq \min\{\mu_1,\mu_2\}$. By lemma 3.1 it follows that $\text{T.V.}(u_\nu(t,\cdot)) < C\mu$ for all $t \geq 0$ and all positive integers ν, for some constant C. It is also easy to prove that the functions $t \mapsto u_\nu(t,\cdot)$ are Lipschitz continuous uniformly in ν, as functions from $[0,+\infty)$ to L^1. Hence by Helly's theorem there exists a subsequence which converges in L^1_{loc} to some function $u(t,x)$. To prove that u is a weak solution of (1) it suffices to show that

$$\lim_{\nu\to\infty}\left[\int_{-\infty}^{+\infty} \bar{u}_\nu(x)\phi(0,x)\,dx + \int_0^{+\infty}\int_{-\infty}^{+\infty}\Big(u_\nu\phi_t + f(u_\nu)\phi_x\Big)\,dxdt\right] = 0, \tag{26}$$

for any smooth function ϕ with compact support.

Fix ϕ with support in $]-\infty,T]\times\mathbb{R}$. By the divergence theorem the expression in brackets in (26) is computed as

$$\begin{aligned}&\int_{-\infty}^{+\infty}\Big(\bar{u}_\nu(x) - \bar{u}(x)\Big)\phi(0,x)\,dx + \\ &\qquad + \int_0^T \sum_\alpha \Big(\dot{x}_\alpha\Big[u_\nu(t,x_\alpha)\Big] - \Big[f(u_\nu(t,x_\alpha))\Big]\Big)\phi(t,x_\alpha)\,dt,\end{aligned}$$

where the $x_\alpha = x_\alpha(t)$ denote the lines of discontinuity of u_ν in the strip $[0,T]\times\mathbb{R}$, and $[\cdot]$ denotes the jump across these discontinuities. By assumption the first term in this equation tends to zero as $\nu\to\infty$.

To estimate the second term, let $\mathcal{R}(t)$ be the set of indices α corresponding to rarefactions at time t, and let σ_α be the strength of the wave at x_α. Given $\rho > 0$, we can choose k_0 as above such that (24) holds for ν large enough. Recalling that the strength of rarefaction fronts is $\leq \delta_\nu$ and proceeding as in [2] we obtain

$$\begin{aligned}
&\left| \int_0^T \sum_\alpha \left(\dot{x}_\alpha \Big[u_\nu(t, x_\alpha) \Big] - \Big[f(u_\nu(t, x_\alpha)) \Big] \right) \phi(t, x_\alpha)\, dt \right| \\
&\quad \leq C(\max|\phi|) \int_0^T \left(\sum_{\alpha \in \mathcal{R}(t)} |\sigma_\alpha(t)|^2 + \sum_{\alpha \in \mathcal{N}(t)} |\sigma_\alpha(t)| \right) dt \\
&\quad \leq CT(\max|\phi|)(\delta_\nu \max_t V(t) + \rho) \\
&\quad \leq CT(\max|\phi|)(\delta_\nu + \rho),
\end{aligned} \tag{27}$$

where C denotes constants independent of ν. By letting $\nu \to \infty$ and since ρ is arbitrary, (26) follows. Hence u is a weak solution of (1).

It remains to show that u is also an entropic solution for μ sufficiently small, i.e. given a flux, entropy-flux pair (η, q), we have

$$\eta(u)_t + q(u)_x \leq 0 \tag{28}$$

in the distributional sense. Also in this case, by doing the same splitting as in (27) we find (see [2])

$$\begin{aligned}
&\int_0^T \sum_\alpha \left(\dot{x}_\alpha \Big[\eta(u_\nu(t, x_\alpha)) \Big] - \Big[q(u_\nu(t, x_\alpha)) \Big] \right) \phi(t, x_\alpha)\, dt \\
&\quad \geq -CT \max|\phi|\, (\delta_\nu + \rho)\,.
\end{aligned} \tag{29}$$

Letting $\nu \to \infty$ in (29), we prove (28). This completes the proof of the theorem.

Remark 1. The approximation error in the above scheme consists of two different parts, due to the approximation of rarefaction waves and the introduction of non-physical waves. The error from approximation of rarefaction waves is $\mathcal{O}(\delta_\nu)$, whereas the error from the non-physical waves is split in two parts. By (20), the non-physical waves of generation $\geq k$ contribute by an amount $\mathcal{O}(2^{-k})$. From (21) the error due to non-physical waves of generation $\leq k$ is bounded by

$$\mathcal{O}\left(k n^{2^k} \delta_\nu^{-2^k} \varepsilon_\nu \right), \tag{30}$$

if we assume that N_ν is $\mathcal{O}(\delta_\nu^{-1})$. By asking the three errors to be of the same order of magnitude, it follows that $k = \mathcal{O}(|\log \delta_\nu|)$ and that an appropriate choice for ε_ν is given by

$$\varepsilon_\nu = \frac{\delta_\nu^{1+1/\delta_\nu}}{|\log \delta_\nu| n^{1/\delta_\nu}}. \tag{31}$$

Remark 2. The second condition in (17) is also necessary. Indeed, assume that both δ_ν and the strength of any wave at time $t = 0$ are $\mathcal{O}(N_\nu^{-1})$. By the interaction estimates one gets that the strength of a wave of generation k is $\mathcal{O}((C_1 N_\nu^{-1})^k)$. If (17) fails, i.e. ε_ν has only polynomial growth w.r.t N_ν^{-1}, then for ν large enough the approximate solution u_ν can contain waves of only a finite number of orders, independent of ν. Hence, in general this cannot yield a weak solution as $\nu \to \infty$.

Remark 3. The algorithm presented in this paper is numerically more efficient than the previous theoretical algorithms since one does not consider the generation order and keeps track only of the "big waves". This reflects what is actually done in practice when one implements front-tracking for systems of conservation laws.

Acknowledgements. We thank A. Bressan for suggesting this problem and for many helpful discussions. The second author thanks A. Bressan and N. H. Risebro for the opportunity of visiting S.I.S.S.A.

References

[1] A. Bressan: *Global Solutions to Systems of Conservation laws by Wave-front tracking*, J. Math. Anal. Appl. **170** (1992), 414–432.

[2] A. Bressan: *Lecture Notes on Conservation Laws*, S.I.S.S.A., Trieste, 1995.

[3] A. Bressan: *The Unique Limit of the Glimm Scheme*, Arch. Rational Mech. Anal. **130** (1995), 205–230.

[4] A. Bressan, G. Crasta and B. Piccoli: *Well-posedness of the Cauchy problem for $n \times n$ systems of conservation laws*, preprint S.I.S.S.A, 1996.

[5] R. J. DiPerna: *Global Existence of Solution to Nonlinear Hyperbolic Systems of Conservation Laws*, J. Diff. Eq. **20** (1976), 187–212.

[6] J. Glimm: *Solutions in the Large for Nonlinear Hyperbolic Systems of Equations*, Comm. Pure. Appl. Math. **18** (1965), 697–715.

[7] P. D. Lax: *Hyperbolic Systems of Conservation Laws II*, Comm. Pure Appl. Math. **10** (1957), 537–566.

[8] N. H. Risebro: *A front-tracking alternative to the random choice method*, Proc. Amer. Math. Soc. **117**, no. 4 (1993), 1125–1139.