Lower Semicontinuity of Weighted Path Length in BV

Paolo Baiti and Alberto Bressan

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy

Abstract. We establish some basic lower semicontinuity properties for a class of weigthed metrics in **BV**. These Riemann-type metrics, uniformly equivalent to the $\mathbf{L}^1$ distance, are defined in terms of the Glimm interaction potential. They are relevant in the study of nonlinear hyperbolic systems of conservation laws, being contractive w.r.t. the corresponding flow of solutions.

1 - Introduction

For a scalar conservation law, it is well known [7, 11] that the entropy-admissible solutions determine a semigroup which is contractive w.r.t. the $\mathbf{L}^1$ distance. This fundamental property plays a key role in the study of uniqueness, stability and perturbations of weak solutions.

On the other hand, for a nonlinear $n \times n$ hyperbolic system

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

the contractivity of the flow is no longer true [14]. For this reason, when $u \in I\!R^n$, to establish the uniqueness and continuous dependence of solutions of (1.1) is a considerably more difficult problem than in the scalar case.

In a recent series of papers [1, 2, 3, 5], it was shown that, restricted to a suitable domain $\mathcal{D}$ of functions with small total variation, the system (1.1) does generate a continuous flow. More precisely, this flow is contractive w.r.t. a suitable Riemann-type metric, uniformly equivalent to the standard $\mathbf{L}^1$ distance. The construction of this weighted distance involves:

- A closed set $\mathcal{D} \subset \mathbf{L}^1(I\!R; I\!R^n)$ consisting of functions with small total variation, positively invariant for the flow of (1.1).
- A dense subset $\mathcal{D}_{PL} \subset \mathcal{D}$ of piecewise Lipschitz functions.
- For each $u \in \mathcal{D}_{PL}$, a space T_u of first order generalized tangent vectors (v, ξ), with weighted norm $\big\|(v,\xi)\big\|_u$. If, say, u is piecewise Lipschitz with N jumps, then $T_u \simeq \mathbf{L}^1 \times I\!R^N$.

Given a suitably regular path $\gamma : [a,b] \mapsto \mathcal{D}_{PL}$, its weighted length is then defined as the integral of the weighted norm of its tangent vector $D\gamma$, i.e.

$$\|\gamma\|_* \doteq \int_a^b \big\|D\gamma(\theta)\big\|_{\gamma(\theta)}\, d\theta. \tag{1.2}$$

In turn, the weighted distance between two functions $u, v \in \mathcal{D}$ is defined as the infimum of the lengths of (suitably regular) paths joining u with v, i.e.

$$d_*(u,v) \doteq \inf\ \big\{\|\gamma\|_*\,; \quad \gamma(a) = u,\ \gamma(b) = v\big\}. \tag{1.3}$$

We remark that this weighted distance does not fit within the standard framework of Riemann or Finsler manifolds [8, p.362], because our tangent spaces T_u are defined not for all u in some open set $\mathcal{U} \subset \mathbf{L}^1$, but only for u in a dense subset with empty interior.

Aim of this paper is to establish some basic properties of the distance d_*. We first study the behavior of the Glimm interaction functional under $\mathbf{L}^1$ convergence, and prove the semicontinuity of the coefficients in the weighted norms. Then we establish the lower semicontinuity of the weighted length (1.2) w.r.t. uniform convergence of paths in $\mathbf{L}^1$. More precisely,

$$\|\gamma\|_* \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_* \tag{1.4}$$

for every sequence of paths γ_ν converging uniformly to γ. This result naturally complements and clarifies the constructions in [3, 5].

We remark that, for any Lipschitz continuous path $\gamma : [a,b] \mapsto \mathbf{L}^1$, the usual $\mathbf{L}^1$ length is defined as

$$\|\gamma\|_{\mathbf{L}^1} \doteq \sup\left\{ \sum_{i=1}^N \big\|\gamma(\theta_i) - \gamma(\theta_{i-1})\big\|_{\mathbf{L}^1}\,; \quad a=\theta_0<\theta_1<\cdots<\theta_N=b, \quad N\geq 1\right\}. \tag{1.5}$$

In this case, given the uniform convergence $\gamma_\nu \to \gamma$, the relation

$$\|\gamma\|_{\mathbf{L}^1} \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_{\mathbf{L}^1}$$

follows from standard convexity arguments. On the other hand, (1.4) cannot be obtained by general weak convergence methods. Indeed, the numerical value of the constants defining the weighted norms $\big\|(v,\xi)\big\|_u$ and the smallness of the total variation of functions $u \in \mathcal{D}$ both play a key role for the validity of (1.4).

All the basic definitions, including generalized tangent vectors and weighted path lengths, are collected in Section 2. Our main lower semicontinuity result is stated in Section 4. The proof relies on the construction of a family of nonlinear functionals, which approximate the weighted distance function in **BV**.

2 - Preliminaries

Let Ω be an open and convex subset of $\mathbb{R}^n$ containing the origin, and let $F : \Omega \mapsto \mathbb{R}^n$ be three times continuously differentiable. For every u, consider the $n\times n$ Jacobian matrix $A(u) \doteq DF(u)$. We assume that the system (1.1) is strictly hyperbolic, so that each $A(u)$ has real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. Given $u, u' \in \Omega$, define

$$A(u,u') \doteq \int_0^1 A\big(\theta u + (1-\theta)u'\big)\,d\theta. \tag{2.1}$$

Of course, $A(u,u) = A(u)$. Call $\lambda_i(u,u')$ the i-th eigenvalue of the matrix $A(u,u')$. By possibly shrinking the size of Ω, there exist n disjoint intervals $[\lambda_i^{min}, \lambda_i^{max}]$ such that

$$\lambda_i(u,u') \in [\lambda_i^{min}, \lambda_i^{max}] \qquad\qquad u, u' \in \Omega, \quad i=1,\ldots,n.$$

We assume that each characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax [12]. So we can choose $\mathcal{C}^2$ families of right and left eigenvectors $r_i(u,u')$, $l_i(u,u')$ of $A(u,u')$, normalized according to

$$\big|r_i(u,u')\big| \equiv 1, \tag{2.2}$$

$$l_j(u,u')\cdot r_i(u,u') = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i\neq j. \end{cases} \tag{2.3}$$

We write also $r_i(u)$ and $l_j(u)$ respectively for $r_i(u,u)$ and $l_j(u,u)$, and choose the orientation of the r_i so that

$$r_i \bullet \lambda_i(u) \doteq \lim_{\varepsilon\to 0} \frac{\lambda_i\big(u+\varepsilon r_i(u)\big)-\lambda_i(u)}{\varepsilon} \geq 0. \tag{2.4}$$

We briefly recall the solution of the Riemann problem [12] and the interaction estimates [10, 13]. Given $u\in\Omega$, call $\sigma\mapsto S_i(\sigma)(u)$ and $\sigma\mapsto R_i(\sigma)(u)$ respectively the i-shock and i-rarefaction curve through u, parametrized by arc-length. As customary, the orientation is chosen so that the i-th characteristic speed is non-decreasing along the curves S_i, R_i. Define the composite curves

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if} \quad \sigma\geq 0, \\ S_i(\sigma)(u) & \text{if} \quad \sigma<0. \end{cases} \tag{2.5}$$

Given two states u^-, u^+ sufficiently close to the origin, by the implicit function theorem there exist unique wave sizes $\sigma_1,\dots,\sigma_n$ such that

$$u^+ = \Psi_n(\sigma_n)\circ\cdots\circ\Psi_1(\sigma_1)(u^-). \tag{2.6}$$

The solution of the Riemann problem with data (u^-,u^+) thus consists of $n+1$ constant states $\omega_0=u^-,\omega_1,\dots,\omega_n=u^+$, where each couple of states ω_{i-1},ω_i is connected by an i-wave of size σ_i (a shock or a rarefaction, depending on the sign of σ_i). We write

$$E_i(u^-,u^+) \doteq \sigma_i, \tag{2.7}$$

for the size of the i-wave determined by the Riemann data (u^-,u^+). This quantity satisfies the well known estimate

$$E_i\big(u^-,u^+\big) = l_i(u^-)\cdot(u^+-u^-) + O\big(|u^+-u^-|^2\big). \tag{2.8}$$

We define also

$$J_i(u^-,u^+) \doteq \big|E_i(u^-,u^+)\big|. \tag{2.9}$$

Let us now recall the basic interaction estimates [10, 13]. Given a left, a middle and a right state $u^l, u^m, u^r\in\Omega$, one has

$$\sum_{i=1}^n \Big|E_i(u^l,u^r)-E_i(u^l,u^m)-E_i(u^m,u^r)\Big| \leq C_3\Delta\big(E(u^l,u^m),E(u^m,u^r)\big), \tag{2.10}$$

where

$$\Delta(\alpha,\beta) \doteq \sum_{k>j} |\alpha_k\beta_j| + \sum_{k\in\mathrm{GNL},\ \min\{\alpha_k,\beta_k\}<0} |\alpha_k\beta_k|, \tag{2.11}$$

and GNL is the set of indices corresponding to genuinely nonlinear families.

