Large Time Behavior of the Solutions to a Hydrodynamic Model for Semiconductors

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Abstract

We establish the global existence of smooth solutions to the Cauchy problem for the one–dimensional isentropic Euler–Poisson (or hydrodynamic) model for semiconductors for small initial data. In particular we show that, as $t \to \infty$, these solutions converge to the stationary solutions of the drift–diffusion equations. The existence and uniqueness of stationary solutions to the drift-diffusion equations is proved without the smallness assumption.

1 Introduction

In this note we are interested in the large time behavior of the solutions for the one–dimensional Euler–Poisson (or hydrodynamic) model for semiconductors in the simple isentropic case, where the energy equation is replaced by a pressure–density relation. Our model equations are given by the one–dimensional system (see e.g. [MRS])

$$\begin{cases} n_t + (nu)_x = 0 \\ (nu)_t + (nu^2 + p(n))_x = nE - \frac{nu}{\tau} \\ E_x = n - b(x) \end{cases} \tag{1.1}$$

for $(x,t) \in I\!R \times [0,+\infty)$. The system is supplemented with the initial conditions

$$n(x,0) = n_0(x) \quad u(x,0) = u_0(x) \tag{1.2}$$

for all $x \in I\!R$. Here n denotes the electron density, u the (average) particle velocity and E the (negative) electric field, which is generated by the Coulomb force of the particles.

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We set $j = nu$ for the current density. The function $b(x)$ stands for the density of fixed, positively charged background ions; $p = p(n)$ is the pressure–density relation, $p'(n) > 0$ for $n > 0$.

Global existence of weak entropy solutions to the Cauchy problem (1.1)–(1.2) has been shown by Marcati and Natalini in [MN1]. Here we are interested in the global existence and large time behavior of smooth solutions.

To give the feeling of the problem, let us introduce the new time scale $s = \tau t$ and denote the corresponding scaled quantities by

$$N^\tau(x,s) = n\Big(x, \frac{s}{\tau}\Big), \; J^\tau(x,s) = \frac{1}{\tau} j\Big(x, \frac{s}{\tau}\Big), \; \hat{E}^\tau(x,s) = E\Big(x, \frac{s}{\tau}\Big). \tag{1.3}$$

The strong limits as $\tau \to 0$ of N^τ, J^τ, $\hat{E}^\tau$ are denoted (if they exist) by N, J and $\hat{E}$ and satisfy the well–known drift–diffusion equations (see again [MRS])

$$\begin{cases} N_t + (N\hat{E} - p(N)_x)_x = 0 \\ J = -p(N)_x + n\hat{E} \\ \hat{E}_x = N - b(x). \end{cases} \tag{1.4}$$

This fact was proved in [MN2] for weak solutions, using the tools of compensated compactness. The general polytropic case has then been investigated in [GN]. It is also clear that the limit $\tau \to 0$ is closely connected with the asymptotic behavior of the solutions. Here we directly study the limit $t \to \infty$. In the following let us assume $\tau = 1$ in (1.1) for simplicity.

Our assumptions on $b(x)$ are

$$\lim_{x \to -\infty} b(x) = b^- > 0 \;, \qquad \lim_{x \to +\infty} b(x) = b^+ > 0; \tag{1.5}$$

$$b(x) > 0 \;, \quad x \in I\!R \;; \tag{1.6}$$

there exists a function $B(x) \in C^2(I\!R)$ such that

$$B(x) > 0 \;, B'(x) \in L^1(I\!R) \cap H^1(I\!R) \;, b(x) - B(x) \in H^1(I\!R) \;. \tag{1.7}$$

Our basic assumptions on the pressure function p are

$$p \in C^3((0, +\infty)) \;, \quad p'(n) > 0 \quad \text{for } n > 0 \;. \tag{1.8}$$

Let us consider now the stationary solution of (1.4), i.e. the solutions of system

$$\begin{cases} (\mathcal{N}\mathcal{E} - p(\mathcal{N})_x)_x = 0 \\ \mathcal{E}_x = \mathcal{N} - b(x), \end{cases} \tag{1.9}$$

under the conditions

$$\mathcal{N}(x) - b(x) \in H^1(I\!R) \tag{1.10}$$

and

$$\mathcal{E}(-\infty) = \mathcal{E}^- \ . \tag{1.11}$$

In section 2 we shall prove the following existence and uniqueness result for these stationary solutions.

Theorem 1.1 *Suppose $b(x)$ satisfies the condition (1.5)–(1.7) and p satisfies (1.8), then there exists a unique solution $(\mathcal{N}, \mathcal{E})$ to the problem (1.9)–(1.11).*

In section 3, we shall investigate the global existence and large time behavior of the solutions to the drift–diffusion equations (1.4). The initial data of (1.4) are given by $N(x,0) = N_0(x)$. Set

$$q(x,0) =: N_0(x) - \mathcal{N}(x), \ v(x,0) = \int_{-\infty}^{x} (N_0(y) - \mathcal{N}(y))dy.$$

The electric field at $t = 0$ is given by $\hat{E}(x,0) = \int_{-\infty}^{x}(N_0(y) - b(y))dy + \mathcal{E}^-$.

Theorem 1.2 *Suppose $osc_{x\in R} b(x)$ is sufficiently small. There exists a positive number δ_0 such that if $\|v(x,0)\|_{H^2} \leq \delta_0$, then there exists a unique global smooth solution $(N(x,t), \hat{E}(x,t))$ of (1.4) with initial data $(N_0(x), \hat{E}(x,0))$. Moreover, let $(\mathcal{N}, \mathcal{E})$ be the (unique) solution of (1.9)–(1.11). Then there holds*

$$\|\hat{E}(\cdot,t) - \mathcal{E}(\cdot)\|_{H^2} \leq C\|\hat{E}(\cdot,0) - \mathcal{E}(\cdot)\|_{H^2} exp(-\alpha_1 t) \ , \tag{1.12}$$

$$\tag{1.13}$$

$$\|N(\cdot,t) - \mathcal{N}(\cdot)\|_{H^1} \leq \hat{C} exp(-\alpha_2 t) \ , \tag{1.14}$$

for some positive constants α_1, α_2, C and $\hat{C}$.

Remark 1.1 Since $H^2(I\!R) \hookrightarrow C^1(I\!R)$ and $H^1(I\!R) \hookrightarrow C^0(I\!R)$, then (1.11) and (1.8) imply

$$\|\hat{E}(\cdot,t) - \mathcal{E}(\cdot)\|_{C^1} \leq K\|\hat{E}(\cdot,0) - \mathcal{E}(\cdot)\|_{H^2} exp(-\alpha_1 t) \ ,$$

$$\|N(\cdot,t) - \mathcal{N}(\cdot)\|_{C^0} \leq \hat{K} exp(-\alpha_2 t) \ .$$

for some positive constants $\hat{K}$ and K.

The main purpose of this paper is to study the global existence and large time behavior of the solutions to (1.1)–(1.2). This is done in Section 4.

Let

$$\varphi(x,0) = n_0(x) - \mathcal{N}(x) \ , \quad j_0(x) = n_0(x)u_0(x)$$

and

$$e(x,0) = \int_{-\infty}^{x} \varphi(y,0)\,\mathrm{d}y \ , \quad E(x,0) = \mathcal{E}^- + \int_{-\infty}^{x} (n_0(y) - b(y))\,\mathrm{d}y \ .$$

Obviously $E(x,\ 0) = \mathcal{E}(x) + e(x,\ 0)$.

Theorem 1.3 *Assume $\mathcal{E}^- = 0$ and $e(\cdot,0) \in H^3(I\!R)$, $j_0 \in H^2(I\!R)$. Then there exists a positive number δ_1 such that if*

$$\begin{aligned}\|e(\cdot,0)\|_{H^3} &+ \|j_0(\cdot)\|_{H^2} + \|B'(\cdot)\|_{L^1} + \|B'(\cdot)\|_{H^1} + \|B''(\cdot)\|_{C^0} \\ &+ \|b(\cdot) - B(\cdot)\|_{C^0} + \|b(\cdot) - B(\cdot)\|_{L^2} + |b^+ - b^-| \le \delta_1 \ ,\end{aligned}$$

then the Cauchy problem (1.1)–(1.2) admits a unique smooth solution $(n(x,t),\ u(x,t),\ E(x,t))$ for all $t \ge 0$. Moreover, let $(\mathcal{N},\mathcal{E})$ be the (unique) solution to (1.9)–(1.11). Then

$$\|n(\cdot,t) - \mathcal{N}(\cdot)\|_{H^2} + \|E(\cdot,t) - \mathcal{E}(\cdot)\|_{H^3} + \|j(\cdot,t)\|_{H^2} \le K exp(-\alpha t),$$

for some positive constants α, K.

Let us remark that a key argument in this work is that, thanks to assumptions (1.6) and (1.7), the density functions n, N and $\mathcal{N}$ are always strictly positive. Problems arising at vacuum are widely beyond the purposes of this paper and will be not considered here.

2 Existence and uniqueness of stationary solutions

The purpose of this section is to prove Theorem 1.1. The proof is based on the following *a priori* estimates.

