Decay of Positive Waves in Nonlinear Systems of Conservation Laws

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1 - Introduction

This paper is concerned with BV solutions to a system of conservation laws in one space dimension:

$$u_t + \Big[f(u)\Big]_x = 0 \qquad\qquad t \in [0, T]. \tag{1.1}$$

Here $f : \Omega \mapsto I\!R^n$ is a smooth function defined on an open set $\Omega \subseteq I\!R^n$. We assume that the system is strictly hyperbolic, and that each characteristic field is either linearly degenerate or genuinely nonlinear [12, 15]. Our aim is to derive certain a priori bounds on the strength of positive waves of genuinely nonlinear families, which extend the classical decay estimates of Oleinik [13].

We recall that, in the scalar case, the assumption $f''(u) > \kappa > 0$ for all $u \in I\!R$ implies

$$u_{xt} + f'(u)u_{xx} = -f''(u)u_x^2 < -\kappa u_x^2\,. \tag{1.2}$$

Regarding (1.2) as a first order equation for the gradient u_x and integrating along characteristics, one obtains the pointwise estimate

$$u_x(\tau, x) < \frac{1}{\kappa\tau} \qquad\qquad \tau \in \,]0, T]\,, \tag{1.3}$$

valid for a sufficiently regular solution defined on the strip $[0, T] \times I\!R$. More generally [13, 15], any bounded entropic solution of (1.1) satisfies

$$u(\tau, x_2) - u(\tau, x_1) < \frac{x_2 - x_1}{\kappa\tau} \qquad\qquad x_1 < x_2\,, \quad \tau > 0\,. \tag{1.4}$$

We will prove an estimate similar to (1.3), valid for general $n \times n$ systems. Namely

$$\mu_\tau^i\big([x_1, x_2]\big) \leq \frac{x_2 - x_1}{\kappa\tau} + C \cdot \big[Q(0) - Q(\tau)\big] \qquad\qquad x_1 < x_2\,, \quad \tau > 0$$

where μ_τ^i is the distributional space derivative of $u(\tau, \cdot)$ along the i-th chracteristic family, Q is the interaction potential, κ and C are suitable constants. For a more precise statement, some more notations are required.

Let $A(u) = Df(u)$ be the Jacobian matrix of f at u. Smooth solutions of (1.1) thus satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0\,. \tag{1.5}$$

Call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of $A(u)$. In what follows, $\hat\lambda$ denotes an upper bound for all characteristic speeds, while $\Delta\lambda$ is a lower bound on the differences between speeds of waves belonging to different characteristic families:

$$\hat\lambda \doteq \sup_{u;i} \big|\lambda_i(u)\big|\,, \qquad \Delta\lambda \doteq \inf_{u,w;i\neq j} \big|\lambda_i(u) - \lambda_j(w)\big|\,. \tag{1.6}$$

For the matrix $A(u)$, choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \dots, n$, normalized so that

$$|r_i| \equiv 1\,, \qquad \langle l_i,\ r_j \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.7}$$

for every $i \in \{1, \dots, n\}$ and all u in the domain of f. The assumption on the genuine nonlinearity of the i-th characteristic family can be written as

$$r_i \bullet \lambda_i(u) \doteq \lim_{h \to 0} \frac{\lambda_i\Big(u + h r_i(u)\Big) - \lambda_i(u)}{h} > \kappa_i > 0\,. \tag{1.8}$$

For a given state $u \in \Omega$ and $i = 1, \dots, n$, we denote by

$$\sigma \mapsto S_i(\sigma)(u), \qquad \sigma \mapsto R_i(\sigma)(u)$$

respectively the i-shock and the i-rarefaction curve through u, parametrized by arc-length. Moreover, we consider the curve

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if} \quad \sigma \geq 0, \\ S_i(\sigma)(u) & \text{if} \quad \sigma < 0. \end{cases} \tag{1.9}$$

