IDENTIFICATION OF MOBILITIES FOR THE BUCKLEY-LEVERETT EQUATION BY FRONT TRACKING

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ABSTRACT. Displacement experiments of oil by water are widely used to investigate the effects of two-phase flow. These one dimensional experiments are used to determine the fractional flow and the total mobility functions that are used as input for reservoir simulations. A new method has been developed to find fractional flow and total mobility functions that will reproduce the discrete measurements. Both constant rate and constant pressure drop experiments are investigated. The calculated fractional flow function is piecewise linear in the range of observed fractional flow values. It is assumed that that the core is homogeneous. Gravity, compressibility, and capillary effects are neglected.

1. INTRODUCTION

Multi–phase flow in porous media is modelled by Darcy's law. This empirical relation relates the macroscopic velocity of each phase to the pressure gradient. The proportionality factor is determined by three factors: absolute permeability is the ability of a porous medium to conduct flow (rock property), viscosity is a fluid property, and relative permeability is a measure of the reduction in ability of a phase to flow due to the presence of other fluids (rock/fluid property).

An overview of relative permeability measurements are given in [4]. Flooding experiments were first analysed by Johnson et al. [5]. Jones and Rozelle [6] developed a graphical technique to determine relative permeability from measurements at discrete times. For a constant pressure drop experiment, interpolation is used to find total velocity at times of measurements. These velocities are used to determine relative permeabilities by a graphical technique. Others have approximated relative permeabilities by multi–parameter families of functions, and determined the parameters by minimisation of the error (see [7]).

A new method to calculate relative permeabilities from measured data will be presented. We will consider the fractional flow and total mobility functions as the unknown functions. Relative permeabilities are easily calculated from fractional flow and total mobility data. We are able to construct a fractional flow function and a total mobility function that will reproduce the measured data with no errors.

The mathematical approach to one dimensional scalar conservation laws is based on front tracking. It is assumed that the flux function is piecewise linear, and that the initial data is piecewise constant. This Cauchy problem may be solved exactly by tracking the positions of the shocks defined by the initial data, and solving the Riemann-problems defined by collisions between shocks. This method was first presented by Dafermos [2], and later analysed by Holden et al. [3]. Front tracking is also used to solve the saturation equation in a commercial reservoir simulator [1].

2. EQUATIONS

The core is assumed to be homogeneous. Gravity, compressibility and diffusive forces will be neglected, and viscosities of the two phases are assumed to be constant. The velocity of a phase

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i, u_i, is described by Darcy's law:

$$u_i = -\frac{k k_{r_i}(s)}{\mu_i} p_x(x), \quad i = w, o,$$

where k is permeability, μ_i is viscosity of phase i, k_{r_i} is relative permeability of phase i, s is the saturation of water, and p is pressure. Conservation of mass for incompressible water and oil gives

$$\begin{aligned} \phi s_t + (u_w)_x &= 0, \\ \phi(1-s)_t + (u_o)_x &= 0, \end{aligned}$$

where ϕ is porosity. Total velocity is found to be constant in space by adding the conservation equations:

$$(u_o + u_w)_x = u_x = 0. \tag{1}$$

Total mobility is defined by $\lambda_T = \lambda_w + \lambda_o$, where $\lambda_i = k_{r_i}/\mu_i$, $i = w, o$. Using equation (1), pressure is given by the formula

$$p(x,t) = p(0,t) - \frac{u(t)}{k} \int_0^x \frac{dx}{\lambda_T(s(x,t))}. \tag{2}$$

By using the expression for the fractional flow curve for water, $f(s) = \frac{u_w}{u} = \frac{\lambda_w}{\lambda_T}$, the conservation equation for water may be written

$$\phi s_t + u f(s)_x = 0. \tag{3}$$

Equation (2) and (3) are decoupled by introducing new dimensionless variables

$$x_D = \frac{x}{L}, \qquad t_D = \frac{\int_0^t u(\tau) d\tau}{\phi L}. \tag{4}$$