Theorem 2.1 *Suppose $b(x)$ satisfies (1.5)–(1.7) and p satisfies (1.8). Let $(\mathcal{N},\mathcal{E})$ be the solution of (1.9)–(1.11) given by Theorem 1.1. Then the following estimates hold:*

$$\inf_{x\in I\!R} b(x) \le \mathcal{N}(x) \le \sup_{x\in I\!R} b(x)\ , \tag{2.1}$$

$$\begin{aligned}\|\mathcal{N}(x) - B(x)\|_{H^2}^2 &\le C_1(\|B'(x)\|_{H^1}^2 + \|B'(x)\|_{L^2}^6 + \|B'(x)\|_{L^1} \\ &+\|b(x) - B(x)\|_{L^2}^2 + |\log b^+ - \log b^-|) =: C_1\gamma\ ,\end{aligned} \tag{2.2}$$

$$\sup_{x\in I\!R}(|\mathcal{N}'(x)|^2 + |\mathcal{N}''(x)|^2) \le C_2(\|B'(x)\|_{C^1}^2 + \|B'(x)\|_{C^0}^4 + \|b_1(x)\|_{C^0}^2 + \gamma^2 + \gamma)\ , \tag{2.3}$$

where $b_1 = b - B$, C_1, C_2 are positive constants which only depend on $\inf b(x)$, $\sup b(x)$, b^+ and b^-.

Proof. Since $\lim_{x\to\pm\infty}\mathcal{N}(x) = b^\pm > 0$, then for any $\delta > 0$ there exists a positive constant $A > 0$ such that

$$\begin{aligned} b^- - \delta \le \mathcal{N}(x) \le b^+ + \delta\ , \qquad x \le -A\ , \\ b^+ - \delta \le \mathcal{N}(x) \le b^+ + \delta\ , \qquad x \ge A\ .\end{aligned} \tag{2.4}$$

Suppose $\mathcal{N}(x)$ takes its positive maximum value in $[-A, A]$ at the point $x_0 \in (-A, A)$. Then

$$\mathcal{N}'(x_0) = 0 \ , \qquad \mathcal{N}''(x_0) \leq 0 \ .$$

Thus the first equation in (1.9) implies

$$\mathcal{N}\mathcal{E}_x(x_0) \leq 0$$

which gives

$$\mathcal{N}(x_0)(\mathcal{N}(x_0) - b(x_0)) \leq 0 \ . \tag{2.5}$$

Therefore we have

$$\mathcal{N}(x_0) \leq \sup_{x \in I\!\!R} b(x). \tag{2.6}$$

The inequalities (2.4) and (2.6) imply

$$\sup_{x \in I\!\!R} \mathcal{N}(x) \leq \text{ Max } \left\{ \sup_{x \in I\!\!R} b(x), \ b^- + \delta, \ b^+ + \delta \right\} \ . \tag{2.7}$$

Now we suppose $\mathcal{N}(x)$ takes its non-negative minimum value in $[-A, A]$ at the point $x_1 \in (-A, A)$. Arguing as above we obtain

$$\mathcal{N}(x_1) \geq \inf_{x \in I\!\!R} b(x) \ . \tag{2.8}$$

Thus

$$\inf_{x \in I\!\!R} \mathcal{N}(x) \geq \inf \left\{ \inf_{x \in I\!\!R} b(x), \ b^- - \delta, \ b^+ - \delta \right\} \tag{2.9}$$

Since δ is a arbitrary number, estimate (2.1) follows.

Let us show (2.2). It follows from (1.9) that

$$\mathcal{N}\mathcal{E} - p(\mathcal{N})_x = \text{const.} = b^- \mathcal{E}^- =: J \ , \tag{2.10}$$

Thus, (1.9) implies

$$\left(\frac{J}{\mathcal{N}} \right)_x + \left(\frac{p(\mathcal{N})_x}{\mathcal{N}} \right)_x = \mathcal{N} - b(x) \ . \tag{2.11}$$

Let $\mathcal{N} = B(x) + \mathcal{N}_1(x)$ and $b(x) = B(x) + b_1(x)$. Then $\mathcal{N}_1(x) \in H^1(I\!\!R)$ and $b_1(x) \in H^1(I\!\!R)$. This yields

$$\left(\frac{J}{\mathcal{N}} \right)_x + \left(\frac{p(B(x) + \mathcal{N}_1)_x}{\mathcal{N}} \right)_x = \mathcal{N}_1 - b_1 \ . \tag{2.12}$$

Multiply (2.12) by $\mathcal{N}_1$ and integrate the resulting equation over $I\!\!R$ to obtain

$$\begin{aligned} &\textstyle\int \mathcal{N}_1^2 \, \mathrm{d}x - \int b_1 \mathcal{N}_1 \, \mathrm{d}x + \int \frac{J}{\mathcal{N}} \mathcal{N}_{1x} dx \\ &+ \textstyle\int \frac{p'(\mathcal{N})}{\mathcal{N}} B'(x) \mathcal{N}_{1x} \, \mathrm{d}x + \int \frac{p'(\mathcal{N})}{\mathcal{N}} \mathcal{N}_{1x}^2 \, \mathrm{d}x = 0. \end{aligned} \tag{2.13}$$

On the other hand,

$$
\begin{aligned}
\textstyle\int \frac{J}{\mathcal{N}}\mathcal{N}_{1x}\,\mathrm{d}x &= \textstyle\int \left(\frac{J}{\mathcal{N}}\mathcal{N}_x - \frac{J}{\mathcal{N}}B'(x)\right)\,\mathrm{d}x \\
&= \textstyle\int J(\log\mathcal{N})_x\,\mathrm{d}x - \int \frac{J}{\mathcal{N}}B'(x)\,\mathrm{d}x \\
&= J(\log b^+ - \log b^-) - \textstyle\int \frac{J}{\mathcal{N}}B'(x)\,\mathrm{d}x\ .
\end{aligned}
\tag{2.14}
$$

In view of (2.1) and $p'(\mathcal{N}) > 0$, there exist some positive constants a_1 and a_2 such that

$$
a_2 > \frac{p'(\mathcal{N})}{\mathcal{N}} > a_1\ . \tag{2.15}
$$

It follows from (2.13), (2.14) and (2.15) that

$$
\begin{aligned}
\textstyle\int \mathcal{N}_1^2\,\mathrm{d}x\ &+a_1\textstyle\int \mathcal{N}_{1x}^2\,\mathrm{d}x \\
&\le |\textstyle\int b_1\mathcal{N}_1\,\mathrm{d}x| + |J(\log b^+ - \log b^-)| \\
&+C\textstyle\int |B'(x)|\,\mathrm{d}x + a_2\int |B'(x)\mathcal{N}_{1x}(x)|\,\mathrm{d}x\ .
\end{aligned}
\tag{2.16}
$$

for a positive constant C. The Cauchy–Schwarz inequality implies

$$
\left|\int b_1\mathcal{N}_1\,\mathrm{d}x\right| \le \frac{1}{2}\int \mathcal{N}_1^2\,\mathrm{d}x + \frac{1}{2}\int b_1^2\,\mathrm{d}x\ , \tag{2.17}
$$

$$
a_2\int |B'(x)\mathcal{N}_{1x}(x)|\,\mathrm{d}x \le \frac{a_1}{2}\int \mathcal{N}_{1x}^2(x)\,\mathrm{d}x + \frac{a_2}{2a_1}\int (B'(x))^2\,\mathrm{d}x\ . \tag{2.18}
$$

Thus, we have

$$
\begin{aligned}
\textstyle\int(\mathcal{N}_1^2 + \mathcal{N}_{1x}^2)\,\mathrm{d}x\ &\le C_3(\|B'(x)\|_{L^2}^2 + \|B'(x)\|_{L^1} \\
&+\|b_1(x)\|_{L^2}^2 + |\log b^+ - \log b^-|) =: C_3\beta\ ,
\end{aligned}
\tag{2.19}
$$

where C_3 is a positive constants only depending on b^+, b^- $\inf b(x)$ and $\sup b(x)$.

Thanks to (2.12), we have

$$
\begin{aligned}
\tfrac{p'(\mathcal{N})}{\mathcal{N}}\mathcal{N}_{1xx}\ &= -\tfrac{p'(\mathcal{N})}{\mathcal{N}}B'' + \left(\tfrac{p'(\mathcal{N})}{\mathcal{N}^2} - \tfrac{p''(\mathcal{N})}{\mathcal{N}}\right)(\mathcal{N}_{1x} + B')^2 \\
&+\tfrac{J}{\mathcal{N}^2}(B' + \mathcal{N}_{1x}) + (\mathcal{N}_1 - b_1)\ .
\end{aligned}
\tag{2.20}
$$

Then it follows that

$$
\int \mathcal{N}_{1xx}^2\,\mathrm{d}x \le C_4\left(\|B''\|_{L^2}^2 + \beta + \int \mathcal{N}_{1x}^4\,\mathrm{d}x + \int (B')^4\,\mathrm{d}x\right)\ , \tag{2.21}
$$

where C_4 is a constant only depending on the same quantities as C_3.