Following [14] and [7], the definition of the Glimm interaction functional can be extended to general BV functions as follows. Let $u : I\!R \mapsto I\!R^n$ have bounded variation. Then the distributional derivative $\mu = Du$ is a vector measure. Let $x_1, x_2, \dots$ be the points where u has a jump, say $\Delta u(x_\alpha) = u(x_\alpha+) - u(x_\alpha-)$. Call $\sigma_\alpha^1, \dots, \sigma_\alpha^n$ the waves generated by the corresponding Riemann problem at x_α. With the notation (1.9), this means

$$u(x_\alpha+) = \Psi_n(\sigma_\alpha^n) \circ \cdots \circ \Psi_1(\sigma_\alpha^1)\Big(u(x_\alpha-)\Big)\,.$$

For $i = 1, \dots, n$ we can now define μ^i as the signed measure such that, for every open interval J,

$$\mu^i(J) = \int_J l_i \cdot Du\,, \tag{1.10}$$

where

$$\begin{cases} l_i(x) = l_i\Big(u(x)\Big) & \text{if} \quad u \text{ is continuous at } x, \\ l_i(x_\alpha) \cdot \Delta u(x_\alpha) = \sigma_\alpha^i & \text{if} \quad u \text{ has a jump at } x_\alpha. \end{cases} \tag{1.11}$$

Observing that

$$\sigma_\alpha^i = l_i\Big(u(x_\alpha-)\Big) \cdot \Delta u(x_\alpha) + O\left(\Big|\Delta u(x_\alpha)\Big|^2\right)\,, \tag{1.12}$$

in the second case of (1.11) we can assume that, for some constant C,

$$\Big|l_i(x_\alpha) - l_i\left(u(x_\alpha-)\right)\Big| \leq C \cdot \Big|\Delta u(x_\alpha)\Big|\,. \tag{1.13}$$

Call μ^{i+}, μ^{i-} the positive and negative parts of the signed measure μ^i, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad\qquad |\mu^i| = \mu^{i+} + \mu^{i-}.$$

The *total strength of waves* in u is defined as

$$V(u) \doteq \sum_{i=1}^{n} V_i(u), \qquad\qquad V_i(u) \doteq |\mu^i|(I\!R), \tag{1.14}$$

Let $\mathcal{N}$ be the set of those indices $i \in \{1, \ldots, n\}$ such that the i-th characteristic family is genuinely nonlinear. The *interaction potential* of waves in u is then defined as

$$Q(u) \doteq \sum_{1 \leq i < j \leq n} \Big(|\mu^j| \times |\mu^i|\Big)\Big(\Big\{(x,y);\quad x < y\Big\}\Big) + \sum_{i \in \mathcal{N}} \Big(\mu^{i-} \times |\mu^i|\Big)\Big(\Big\{(x,y);\quad x \neq y\Big\}\Big). \tag{1.15}$$

Throughout this paper we deal with solutions of (1.1) satisfying the uniform bounds

$$V(u) \leq \delta_0, \qquad\qquad Q(u) \leq \delta_0, \tag{1.16}$$

for some small positive δ_0. For such solutions, the classical interaction estimates of Glimm [11] state that, for some constant C_0, one has

$$V\Big(u(t')\Big) + C_0 \cdot Q\Big(u(t')\Big) \leq V\Big(u(t)\Big) + C_0 \cdot Q\Big(u(t)\Big), \qquad\qquad 0 \leq t < t'. \tag{1.17}$$

In the following, the positive and negative part of a number s are written $[\![s]\!]_+ \doteq \max\{s, 0\}$, $[\![s]\!]_- \doteq \max\{-s, 0\}$.

Consider a smooth solution $u = u(t,x)$ of (1.1) defined for $t \in [0, T]$. In the vector-valued case, introduce the gradient components

$$u_x^i \doteq \Big\langle l_i(u),\ u_x \Big\rangle \qquad\qquad i = 1, \ldots, n\,. \tag{1.18}$$

Differentiating (1.5) one finds

$$(u_x^i)_t + \Big[\lambda_i(u)\Big](u_x^i)_x = -\Big[r_i \bullet \lambda_i(u)\Big](u_x^i)^2 + \sum_{j \neq k} G_{ijk}(u) u_x^j u_x^k, \tag{1.19}$$

where the coefficients G_{ijk} are determined in terms of the Lie brackets of the vector fields r_j, r_k. If $G_{ijk} \equiv 0$ for all j, k, then the gradient component u_x^i would satisfy the same estimate (1.3) as in the scalar case, for a suitable constant κ. In general, (1.3) may fail because of the last term on the