As in [4], the Glimm functional can be defined also for a general **BV** function. Let $u : \mathbb{R} \mapsto \Omega$ have bounded variation. By possibly changing the values of u at countably many points, we can assume that u is right continuous. Its distributional derivative $\mu \doteq D_x u$ is then a vector measure, which can be decomposed into a continuous and an atomic part: $\mu = \mu^c + \mu^a$. For $i = 1,\ldots,n$ define the signed measure $\mu_i = \mu_i^c + \mu_i^a$ as follows. The continuous part of μ_i is the Radon measure such that

$$\int \phi \, d\mu_i^c = \int l_i(u)\cdot\phi \, d\mu^c \tag{2.12}$$

for every scalar continuous function ϕ with compact support. The atomic part of μ_i is the measure concentrated on the countable set $\{x_\alpha;\ \alpha = 1,2,\ldots\}$ where u has a jump, such that

$$\mu_i^a\big(\{x_\alpha\}\big) = E_i\big(u(x_\alpha-),\ u(x_\alpha+)\big) \tag{2.13}$$

is the size of the i-th wave in the solution of the corresponding Riemann problem at x_α. Call μ_i^+, μ_i^- the positive and negative parts of the signed measure μ_i, so that

$$\mu_i = \mu_i^+ - \mu_i^-, \qquad\qquad |\mu_i| = \mu_i^+ + \mu_i^-.$$

The *total strength of waves* in u is then defined as

$$V(u) \doteq \sum_{i=1}^n V_i(u), \qquad\qquad V_i(u) \doteq |\mu_i|(\mathbb{R}), \tag{2.14}$$

while the *interaction potential* of waves in u is

$$Q(u) \doteq \sum_{i<j} \big(|\mu_j|\times|\mu_i|\big)\Big(\big\{(x,y);\ x<y\big\}\Big) + \sum_i \big(\mu_i^-\times|\mu_i|\big)\Big(\big\{(x,y);\ x\neq y\big\}\Big). \tag{2.15}$$

When u is piecewise constant, one easily checks that the definitions (2.14)-(2.15) reduce to the usual ones. On the other hand, if u is Lipschitz continuous, then its derivative $u_x(x)$ exists at almost every point $x\in\mathbb{R}$. In this case, setting

$$u_x^i(x) \doteq l_i\big(u(x)\big)\cdot u_x(x), \tag{2.16}$$

one has

$$V_i(u) = \int_{-\infty}^{\infty} \big|u_x^i(x)\big| \, dx.$$

The quantities V, Q satisfy two basic properties. The first is a straightforward consequence of the definitions:

(P1) Let $\varphi : I\!\!R \mapsto I\!\!R$ be a continuous, increasing one-to-one mapping. Then, for every $u \in \mathbf{BV}$, the composed function $v(x) \doteq u\big(\varphi(x)\big)$ satisfies

$$Q(v) = Q(u), \qquad V_i(v) = V_i(u), \qquad i = 1, \ldots, n. \tag{2.17}$$

The second property follows from the Glimm interaction estimates. It can be proved first for piecewise constant functions, then extended to all **BV** functions by an approximation argument.

(P2) There exist constants $\kappa_0, \delta_0 > 0$ such that the following holds. Assume $u \in \mathbf{L}^1$, with Tot.Var.$(u) \leq \delta_0$. Then, for any $a < b$ and $\bar{x} \in [a, b[$, the function

$$w(x) \doteq \begin{cases} u(x) & \text{if} \quad x \notin [a, b[, \\ u(\bar{x}) & \text{if} \quad x \in [a, b], \end{cases} \tag{2.18}$$

satisfies

$$Q(w) \leq Q(u), \qquad\qquad V(w) + \kappa_0 Q(w) \leq V(u) + \kappa_0 Q(u), \tag{2.19}$$

$$V_i(w) + \kappa_0 Q(w) \leq V_i(u) + \kappa_0 Q(u), \qquad i = 1, \ldots, n. \tag{2.20}$$

Observe that the function w in (2.18) is obtained from u by collapsing all wave-fronts in $[a, \bar{x}]$ onto the point a, and all wave-fronts in $]\bar{x}, b]$ onto the point b.

With the same constants κ_0, δ_0 as in **(P2)**, we now define the domains

$$\mathcal{D} \doteq \big\{ u \in \mathbf{L}^1(I\!\!R;\, I\!\!R^n), \quad V(u) + \kappa_0 Q(u) \leq \delta_0 \big\}, \tag{2.21}$$

$$\mathcal{D}_{PL} \doteq \big\{ u \in \mathcal{D}, \quad u \text{ is Piecewise Lipschitz continuous} \big\}. \tag{2.22}$$

We recall below the definition of generalized differential of a path $\gamma : [a, b] \mapsto \mathbf{L}^1$, introduced in [6]. For any $u \in \mathbf{L}^1$, on the family Σ_u of all continuous paths $\gamma : [0, \theta_0] \mapsto \mathbf{L}^1$ such that $\gamma(0) = u$, consider the equivalence relation

$$\gamma \sim \gamma' \qquad \text{iff} \qquad \lim_{\theta \to 0+} \frac{1}{\theta} \big\| \gamma(\theta) - \gamma'(\theta) \big\|_{\mathbf{L}^1} = 0 \qquad\qquad \big(\gamma, \gamma' \in \Sigma_u\big). \tag{2.23}$$

Now assume that u is piecewise Lipschitz, say with jumps at the points $x_1 < \cdots < x_N$. The space of *generalized tangent vectors at* u is then defined as $T_u \doteq \mathbf{L}^1 \times I\!\!R^N$. To each $(v, \xi) \in T_u$, with $\xi = (\xi_1, \ldots, \xi_N)$, we associate the path $\gamma_{(v,\xi;u)} \in \Sigma_u$ defined by

$$\gamma_{(v,\xi;u)}(\theta) \doteq u + \theta v + \sum_{\xi_\alpha < 0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\, \chi_{[x_\alpha + \theta\xi_\alpha,\, x_\alpha]} - \sum_{\xi_\alpha > 0} \big[u(x_\alpha^+) - u(x_\alpha^-)\big]\, \chi_{[x_\alpha,\, x_\alpha + \theta\xi_\alpha]}. \tag{2.24}$$

More generally, we say that a path $\gamma \in \Sigma_u$ *generates the generalized tangent vector* $(v, \xi) \in T_u$, if γ is equivalent to $\gamma_{(v,\xi;u)}$, under the relation (2.23).

In other words, for small values of θ, the function $u^\theta \doteq \gamma(\theta)$ can be obtained from u by adding θv and by shifting the positions of the jumps from x_α to $x_\alpha + \theta\xi_\alpha$. As $\theta \to 0+$, this procedure yields a first order approximation to u^θ, with an error $o(\theta)$ in the $\mathbf{L}^1$ norm, with $o(\theta)/\theta \to 0$ as $\theta \to 0$. In connection with the above differential structure, one can define a kind of continuous differentiability property for maps $\gamma : \theta \mapsto u^\theta \in \mathbf{L}^1$, with piecewise Lipschitz values. Following [6], we say that a map $\gamma :]a, b[\mapsto \mathbf{L}^1$ is a *regular path* if there exists an integer N such that:

(i) Every function $u^\theta \doteq \gamma(\theta)$ is piecewise Lipschitz continuous with jumps at points $x_1^\theta < \cdots < x_N^\theta$ continuously depending on θ. Outside the jumps, each u^θ is continuous with a Lipschitz constant L independent of θ. All functions u^θ coincide outside some interval $[-M, M]$.

(ii) The map $\theta \mapsto u_x^\theta$ is continuous with values in $\mathbf{L}^1$.

(iii) There exists a continuous map $\theta \mapsto (v^\theta, \xi^\theta) \in \mathbf{L}^1 \times I\!\!R^N$ such that for every θ

$$\lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \Big\| \gamma(\theta + \varepsilon) - \gamma_{(v^\theta, \xi^\theta; u^\theta)}(\varepsilon) \Big\|_{\mathbf{L}^1} = 0. \tag{2.25}$$

More generally, we say that a continuous map $\gamma : [a, b] \mapsto \mathbf{L}^1$ is a *piecewise regular path* if there exist points $a = \theta_0 < \theta_1 < \cdots < \theta_\nu = b$ such that the restriction of γ to each open subinterval $]\theta_{j-1}, \theta_j[$ is a regular path.