By Sobolev's inequality

$$\sup_{x\in R}\mathcal{N}_{1x}^2(x)\le 2\left(\int\mathcal{N}_{1x}^2\,\mathrm{d}x\right)^{1/2}\left(\int\mathcal{N}_{1xx}^2\,\mathrm{d}x\right)^{1/2}, \tag{2.22}$$

one has

$$\begin{aligned} C_4\int\mathcal{N}_{1x}^4\,\mathrm{d}x &\le C_4\sup_{x\in\mathbb{R}}\mathcal{N}_{1x}^2(x)\int\mathcal{N}_{1x}^2\,\mathrm{d}x \\ &\le 2C_4^2\left(\int\mathcal{N}_{1x}^2\,\mathrm{d}x\right)^3+\tfrac{1}{2}\int\mathcal{N}_{1xx}^2\,\mathrm{d}x \\ &\le 2C_4^2C_1^3\beta^3+\tfrac{1}{2}\int\mathcal{N}_{1xx}^2\,\mathrm{d}x\ . \end{aligned} \tag{2.23}$$

The same argument leads to

$$\int(B')^4\,\mathrm{d}x\le 2\left(\int(B')^2\,\mathrm{d}x\right)^3+\frac{1}{2}\int(B'')^2\,\mathrm{d}x\ . \tag{2.24}$$

Hence (2.21), (2.23) and (2.24) imply

$$\int\mathcal{N}_{1xx}^2\,\mathrm{d}x\le C_5(\|B'\|_{H^1}^2+\|B'\|_{L^2}^6+\beta)=C_5\gamma\ . \tag{2.25}$$

It then follows (2.22), (2.19) and (2.25) that

$$\sup_{x\in\mathbb{R}}\mathcal{N}_{1x}^2(x)\le C_6\gamma\ . \tag{2.26}$$

On the other hand,

$$\sup_{x\in R}\mathcal{N}_1^2\le 2\left(\int\mathcal{N}_1^2\,\mathrm{d}x\right)^{1/2}\left(\int\mathcal{N}_{1x}^2\,\mathrm{d}x\right)^{1/2}\le 2C_1\beta\ . \tag{2.27}$$

We conclude from (2.15), (2.20), (2.26), (2.27) and the bound of $\mathcal{N}$ that

$$\sup_{x\in R}\mathcal{N}_{1xx}^2\le C_7\left\{\sup_{x\in R}\left[(B''(x))^2+(B'(x))^4+(B'(x))^2+(b_1(x))^2\right]+\gamma^2+\gamma\right\} \tag{2.28}$$

which completes the proof of Theorem 2.1. □

Proof of Theorem 1.1. Based on Theorem 2.1, the existence of $(\mathcal{N},\mathcal{E})$ can be proved by a fixed point argument. Indeed, we consider the following linear problem:

$$\begin{cases}(\mathcal{N}\hat{\mathcal{E}}-p'(M)\mathcal{N}_x)_x=0,\\ \hat{\mathcal{E}}_x=M(x)-b(x),\end{cases}$$

where $M(x)$ satisfies

$$M(x)>0\ ,\quad M(x)-b(x)\in H^1(\mathbb{R})$$

and $\hat{\mathcal{E}}$ satisfies

$$\hat{\mathcal{E}}(-\infty) = \mathcal{E}^- \ .$$

The above linear problem gives a map

$$\mathcal{N} = \Phi(M) \ .$$

We choose $\mathcal{N}_0$ such that $\mathcal{N}_0(x) > 0$, $\mathcal{N}_0(x) - b(x) \in H^1(I\!R)$, and get $\mathcal{N}_1 = \Phi(\mathcal{N}_0)$. Using the methods of the proof of Theorem 2.1, we find that $\mathcal{N}_1(x) > 0$, $\mathcal{N}_1(x) - b(x) \in H^1(I\!R)$. Define the sequence, for $k \geq 2$,

$$\mathcal{N}_k = \Phi(\mathcal{N}_{k-1}).$$

Then, by arguing as above, we have that this sequence $\{\mathcal{N}_k\}$ is uniformly bounded in C^2, and $\{\mathcal{N}_k - B(x)\}$ is uniformly bounded in H^2. Therefore, for any $K > 0$, there exists a subsequence of $\{\mathcal{N}_k\}$ such that this subsequence converges in $C^{1+\mu}([-K, K])$, for $0 < \mu < 1$. Standard diagonal arguments lead to the existence result, with the help of the fact $p' > 0$, which implies the needed regularity.

Let us prove now the uniqueness result.

Suppose $(\overline{\mathcal{N}}, \overline{\mathcal{E}})$ and $(\hat{\mathcal{N}}, \hat{\mathcal{E}})$ are two different smooth solutions to (1.9)–(1.11).

Set $\theta = \overline{\mathcal{N}} - \hat{\mathcal{N}}$ and $\alpha = \overline{\mathcal{E}} - \hat{\mathcal{E}}$. Then

$$\begin{cases} \alpha\hat{\mathcal{N}} + \theta\hat{\mathcal{E}} + \alpha\theta - (p(\hat{\mathcal{N}} + \theta) - p(\hat{\mathcal{N}}))_x = 0 \ , \\ \alpha_x = \theta \ . \end{cases} \tag{2.29}$$

clearly $\alpha(-\infty) = 0$.

Due to (1.9), one has that

$$\overline{\mathcal{N}}\hat{\mathcal{E}} - p(\hat{\mathcal{N}})_x = C = b^-\mathcal{E}^- \ .$$

Thus, $\hat{\mathcal{E}}(+\infty) = \frac{b^-}{b^+}\mathcal{E}^-$. Similarly, $\mathcal{E}(+\infty) = \frac{b^-}{b^+}\mathcal{E}^-$. Hence $\alpha(+\infty) = 0$.

For any $\delta > 0$, there exists a positive number $K > 0$ such that

$$|\alpha(x)| < \delta \ , \qquad |x| \geq K \ . \tag{2.30}$$

We claim that there holds also

$$|\alpha(x)| \leq \delta \ , \quad |x| < K \ . \tag{2.31}$$

Since δ is an arbitrary positive number, (2.30) and (2.31) imply $\alpha(x) \equiv 0$, for all $x \in I\!R$ and $\theta(x) = 0$ holds as a consequence of (2.29).

(2.31) follows from a maximum principle argument. Indeed, suppose $\alpha(x)$ takes its positive maximum value in $[-K, K]$ at the point $x_0 \in (-K, K)$. Then

$$\alpha_x(x_0) = 0 \ , \quad \alpha_{xx}(x_0) \leq 0 \tag{2.32}$$

and the system (2.29) yields

$$\alpha\hat{\mathcal{N}} + \alpha_x\hat{\mathcal{E}} + \alpha\alpha_x - (p(\hat{\mathcal{N}} + \alpha_x) - p(\hat{\mathcal{N}}))_x = 0, \tag{2.33}$$

which implies

$$\alpha(x_0)\hat{\mathcal{N}}(x_0) \le 0 \ . \tag{2.34}$$

Since $\hat{\mathcal{N}}(x_0) > 0$, so (2.34) is a contradiction. Thus, we have proved $\alpha(x) \le \delta$ for $x \in (-K, K)$. The same argument leads to $\alpha(x) \ge -\delta$ for $x \in (-K, K)$, which completes the proof of (2.31). □

3 Large time behavior of the solutions to the drift–diffusion model

In this section we prove Theorem 1.2. The proof is based on an *a priori* estimate. Suppose (N, E) is the solution of (1.4) with the initial data $(N_0(x), E_0(x))$ and let $(\mathcal{N}, \mathcal{E})$ be the (unique) solution of (1.9)–(1.11).

Set

$$\begin{aligned} q(x,t) &= N(x,t) - \mathcal{N}(x) && (3.1)\\ v(x,t) &= \hat{E}(x,t) - \mathcal{E}(x) && (3.2) \end{aligned}$$

and assume

$$q(x,0) \in H^1(I\!R), \ \ v(x,0) \in H^2(I\!R). \tag{3.3}$$

The functions (q, v) satisfy the system

$$\begin{cases} q_t + (qv)_x + (q\mathcal{E})_x + (\mathcal{N}v)_x - (p(\mathcal{N} + q) - p(\mathcal{N}))_{xx} = 0 \\ v_x = q \ . \end{cases} \tag{3.4}$$

System (3.4) can be reduced to a scalar equation about v as follows. Differentiating the second equation of (3.4) with respect to t and using the second equation give

$$v_{xt} + (\frac{1}{2}v^2)_{xx} + (v_x\mathcal{E})_x + (\mathcal{N}v)_x - (p(\mathcal{N} + v_x) - p(\mathcal{N}))_{xx} = 0 \ . \tag{3.5}$$

Integrating (3.5) with respect to x shows that

$$v_t + \frac{1}{2}(v^2)_x + v_x\mathcal{E} + \mathcal{N}v - (p(\mathcal{N} + v_x) - p(\mathcal{N})_x = 0 \ , \tag{3.6}$$

if $v(\cdot, t) \in H^2$ for $t \ge 0$. This is the desired equation.

The existence and large time behavior of solutions to the Cauchy problem for (3.6) will follow from the following *a priori* estimate which will be proved by an energy method.