right hand side of (1.19). Observe that this summation essentially depends on the (instantaneous) amount of wave interaction. This suggests that the amount by which (1.3) fails, measured by

$$\int \Big[\Big[u_x^i(\tau,x) - \frac{1}{\kappa\tau}\Big]\Big]_+ dx\,,$$

can be estimated in terms of the total amount of interaction taking place during the interval $[0,\tau]$. This quantity, in turn, can be bounded by $Q(u(0)) - Q(u(\tau))$, i.e. by the decay in the wave interaction potential. In the following, the measure μ_τ^i is defined as the i-th component of the distributional derivative of $u(\tau,\cdot)$, as in (1.10). For simplicity, we shall often write $Q(t) \doteq Q(u(t))$. The Lebesgue measure of a set J is $\mathrm{meas}(J)$.

Theorem 1. *Let the system (1.1) be strictly hyperbolic and let the i-th characteristic field be genuinely nonlinear. Then there exist constants $C_1, \kappa > 0$ such that, for every Borel set $J \subset \mathbb{R}$ and every solution u with small total variation obtained as limit of wave-front tracking approximations, one has*

$$\mu_\tau^i(J) \leq \frac{\mathrm{meas}(J)}{\kappa\tau} + C_1 \cdot \Big[Q(0) - Q(\tau)\Big]. \tag{1.20}$$

Remark 1. It was recently proved in [4] that the solutions of (1.1) obtained as limits of wave-front tracking approximations are precisely the same as those generated by the Glimm scheme. Indeed, they are the trajectories of the unique Standard Riemann Semigroup generated by (1.1). For the definition and main properties of SRS we refer to [4, 5]. The reason for stating Theorem 1 only for solutions generated by the wave-front tracking algorithm is that, in this way, we can provide a self-contained proof, entirely independent of the results in [6, 8].

Corollary 1. *Under the same assumptions of Theorem 1, for every piecewise Lipschitz solution of (1.1) and every $\eta > 1/\kappa\tau$ one has*

$$\int_{\{u_x^i > \eta\}} u_x^i(\tau,x)\,dx \leq C_1 \cdot \big[Q(0) - Q(\tau)\big] \cdot \left(1 - \frac{1}{\eta\kappa\tau}\right)^{-1}. \tag{1.21}$$

Indeed, calling $J = \Big\{x;\ u_x^i(\tau,x) > \eta\Big\}$, one has

$$\mathrm{meas}(J) \leq \frac{1}{\eta} \cdot \mu_\tau^i(J)\,.$$

Hence, by (1.20),

$$\int_J u_x^i(t,x)\,dx = \mu_\tau^i(J) \leq \frac{\mathrm{meas}(J)}{\kappa\tau} + C_1 \cdot \Big[Q(0) - Q(\tau)\Big] \leq \frac{1}{\eta\kappa\tau} \cdot \mu_\tau^i(J) + C_1 \cdot \Big[Q(0) - Q(\tau)\Big]\,,$$

$$\left(1-\frac{1}{\eta\kappa\tau}\right)\mu^i_\tau(J)\leq C_1\cdot\Big[Q(0)-Q(\tau)\Big]\,.$$

This yields (1.21).

Remark 2. If the shock and rarefaction curves of the i-th family coincide [16], one can redefine the interaction potential Q in (1.15), omitting the term $\mu^{i-}\times|\mu_i|\Big(\{(x,y);\ x\neq y\}\Big)$ from the second summation. In other words, one can regard two i-waves as never approaching, regardless of their sign. All results then remain valid. In particular, in the scalar case one can take $Q\equiv 0$, so that (1.21) is equivalent to the Oleinik estimate (1.3), except for a possibly worse constant κ.

In Sections 2 and 3 we review the wave-front tracking algorithm [2, 3] and establish a technical result, showing that the local interaction potential of approximate solutions decays quickly, in a forward neighborhood of any point in the (t,x)-plane. This result, besides its application in the present paper, plays a major role in the construction of ε-approximate semigroups for $n\times n$ systems [8]. Theorem 1 is then proved in Section 4.

2 - Wave-Front Tracking Approximations.