Now consider any $u \in \mathcal{D}_{PL}$, say with jumps at $x_1 < \cdots < x_N$. For every $\alpha = 1, \ldots, N$, $i = 1, \ldots, n$, define the strength of the i-th wave in the Riemann problem at x_α as

$$J^i_{x_\alpha} \doteq \Big| E_i\big(u(x_\alpha-), u(x_\alpha+)\big) \Big|.$$

Given a generalized tangent vector $(v, \xi) \in T_u = \mathbf{L}^1 \times I\!\!R^N$, recalling (2.16) we define its weighted norm as

$$\big\|(v, \xi)\big\|_u \doteq \sum_{\alpha=1}^{N} \sum_{i=1}^{n} J^i_{x_\alpha} |\xi_\alpha| W_i^u(x_\alpha) + \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|v_i(x)\big| W_i^u(x)\, dx. \tag{2.26}$$

Here $v_i(x) \doteq l_i\big(u(x)\big) \cdot v(x)$ is the i-th component of v, Q is the interaction potential (2.15) and the weights W_i^u are defined by

$$W_i^u(x) \doteq 1 + \kappa_1 R_i^u(x) + \kappa_1 \kappa_2 Q(u), \tag{2.27}$$

$$\begin{aligned} R_i^u(x) \doteq {} & \left[\sum_{j \le i} \int_x^\infty + \sum_{j \ge i} \int_{-\infty}^x \right] \big|u_x^j(y)\big|\, dy + \left[\sum_{\substack{k \le i \\ x_\alpha > x}} + \sum_{\substack{k \ge i \\ x_\alpha < x}} \right] J^k_{x_\alpha} + \\ & + \begin{cases} J^i_{x_\beta} & \text{if } \ x = x_\beta \ \text{ and } \ E_i\big(u(x_\beta-), u(x_\beta+)\big) > 0, \\ 0 & \text{otherwise,} \end{cases} \end{aligned} \tag{2.28}$$

for suitably large constants κ_1, κ_2. Intuitively, $R_i^u(x)$ can be regarded as the total strength of all waves in u which approach an infinitesimal i-shock located at x. Finally, let $\gamma : \theta \mapsto u^\theta$ be a piecewise regular path defined on $[a,b]$, and let (v^θ, ξ^θ) be its generalized tangent vector at u^θ. The *weighted length* of γ is then defined as

$$\|\gamma\|_* \doteq \int_a^b \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}\, d\theta. \tag{2.29}$$

By the standard interaction estimates (2.10), one can choose constants $\delta_0 > 0$ small and $\kappa_0, \kappa_1, \kappa_2$ large enough in (2.21), (2.27) so that the following property holds.

(P3) Let $u \in \mathcal{D}_{PL}$. For any $a < b$ and $\bar{x} \in [a,b]$, the function w in (2.18) satisfies

$$W_i^w(x) \le W_i^u(x) \qquad x \notin \ [a,b[, \quad i = 1, \ldots, n. \tag{2.30}$$

From the definition of regular path, the following continuity properties can be easily derived:

Lemma 1. *Let $\gamma :]a,b[\mapsto \mathcal{D}_{PL}$ be a regular path. Let $u^\theta = \gamma(\theta)$ have jumps at the points $x_1^\theta < \cdots < x_N^\theta$. Then the following holds.*

a) The map $(\theta, x) \mapsto u^\theta(x)$ is continuous outside the jump set $\mathcal{J} \doteq \big\{(\theta, x_\alpha^\theta), \quad \alpha = 1, \ldots, N, \quad \theta \in\ ]a,b[\big\}$. For every $\bar\theta$, at each jump point $x_\alpha^{\bar\theta}$ one has

$$u^\theta\Big(x_\alpha^\theta \pm\Big) \to u^{\bar\theta}\Big(x_\alpha^{\bar\theta} \pm\Big) \quad as \quad \theta \to \bar\theta. \tag{2.31}$$

b) The map $(\theta, x) \mapsto W_i^{u^\theta}(x)$ is continuous outside the jump set $\mathcal{J}$. For every $\bar\theta$, at each jump point $x_\alpha^{\bar\theta}$ one has

$$W_i^{u^\theta}(x_\alpha^\theta) \to W_i^{u^{\bar\theta}}(x_\alpha^{\bar\theta}) \quad as \quad \theta \to \bar\theta. \tag{2.32}$$

c) The map $\theta \mapsto \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}$ is continuous.

We conclude this section with a useful approximation lemma. By an *elementary path* we mean a path of the form

$$\theta \mapsto w^\theta = w_1 \cdot \chi_{]-\infty,\ \alpha\theta+\beta]} + w_2 \cdot \chi_{]\alpha\theta+\beta, +\infty[}, \qquad \theta \in [a,b],$$

where the functions w_1, w_2 are piecewise constant. A *piecewise elementary path* is a finite concatenation of elementary paths.

Lemma 2. *Let $\gamma : [a,b] \mapsto \mathbf{BV}$ be a piecewise regular path. Then there exists a sequence of piecewise elementary paths γ_ν such that, as $\nu \to \infty$,*

$$\sup_{\theta \in [a,b]} \big\|\gamma_\nu(\theta) - \gamma(\theta)\big\|_{\mathbf{L}^1} \to 0, \qquad \big\|\gamma_\nu\big\|_* \to \|\gamma\|_*. \tag{2.33}$$

The construction of the paths γ_ν goes as follows. For simplicity, assume that γ is a regular path, with each $u^\theta = \gamma(\theta)$ having the same number of jumps, say at $x_1^\theta < \cdots < x_N^\theta$. Let all functions u^θ coincide when $x \notin [-M, M]$. Fix $\nu \geq 1$ and define

$$\theta_m \doteq a + \frac{m}{\nu}(b-a) \qquad\qquad m = 1, \ldots, \nu.$$

For ν sufficiently large, for each m we can choose points p_i such that

$$-M < p_0 < x_1^\theta < p_1 < \cdots < x_N^\theta < p_N < M \qquad\qquad \theta \in [\theta_{m-1}, \theta_m].$$

We now approximate each u^{θ_m} with a piecewise constant function $\tilde{u}^m$. The restriction of the original path γ to the subinterval $[\theta_{m-1}, \theta_m]$ is then replaced by a new path γ' defined as follows. If $\vartheta \in [-M, p_0] \cup [p_N, M]$ we set

$$\gamma'(\vartheta) \doteq u^m \cdot \chi_{]-\infty, \vartheta]} + u^{m-1} \cdot \chi_{]\vartheta, +\infty[} \,. \tag{2.34}$$

The same definition (2.34) is valid if $\vartheta \in]p_{i-1}, p_i]$ and $x_i^{\theta_{m-1}} \leq x_i^{\theta_m}$. On the other hand, if $\vartheta \in]p_{i-1}, p_i]$ but $x_i^{\theta_{m-1}} > x_i^{\theta_m}$, we set

$$\gamma'(\vartheta) \doteq u^m \cdot \chi_{]-\infty, p_{i-1}] \cup]p_{i-1}+p_i-\vartheta,\ p_i]} + u^{m-1} \cdot \chi_{]p_{i-1},\ p_{i-1}+p_i-\vartheta] \cup]p_i, +\infty[} \,. \tag{2.35}$$

Clearly, γ' is piecewise elementary, with $\gamma'(-M) = u^{m-1}$, $\gamma'(M) = u^m$. We now perform a suitable parameter rescaling $\theta \mapsto \vartheta(\theta)$ mapping $[\theta_{m-1}, \theta_m]$ onto $[-M, M]$ and define the path $\gamma_\nu(\theta) \doteq \gamma'\big(\vartheta(\theta)\big)$. Applying the same procedure to each subinterval $[\theta_{m-1}, \theta_m]$ we obtain the piecewise elementary path $\gamma_\nu : [a, b] \mapsto \mathbf{BV}$. If the approximations u^m of u^{θ_m} are chosen in a suitably accurate way, the properties (2.33) follow.

3 - Lower semicontinuity of the Glimm functionals

In this section we establish the lower semicontinuity of the functionals Q and $V + \kappa_0 Q$ on the domain $\mathcal{D}$ defined at (2.21).

Theorem 1. *Consider a sequence of functions $u_\nu \in \mathcal{D}$, with $u_\nu \to u$ in $\mathbf{L}^1$, as $\nu \to \infty$. Then*

$$Q(u) \leq \liminf_{\nu \to \infty} Q(u_\nu), \tag{3.1}$$

$$V(u) + \kappa_0 Q(u) \leq \liminf_{\nu \to \infty} \big\{ V(u_\nu) + \kappa_0 Q(u_\nu) \big\}. \tag{3.2}$$

In particular, the functional $V + \kappa_0 Q$ is lower semicontinuous on $\mathcal{D}$ and $\mathcal{D}$ is closed in $\mathbf{L}^1$.

Toward the proof, we shall need

Lemma 3. *For some constant C_0 and every $\varepsilon > 0$ the following holds. If $u \in \mathcal{D}$ satisfies $\big|u(x) - \bar u\big| \le \varepsilon$ for some constant state $\bar u$ and all x in an open interval I, then*

$$\left| \int_I l_i(\bar u) \cdot \varphi \, Du - \int_I \varphi \, d\mu_i \right| \le C_0 \varepsilon \int_I |\varphi| \, |Du|, \tag{3.3}$$

for every $\varphi \in \mathcal{C}_c(I; I\!R)$, $i = 1, \ldots, n$.