Note that dimensionless time t_D is equal to the number of pore volumes injected. Equation (3) is now simplified to a scalar hyperbolic conservation law

$$s_{t_D} + f(s)_{x_D} = 0,$$

and the pressure drop in the core is given by

$$p(x_D, t_D) = p(0, t_D) - \frac{u(t_D) L}{k} \int_0^{x_D} \frac{dx_D}{\lambda_T(s(x_D, t_D))}.$$

3. Front Tracking Solution with Constant Rate

We will analyse a unsteady-state displacement of oil with water. Boundary conditions are given by

$$s(x,0) = s_R, \qquad s(0,t) = s_L > s_R, \qquad p(0,t) = p_L, \qquad u(t) = u_T > 0. \tag{5}$$

Assume that measurements are made at times $t_0, t_1, \cdots, t_M$, where $0 = t_0 < t_1 < \cdots < t_M$ and t_1 is the time of breakthrough. Let p_i be the pressure drop over the core at time t_i, and f_i the average fractional flow values at the outlet during time interval $[t_i, t_{i+1}]$. We will assume that $f_i < f_{i+1}$, $i = 0, \cdots, M-2$. The required measurements are indicated in Figure 1.

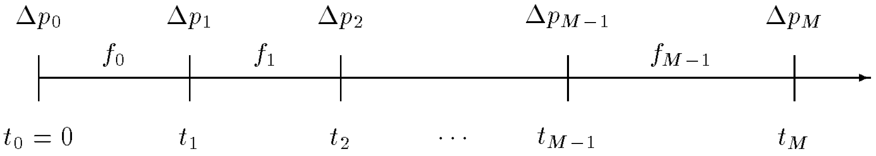

Figure 1. Measurements of pressure drop p_i and average fractional flow f_i are made at times t_i.

Define dimensionless time $t_{D_i} = u_T t_i/(\phi L)$. We will make the following assumptions on the fractional flow function:

> f is continuous, increasing, and piecewise linear in saturation for fractional flow values in the range $f_0 < f < f_{M-1}$. Breakpoints[1] are given by (s_i, f_i). No assumptions are made on f for fractional flow values larger than f_{M-1}.

The structure of the solution is simple in this case: The saturation profile is piecewise constant for saturations corresponding to fractional flow values in the interval $[f_0, f_{M-1}]$. Let s_i be the saturation corresponding to f_i. The dimensionless velocity of the front between f_{i-1} and f_i is given by the Rankine–Hugoniot condition:

$$v_{i-1,i} = \frac{f_i - f_{i-1}}{s_i - s_{i-1}}.$$

Note that average fractional flow values are reproduced if producing fractional flow in the interval $[t_{i-1}, t_i]$ is given by the constant f_i. We will therefore assume that the front between s_{i-1} and s_i reach the outlet at dimensionless time t_{D_i}:

$$\frac{f_i - f_{i-1}}{s_i - s_{i-1}} t_{D_i} = 1,$$

Note that saturations may be calculated sequentially

$$s_i = s_{i-1} + (f_i - f_{i-1}) t_{D_i}, \; i = 1, \cdots, M-1, \qquad s_0 = s_R. \tag{6}$$

Let λ_{T_i} be the total mobility corresponding to s_i. We will calculate λ_{T_i}, $i = 1, \cdots, M-1$, by using the measured pressure drops. The saturation profile is piecewise constant for saturations between s_0 and s_{M-1}. Thus, total mobility will also be piecewise constant in the corresponding part of space at a given time. The pressure drop may be rewritten as

$$\Delta p = \Delta p^{(M)} + \Delta p^{(M-1)} + \cdots + \Delta p^{(1)} + \Delta p^{(0)},$$

where $\Delta p^{(M)}$ is the pressure drop due to saturations not observed at the outlet, and $\Delta p^{(i)}$ is the pressure drop due to total mobility λ_{T_i}. Since the saturation profile is a similarity solution, $s(x_D, t_D) = s(x_D/t_D) = s(\xi)$, $\xi = x_D/t_D$, we have that

$$\Delta p^{(M)}(t_D) = \frac{u_T L}{k} \int_0^{t_D/t_{D_M}} \frac{dx_D}{\lambda_T(x_D/t_D)} = \frac{u_T L}{k} t_D \int_0^{1/t_{D_M}} \frac{d\xi}{\lambda_T(\xi)} \equiv \frac{u_T L}{k} t_D I_M.$$