We recall below the algorithm of wave–front tracking [3]. Let an initial data

$$u(0,x)=\bar u(x) \tag{2.1}$$

be given, with small total variation. A sequence of piecewise constant approximate solutions $u_\nu=u_\nu(t,x)$, is then constructed as follows. Start with a piecewise constant function $u_\nu(0,\cdot)$ close to $\bar u$. This initial condition is chosen so that, as $\nu\to\infty$, one has

$$\Big\|u_\nu(0,\cdot)-\bar u\Big\|_{\mathbf{L}^1}\to 0\,,\qquad\qquad V\Big(u_\nu(0,\cdot)\Big)\to V(\bar u)\,,\qquad\qquad Q\Big(u_\nu(0,\cdot)\Big)\to Q(\bar u)\,. \tag{2.2}$$

At each point of jump in $u_\nu(0,\cdot)$, one approximately solves the corresponding Riemann problem within the class of piecewise constant functions. This yields an approximate solution defined up to the first time t_1 where a wave-front interaction takes place. The new Riemann problem is then solved again within the class of piecewise constant functions, prolonging the solution up to some time t_2 where the second set of interactions takes place, etc. In order to keep finite the total number of wave-fronts, two distinct procedure are used for solving a Riemann problem: an accurate method, which possibly introduces several new fronts, and a simplified method, which minimizes the number of new wave-fronts. For a given integer $\nu\geq 1$, these are described below.

Accurate Riemann Solver: Consider a Riemann problem with data u^-, u^+, say generated by the interaction of an i-wave with a j-wave. Let $\omega_0 = u^-$, $\omega_1, \ldots, \omega_n = u^+$ be the constant states present in the exact solution of the Riemann problem. The piecewise constant approximation u_ν is then obtained by replacing each rarefaction wave of a characteristic family $k \neq i, j$ with a rarefaction fan. This is done by dividing the jump (ω_{k-1}, ω_k) into ν equal jumps, inserting the intermediate states

$$\omega_{k,0} = \omega_{k-1}, \omega_{k,1}, \ldots, \omega_{k,\nu} = \omega_k.$$

Each small jump $(\omega_{k,\ell-1}, \omega_{k,\ell})$ will travel with speed $\lambda_k(\omega_{k,\ell})$, i.e. with the characteristic speed of its right state. When $k = i$ or $k = j$, a rarefaction wave of the same family of an incoming front is not further partitioned: it is propagated as a single wave-front, travelling with the characteristic speed $\lambda(\omega_k)$ of its right state. It is understood that, at the initial time $t = 0$, there are no incoming fronts and hence every centered rarefaction wave is partitioned into ν equal jumps.

Simplified Riemann Solver: Assume that the Riemann problem is determined by the interaction of two waves of distinct families, say $i < j$, with sizes σ_i, σ_j. Call u^l, u^m, u^r the left, middle and right states before the interaction. With the notation introduced in (1.9) we thus have $u^m = \Psi_j(\sigma_j)(u^l)$ and $u^r = \Psi_i(\sigma_i)(u^m)$. We then solve the Riemann problem in terms of two outgoing wave-fronts of the same families, still with sizes σ_i, σ_j. The solution will thus involve a middle state $\tilde{u}^m = \Psi_i(\sigma_i)(u^l)$ and a new right state $\tilde{u}^r = \Psi_j(\sigma_j)(\tilde{u}^m)$. In general, $\tilde{u}^r \neq u^r$. The jump $(\tilde{u}^r, u^r)$ is then propagated along a "non-physical wave-front", travelling with the fixed speed $\hat{\lambda}$ in (1.6), greater than all characteristic speeds. In the case where both incoming wave-fronts belong to the same i-th family and have sizes σ', σ'', the Riemann problem is solved by a single outgoing wave of size $\sigma' + \sigma''$ together with a non-physical wave-front connecting the states $\Psi_i(\sigma' + \sigma'')(u^l)$ and u^r, travelling with speed $\hat{\lambda}$. Finally, if a non-physical front meets an i-wave of size σ_i, then the Riemann problem is solved in terms of an i-wave, still of size σ_i, and an outgoing non-physical front, always travelling with speed $\hat{\lambda}$.