Proof of Lemma 3. By (2.8), at each point x where u has a jump there exists a vector $\tilde l_i(x)$ such that

$$\Big|\tilde l_i(x) - l_i\big(u(x-)\big)\Big| \le C \cdot \big|u(x+) - u(x-)\big|, \tag{3.4}$$

$$E_i\big(u(x-), u(x+)\big) = \tilde l_i(x) \cdot \big(u(x+) - u(x-)\big), \tag{3.5}$$

for some constant C depending only on the system (1.1). We can now write

$$\int \varphi \, d\mu_i = \int \tilde l_i \cdot \varphi \, Du, \tag{3.6}$$

where $\tilde l_i(x) = l_i\big(u(x)\big)$ at points where u is continuous, while $\tilde l_i(x)$ is some vector which satisfies (3.4)-(3.5) at points of jump. In all cases, the assumptions of the lemma imply an estimate of the form

$$\big|l_i(\bar u) - \tilde l_i(x)\big| \le C_0 \varepsilon.$$

Hence,

$$\left| \int l_i(\bar u) \cdot \varphi \, Du - \int \varphi \, d\mu_i \right| \le \int \Big|l_i(\bar u) - \tilde l_i(x)\Big| \cdot \big|\varphi(x)\big| \, \big|Du(x)\big| \le C_0 \varepsilon \int |\varphi| \, |Du|.$$

□

We can now prove Theorem 1. Let μ_i be as in (2.12)-(2.13) and let $\mu_{\nu,i}$ be analogously defined, with u replaced by u_ν. By passing to a subsequence we can assume that $\lim_{\nu \to \infty} \big\{V(u_\nu) + \kappa_0 Q(u_\nu)\big\}$ exists, that $u_\nu(x) \to u(x)$ for all $x \in I\!R$ and that $\big|Du_\nu\big| \rightharpoonup \tilde\mu$ weakly in the sense of measures as $\nu \to \infty$, where $\tilde\mu$ is a non negative Radon measure. Now fix $\varepsilon > 0$. Since the total mass of $\tilde\mu$ is finite, one can select finitely many points $y_1, \ldots, y_N$ such that

$$\tilde\mu\big(\{x\}\big) < \varepsilon, \qquad \forall\, x \notin \{y_1, \ldots y_N\}. \tag{3.7}$$

We now choose disjoint intervals $I_k \doteq (y_k - \rho, y_k + \rho)$ such that

$$\sum_{k=1}^N \tilde\mu\big(I_k \setminus \{y_k\}\big) < \frac{\varepsilon}{N}. \tag{3.8}$$

Moreover, there exists $R > 0$ such that

$$\bigcup_{k=1}^{N} I_k \subset [-R, R], \qquad \tilde\mu\big(I\!R \setminus [-R, R]\big) < \varepsilon. \tag{3.9}$$

Because of (3.7), we can now choose points $p_0 < -R < p_1 < \cdots < R < p_r$ which are continuity points for u and for every u_ν, and such that either

$$p_{h-1} < y_k < p_h, \qquad [p_{h-1}, p_h] \subset I_k, \tag{3.10}$$

for some k, or else

$$\tilde\mu\big([p_{h-1}, p_h]\big) < \varepsilon. \tag{3.11}$$

Call $J_h \doteq [p_{h-1}, p_h]$. By weak convergence, for some ν_0 sufficiently large one has

$$\big|Du_\nu\big|(J_h) < \varepsilon \qquad \text{whenever} \quad J_h \cap \{y_1, \ldots, y_N\} = \emptyset, \quad \nu \geq \nu_0. \tag{3.12}$$

Moreover, if (3.10) holds, from (3.8) it follows

$$|Du|\big(J_h \setminus \{y_k\}\big) \leq \tilde\mu\big(J_h \setminus \{y_k\}\big) < \frac{\varepsilon}{N}. \tag{3.13}$$

On the other hand, if (3.12) holds, then the oscillation of u_ν on the interval J_h is very small. Indeed, for every $x, y \in J_h$ and ν sufficiently large,

$$\big|u_\nu(x) - u_\nu(y)\big| \leq C\big|Du_\nu\big|(J_h) \leq 2\tilde\mu(J_h) < 2\varepsilon. \tag{3.14}$$

The same is also true for u. Set $\bar u_h \doteq u(p_h)$. By pointwise convergence and (3.14) it follows that

$$\big|u_\nu(x) - \bar u_h\big| < C\varepsilon, \qquad \big|u(x) - \bar u_h\big| < C\varepsilon, \qquad x \in J_h, \tag{3.15}$$

for all ν sufficiently large. Hence, by (3.3) we get

$$\left| \int_{J_h} \varphi\, d\mu_i - \int_{J_h} \varphi\, d\mu_{\nu,i} \right| \leq C_0\varepsilon \int_{J_h} |\varphi|\big(|Du| + |Du_\nu|\big) + \left| \int_{J_h} l_i(\bar u_h) \cdot \varphi\big(Du - Du_\nu\big) \right|, \tag{3.16}$$

for all $\varphi \in \mathcal{C}\big(J_h; I\!R\big)$. By weak convergence and by taking the supremum over all $|\varphi| \leq 1$, we obtain

$$\big|\mu_i\big|(J_h) \leq \liminf_{\nu\to\infty} \big|\mu_{\nu,i}\big|(J_h) + 2C_0\varepsilon\tilde\mu(J_h). \tag{3.17}$$

On the other hand, inserting $\varphi \equiv 1$ in (3.16) we obtain

$$\big|\mu_i(J_h) - \mu_{\nu,i}(J_h)\big| \leq C_0\varepsilon \int_{J_h} \big(|Du| + |Du_\nu|\big) + \left| \int_{J_h} l_i(\bar u) \cdot (Du - Du_\nu) \right|. \tag{3.18}$$

Letting $\nu \to \infty$ and using (3.17) this yields

$$\begin{aligned}\mu_i^-(J_h) &= \frac{1}{2}\Big[|\mu_i|(J_h) - \mu_i(J_h)\Big]\\ &\leq \liminf_{\nu\to\infty} \mu_{\nu,i}^-(J_h) + 2C_0\varepsilon\tilde{\mu}(J_h).\end{aligned} \tag{3.19}$$

We now take care of the intervals J_h containing a point y_k of large oscillation. For each $k = 1, \ldots, N$, let $h = h(k) \in \{1, \ldots, r\}$ be the index such that

$$y_k \in J_h \doteq [p_{h-1},\ p_h].$$

For each $\nu \geq 1$ consider the function

$$\hat{u}_\nu(x) \doteq \begin{cases} u_\nu(x) & \text{if } \ x \notin \cup_k J_{h(k)}, \\ u_\nu(p_{h(k)-1}) & \text{if } \ x \in [p_{h(k)-1},\ y_k]\,, \\ u_\nu(p_h) & \text{if } \ x \in]y_k,\ p_{h(k)}]. \end{cases} \tag{3.20}$$

Observe that each $\hat{u}_\nu$ is continuous at all points $p_0, \ldots, p_r$. Call $\hat{\mu}_{\nu,i}$, $i = 1, \ldots, n$, the corresponding measures, defined as in (2.12)-(2.13) with u replaced by $\hat{u}_\nu$. Clearly $\hat{\mu}_{\nu,i} = \mu_{\nu,i}$ outside the intervals $J_{h(k)}$. By property **(P2)** it follows

$$Q(\hat{u}_\nu) \leq Q(u_\nu), \qquad\qquad V(\hat{u}_\nu) + \kappa_0 Q(\hat{u}_\nu) \leq V(u_\nu) + \kappa_0 Q(u_\nu). \tag{3.21}$$

Since $u_\nu \to u$ pointwise, by (3.13) for each k one has

$$\begin{aligned}\Big|\mu_i(\{y_k\}) - \hat{\mu}_{\nu,i}(\{y_k\})\Big| &= \Big|E_i\big(u(y_k-), u(y_k+)\big) - E_i\big(u_\nu(p_{h(k)-1}), u_\nu(p_{h(k)})\big)\Big|\\ &\leq C\cdot\Big\{\Big|u(y_k-) - u(p_{h(k)-1})\Big| + \Big|u(p_{h(k)-1}) - u_\nu(p_{h(k)-1})\Big|\\ &\qquad + \Big|u(y_k+) - u(p_{h(k)})\Big| + \Big|u(p_{h(k)}) - u_\nu(p_{h(k)})\Big|\Big\}\\ &\leq C\cdot\frac{\varepsilon}{N}\end{aligned} \tag{3.22}$$

for each $k = 1, \ldots, N$ and all ν sufficiently large. By construction we also have

$$|\hat{\mu}_{\nu,i}|\big(J_{h(k)} \setminus \{y_k\}\big) = 0, \qquad\qquad |\mu_i|\big(J_{h(k)} \setminus \{y_k\}\big) \leq \frac{\varepsilon}{N}. \tag{3.23}$$

Recalling the definition (2.15) and using (3.17), (3.19), (3.22), (3.23) and (3.9), we obtain an estimate of the form

$$\begin{aligned}Q(u_\nu) &\geq Q(\hat{u}_\nu)\\ &\geq \sum_{i<j}\sum_{h<\ell}\big(|\hat{\mu}_{\nu,j}| \times |\hat{\mu}_{\nu,i}|\big)(J_h \times J_\ell) + \sum_i\sum_{h\neq\ell}\big(\hat{\mu}_{\nu,i}^- \times |\hat{\mu}_{\nu,i}|\big)(J_h \times J_\ell)\\ &\geq \sum_{i<j}\sum_{h<\ell}\big(|\mu_j| \times |\mu_i|\big)(J_h \times J_\ell) + \sum_i\sum_{h\neq\ell}\big(\mu_i^- \times |\mu_i|\big)(J_h \times J_\ell) - C\varepsilon\\ &\geq Q(u) - C'\varepsilon,\end{aligned} \tag{3.24}$$

for suitable constants C, C' and all ν sufficiently large. Since $\varepsilon > 0$ was arbitrary, this establishes (3.1).