For $\Delta p^{(i)}$, $i = 1, \cdots, M-1$ we find that

$$\Delta p^{(i)}(t_D) = \frac{u_T L}{k} \int_{t_D/t_{D_{i+1}}}^{t_D/t_{D_i}} \frac{dx_D}{\lambda_{T_i}} = \frac{u_T L}{k} t_D \frac{1/t_{D_i} - 1/t_{D_{i+1}}}{\lambda_{T_i}} \equiv \frac{u_T L}{k} t_D I_i.$$

Initial total mobility λ_{T_0} is calculated from $\Delta p_0 = \frac{u_T L}{k \lambda_{T_0}}$. The integral I_M and the total mobilities λ_{T_i}, $i = M-1, \cdots, 1$ are calculated sequentially by matching the observed pressure drops Δp_i. Since $\Delta p_i = \Delta p^{(M)}(t_{D_i}) + \cdots + \Delta p^{(i)}(t_{D_i})$, we have

$$\frac{\Delta p_i \, k}{u_T L t_{D_i}} = I_M + I_{M-1} + \cdots + I_i, \qquad i = 1, \cdots, M. \tag{7}$$

Theorem 1. *Let the fractional flow function f be calculated from equation (6), and the total mobility from equation (7). Then these functions will reproduce the measured data Δp_i and f_i.*

Numerical results for this algorithm are given in section 5.

[1] We will use the following definition: The point (s_b, f_b) is a *breakpoint* of the function f if f' is discontinuous at s_b and $f(s_b) = f_b$.

4. Front Tracking Solution with Constant Pressure Drop

We will now study the displacement of oil with the following boundary and initial conditions

$$s(x,0) = s_R, \qquad s(0,t) = s_L > s_R, \qquad p(0,t) = p_L, \qquad p(L,t) = p_R < p_L. \tag{8}$$

Since total velocity is not constant in time, dimensionless time will now depend on the pressure solution.

Let u_0 be the total velocity at time 0, $\bar{u}_i$ be the measured average total velocity during time interval $[t_i, t_{i+1}]$,

$$\bar{u}_i = \frac{1}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} u(t)dt,$$

and f_i average effluent fractional flow values in the same time interval

$$f_i = \frac{1}{\bar{u}_i(t_{i+1} - t_i)} \int_{t_i}^{t_{i+1}} u(t) f(L,t)dt.$$

The available measured quantities are illustrated in Figure 2.

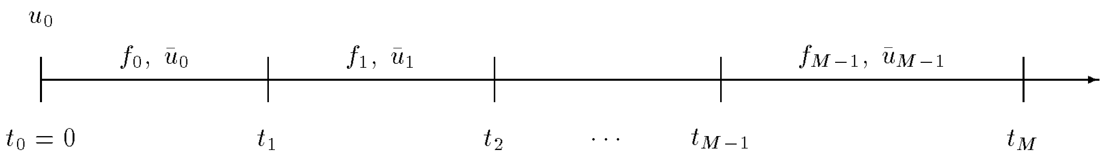

Figure 2. Measurements of average total velocity $\bar{u}_i$ and average fractional flow f_i are made at times t_i.

Dimensionless time t_{D_i} is the number of pore volumes injected at time t_i:

$$t_{D_i} = \frac{1}{\phi L} \sum_{j=1}^{i} (t_j - t_{j-1}) \bar{u}_{j-1}.$$

Since the saturation equation depends on dimensionless time only, s_i is given by the result in the previous section, see equation (6).