Lemma 3.1 *There exist some positive constants $\varepsilon_0 > 0$ and $c > 0$ such that, for any $T > 0$, if*

$$\sup_{0\le t\le T} \|v(\cdot,t)\|_{H^2} \le \varepsilon_0,$$

then there holds, for any $t \in [0,T]$

$$\|v(\cdot,t)\|_{H^2}^2 \le exp(-ct)\|v(\cdot,0)\|_{H^2}^2 \ . \tag{3.7}$$

Proof. The inequalities (2.1) imply

$$a_0 \le \mathcal{N}(x) \le a_1, \qquad x \in I\!\!R \ , \tag{3.8}$$

for some positive constants $a_0, a_1 > 0$.

Multiply (3.6) by v and integrate over $I\!\!R$ to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}v^2\,\mathrm{d}x + \int \mathcal{N}v^2\,\mathrm{d}x + \int (p(\mathcal{N}+v_x) - p(\mathcal{N}))v_x\,\mathrm{d}x + \int v_x\mathcal{E}v\,\mathrm{d}x = 0 \ . \tag{3.9}$$

On the other hand, since $\mathcal{E}$ is bounded (see (2.10),

$$\begin{aligned} \textstyle\int v_x\mathcal{E}v\,\mathrm{d}x &= \textstyle\int \left(\frac{1}{2}v^2\right)_x \mathcal{E}\,\mathrm{d}x \\ &= -\textstyle\int \frac{1}{2}v^2\mathcal{E}_x\,\mathrm{d}x = -\int \frac{1}{2}v^2(\mathcal{N}(x) - b(x))\,\mathrm{d}x \ . \end{aligned} \tag{3.10}$$

In view of (2.1), there holds $|\mathcal{N}(x) - b(x)| \le \mathrm{osc}_{x\in I\!\!R} b(x)$, and (3.10) implies

$$\left|\int v_x\mathcal{E}v\,\mathrm{d}x\right| \le \frac{1}{2}\mathrm{osc}_{x\in I\!\!R} b(x)\int v^2\,\mathrm{d}x \ . \tag{3.11}$$

Assume now $|v_x| \le \frac{1}{2}\inf_{x\in I\!\!R}\mathcal{N}(x)$. Then $\mathcal{N} + v_x \ge \frac{1}{2}\inf_{x\in I\!\!R}\mathcal{N}(x) > 0$, which implies that there exists a positive constant $a_3 > 0$ such that

$$(p(\mathcal{N}+v_x) - p(\mathcal{N}))v_x \ge a_3 v_x^2 \ . \tag{3.12}$$

Hence, from (3.9), (3.11), and (3.12), we have that if $|v_x|$ and $\mathrm{osc}_{x\in I\!\!R} b(x)$ are small enough, then

$$\frac{d}{\mathrm{d}t}\int v^2\,\mathrm{d}x + a_4\left(\int v^2\,\mathrm{d}x + \int v_x^2\,\mathrm{d}x\right) \le 0 \tag{3.13}$$

for some positive constant $a_4 > 0$.

Multiply equation (3.5) by v_x to give

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\ \left(\textstyle\int \frac{1}{2}v_x^2\,\mathrm{d}x\right) + \textstyle\int (\frac{1}{2}v^2)_{xx}v_x dx + \int (v_x\mathcal{E})_x v_x\,\mathrm{d}x \\ &+ \textstyle\int \mathcal{N}v_x^2\,\mathrm{d}x + \int \mathcal{N}_x v v_x + \int (p(\mathcal{N}+v_x) - p(\mathcal{N}))_x v_{xx}\,\mathrm{d}x = 0. \end{aligned} \tag{3.14}$$

Then

$$\int \mathcal{N} v_x^2 \,\mathrm{d}x \geq a_0 \int v_x^2 \,\mathrm{d}x \ , \tag{3.15}$$

$$\int (p(\mathcal{N}+v_x)-p(\mathcal{N}))_x v_{xx} \,\mathrm{d}x = \int (p'(\mathcal{N}+v_x)-p'(\mathcal{N}))\mathcal{N}_x v_{xx} + \int p'(\mathcal{N}+v_x) v_{xx}^2 \,\mathrm{d}x \tag{3.16}$$

and

$$|\int (\frac{1}{2}v^2)_{xx} v_x dx| = |\int (\frac{1}{2}v^2)_x v_{xx} dx| \leq \frac{1}{2}\sup |v_x| \int (v^2+v_{xx}^2) dx. \tag{3.17}$$

Therefore, there exists a positive number a_5 such that $p'(\mathcal{N}+v_x) \geq a_5$, if $|v_x|$ is small enough. Thus

$$\int p'(\mathcal{N}+v_x) v_{xx}^2 \,\mathrm{d}x \geq a_5 \int v_{xx}^2 \,\mathrm{d}x \ , \tag{3.18}$$

Since $\mathcal{E}$ is bounded, it follows from the Cauchy–Schwarz inequality that

$$\left|\int (v_x \mathcal{E})_x v_x \,\mathrm{d}x\right| = \left|\int v_x \mathcal{E} v_{xx} \,\mathrm{d}x\right| \leq \frac{a_5}{4} \int v_{xx}^2 \,\mathrm{d}x + C \int v_x^2 \,\mathrm{d}x \ , \tag{3.19}$$

for some constant $C > 0$.

Similarly

$$\left|\int \mathcal{N}_x v v_x \,\mathrm{d}x\right| \leq C \left(\int v^2 \,\mathrm{d}x + \int v_x^2 \,\mathrm{d}x\right) \tag{3.20}$$

and

$$\left|\int (p'(\mathcal{N}+v_x)-p'(\mathcal{N}))\mathcal{N}_x v_{xx} \,\mathrm{d}x\right| \leq \frac{a_5}{4} \int v_{xx}^2 \,\mathrm{d}x + C \int v_x^2 \,\mathrm{d}x \tag{3.21}$$

for some constant $C > 0$. Thus, if $|v_x|$ is small enough, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \int v_x^2 dx + \int v_{xx}^2 \,\mathrm{d}x \leq C \left(\int v^2 \,\mathrm{d}x + \int v_x^2 \,\mathrm{d}x\right). \tag{3.22}$$

Then, using (3.13) and (3.22) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int (v^2+v_x^2) \,\mathrm{d}x\right) + a_6 \int (v^2+v_x^2+v_{xx}^2) \,\mathrm{d}x \leq 0 \ , \tag{3.23}$$

if $\mathrm{osc}_{x\in \mathbb{R}} b(x)$ and $|v_x|$ are small enough and for some constant a_6. Now, differentiate (3.5) with respect to x and multiply the resulting equation by v_{xx}. We have, by arguing as above

$$\frac{\mathrm{d}}{\mathrm{d}t} \int (v^2+v_x^2+v_{xx}^2) \,\mathrm{d}x + a_7 \int (v^2+v_x^2+v_{xx}^2+v_{xxx}^2) \,\mathrm{d}x \leq 0 \ , \tag{3.24}$$

for some positive constant $a_7 > 0$, which implies

$$\int (v_x^2+v_x^2+v_{xx}^2)(x,t) \,\mathrm{d}x \leq exp(-a_7 t) \int (v_x^2+v_x^2+v_{xx}^2)(x,0) \,\mathrm{d}x. \tag{3.25}$$

This complete the proof of Lemma 3.1. □

Proof of Theorem 1.2. If the initial perturbation $\|v(\cdot,0)\|_{H^2} + \|q(\cdot,0)\|_{H^3}$ and $\mathrm{osc}_{x\in \mathbb{R}} b(x)$ are small enough, then the standard local existence results and Lemma 3.1 imply the global existence of (v,q), and the estimate in Lemma 3.1 holds for all $t \geq 0$. The smallness of $|v_x|$ can be ensured by the embedding result $H^2(\mathbb{R}) \subset C^1(\mathbb{R})$. Thus the proof of Theorem 1.2 is completed. □

4 Global existence and large time behavior of the solutions to the hydrodynamic model

In the section we prove Theorem 1.3 by using an energy method. Let $(\mathcal{N}, \mathcal{E})$ be the solution of (1.9)–(1.11) given by Theorem 1.1. Let us write a solution to (1.1)–(1.2) as

$$n(x,t) = \mathcal{N}(x) + \varphi(x,t), \tag{4.1}$$

$$j(x,t) = nu(x,t) \ , \tag{4.2}$$

$$E(x,t) = \mathcal{E}(x) + e(x,t) \ . \tag{4.3}$$

Since $\mathcal{E}^- = 0$, we have, from (1.9)

$$J := \mathcal{N}\mathcal{E} - p(\mathcal{N})_x = \text{const.} = b^- \mathcal{E}^- = 0 \ . \tag{4.4}$$

Thus the functions (φ, j, e) satisfy the system

$$\begin{cases} \varphi_t + j_x = 0 \\ j_t + (p(\mathcal{N}+\varphi) - p(\mathcal{N}))_x + j + \left(\frac{j^2}{\mathcal{N}+\varphi}\right)_x = \mathcal{N}e + \varphi\mathcal{E} + \varphi e \ , \\ e_x = \varphi \ . \end{cases} \tag{4.5}$$

Our assumptions in Theorem 1.3 imply

$$e(x,0) = \int_{-\infty}^{x} \varphi(y,0)\,\mathrm{d}y = \int_{-\infty}^{x} (n_0(y) - \mathcal{N}(y))\,\mathrm{d}y \in H^3 \tag{4.6}$$

and

$$j(x,0) = n_0(x)u_0(x) \in H^2 \ . \tag{4.7}$$

The system (4.5) is equivalent to a nonlinear equation

$$e_{tt} - (p(\mathcal{N}+e_x) - p(\mathcal{N}))_x + e_t + \mathcal{N}e - \left(\frac{e_t^2}{\mathcal{N}+e_x}\right)_x + e_x\mathcal{E} + e_x e = 0 \ , \tag{4.8}$$

since

$$e_x = \varphi \ , \qquad e_t = -j \ . \tag{4.9}$$

We notice that (4.8) is a second order hyperbolic equation with strong dissipation, due to the friction term e_t and the fact that $\mathcal{N} > 0$. Therefore, an energy method will lead to the following crucial *a priori* estimate.