By the second inequality in (3.21) together with (3.17), an entirely similar argument yields (3.2). □

By the property **(P3)**, the above arguments imply also the lower semicontinuity of the weight functions W_i^u in (2.27).

Lemma 4. *Consider a sequence $u_\nu \in \mathcal{D}_{PL}$ converging to some function $u \in \mathcal{D}_{PL}$ in the $\mathbf{L}^1$ norm. Then for every $x \in I\!R$ and $i = 1, \ldots, n$ one has*

$$W_i^u(x) \leq \liminf_{\nu\to\infty} W_i^{u_\nu}(x). \tag{3.25}$$

The limit in (3.25) *is uniform for x bounded away from the jumps of u. More precisely let J be an open set containing all points $x_1 < \cdots < x_N$ where u has a jump. Then, for each $\varepsilon > 0$, there exists $\rho > 0$ such that every $w \in \mathcal{D}_{PL}$ with $\|w - u\|_{\mathbf{L}^1} < \rho$ satisfies*

$$W_i^u(x) \leq W_i^w(x) + \varepsilon \qquad\qquad x \notin J, \quad i = 1, \ldots, n. \tag{3.26}$$

4 - Weighted path lengths

In the following we consider the domain $\mathcal{D}$ of functions with small variation, introduced at (2.21), and the weighted norms on generalized tangent vectors, introduced at (2.26). We assume that the constants δ_0, κ_i in (2.21) and (2.27) are chosen so that the properties **(P2)-(P3)** hold.

Theorem 2. *Let $\gamma_\nu : [a, b] \mapsto \mathcal{D}$, $\nu \geq 1$, be a sequence of piecewise regular paths, uniformly converging to a piecewise regular path γ. Then the corresponding weighted lengths satisfy*

$$\|\gamma\|_* \leq \liminf_{\nu\to\infty} \|\gamma_\nu\|_*. \tag{4.1}$$

To prove the theorem, it not restrictive to assume that γ is a regular path on $]a, b[$. Let the functions $u^\theta \doteq \gamma(\theta)$ have jumps at the points $x_1^\theta < \cdots < x_N^\theta$, and uniform Lipschitz constant L outside the jumps, so that

$$\left|u^\theta(x) - u^\theta(y)\right| \leq L|x - y| \qquad \text{whenever} \quad [x, y] \cap \{x_1^\theta, \ldots, x_N^\theta\} = \emptyset. \tag{4.2}$$

Moreover, by the approximation Lemma 2, we can assume that each γ_ν is a finite concatenation of elementary paths.

Let $\varepsilon > 0$ be given. The proof will be achieved by constructing a family of continuous functionals $\Phi_\theta : \mathcal{D}_{PL} \mapsto I\!R$, $\theta \in]a, b[$, satisfying the two conditions:

(C1) For some constant C_1, independent of ε, the following holds. For every $\bar\theta$ there exists $\rho_1 = \rho_1(\bar\theta) \in]0, \varepsilon]$ such that

$$\Phi_{\bar\theta}(u^{\bar\theta+\zeta}) - \Phi_{\bar\theta}(u^{\bar\theta}) \geq \int_{\bar\theta}^{\bar\theta+\zeta} \left\|(v^\theta, \xi^\theta)\right\|_{u^\theta} d\theta - C_1\varepsilon\zeta, \qquad\qquad \zeta \in [0, \rho_1]. \tag{4.3}$$

(C2) For some constant C_2, independent of ε, the following holds. For every $\bar\theta$, there exist constants $\delta, \rho_2 \in]0, \varepsilon]$ such that

$$\Phi_{\bar\theta}\big(\tilde u^{\bar\theta+\zeta}\big) - \Phi_{\bar\theta}\big(\tilde u^{\bar\theta}\big) \le (1 + C_2\varepsilon) \int_{\bar\theta}^{\bar\theta+\zeta} \Big\| \big(\tilde v^\theta, \tilde\xi^\theta\big)\Big\|_{\tilde u^\theta}\, d\theta, \tag{4.4}$$

for every $\zeta \in [0, \rho_2]$ and every piecewise elementary path $\tilde\gamma : \theta \mapsto \tilde u^\theta$ with generalized tangent vector $\big(\tilde v, \tilde\xi\big)$, satisfying $\big\|u^\theta - \tilde u^\theta\big\|_{\mathbf{L}^1} < \delta$ for all θ.

Let us show that the existence of these functionals implies (4.1). The family of intervals

$$\Big\{ \big[\bar\theta,\ \bar\theta + \zeta\big]; \quad \theta \in]a, b[\,, \quad 0 < \zeta < \min\big\{\rho_1(\bar\theta), \rho_2(\bar\theta)\big\}\Big\} \tag{4.5}$$

is a fine covering of $]a, b[$. Hence by Vitali's theorem [9] there exist finitely many pairwise disjoint intervals $[\theta_j, \theta_j + \zeta_j]$, $j = 1, \ldots, M$, with $\zeta_j \in \big]0, \rho(\theta_j)\big]$, such that

$$\sum_{j=1}^M \int_{\theta_j}^{\theta_j+\zeta_j} \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}\, d\theta > \int_a^b \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}\, d\theta - \varepsilon. \tag{4.6}$$

Let now $\gamma_\nu : \theta \mapsto u_\nu^\theta$ be a sequence of piecewise elementary paths with tangent vector $\big(v_\nu^\theta, \xi_\nu^\theta\big)$, uniformly converging to γ. By the continuity of the functionals Φ_{θ_j} and the uniform convergence $u_\nu^\theta \to u^\theta$, for all ν sufficiently large we have

$$\Big|\Phi_{\theta_j}\big(u_\nu^{\theta_j}\big) - \Phi_{\theta_j}\big(u^{\theta_j}\big)\Big| < \frac{\varepsilon}{M}, \qquad \Big|\Phi_{\theta_j}\big(u_\nu^{\theta_j+\zeta_j}\big) - \Phi_{\theta_j}\big(u^{\theta_j+\zeta_j}\big)\Big| < \frac{\varepsilon}{M}, \tag{4.7}$$

$$\big\|u_\nu^\theta - u^\theta\big\|_{\mathbf{L}^1} < \min\big\{\delta(\theta_1), \ldots, \delta(\theta_M)\big\}, \qquad \theta \in [a, b]. \tag{4.8}$$

Using (4.4), (4.7), (4.3) and (4.6) we now obtain

$$\begin{aligned}
\|\gamma_\nu\|_* &\ge \sum_{j=1}^M \int_{\theta_j}^{\theta_j+\zeta_j} \Big\|\big(v_\nu^\theta, \xi_\nu^\theta\big)\Big\|_{u_\nu^\theta}\, d\theta \\
&\ge \frac{1}{1 + C_2\varepsilon} \sum_{j=1}^M \Big[\Phi_{\theta_j}\big(u_\nu^{\theta_j+\zeta_j}\big) - \Phi_{\theta_j}\big(u_\nu^{\theta_j}\big)\Big] \\
&\ge \frac{1}{1 + C_2\varepsilon} \sum_{j=1}^M \Big[\Phi_{\theta_j}\big(u^{\theta_j+\zeta_j}\big) - \Phi_{\theta_j}\big(u^{\theta_j}\big) - \frac{2\varepsilon}{M}\Big] \\
&\ge \frac{1}{1 + C_2\varepsilon} \sum_{j=1}^M \Big[\int_{\theta_j}^{\theta_j+\zeta_j} \big\|(v^\theta, \xi^\theta)\big\|_{u^\theta}\, d\theta - C_1\varepsilon\zeta_j - \frac{2\varepsilon}{M}\Big] \\
&\ge \frac{1}{1 + C_2\varepsilon} \Big[\big(\|\gamma\|_* - \varepsilon\big) - C_1\varepsilon(b - a) - 2\varepsilon\Big].
\end{aligned} \tag{4.9}$$

Since $\varepsilon > 0$ was arbitrary and the constants C_1, C_2 do not depend on ε, the relation (4.1) is proved.

5 - Completion of the proof

To complete the proof of Theorem 2, we need to construct suitable functionals Φ_θ satisfying the conditions **(C1)**, **(C2)**. Roughly speaking, $\Phi_\theta(w)$ should measure the weighted distance between w and u^θ. Since we do not have an explicit formula for this distance, we resort to a suitable approximation.