Initial mobility λ_{T_0} is calculated from the equation $\frac{\Delta p\, k}{L} = \frac{u_0}{\lambda_{T_0}}$. At time t_i we have the following expression for the pressure drop

$$\frac{\Delta p\, k}{L} = u_i t_{D_i} (I_M + I_{M-1} + \cdots + I_i), \quad i = 1, \cdots, M, \tag{9}$$

where u_i is the unknown total velocity at time t_i. These velocities are not measured, and must be calculated from the observed initial velocity u_0 and average total velocities $\bar{u}_i$, $i = 0, \cdots, M-1$. Pressure drop for $0 < t < t_1$ is described by:

$$\frac{\Delta p\, k}{L} = u(t) \left(t_D (I_M + I_{M-1} + \cdots + I_1) + \frac{1 - t_D/t_{D_1}}{\lambda_{T_0}} \right).$$

By using the expression for initial mobility, total velocity may be written

$$u(t) = \frac{u_0}{1 + b_0 t_D}, \tag{10}$$

where $b_0 = \lambda_{T_0}(I_M + \cdots + I_1) - 1/t_{D_1}$. Total velocity for $t_i < t < t_{i+1}$ may be expressed by the integral equation

$$u(t) = \frac{u_i}{1 + b_i (t_D(t) - t_{D_i})}, \qquad t_i < t < t_{i+1}, \tag{11}$$

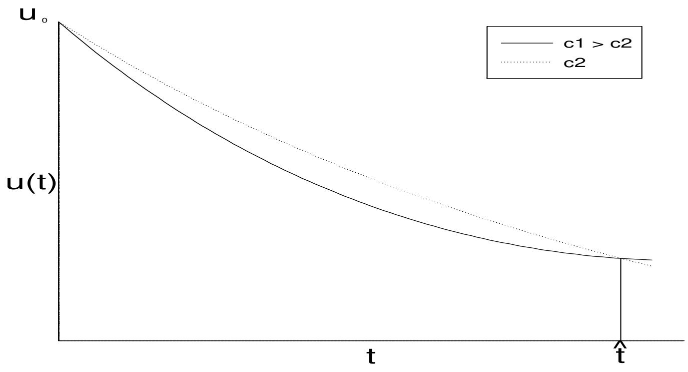

FIGURE 3. The solutions $u_{b_1}(t)$ and $u_{b_2}(t)$ are assumed to have a point of intersection at time $\hat{t}$.

where

$$b_i = \frac{(I_M + \cdots + I_{i+1})/t_{D_i} - (I_M + \cdots + I_i)/t_{D_{i+1}}}{(I_M + \cdots + I_i)t_{D_i}(1/t_{D_i} - 1/t_{D_{i+1}})}, \qquad i = 1, \cdots, M-1.$$

Lemma 1. *Let $u(t)$ be the solution of equation (11) on the interval $t_i < t < t_{i+1}$, where t_D is given by equation (4). If u_i and t_{i+1} are fixed, $\bar{u}(b_i) = \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}} u(\tau)d\tau$ is a strictly decreasing function of b_i.*

Proof It is sufficient to prove the theorem in the case $t_{D_i} = 0$. Equation (10) may be written

$$u(t) = \frac{u_0}{1 + c\int_0^t u(\tau)d\tau}, \quad c = \frac{b_0}{\phi L},$$

and by differentiation with respect to time we find that

$$u'(t) = -c\,\frac{u(t)^2}{1 + c\int_0^t u(\tau)d\tau}, \quad u(0) = u_0. \tag{12}$$

Notice that $u'(0) = -cu_0^2$. The solution $u(t)$ will be less than initial velocity u_0, $u(t) < u_0$, if $c > 0$. Also, $u(t) > u_0$ if $c < 0$. Assume there are constants c_1 and c_2, $c_1 > c_2 > 0$, such that the corresponding solutions $u_{c_1}(t)$ and $u_{c_2}(t)$ intersect. Let $\hat{t}$ be the first time of intersection. The following inequality connects the slopes of the curves at $\hat{t}$:

$$u'_{c_1}(\hat{t}) = -c_1\frac{u_{c_1}(\hat{t})^2}{1 + c_1\int_0^{\hat{t}} u_{c_1}(\tau)d\tau} \geq -c_2\frac{u_{c_2}(\hat{t})^2}{1 + c_2\int_0^{\hat{t}} u_{c_2}(\tau)d\tau} = u'_{c_2}(\hat{t}).$$

Write this inequality

$$c_1c_2\int_0^{\hat{t}}(u_{c_1} - u_{c_2})(\tau)d\tau \geq c_1 - c_2 > 0.$$

We have reached a contradiction since $c_1c_2 > 0$ and $\int_0^{\hat{t}}(u_{c_1} - u_{c_2})(\tau)d\tau < 0$ (see figure 3). A similar argument will prove the lemma if $c_2 < c_1 < 0$. ■

By Lemma 1 there is a unique b_i such that $\bar{u}(b_i) = \bar{u}_i$, $i = 0, \cdots, M-1$. By this procedure we are able to find total velocities $u(t_i)$, $i = 1, \cdots, M$, and total mobilities λ_{T_i} may be calculated from equation (7). The results in this section are summarised in the following theorem.