Lemma 4.1 *There exist positive constants δ_2, C and α such that, for any $T > 0$, if*

$$\sup_{0\le t\le T} (\|e(\cdot,t)\|_{H^3} + \|e_t(\cdot,t)\|_{H^2}) + |\mathcal{N}_x|_{C^1} \le \delta_2 \ , \tag{4.10}$$

then, for any $t \in [0,T]$,

$$\left(\|e(\cdot,t)\|_{H^3}^2 + \|e_t(\cdot,t)\|_{H^2}^2\right) \le C\left(\|e(\cdot,0)\|_{H^3}^2 + \|e_t(\cdot,0)\|_{H^2}^2\right)\exp(-\alpha t). \tag{4.11}$$

Proof of Theorem 1.3. Based on Lemma 4.1, the proof of Theorem 1.3 is standard. In fact, combining the standard theory of existence and uniqueness of local (in time) solutions with the estimates (4.11), one can extend the local solution by usual continuation arguments and show that the estimates (4.11) holds globally (see for instance [HL]), if the initial perturbation $\|e(\cdot,0)\|_{H^3} + \|e_t(\cdot,0)\|_{H^2}$ is sufficiently small.

The smallness of $|\mathcal{N}_x|$, $|\mathcal{N}_{xx}|$ can be ensured by Theorem 2.1 and the assumptions of Theorem 1.3. $\square$

We thus focus on the proof of Lemma 4.1. Set

$$M(T) = \sup_{0\le t\le T} \left(\|e(\cdot,t)\|_{H^3}^2 + \|e_t(\cdot,t)\|_{H^2}^2 \right) \ . \tag{4.12}$$

Then the standard Sobolev inequalities imply

$$\sup_{0\le t\le T} \|e(\cdot,t)\|_{C^2} \le C\, M(T) \ , \tag{4.13}$$

$$\sup_{0\le t\le T} \|e_t(\cdot,t)\|_{C^1} \le C\, M(T) \ , \tag{4.14}$$

with a uniform generic positive constant C. By means of (4.4), $\mathcal{E}$ can be bounded as

$$|\mathcal{E}| \le C|\mathcal{N}_x| \ . \tag{4.15}$$

Due to (2.1), we know that $\mathcal{N} + e_x$ can be bounded in an interval $[\underline{N}, \overline{N}]$ with positive $\underline{N}$ and $\overline{N}$ if e_x is sufficiently small.

Proof of Lemma 4.1. Multiply (4.8) with e and e_t respectively, and integrate the resulting equations over $I\!R$. Integration by parts leads to

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \left(e_t e + \tfrac{1}{2}e^2\right) \mathrm{d}x \quad &+ \int (p(\mathcal{N}+e_x) - p(\mathcal{N}))e_x \,\mathrm{d}x - \int e_t^2 \,\mathrm{d}x + \int \mathcal{N} e^2 \,\mathrm{d}x \\ &+ \int \tfrac{e_t^2}{\mathcal{N}+e_x} e_x \,\mathrm{d}x + \int e_x \mathcal{E} e \ \mathrm{d}x + \int e_x e^2 \,\mathrm{d}x = 0 \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \left(\tfrac{1}{2}e_t^2 + \tfrac{1}{2}Ne^2\right) \mathrm{d}x \quad &+ \int (p(\mathcal{N}+e_x) - p(\mathcal{N}))e_{xt} \,\mathrm{d}x + \int e_t^2 \,\mathrm{d}x \\ &+ \int \left(\tfrac{e_t^2}{\mathcal{N}+e_x}\right) e_{xt} \,\mathrm{d}x + \int e_x \mathcal{E} e_t \,\mathrm{d}x + \int e_x e e_t \,\mathrm{d}x = 0 \ . \end{aligned} \tag{4.17}$$

Define

$$\overline{q}(x,t) = \int_0^{e_x} (p(\mathcal{N}+\theta) - p(\mathcal{N}))\,\mathrm{d}\theta \ . \tag{4.18}$$

Since $p' > 0$, and

$$\mathcal{N}(x) \ge a_0 > 0 \ , \qquad x \in I\!R \tag{4.19}$$

for some positive constant a_0, one has

$$d_1 e_x^2 \le \overline{q} \le d_2 e_x^2 \tag{4.20}$$

for some positive constants $0 < d_1 < d_2$.

On the other hand, we have

$$\overline{q}_t = (p(\mathcal{N} + e_x) - p(\mathcal{N}))e_{xt} \ . \tag{4.21}$$

Hence

$$\int [p(\mathcal{N} + e_x) - p(\mathcal{N})]e_{xt} = \frac{\mathrm{d}}{\mathrm{d}t}\int \overline{q}(x,t)\,\mathrm{d}x \ . \tag{4.22}$$

In addition,

$$\int \left(\frac{e_t^2}{\mathcal{N} + e_x}\right) e_{xt}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{e_t^2}{\mathcal{N} + e_x} e_x\,\mathrm{d}x - \int \frac{2e_t e_{tt}(\mathcal{N} + e_x) - e_t^2 e_{xt}}{(\mathcal{N} + e_x)^2} e_x\,\mathrm{d}x \ . \tag{4.23}$$

Then (4.17), together with (4.22)-(4.23) imply that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int \Big(\tfrac{1}{2}e_t^2 & + \tfrac{1}{2}\mathcal{N}e^2 + \overline{q} + \tfrac{e_t^2 e_x}{\mathcal{N}+e_x}\Big)\,\mathrm{d}x + \int e_t^2\,\mathrm{d}x \\ & - \int \tfrac{2e_t e_{tt} e_x}{(\mathcal{N}+e_x)}\,\mathrm{d}x + \int \tfrac{e_t^2 e_{xt} e_x}{(\mathcal{N}+e_x)^2}\,\mathrm{d}x + \int e_x \mathcal{E} e_t\,\mathrm{d}x + \int e_x e\, e_t\,\mathrm{d}x = 0 \ . \end{aligned} \tag{4.24}$$

Combining (4.16) with (4.24) shows that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int e_t e & + \left(\tfrac{1}{2} + 2\mathcal{N}\right) e^2 + 2e_t^2 + 4\overline{q} + \tfrac{4e_t^2 e_x}{\mathcal{N}+e_x}\,\mathrm{d}x + 3\int e_t^2\,\mathrm{d}x \\ & + \int \mathcal{N}e^2\,\mathrm{d}x + \int [p(\mathcal{N} + e_x) - p(\mathcal{N})]e_x\,\mathrm{d}x \\ & + \int \left[\tfrac{e_t^2 e_x}{\mathcal{N}+e_x} - \tfrac{8e_t e_{tt} e_x}{(\mathcal{N}+e_x)} + \tfrac{4e_t^2 e_{xt} e_x}{(\mathcal{N}+e_x)^2} + e_x\mathcal{E}e + e_x e^2 + 4e_x\mathcal{E}e_t + 4e_x e\, e_t\right]\,\mathrm{d}x = 0 \ . \end{aligned} \tag{4.25}$$

On the other hand, since $p' > 0$,

$$\int [p(\mathcal{N} + e_x) - p(\mathcal{N})]e_x\,\mathrm{d}x = \int p'(\mathcal{N} + \theta_1 e_x)e_x^2\,\mathrm{d}x \ge d_3 \int e_x^2\,\mathrm{d}x \tag{4.26}$$

where $d_3 > 0$ is a constant, if $|e_x|$ is small enough.