To complete the description of the algorithm, it remains to specify which Riemann solver is used at any given interaction. For this purpose, to each wave-front we attach a positive integer, keeping track of how many interactions were needed to generate that particular front. Waves originating from the Riemann problems at time $t = 0$ are assigned order 1. To the newly born waves, generated by the interaction of fronts of orders p_1, p_2, we assign order $1 + \max\{p_1, p_2\}$. In the construction of the ν-th approximation u_ν, the Riemann problems generated by the interaction

of two fronts of order $p_1, p_2 < \nu$ are solved accurately. On the other hand, if one of the incoming fronts has order $\geq \nu$, then the Simplified Riemann solver is used. In the above, we tacitly assumed that only two wave-fronts interact at any given time. This can always be achieved by an arbitrarily small decrease in the speed of one of the interacting fronts.

Given any initial data $\bar{u}$ with sufficiently small total variation, using the above algorithm we obtain a sequence $(u_\nu)_{\nu\geq 1}$ of piecewise constant approximate solutions, with $u_\nu(0) \to \bar{u}$ in $\mathbf{L}^1$. Each u_ν is defined for all $t \in [0, \infty[$ and has a finite number of lines of discontinuity in the (t, x)-plane. At any fixed time τ, jumps in $u_\nu(\tau, \cdot)$ can be of two different types.

1. Physical wave-fronts, of order $\leq \nu$. At any such point of jump x_α, by construction we have

$$u_\nu(\tau, x_\alpha+) = \Psi_{k_\alpha}(\sigma_\alpha)\Big(u_\nu(\tau, \xi_\alpha-)\Big) \tag{2.3}$$

for some $\sigma_\alpha \in I\!R$ and $k_\alpha \in \{1, \ldots, n\}$, with Ψ as in (1.9). In this case we say that $u_\nu(\tau)$ has a wave-front of strength $|\sigma_\alpha|$ at x_α, of the k_α-th family.

2. Non-physical wave-fronts, of order $\nu + 1$. At any such point x_α, we define the strength of the jump as

$$|\sigma_\alpha| \doteq \Big|u_\nu(\tau, x_\alpha+) - u_\nu(\tau, x_\alpha-)\Big|. \tag{2.4}$$

For notational convenience, in this case we set $k_\alpha \doteq n + 1$.

A priori bounds on the functions u_ν are obtained as in [2], introducing suitable functionals measuring the total strength of waves and the interaction potential:

$$V\Big(u_\nu(\tau)\Big) \doteq \sum_\alpha |\sigma_\alpha|, \qquad\qquad Q\Big(u_\nu(\tau)\Big) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha\sigma_\beta|, \tag{2.5}$$

where $\mathcal{A}$ denotes the set of all couples of approaching wave-fronts. More precisely, two wave-fronts, located at $x_\alpha < x_\beta$ are said to be *approaching* if either $k_\alpha > k_\beta$, or else if the two fronts belong to the same genuinely nonlinear family $k_\alpha = k_\beta \in \{1, \ldots, n\}$ and at least one of them is a shock. Following [2] one then proves that, as $\nu \to \infty$,

(i) The total variation of $u_\nu(t, \cdot)$ remains uniformly small,

(ii) The maximum size of rarefaction fronts in u_ν approaches zero,

(iii) The total strength of all non-physical waves approaches zero.

By (i), Helly's theorem guarantees the existence of a subsequence strongly convergent in $\mathbf{L}^1_{\rm loc}$. By (ii) and (iii), this limit provides a weak solution to (1.1). See [3] for details.

The next proposition establishes the local decay of the interaction potential, for wave-front tracking approximations. In analogy with (1.14), (1.15), the total strength of waves in u inside a set $J \subset I\!R$ is written

$$V(u;J) \doteq \sum_{x_\alpha \in J} |\sigma_\alpha|\,,$$

while $Q(u;J)$ denotes the interaction potential of u restricted to couples of approaching waves both contained inside J. As in (1.6), $\hat\lambda$ is an upper bound for all wave speeds. Observe that, for every point (τ,ξ) in the (t,x)-plane, since $u(\tau,\cdot)$ has bounded variation there exists $\rho > 0$ such that $Q\Big(u(\tau);\ [\xi-\rho,\xi+\rho]\Big) < \varepsilon$. In turn, this implies

$$Q\Big(u(t);\ J(t)\Big) < \varepsilon \qquad\qquad t \in [\tau,\ \tau+\rho/\hat\lambda]\,,$$

where $J(t)$ is the interval of dependence

$$J(t) \doteq \Big[\xi-\rho+(t-\tau)\hat\lambda,\quad \xi+\rho-(t-\tau)\hat\lambda\Big]\,, \tag{2.6}$$

A similar estimate holds for approximate solutions u_ν, uniformly w.r.t. ν.