Let $\varepsilon > 0$ be given. For any $\bar\theta \in]a,b[$, let $\bar x_1 < \cdots < \bar x_N$ be the jumps in $u^{\bar\theta}$ and define $u_\alpha^\pm \doteq u^{\bar\theta}(x_\alpha\pm)$. Choose

$$\eta = \eta(\bar\theta) \in]0,\varepsilon] \tag{5.1}$$

so that the intervals $I_\alpha \doteq [\bar x_\alpha - \eta,\ \bar x_\alpha + \eta]$ are mutually disjoint and

$$\sum_{i=1}^n \sum_{\alpha=1}^N \int_{I_\alpha} \left| v_i^{\bar\theta}(x)\right| dx < \varepsilon. \tag{5.2}$$

Define the functional $\Phi_{\bar\theta} : \mathcal{D}_{PL} \mapsto I\!R$ by setting

$$\Phi_{\bar\theta}(w) \doteq \int g\big(x, w(x)\big)\, dx, \tag{5.3}$$

$$g(x,w) \doteq \begin{cases} \sum_{i=1}^n J_i^\alpha\big(u^{\bar\theta}(x), w\big) W_i^{u^{\bar\theta}}(\bar x_\alpha) & \text{if} \quad x \in I_\alpha^- \doteq [\bar x_\alpha - \eta,\ \bar x_\alpha[\,, \\[2ex] \sum_{i=1}^n J_i^\alpha\big(w, u^{\bar\theta}(x)\big) W_i^{u^{\bar\theta}}(\bar x_\alpha) & \text{if} \quad x \in I_\alpha^+ \doteq]\bar x_\alpha,\ \bar x_\alpha + \eta]\,, \\[2ex] \sum_{i=1}^n J_i\big(u^{\bar\theta}(x), w\big) W_i^{u^{\bar\theta}}(x) & \text{if} \quad x \notin \cup_\alpha I_\alpha, \end{cases} \tag{5.4}$$

where $J_i(u,v) \doteq \big|E_i(u,v)\big|$ and

- if $\big|E_i(u_\alpha^-, u_\alpha^+)\big| \le \varepsilon$, we let $J_i^\alpha(w_1,w_2) \equiv 0$;
- if $\big|E_i(u_\alpha^-, u_\alpha^+)\big| > \varepsilon$, we define

$$J_i^\alpha(w_1,w_2) \doteq \begin{cases} 0 & \text{if} \quad \mathrm{sgn}\big(E_i(w_1,w_2)\big) \neq \mathrm{sgn}\big(E_i(u_\alpha^-, u_\alpha^+)\big), \\ [\![J_i(w_1,w_2) - \varepsilon]\!]_+ & \text{if} \quad \mathrm{sgn}\big(E_i(w_1,w_2)\big) = \mathrm{sgn}\big(E_i(u_\alpha^-, u_\alpha^+)\big). \end{cases} \tag{5.5}$$

Here $[\![t]\!]_+$ denotes the positive part of t. Observe that $\Phi_{\bar\theta}\big(u^{\bar\theta}\big) = 0$.

Proof of Property **(C1)**. First of all take $\rho_1(\bar\theta)$ small so that $x_\alpha\big(\bar\theta + \zeta\big) \in I_\alpha$ for all $\zeta \in [0, \rho_1]$. Outside the intervals I_α, from part a) of Lemma 1, if ρ_1 is small enough, we have

$$\begin{aligned} &\sum_{i=1}^n \int_{I\!R\setminus\cup_\alpha I_\alpha} J_i\Big(u^{\bar\theta}(x), u^{\bar\theta+\zeta}(x)\Big) W_i^{u^{\bar\theta}}(x)\, dx \\ &\qquad \ge \zeta \sum_{i=1}^n \int_{I\!R\setminus\cup_\alpha I_\alpha} \left| v_i^{\bar\theta}(x)\right| W_i^{u^{\bar\theta}}(x)\, dx - C \int_{I\!R\setminus\cup_\alpha I_\alpha} \left| u^{\bar\theta+\zeta}(x) - u^{\bar\theta}(x)\right|^2 dx + \end{aligned}$$

$$
\begin{aligned}
&-C\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\left|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)-\zeta v^{\bar\theta}(x)\right|dx\\
&\geq \zeta\sum_{i=1}^n\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\left|v_i^{\bar\theta}(x)\right|W_i^{u^{\bar\theta}}(x)\,dx - C\sup_{x\in\mathbb{R}\setminus\cup_\alpha I_\alpha}\left|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)\right|\times\\
&\quad\times\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\left[\left|u^{\bar\theta+\zeta}(x)-u^{\bar\theta}(x)-\zeta v^{\bar\theta}(x)\right|+\zeta\sum_{i=1}^n\left|v_i^{\bar\theta}(x)\right|\right]dx - C\varepsilon\zeta\\
&\geq \zeta\sum_{i=1}^n\int_{\mathbb{R}\setminus\cup_\alpha I_\alpha}\left|v_i^{\bar\theta}(x)\right|W_i^{u^{\bar\theta}}(x)\,dx - C\varepsilon\zeta.
\end{aligned}
\tag{5.6}
$$

Denote with G_α the set of indices i such that $\left|E_i(u_\alpha^-,u_\alpha^+)\right|>\varepsilon$.

Assume that $\bar\xi_\alpha \doteq \xi_\alpha(\bar\theta)>0$, the case $\bar\xi_\alpha<0$ being similar. We can choose $\rho_1(\bar\theta)$ small enough such that $J_i\Big(u^{\bar\theta+\zeta}(x),u^{\bar\theta}(x)\Big)>\varepsilon$ for all $i\in G_\alpha$, $x\in[\bar x_\alpha,\bar x_\alpha+\bar\xi_\alpha\zeta]$ and $\zeta\leq\rho_1$. By (5.2), and by making (if necessary) ρ_1 smaller, it follows

$$
\begin{aligned}
&\sum_{i\in G_\alpha}\int_{\bar x_\alpha}^{\bar x_\alpha+\bar\xi_\alpha\zeta}J_i^\alpha\Big(u^{\bar\theta+\zeta}(x),u^{\bar\theta}(x)\Big)W_i^{u^{\bar\theta}}(\bar x_\alpha)\,dx\\
&\quad\geq\sum_{i\in G_\alpha}\int_{\bar x_\alpha}^{\bar x_\alpha+\bar\xi_\alpha\zeta}J_i(u_\alpha^-,u_\alpha^+)W_i^{u^{\bar\theta}}(\bar x_\alpha)\,dx - C\zeta\sum_{i\in G_\alpha}\int_{\bar x_\alpha}^{\bar x_\alpha+\bar\xi_\alpha\zeta}\left|v_i^{\bar\theta}(x)\right|dx+\\
&\qquad -C\int_{\bar x_\alpha}^{\bar x_\alpha+\bar\xi_\alpha\zeta}\left(\left|u^{\bar\theta+\zeta}(x)-u_\alpha^-\right|+\left|u^{\bar\theta}(x)-u_\alpha^+\right|\right)dx - C\varepsilon\zeta\\
&\quad\geq\zeta\,\bar\xi_\alpha\sum_{i=1}^n J_i(u_\alpha^-,u_\alpha^+)W_i^{u^{\bar\theta}}(\bar x_\alpha) - C\zeta\sum_{i=1}^n\int_{I_\alpha}\left|v_i^{\bar\theta}(x)\right|dx - C\varepsilon\zeta.
\end{aligned}
\tag{5.7}
$$

By (5.6)-(5.7) it follows

$$
\Phi_{\bar\theta}(u^{\bar\theta+\zeta})\geq\zeta\left\|(v^{\bar\theta},\xi^{\bar\theta})\right\|_{u^{\bar\theta}} - C\zeta\sum_{i=1}^n\sum_{\alpha=1}^N\int_{I_\alpha}\left|v_i^{\bar\theta}(x)\right|dx - C\varepsilon\zeta,
\tag{5.8}
$$

hence by (5.2) and part c) of Lemma 1, (4.3) follows. □

Proof of Property **(C2)**. It is sufficient to prove that, for some $\rho_2(\bar\theta)$ small enough and all $\theta\in\big[\bar\theta,\bar\theta+\rho_2(\bar\theta)\big]$, one has

$$
\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta}\leq\left\|\left(\tilde v^\theta,\tilde\xi^\theta\right)\right\|_{\tilde u^\theta}(1+C\varepsilon),
\tag{5.9}
$$

provided that the distance $\|\tilde u^\theta - u^\theta\|$ remains sufficiently small. For a.e. point θ the map $\theta\mapsto\tilde u^\theta$ is an elementary path in a neighborhood of θ. So fix one of these points θ. Assume that $\tilde\xi^\theta = \big(0,\ldots,0,\tilde\xi_\beta,0,\ldots,0\big)$, with $\tilde\xi_\beta$ being the shift corresponding to a discontinuity of $\tilde u^\theta$ located at point x_β. There are three cases.