From (4.13) and (4.14), we have

$$\begin{aligned} \Big|\int \tfrac{e_t^2 e_x}{\mathcal{N}+e_x} & - \tfrac{8e_t e_{tt} e_x}{\mathcal{N}+e_x} + \tfrac{4e_x^2 e_{xt} e_x}{(\mathcal{N}+e_x)^2} + 4e_x e\, e_t + e_x e^2\,\mathrm{d}x\Big| \\ & \le O(1)M(T)\int (e_t^2 + e_x^2 + e^2)\,\mathrm{d}x \ , \end{aligned} \tag{4.27}$$

for small $|e_x|$, since $\mathcal{N} > a_0 > 0$. Here and in the following $O(1)$ stands for a constant which does not depend on T. ¿From (4.15), we get

$$|\int e_x \mathcal{E} e dx| \leq O(1)|\mathcal{N}_x|_{C^0} \int (e_x^2 + e^2) dx$$

and

$$|\int 4 e_x \mathcal{E} e_t dx| \leq O(1)|\mathcal{N}_x|_{C^0} \int (e_x^2 + e_t^2) dx.$$

Thus, if $M(T)$ and $|\mathcal{N}_x|_{C^0}$ are small enough, there holds

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(t) + c \int (e^2 + e_t^2 + e_x^2)\, \mathrm{d}x \leq 0 \ , \tag{4.28}$$

for some $c > 0$, where

$$F_1(t) := \int \left(e_t e + \left(\frac{1}{2} + 2\mathcal{N} \right) e^2 + 2 e_t^2 + 4\overline{q} + \frac{4 e_t^2 e_x}{\mathcal{N} + e_x} \right) \, \mathrm{d}x \ . \tag{4.29}$$

Now we have, again from the Cauchy–Schwarz inequality, that there exist some positive constants $d_4, d_5 > 0$ such that

$$d_4 \int (e^2 + e_t^2 + e_x^2)(\cdot, t)\, \mathrm{d}x \leq F_1(t) \leq d_5 \int (e^2 + e_t^2 + e_x^2)(\cdot, t)\, \mathrm{d}x \tag{4.30}$$

if $|e_x|$ is small enough.

Thus (4.28), (4.30) imply that

$$\frac{\mathrm{d}}{\mathrm{d}t} F_1(t) + d_6 F_1(t) \leq 0 \tag{4.31}$$

for some positive constant $d_6 > 0$.

Hence,

$$F_1(t) \leq exp(-d_6 t) F_1(0) \ , \tag{4.32}$$

which yields the desired basic energy estimate. The next step is to get the estimates for the higher order derivatives.

Differentiate (4.8) with respect to x and then multiply the resulting equation by e_x and e_{xt} respectively. Therefore there holds

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t} \int \left(e_{tx} e_x \quad + \tfrac{1}{2} e_x^2 \right) \mathrm{d}x + \int [\mathcal{N} e_x^2 + (p(\mathcal{N} + e_x) - p(\mathcal{N}))_x e_{xx}]\, \mathrm{d}x - \int e_{tx}^2 \, \mathrm{d}x \\ &\qquad + \int \left[\mathcal{N}_x e e_x + e_{xx} \mathcal{E} e_x + e_x^2 \mathcal{E}_x + \left(\tfrac{e_t^2}{\mathcal{N} + e_x} \right)_x e_{xx} + \left(\tfrac{1}{2} e^2 \right)_{xx} e_x \right] \, \mathrm{d}x = 0 \end{aligned} \tag{4.33}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\tfrac{1}{2}e_{tx}^2 \quad +\tfrac{1}{2}\mathcal{N}e_x^2\right)\mathrm{d}x+\int e_{tx}^2\,\mathrm{d}x+\int[p(\mathcal{N}+e_x)-p(\mathcal{N})]_x e_{txx}\,\mathrm{d}x$$
$$+\int\left[\mathcal{N}_x e e_{tx}-\left(\tfrac{e_t^2}{\mathcal{N}+e_x}\right)_{xx}e_{tx}+e_{xx}\mathcal{E}e_{tx}+e_x\mathcal{E}_x e_{tx}+\left(\tfrac{1}{2}e^2\right)_{xx}e_{tx}\right]\,\mathrm{d}x=0. \tag{4.34}$$

Let us estimate the terms appearing in the equalities (4.33) and (4.34).

First we have

$$\begin{aligned}\int[p(\mathcal{N}+e_x)-p(\mathcal{N})]_x e_{xx}\,\mathrm{d}x &= \int[p'(\mathcal{N}+e_x)e_{xx}^2+(p'(\mathcal{N}+e_x)-p'(\mathcal{N}))\mathcal{N}_x e_{xx}]\,\mathrm{d}x\\ &\geq d_3\int e_{xx}^2\,\mathrm{d}x-O(1)\int|e_x|\;|e_{xx}|\,\mathrm{d}x\geq\tfrac{d_3}{2}\int e_{xx}^2\,\mathrm{d}x-O(1)\int e_x^2\,\mathrm{d}x.\end{aligned} \tag{4.35}$$

Integration by parts gives

$$\begin{aligned}&2\int[p(\mathcal{N}+e_x)-p(\mathcal{N})]_x e_{txx}\,\mathrm{d}x\\ &=2\int p'(\mathcal{N}+e_x)e_{xx}e_{txx}\,\mathrm{d}x+2\int[p'(\mathcal{N}+e_x)-p'(\mathcal{N})]\mathcal{N}_x e_{txx}\,\mathrm{d}x\\ &=\int p'(\mathcal{N}+e_x)(e_{xx}^2)_t\,\mathrm{d}x-2\int p''(\mathcal{N}+e_x)e_{xx}\mathcal{N}_x e_{tx}\,\mathrm{d}x\\ &-2\int(p''(\mathcal{N}+e_x)-p''(\mathcal{N}))\mathcal{N}_x^2 e_{tx}\,\mathrm{d}x-2\int(p'(\mathcal{N}+e_x)-p'(\mathcal{N}))\mathcal{N}_{xx}e_{tx}\,\mathrm{d}x\\ &=\frac{\mathrm{d}}{\mathrm{d}t}\int p'(\mathcal{N}+e_x)e_{xx}^2\,\mathrm{d}x-\int p''(\mathcal{N}+e_x)e_{xt}e_{xx}^2\,\mathrm{d}x\\ &-2\int p''(\mathcal{N}+e_x)\mathcal{N}_x e_{xx}e_{tx}\,\mathrm{d}x-2\int(p''(\mathcal{N}+e_x)-p''(\mathcal{N}))\mathcal{N}_x^2 e_{tx}\,\mathrm{d}x\\ &-2\int(p'(\mathcal{N}+e_x)-p'(\mathcal{N}))\mathcal{N}_{xx}e_{tx}\,\mathrm{d}x\;.\end{aligned} \tag{4.36}$$

For small $M(T)$ we have the following inequalities

$$\left|\int p''(\mathcal{N}+e_x)e_{xt}e_{xx}^2\,\mathrm{d}x\right|\leq O(1)\sup|e_{xx}|\left(\int e_{xt}^2\,\mathrm{d}x+\int e_{xx}^2\,\mathrm{d}x\right), \tag{4.37}$$

$$\left|2\int p''(\mathcal{N}+e_x)\mathcal{N}_x e_{xx}e_{tx}^2\,\mathrm{d}x\right|\leq O(1)\sup|\mathcal{N}_x|\left(\int(e_{xx}^2+e_{xt}^2)\,\mathrm{d}x\right)\;, \tag{4.38}$$

$$\left|2\int(p''(\mathcal{N}+e_x)-p''(\mathcal{N}))\mathcal{N}_x^2 e_{tx}\,\mathrm{d}x\right|\leq O(1)\sup|\mathcal{N}_x^2|\left(\int e_{tx}^2\,\mathrm{d}x+\int e_x^2\,\mathrm{d}x\right)\;, \tag{4.39}$$

$$\left|2\int(p'(\mathcal{N}+e_x)-p'(\mathcal{N}))\mathcal{N}_{xx}e_{tx}\,\mathrm{d}x\right|\leq O(1)\sup|\mathcal{N}_{xx}|\left(\int e_{tx}^2\,\mathrm{d}x+\int e_x^2\,\mathrm{d}x\right)\;, \tag{4.40}$$

$$\left|\int\mathcal{N}_x e\,e_x\,\mathrm{d}x\right|\leq O(1)\int(e^2+e_x^2)\,\mathrm{d}x\;, \tag{4.41}$$

$$\left|\int e_{xx}\mathcal{E}e_x \,\mathrm{d}x\right| \le O(1)|\mathcal{N}_x|_{C^0}\int (e_{xx}^2+e_x^2)\,\mathrm{d}x \ , \tag{4.42}$$

$$\begin{aligned}\left|\int\left(\frac{e_t^2}{\mathcal{N}+e_x}\right)_x e_{xx}\,\mathrm{d}x\right| &= \left|\int\left(\frac{2e_te_tx}{\mathcal{N}+e_x}e_{xx}-\frac{e_t^2(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2}e_{xx}\right)\,\mathrm{d}x\right| \\ &\le O(1)\left(\sup(|e_t|+|e_t^2|)\right)\int(e_{tx}^2+e_{xx}^2)\,\mathrm{d}x \ ,\end{aligned} \tag{4.43}$$

$$\left|2\int \mathcal{N}_x e e_{tx}\,\mathrm{d}x\right| \le \frac{1}{2}\int e_{tx}^2\,\mathrm{d}x + O(1)\int e^2\,\mathrm{d}x \ . \tag{4.44}$$