Theorem 2. *Let the fractional flow function f be calculated from equation (6), and total mobility from equation (7), with instantaneous total velocities from equation (11). Then these functions will reproduce the measured data $\bar{u}_i$ and f_i.*

W_i (ml)	N_p (ml)	Δp (psi)
0.00	0.00	138.6
7.00	7.00	97.5
11.20	7.84	91.9
16.28	8.43	87.9
24.27	8.93	83.7
39.2	9.30	78.5
62.3	9.65	74.2
108.9	9.96	70.0
155.6	10.11	68.1
311.3	10.30	65.4

TABLE 1. Data for calculation of relative permeabilities from a constant rate waterflood example.

f	s	λ_T	k_{r_o}	k_{r_w}
.000000	.350000	.074149	.774859	.000000
.800000	.529819	.096197	.201052	.074649
.883858	.559977	.102037	.123841	.087481
.937422	.587978	.106545	.069674	.096881
.975218	.617433	.111522	.028881	.105495
.984848	.629555	.119828	.018973	.114472
.993348	.646586	.128727	.008948	.124035
.996788	.658616	.138076	.004635	.133503
.998780	.668568	.145160	.001851	.140633

TABLE 2. Calculated results from data in Table 1.

5. Examples

We will present two examples. The first is a constant rate waterflood example from Jones and Rozelle [6]. Data for the second example is generated by solving the flow equations with given relative permeabilities. Average total velocity and fractional flow are calculated in some given time intervals. Relative permeabilities from "measured" data are compared with true relative permeability.

5.1. **Example 1.** The input data from Jones and Rozelle [6] are as follows:

$$\begin{array}{llll} \mu_o = 10.45 \text{ cp} & \mu_w = 0.97 \text{ cp} & s_0 = 0.35 & L = 12.705 \text{ cm} \\ k = 35.4 \text{ mD} & \phi = 0.215 & u = 1.95 \cdot 10^{-3} \text{ cm/s} & \end{array}$$

Production data are given in Table 1. W_i is total water volume injected, and N_p is cumulative production of oil. The fractional flow values given are calculated from W_i and N_p: $f = 1 - \Delta N_p / \Delta W_i$. Pressure drop is monotonically decreasing since the viscosity of the oil is an order of magnitude larger than the viscosity of water. Also, there is no clear breakpoint in pressure drop at the time of breakthrough.

Calculated relative permeability curves are close to the results given by Jones and Rozelle, and pressure drop and production are matched exactly at the points of measurement.

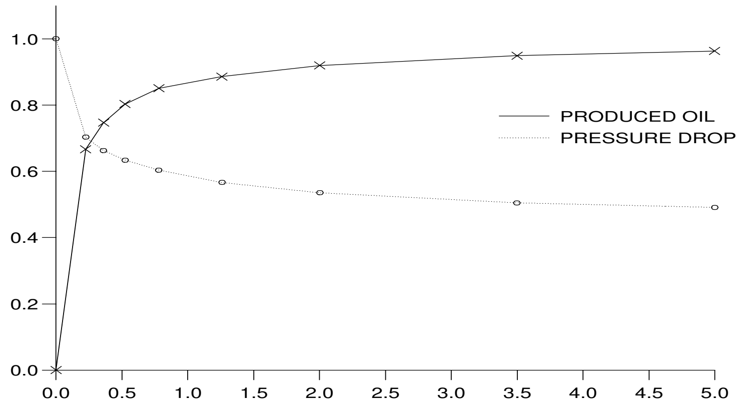

FIGURE 4. Normalised pressure drop $\Delta p/\Delta p_0$ and normalised cumulative oil production vs. dimensionless time. Markers indicate measured points.