Case 1: $x_\beta\in I_\alpha^+$ for some $\alpha=1,\ldots,N$. Assume $\tilde\xi_\beta>0$. Hence we can see that on a right neighborhood of θ we have

$$
\tilde u^{\theta+\Delta\theta}(x)=\tilde u^\theta(x)+(\tilde u^- - \tilde u^+)\,\chi_{[x_\beta,x_\beta+\tilde\xi_\beta\Delta\theta]}(x),
\tag{5.10}
$$

for suitable states $\tilde u^-, \tilde u^+$. Notice that we can also assume that the Riemann problem $(\tilde u^-, \tilde u^+)$ is solved only by one wave belonging to some characteristic family, say the j-th one. Moreover,

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} = \tilde\xi_\beta \sum_{i=1}^n \Big(J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha). \tag{5.11}$$

Observe that $\big\|(\tilde v^\theta, \tilde\xi^\theta)\big\|_{\tilde u^\theta} = \tilde\xi_\beta\, J_j(\tilde u^-, \tilde u^+) W_j^{\tilde u^\theta}(x_\beta)$, so we will show that

$$\sum_{i=1}^n \Big(J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha) \le (1 + C\varepsilon)\, J_j(\tilde u^-, \tilde u^+) W_j^{\tilde u^\theta}(x_\beta). \tag{5.12}$$

Call B_α the set of indices i which belong to G_α and satisfy $J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) > 0$. Notice that the terms corresponding to indices $i \notin B_\alpha$ do not give any positive contribution to the sum in (5.12). So take $i \in B_\alpha$. Moreover, given $\varepsilon_1 \in]0, \varepsilon]$, it is easy to see that in any fixed small interval there exists a point y^θ such that

$$\Big|\tilde u^\theta(y^\theta) - u^{\bar\theta}(y^\theta)\Big| \le \varepsilon_1, \qquad \theta \in [\bar\theta, \bar\theta + \rho_2], \tag{5.13}$$

if $\big\|\tilde u^\theta - u^\theta\big\|_{\mathbf{L}^1} < \delta$, with ρ_2 and δ sufficiently small. So we can find y_β^+ close to the right of x_β and y_β^- close to the left of $\bar x_\alpha$ such that they are continuity points for $\tilde u^\theta$, such that $\Big|\tilde u^\theta(y_\beta^+) - u^{\bar\theta}(x_\beta)\Big| \le \varepsilon_1$, $\Big|\tilde u^\theta(y_\beta^-) - u_\alpha^-\Big| \le \varepsilon_1$, and

$$\Big| E_k\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - E_k\big(\tilde u^-, \tilde u^\theta(y_\beta^+)\big)\Big| \le \varepsilon/4, \tag{5.14}$$

$$\Big| E_k\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) - E_k\big(\tilde u^+, \tilde u^\theta(y_\beta^+)\big)\Big| \le \varepsilon/4, \tag{5.15}$$

for all $k = 1, \ldots, n$. Call $\tilde u_\beta^\pm \doteq \tilde u^\theta\big(y_\beta^\pm\big)$.

Assume now that $E_i(u_\alpha^-, u_\alpha^+) < 0$. Since $J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) > 0$, it follows that $E_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) < -\varepsilon$. There are two subcases.
Subcase a): $E_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) < -\varepsilon/4$. Hence $E_i\big(\tilde u^+, \tilde u^\theta(y_\beta^+)\big) < 0$ and one can prove that

$$J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) \le J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big). \tag{5.16}$$

In connection with $\tilde u^\theta$, define the following functions

$$u^*(x) \doteq \begin{cases} \tilde u^\theta(x) & \text{if } \ x \le y_\beta^- \ \text{ or } \ x \ge y_\beta^+, \\ \tilde u^\theta\big(y_\beta^-\big) & \text{if } \ y_\beta^- \le x < \bar x_\alpha, \\ \tilde u^\theta\big(y_\beta^+\big) & \text{if } \ \bar x_\alpha < x \le y_\beta^+, \end{cases} \tag{5.17}$$

$$u^\star(x) \doteq \begin{cases} \tilde u^\theta(x) & \text{if } \ x \le y_\beta^- \ \text{ or } \ x \ge y_\beta^+, \\ \tilde u^\theta(x_\beta -) & \text{if } \ y_\beta^- < x < x_\beta, \\ \tilde u^\theta(x_\beta +) & \text{if } \ x_\beta < x < y_\beta^+, \end{cases} \tag{5.18}$$

By the semicontinuity of the weights and properties **(P1)**-**(P3)**, if we choose δ and ρ_2 sufficiently small, we can see that for every $i = 1, \ldots, n$

$$W_i^{u^{\bar\theta}}(\bar x_\alpha) \le W_i^{u^*}(\bar x_\alpha) + \varepsilon, \tag{5.19}$$

$$W_i^{u^*}(\bar x_\alpha) \le W_i^{u^*}(x_\beta) \le W_i^{\tilde u^\theta}(x_\beta). \tag{5.20}$$

Then from (5.11) and (5.19)

$$\begin{aligned}
&\Big(J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha)\\
&\qquad = \tilde\xi_\beta\Big(J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^*}(\bar x_\alpha) + \\
&\qquad\quad + \tilde\xi_\beta\Big(J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big)\Big)\Big(W_i^{u^{\bar\theta}}(\bar x_\alpha) - W_i^{u^*}(\bar x_\alpha)\Big)\\
&\qquad \doteq B_1 + B_2,
\end{aligned} \tag{5.21}$$

and

$$B_2 \le C\tilde\xi_\beta\big|\tilde u^- - \tilde u^+\big|\varepsilon \le C\Big\|\Big(\tilde v^\theta, \tilde\xi^\theta\Big)\Big\|_{\tilde u^\theta}\varepsilon, \tag{5.22}$$

$$\begin{aligned}
B_1 &= \tilde\xi_\beta\Big(J_i\Big(\tilde u^-, \tilde u_\beta^+\Big) - J_i\Big(\tilde u^+, \tilde u_\beta^+\Big)\Big) W_i^{u^*}(\bar x_\alpha) + \\
&\quad + \tilde\xi_\beta\Big[\Big(J_i\Big(\tilde u^-, u^{\bar\theta}(x_\beta)\Big) - J_i\Big(\tilde u^-, \tilde u_\beta^+\Big)\Big) + \Big(J_i\Big(\tilde u^+, \tilde u_\beta^+\Big) - J_i\Big(\tilde u^+, u^{\bar\theta}(x_\beta)\Big)\Big)\Big] W_i^{u^*}(\bar x_\alpha)\\
&\doteq B_1' + B_1''.
\end{aligned} \tag{5.23}$$

We want to estimate B_1''. First we notice that for $u_1, u_2, u_3 \in \Omega$ and $i = 1, \ldots, n$, we have

$$\begin{aligned}
J_i(u_1, u_2) - J_i(u_1, u_3) &= \int_0^1 \frac{d}{ds}\Big[J_i\big(u_1, su_2 + (1-s)\,u_3\big)\Big]\,ds\\
&= (u_2 - u_3)\int_0^1 \partial_2 J_i\left(u_1, su_2 + (1-s)\,u_3\right) ds.
\end{aligned} \tag{5.24}$$

This implies

$$\begin{aligned}
B_1'' &\le \tilde\xi_\beta\Big|u^{\bar\theta}(x_\beta) - \tilde u^\theta\big(y_\beta^+\big)\Big|\; W_i^{u^*}(\bar x_\alpha)\bigg|\int_0^1\Big[\partial_2 J_i\Big(\tilde u^-, su^{\bar\theta}(x_\beta) + (1-s)\,\tilde u^\theta\big(y_\beta^+\big)\Big) + \\
&\qquad\qquad - \partial_2 J_i\Big(\tilde u^+, su^{\bar\theta}(x_\beta) + (1-s)\,\tilde u^\theta\big(y_\beta^+\big)\Big)\Big]\,ds\bigg|\\
&= \tilde\xi_\beta\Big|u^{\bar\theta}(x_\beta) - \tilde u^\theta\big(y_\beta^+\big)\Big|\,\big|\tilde u^- - \tilde u^+\big|\, W_i^{u^*}(\bar x_\alpha)\times\\
&\quad\times\bigg|\int_0^1\int_0^1 \partial_1\partial_2 J_i\Big(\sigma\tilde u^- + (1-\sigma)\,\tilde u^+, su^{\bar\theta}(x_\beta) + (1-s)\,\tilde u^\theta\big(y_\beta^+\big)\Big)\,ds\,d\sigma\bigg|\\
&\le C\tilde\xi_\beta\big|\tilde u^- - \tilde u^+\big|\,\Big|u^{\bar\theta}(x_\beta) - \tilde u^\theta\big(y_\beta^+\big)\Big| \le C\Big\|\Big(\tilde v^\theta, \tilde\xi^\theta\Big)\Big\|_{\tilde u^\theta}\varepsilon.
\end{aligned} \tag{5.25}$$

Finally we want to estimate B_1'. Call $E_i \doteq E_i(\tilde u^-, \tilde u_\beta^+)$, $E_i' \doteq E_i(\tilde u^-, \tilde u^+)$, $E_i'' \doteq E_i(\tilde u^+, \tilde u_\beta^+)$, and the same for the J's. Define also $W_i \doteq W_i^{u^*}(\bar x_\alpha)$ and $W_i' \doteq W_i^{u^*}(x_\beta)$. Recall now that $E_i, E_i'' < 0$, $J_i - J_i'' > 0$ and that $E_i' = 0$ for all $i \neq j$. Hence if $i = j$ from Glimm estimates we get $0 < J_j - J_j'' \leq -E_j' + C_3\Delta(E', E'')$. If δ_0 is sufficiently small, this implies that $E_j' < 0$ hence $J_j' = -E_j'$. If, instead, $i \neq j$ we get $J_i - J_i'' \leq C_3\Delta(E', E'')$.