A straightforward calculation yields

$$\begin{aligned}2\int\left(\frac{e_t^2}{\mathcal{N}+e_x}\right)_{xx} e_{tx}\,\mathrm{d}x &= 2\int\Bigg[\frac{2e_{tx}^2+2e_te_{txx}}{\mathcal{N}+e_x}-\frac{2e_te_{tx}(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2} \\ &\quad -\frac{2e_te_{tx}(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2}-\frac{e_t^2(\mathcal{N}_{xx}+e_{xxx})}{(\mathcal{N}+e_x)^2}+2\frac{e_t^2(\mathcal{N}_x+e_{xx})^2}{(\mathcal{N}+e_x)^3}\Bigg]e_{tx}\,\mathrm{d}x \\ &=: I + 4\int\left(\frac{e_te_{txx}}{\mathcal{N}+e_x}e_{tx}-\frac{e_t^2e_{xxx}}{(\mathcal{N}+e_x)^2}e_{tx}\right)\,\mathrm{d}x \ .\end{aligned} \tag{4.45}$$

The term I can be handled as follows

$$|I| \le O(1)\sup\left(|e_{tx}+|e_t|+|e_{xx}|\right)\int\left(e_{tx}^2+e_{xx}^2+e_t^2\right)\,\mathrm{d}x \tag{4.46}$$

On the other hand,

$$\begin{aligned}\left|4\int\frac{e_te_{txx}}{\mathcal{N}+e_x}e_{tx}\,\mathrm{d}x\right| &= \left|2\int\frac{e_t}{\mathcal{N}+e_x}(e_{tx}^2)_x\,\mathrm{d}x\right| \\ &= \left|2\int e_{tx}^2\left(\frac{e_{tx}}{\mathcal{N}+e_x}-\frac{e_t(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2}\right)\,\mathrm{d}x\right| \\ &\le O(1)\sup(|e_{tx}|+|e_t|)\int(e_{tx}^2+e_{xx}^2)\,\mathrm{d}x\end{aligned} \tag{4.47}$$

and

$$\begin{aligned}-\int\frac{e_te_{xxx}}{(\mathcal{N}+e_x)^2}e_{tx} &= \int e_{xx}\left[\frac{2e_te_{tx}^2}{(\mathcal{N}+e_x)^2}-\frac{2e_t^2(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^3}e_{tx}\right]\,\mathrm{d}x \\ &\quad +\int\frac{e_t^2e_{xx}}{(\mathcal{N}+e_x)^2}e_{txx}\,\mathrm{d}x \ .\end{aligned} \tag{4.48}$$

Integration by parts gives

$$\begin{aligned}4\int\frac{e_t^2e_{xx}}{(\mathcal{N}+e_x)^2}e_{txx}\,\mathrm{d}x &= 2\int\frac{e_t^2}{(\mathcal{N}+e_x)^2}(e_{xx}^2)_t\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int\frac{2e_t^2e_{xx}^2}{(\mathcal{N}+e_x)^2}\,\mathrm{d}x-2\int e_{xx}^2\left(\frac{2e_te_{tt}}{(\mathcal{N}+e_x)^2}-\frac{2e_t^2e_{tx}}{(\mathcal{N}+e_x)^3}\right)\,\mathrm{d}x \ .\end{aligned} \tag{4.49}$$

Also, the equation (4.8) yields

$$|e_{tt}| \le O(1)(|e_{xx}|+|e_x|+|\mathcal{N}_x|+|e_t|+|e|+|e_{tx}|+|e|) \ . \tag{4.50}$$

Thus, we conclude

$$
\begin{aligned}
2\int \left(\frac{e_t^2}{\mathcal{N}+e_x}\right)_{xx} e_{tx}\,\mathrm{d}x \geq \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{2e_t^2 e_{xx}}{(\mathcal{N}+e_x)^2}\,\mathrm{d}x \\
+O(1)\sup\left(|e_t|_{C^1}+|e|_{C^2}+|\mathcal{N}_x|\right)\int (e_{xx}^2+e_{xt}^2+e_t^2)\,\mathrm{d}x\ .
\end{aligned}
\tag{4.51}
$$

There holds from (4.15)

$$
\left|\int e_{xx}\mathcal{E}e_{tx}\,\mathrm{d}x\right| \leq O(1)\sup|\mathcal{N}_x|\int (e_{xx}^2+e_{tx}^2)\,\mathrm{d}x \tag{4.52}
$$

and the Cauchy–Schwarz inequality implies

$$
\left|\int e_x\mathcal{E}_x e_{tx}\,\mathrm{d}x\right| \leq \frac{1}{4}\int e_{tx}^2\,\mathrm{d}x + O(1)\int e_x^2\,\mathrm{d}x\ , \tag{4.53}
$$

$$
\left|\int \left(\frac{1}{2}e^2\right)_{xx} e_{tx}\,\mathrm{d}x\right| \leq O(1)\sup(|e_x|+|e|)\int (e_{tx}^2+e_{xx}^2)\,\mathrm{d}x\ . \tag{4.54}
$$

Thus, if we put all the estimates together, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}F_3 + d_7\int (e_{xx}^2+e_{tx}^2)\,\mathrm{d}x \leq O(1)\int (e^2+e_x^2+e_t^2)\,\mathrm{d}x\ , \tag{4.55}
$$

if $|e_t|_{C^1}$, $|e|_{C^2}$ and $|\mathcal{N}_x|$, $|\mathcal{N}_{xx}|$ are sufficiently small, d_7 is a positive constant and

$$
F_3(t) := \int \left(e_{tx}e_x + \frac{1}{2}e_x^2 + e_{tx}^2 + \mathcal{N}e_x^2 + p'(\mathcal{N}+e_x)e_{xx}^2 + 2\frac{e_t^2 e_{xx}^2}{(\mathcal{N}+e_x)^2}\right)(x,t)\,\mathrm{d}x \tag{4.56}
$$

It is easy to show that there exists a positive constant d_8 such that

$$
F_3(t) \geq d_8\int (e_x^2+e_{tx}^2+e_{xx}^2)(x,t)\,\mathrm{d}x\ . \tag{4.57}
$$

Hence (4.23), (4.55) and (4.57) imply

$$
\int (e^2+e_x^2+e_{xx}^2+e_t^2+e_{tx}^2)(x,t)\,\mathrm{d}x \leq d_{10}exp(-d_9 t)\int (e^2+e_x^2+e_{xx}^2+e_t^2+e_{xt}^2)(x,0)\,\mathrm{d}x \tag{4.58}
$$

if $|\mathcal{N}_x|$, $|\mathcal{N}_{xx}|$ and $|e|_{C^2}$, $|e_t|_{C^1}$ are sufficiently small.

We turn now to the estimates of the third derivatives.

First differentiate (4.8) with respect to x twice and multiply the resulting equation by e_{xx} and e_{txx} respectively. Then integrate over $I\!R$ to get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(e_{txx}e_{xx}+\tfrac{1}{2}e_{xx}^2\right)(x,t)\,\mathrm{d}x - \int e_{txx}^2\,\mathrm{d}x \\
&+\int \mathcal{N}e_{xx}^2\,\mathrm{d}x + \int (p(\mathcal{N}+e_x)-p(\mathcal{N}))_{xx}e_{xxx}\,\mathrm{d}x + \int \left(\frac{e_t^2}{\mathcal{N}+e_x}\right)_{xx} e_{xxx}\,\mathrm{d}x \\
&\leq \left|\int \mathcal{E}e_{xxx}e_{xx}\,\mathrm{d}x\right| + \left|\int \mathcal{E}_{xx}e_x e_{xx}\,\mathrm{d}x\right| + O(1)\left(\|e(\cdot,t)\|_{H^2}^2+\|e_t(\cdot,t)\|_{H^1}^2\right)\ ,
\end{aligned}
\tag{4.59}
$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(e_{txx}^2+\tfrac{1}{2}\mathcal{N}e_{xx}^2\right)(x,t)\,\mathrm{d}x+\int(p(\mathcal{N}+e_x)-p(\mathcal{N}))_{xx}e_{txxx}\,\mathrm{d}x$$
$$\int e_{txx}^2\,\mathrm{d}x+\int\left(\tfrac{e_t^2}{\mathcal{N}e_x}\right)_{xx}e_{txxx}\,\mathrm{d}x+\int(e_x\mathcal{E})_{xx}e_{xxt} \tag{4.60}$$
$$+\int(\mathcal{N}_{xx}e+2\mathcal{N}_xe_x)e_{xxt}+\int(e_xe)_{xx}e_{xxt}\,\mathrm{d}x=0\ .$$

In (4.60), the fourth derivatives of e appears, but we only assume $e(\cdot,t)\in H^3(I\!R)$ and $e_t(\cdot,t)\in H^2(I\!R)$. In fact, we can assume $e_t(\cdot,t)\in H^4(I\!R)$ and $e_t(\cdot,t)\in H^3(I\!R)$, by stqandard regularization arguments. The same result holds for $e(\cdot,t)\in H^3(I\!R)$ and $e_t(\cdot,t)\in H^2(I\!R)$. We estimate the terms appearing in (4.59) and (4.60).