5.2. **Example 2.** The algorithms in section 4 is tested on data from a given model. Equation (12) is solved by a simple difference scheme, and the integral is estimated by the trapezoidal method. The input data are

$$\begin{array}{lll} \mu_o = 1.0 \text{ cp} & \mu_w = 1.0 \text{ cp} & L = 10.0 \text{ cm} \\ k = 10 \text{ mD} & \phi = 0.3 & \Delta p = 1.0 \text{ atm.} \\ k_{r_w}(s) = (s-0.1)^2 & k_{r_o}(s) = (0.9-s)^2 & s_0 = 0.1. \end{array}$$

Production for this model is shown in Figure 5. Notice the breakpoint in total velocity at the time of breakthrough. Average total velocity and fractional flow values are calculated for the time intervals given in Table 3.

t_i (s)	$\bar{u}_i$ (cm/s)	f_i
373.16	0.0053280376	0.0000000000
400.00	0.0045935717	0.8607505311
500.00	0.0047227435	0.8882508802
700.00	0.0049687473	0.9274192642
1000.0	0.0052568970	0.9585494458
1500.0	0.0055403794	0.9784913693
2000.0	0.0057515825	0.9884718845

TABLE 3. Averaged total velocities $\bar{u}_i$ and averaged fractional flow values f_i in some given time intervals.

From these "measured" data, fractional flow and total mobility values are calculated. The results are given in Figure 6. It is seen that the difference between the correct and calculated relative permeabilities is small.

Previous publications on determination of relative permeability have only presented relative permeability curves from data measured in the laboratory, where the "true" relative permeability curves are unknown. The author suggests that algorithms should be tested on data calculated from given relative permeability curves before they are used to analyse laboratory experiments.

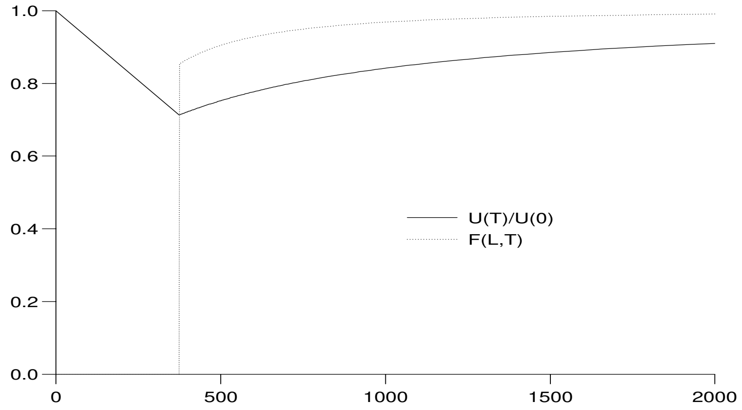

FIGURE 5. Normalised total velocity $u(t)/u_0$ and producing fractional flow vs. time.

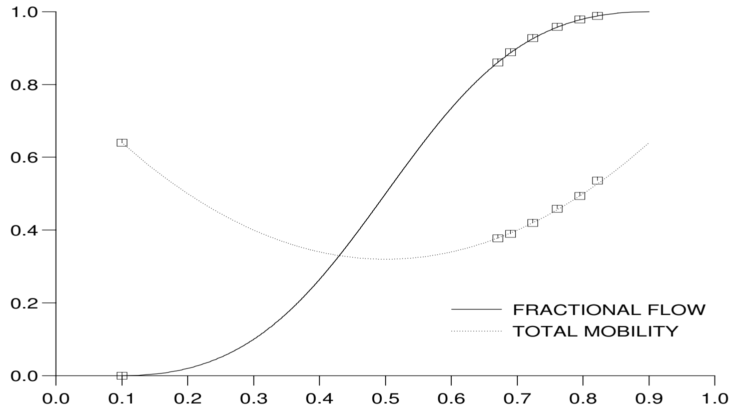

FIGURE 6. Fractional flow and total mobility function vs. saturation. Markers indicate measured points.

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