Subcase b): $E_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) > -\varepsilon/4$. Hence $E_i\big(\tilde u^+, \tilde u^\theta(y_\beta^+)\big) > -\varepsilon/2$. If it is positive then

$$J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) = J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - \varepsilon \leq J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big). \tag{5.26}$$

With the notations introduced before if $i = j$ we have $J_j \leq -E_j' - E_j'' + C_3\Delta(E', E'') \leq -E_j' + C_3\Delta(E', E'')$ hence $E_j' < 0$, while $J_i \leq C_3\Delta(E', E'')$ if $i \neq j$.

If, instead, $0 \geq E_i\big(\tilde u^+, \tilde u^\theta(y_\beta^+)\big) > -\varepsilon/2$, we get

$$J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) \leq J_i\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big) - J_i\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big), \tag{5.27}$$

so we recover (5.16) and we can conclude as in subcase a).
The case $E_i(u_\alpha^-, u_\alpha^+) > 0$ is treated in a similar way.

In both cases, one can prove that $J_j'\big(W_j - W_j'\big) \leq -\kappa_1\Delta(E', E'')$, if $\kappa_2 > C_3$. By the previous analysis it follows

$$\begin{aligned}\sum_{i\in B_\alpha} (J_i - J_i'')W_i - J_j'W_j' &\leq \sum_{i\neq j} C_3\Delta(E', E'')W_i + \big(J_j - J_j'' - J_j'\big)W_j + J_j'\big(W_j - W_j'\big)\\ &\leq \Delta(E', E'')\big(C_4 + C_5\kappa_1\delta_0 - \kappa_1\big) < 0,\end{aligned} \tag{5.28}$$

if $\kappa_1 > 2C_4$ and $\delta_0 < (2C_5)^{-1}$. Finally, this together with (5.20) implies (4.4).

Assume now that $\tilde\xi_\beta < 0$, hence in a neighborhood of θ

$$\tilde u^{\theta+\Delta\theta}(x) = \tilde u^\theta(x) + (\tilde u^+ - \tilde u^-)\,\chi_{[x_\beta + \tilde\xi_\beta\Delta\theta, x_\beta]}(x), \tag{5.29}$$

and

$$\frac{d\Phi_{\bar\theta}\big(\tilde u^\theta\big)}{d\theta} = \big|\tilde\xi_\beta\big| \sum_{i=1}^n \Big(J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha). \tag{5.30}$$

Proceeding as before, we will show that

$$\sum_{i=1}^n \Big(J_i^\alpha\big(\tilde u^+, u^{\bar\theta}(x_\beta)\big) - J_i^\alpha\big(\tilde u^-, u^{\bar\theta}(x_\beta)\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha) \leq (1 + C\varepsilon)\, J_j(\tilde u^-, \tilde u^+) W_j^{\tilde u^\theta}(x_\beta). \tag{5.31}$$

We can made an analysis entirely similar to that before and show again that (5.9) holds.

Case 2: $x_\beta \in I_\alpha^-$. Then, if for example $\tilde\xi_\beta > 0$, it is easy to show that

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} = \tilde\xi_\beta \sum_{i=1}^n \Big(J_i^\alpha\big(u^{\bar\theta}(x_\beta), \tilde u^-\big) - J_i^\alpha\big(u^{\bar\theta}(x_\beta), \tilde u^+\big)\Big) W_i^{u^{\bar\theta}}(\bar x_\alpha). \tag{5.32}$$

We can treat this case as Case 1.

Case 3: $x_\beta \in \mathcal{F} \doteq I\!\!R \setminus \cup_\alpha I_\alpha$. We want to show that also in this case (5.9) holds. Since $u^{\bar\theta}$ and $W_i^{u^{\bar\theta}}(x)$ are continuous on $\mathcal{F}$, by assuming for example that $\tilde\xi_\beta > 0$, we get

$$\begin{aligned}\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} &\le \lim_{h\to 0}\frac{1}{h}\int_{x_\beta}^{x_\beta+\tilde\xi_\beta h}\sum_{i=1}^n \Big|J_i\Big(u^{\bar\theta}(x), \tilde u^-\Big) - J_i\Big(u^{\bar\theta}(x), \tilde u^+\Big)\Big| W_i^{u^{\bar\theta}}(x)\,dx \\ &= |\tilde\xi_\beta| \sum_{i=1}^n \Big|J_i\Big(u^{\bar\theta}(x_\beta), \tilde u^-\Big) - J_i\Big(u^{\bar\theta}(x_\beta), \tilde u^+\Big)\Big| W_i^{u^{\bar\theta}}(x_\beta),\end{aligned} \tag{5.33}$$

and the same is true if instead $\tilde\xi_\beta < 0$.

As in (5.13) we can prove there exist points $y_\beta^- < x_\beta$ and $y_\beta^+ > x_\beta$, close to x_β such that

$$\Big|\tilde u^\theta(y_\beta^\pm) - u^{\bar\theta}(x_\beta)\Big| \le \varepsilon_1, \tag{5.34}$$

Define $u^{\boldsymbol *}$ and $u^\star$ as in (5.17)-(5.18) with $\bar x_\alpha$ replaced by x_β. By **(P2)-(P3)** and part a) of Lemma 1, if we choose δ and ρ_2 sufficiently small, we can see that

$$W_i^{u^{\bar\theta}}(x_\beta) \le W_i^{u^*}(x_\beta) + \varepsilon, \qquad\qquad W_i^{u^*}(x_\beta) \le W_i^{u^\star}(x_\beta) \le W_i^{\tilde u^\theta}(x_\beta). \tag{5.35}$$

We observe that since $u^{\bar\theta}$ is continuous at x_β, if ε_1 is small enough, the Riemann problem $\Big(\tilde u^\theta(y_\beta^-), \tilde u^\theta(y_\beta^+)\Big)$ is solved by waves with small strength, say less than ε. Call now $\tilde u_\beta^\pm \doteq \tilde u^\theta(y_\beta^\pm)$.

By proceeding as in (5.21)-(5.25) one gets

$$\frac{d\Phi_{\bar\theta}(\tilde u^\theta)}{d\theta} \le |\tilde\xi_\beta| \sum_{i=1}^n \Big|J_i\Big(\tilde u_\beta^-, \tilde u^-\Big) - J_i\Big(\tilde u_\beta^-, \tilde u^+\Big)\Big| W_i^{u^*}(x_\beta) + C\Big\|\Big(\tilde v^\theta, \tilde\xi^\theta\Big)\Big\|_{\tilde u^\theta}\varepsilon. \tag{5.36}$$

Define $E_i \doteq E^i(\tilde u_\beta^-, \tilde u^+)$, $E_i' \doteq E^i(\tilde u_\beta^-, \tilde u^-)$, $E_i'' \doteq E^i(\tilde u^-, \tilde u^+)$ and the same for the J's. Also call $W_i \doteq W_i^{u^*}(x_\beta)$ and $W_i'' \doteq W_i^{u^\star}(x_\beta)$. Recall that we can assume $J_i'' = 0$ for $i \neq j$. Now we want to show that, for a suitable choice of κ_1 and δ_0, the following holds

$$\sum_{i=1}^n |J_i' - J_i| W_i \le J_j'' W_j''(1 + C\varepsilon). \tag{5.37}$$

By (2.10) one gets $|J_i' - J_i| \le J_i'' + C_3\Delta(E', E'')$ for all $i = 1, \ldots, n$, and since the waves in the Riemann problem $\Big(\tilde u^\theta(y_\beta^-), \tilde u^\theta(y_\beta^+)\Big)$ are very small, it follows

$$J_j''(W_j - W_j'') \le C\varepsilon J_j'' - \kappa_1\Delta(E', E''). \tag{5.38}$$

By (5.37)-(5.38) we get

$$\begin{aligned}\sum_{i=1}^n |J_i' - J_i| W_i - J_j'' W_j'' &\le \sum_{i\neq j} |J_i' - J_i| W_i + \Big(|J_j' - J_j| - J_j''\Big) W_j + J_j''(W_j - W_j'') \\ &\le C\varepsilon J_j'' W_j'' + \Delta(E', E'')(C_4 + C_5\kappa_1\delta_0 - \kappa_1) \le C\varepsilon J_j'' W_j'',\end{aligned} \tag{5.39}$$

if $\kappa_1 > 2C_4$ and $\delta_0 < (2C_5)^{-1}$. This together with (5.35) implies (5.9). $\square$

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