First we have

$$\int[p(\mathcal{N}+e_x)-p(\mathcal{N})]_{xx}e_{xxx}\,\mathrm{d}x\geq\int p'(\mathcal{N}+e_x)e_{xxx}^2\,\mathrm{d}x-O(1)\int|e_x+e_{xx}|\,|e_{xxx}|\,\mathrm{d}x \tag{4.61}$$

Since $p'(\mathcal{N}+e_x)\geq a_0>0$, as $|e_x|$ is small, thus

$$\int[p(\mathcal{N}+e_x)-p(\mathcal{N})]_{xx}e_{xxx}\,\mathrm{d}x\geq\frac{a_0}{2}\int e_{xxx}^2\,\mathrm{d}x-O(1)\|e(\cdot,t)\|_{H^2}^2\ . \tag{4.62}$$

The Cauchy–Schwarz inequality implies

$$\left|\int\left(\tfrac{e_t^2}{\mathcal{N}+e_x}\right)_{xx}e_{xxx}\,\mathrm{d}x\right|\leq O(1)\sup|e_t|\int\left(e_{txx}^2+e_{xxx}^2\right)\,\mathrm{d}x$$
$$+O(1)\sup|e_{tx}|\int e_{txx}^2\,\mathrm{d}x+O(1)\left(\|e(\cdot,t)\|_{H^2}^2+\|e_t(\cdot,t)\|_{H^1}^2\right)\ , \tag{4.63}$$
$$\left|\int\mathcal{E}e_{xxx}e_{xx}\,\mathrm{d}x\right|\leq\frac{a_0}{4}\int e_{xxx}^2\,\mathrm{d}x+O(1)\int e_{xx}^2\,\mathrm{d}x\ , \tag{4.64}$$
$$\left|\int\mathcal{E}_{xx}e_xe_{xx}\,\mathrm{d}x\right|=\int\mathcal{E}_x(e_{xx}^2+e_xe_{xxx})\,\mathrm{d}x\leq\frac{a_0}{8}\int e_{xxx}^2\,\mathrm{d}x+O(1)\int(e_x^2+e_{xx}^2)\,\mathrm{d}x\ . \tag{4.65}$$

Thus, the above estimates give us, in view of (4.13), (4.14),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\left(e_{txx}e_{xx}+\tfrac{1}{2}e_{xx}^2\right)(x,t)\,\mathrm{d}x-\int e_{txx}^2\,\mathrm{d}x+\left(\tfrac{a_0}{8}-M(T)\right)\int e_{xxx}^2\,\mathrm{d}x$$
$$\leq O(1)M(T)\int e_{txx}^2\,\mathrm{d}x+O(1)(\|e(\cdot,t)\|_{H^2}^2+\|e_t(\cdot,t)\|_{H^1}^2) \tag{4.66}$$

We turn now to the estimates of the terms in (4.60).

Since we do not assume that $b(x)\in C^1$, so we can not get the bound of $\mathcal{N}_{xxx}$. Thus, we must pay attention to the second and fourth terms in (4.60).

Integration by parts gives, in view of (4.13), (4.14),

$$\begin{aligned}\int (p(\mathcal{N}+e_x) & -p(\mathcal{N}))_{xx} e_{txxx}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int (p(\mathcal{N}+e_x)-p(\mathcal{N}))_{xx} e_{xxx}\,\mathrm{d}x \\ & -\int (p(\mathcal{N}+e_x)-p(\mathcal{N}))_{txx} e_{xxx}\,\mathrm{d}x \geq \frac{\mathrm{d}}{\mathrm{d}t}\int (p(\mathcal{N}+e_x)-p(\mathcal{N}))_{xx} e_{xxx}\,\mathrm{d}x \\ & -\int p'(\mathcal{N}+e_x) e_{txxx} e_{xxx}\,\mathrm{d}x - O(1)(M(T)+|\mathcal{N}_x|_{C^1})\int e_{xxx}^2\,\mathrm{d}x - O(1)w(t) \\ & \geq \frac{\mathrm{d}}{\mathrm{d}t}\int \left[(p(\mathcal{N}+e_x)-p(\mathcal{N}))_{xx} e_{xxx} - \tfrac{1}{2}p'(\mathcal{N}+e_x)e_{xxx}^3\right]\mathrm{d}x \\ & -O(1)(M(T)+|\mathcal{N}_x|_{C^1})\int e_{xxx}^2\,\mathrm{d}x - O(1)w(t)\ ,\end{aligned} \tag{4.67}$$

where and in the following

$$w(t) =: \|e(\cdot,t)\|_{H^2}^2 + \|e_t(\cdot,t)\|_{H^1}^2\ .$$

A standard calculation gives

$$\begin{aligned}\int \left(\tfrac{e_t^2}{\mathcal{N}+e_x}\right)_{xx} e_{xxxt}\,\mathrm{d}x = \int \Big[& \tfrac{(e_t^2)_{xx}}{(\mathcal{N}+e_x)} - \tfrac{2(e_t^2)_x(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2} \\ -e_t^2\left(\tfrac{\mathcal{N}_x}{(\mathcal{N}+e_x)^2}\right)_x & - e_t^2\left(\tfrac{e_{xx}}{(\mathcal{N}+e_x)^2}\right)_x\Big] e_{txxx}\,\mathrm{d}x\ .\end{aligned} \tag{4.68}$$

We estimate each term on the RHS of (4.68) as follows. First we have

$$\begin{aligned}\int \tfrac{(e_t^2)_{xx}}{\mathcal{N}+e_x} e_{txxx} &= \int \tfrac{2e_{tx}^2}{\mathcal{N}+e_x} e_{txxx}\,\mathrm{d}t + \int \tfrac{e_t(e_{txx}^2)_x}{\mathcal{N}+e_x}\,\mathrm{d}x = -\int e_{txx}\left(\tfrac{2e_{tx}^2}{\mathcal{N}+e_x}\right)_x\,\mathrm{d}x \\ -\int e_{txx}^2\left(\tfrac{e_t}{\mathcal{N}+e_x}\right)_x\,\mathrm{d}x &\geq -O(1)(M(t)+|\mathcal{N}_x|_{C^1})\int e_{txx}^2\,\mathrm{d}x - O(1)w(t)\ .\end{aligned} \tag{4.69}$$

Then

$$\begin{aligned}-\int \tfrac{2(e_t^2)_x(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2} e_{txxx}\,\mathrm{d}x &= \int e_{txx}\left[\tfrac{2(e_t^2)_x(\mathcal{N}_x+e_{xx})}{(\mathcal{N}+e_x)^2}\right]_x\,\mathrm{d}x \\ \geq -O(1)(M(T)+|\mathcal{N}_x|_{C^1}) & \int (e_{txx}^2+e_{xxx}^2)\,\mathrm{d}x - O(1)w(t)\ .\end{aligned} \tag{4.70}$$

$$\begin{aligned}-\int e_t^2 & \left(\tfrac{\mathcal{N}_x}{(\mathcal{N}+e_x)^2}\right)_x e_{txxx} = -\frac{\mathrm{d}}{\mathrm{d}t}\int e_t^2\left(\tfrac{\mathcal{N}_x}{(\mathcal{N}+e_x)^2}\right)_x e_{xxx} \\ & +\int e_{xxx}\left[e_t^2\left(\tfrac{\mathcal{N}_x}{(\mathcal{N}+e_x)^2}\right)_x\right]_t\,\mathrm{d}x \geq -\frac{\mathrm{d}}{\mathrm{d}t}\int e_t^2\left(\tfrac{\mathcal{N}_x}{(\mathcal{N}+e_x)^2}\right)_x \\ & -O(1)(M(T)+|\mathcal{N}_x|_{C^1})\int (e_{txx}^2+e_{xxx}^2)\,\mathrm{d}x - O(1)w(t)\ .\end{aligned} \tag{4.71}$$

$$
\begin{aligned}
-\int e_t^2\left(\frac{e_{xx}}{(\mathcal{N}+e_x)^2}\right)_x \quad e_{txxx}\,\mathrm{d}x &= -\int e_x^2\frac{\left(\frac{1}{2}e_{xxx}^2\right)_t}{(\mathcal{N}+e_x)^2} - \int e_t^2\left(\frac{1}{(\mathcal{N}+e_x^2)}\right)_x e_{xx}e_{txxx}\,\mathrm{d}x \\
&\geq \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\frac{e_t^2}{(\mathcal{N}+e_x)^2}e_{xxx}^2\,\mathrm{d}x + \int \frac{1}{2}e_{xxx}^2\left[\frac{e_t^2}{(\mathcal{N}+e_x)^2}\right]_t\,\mathrm{d}x \\
&+\int e_{xxx}\left[e_t^2\left(\frac{1}{\mathcal{N}+e_x^2}\right)_x\right]_x\,\mathrm{d}x \geq -\frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}\frac{e_t^2}{(\mathcal{N}+e_x)^2}e_{xxx}^2\,\mathrm{d}x \\
&-O(1)(M(T)+|\mathcal{N}_x|_{C^1})\int(e_{xxx}^2+e_{txx}^2)\,\mathrm{d}x - O(1)w(t)\ .
\end{aligned}
\tag{4.72}
$$

The other terms on LHS of (4.60) can be handled easily by using the Cauchy–Schwarz inequality.

Thus, with a similar approach as before, we can get

$$
\int(e_{xxx}^2+e_{xxt}^2)(x,t)\,\mathrm{d}x \leq O(1)M(0)\exp(-\,\mathrm{d}t)\ , \tag{4.73}
$$

for a positive constant d, if the quantity $|\mathcal{N}_x|_{C^1}+M(T)$ is small. Then the conclusion follows. $\square$

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