Proposition 1. *Assume that the system (1.1) is genuinely nonlinear, so that (1.8) holds for every $i = 1,\ldots,n$. Let $(u_\nu)_{\nu\geq 1}$ be a sequence of approximate solutions of (1.1) generated by the above wave-front tracking algorithm. Let a point (τ,ξ) and $\varepsilon > 0$ be given. Then, there exists $\rho > 0$ such that, letting $J(t)$ be as in (2.6), by possibly taking a subsequence one has*

$$\limsup_{\nu\to\infty}\ Q\Big(u_\nu(t);\ J(t)\Big) < \varepsilon \qquad\qquad \forall t \in \Big]\tau,\tau+\rho/\hat\lambda\Big]\ . \tag{2.7}$$

The proof is deferred to Section 3.

Remark 3. While waves of different families cross each other very quickly, positive and negative wave-fronts of the same family could, in principle, travel for a long time with almost the same speed and not interact with each other. Proposition 1 states that, in the genuinely nonlinear case, such a situation cannot happen. The validity of the result is essentially due to a specific provision made in our algorithm: the speed of every rarefaction front is set to be equal to the characteristic speed of the state at the right of the front. With other definitions, (2.7) may fail.

Example 1. For the following Cauchy problem for the Burgers' equation

$$\begin{cases} u_t + (u^2/2)_x = 0 \\ u(0,x) = 0 \end{cases} \tag{2.8}$$

consider the sequence of approximate solutions

$$u_\nu(t,x) \doteq \begin{cases} 1/\nu & \text{if} \quad x \in I, \\ -1/\nu & \text{if} \quad x \in I\!R \setminus I, \end{cases} \tag{2.9}$$

where

$$I \doteq \left(\bigcup_{k=0}^{\nu-1} \left] \frac{2k}{\nu^2}, \ \frac{2k+1}{\nu^2} \right] \right) \cup \left] \frac{2}{\nu}, +\infty \right[. \tag{2.10}$$

Observe that $u_\nu(t)$ contains $\nu+1$ positive and ν negative fronts, all with strength $2/\nu$. In this case, there is no cancellation between positive and negative wave-fronts. Indeed, taking $(\tau,\xi) = (0,0)$, for any $\rho > 0$ we have

$$\lim_{\nu\to\infty} Q\Big(u_\nu(t);\ [-\rho + \hat\lambda t,\ \rho - \hat\lambda t]\Big) = \nu\Big[(\nu-1) + (\nu+1)\Big] \cdot \frac{2}{\nu} \cdot \frac{2}{\nu} = 8$$

for all $t \in [0, \rho[$. Of course, this is in contrast with the conclusion of Proposition 1. Observe that one key assumption is here violated. Namely, every rarefaction front in u_ν travels with speed $0 = \nu^{-1} - \nu^{-1}$ equal to the average between the left and right characteristic speed. Instead, our algorithm would assign the speed ν^{-1} to rarefactions and the speed 0 to shock fronts, thus determining a substantial amount of cancellation between positive and negative waves within a very short time.

For future reference, we introduce here some notations in connection with a piecewise constant approximate solution $u = u(t,x)$ obtained by wave-front tracking. Consider two incoming fronts, with sizes σ, σ' in the characteristic families j, j' respectively, which interact at a point (t,x). The instantaneous amounts of interaction and cancellation are then defined as

$$[\text{Interaction}] \ \doteq \ |\sigma\sigma'|\,, \tag{2.11}$$

$$[\text{Cancellation}] \ \doteq \ \begin{cases} \min\Big\{|\sigma|, |\sigma'|\Big\} & \text{if } j = j' \text{ and } \sigma\cdot\sigma' < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

Recalling (2.5), consider the quantity

$$\Upsilon \doteq V + C_0 \cdot Q. \tag{2.13}$$

By (1.17), Υ is non-increasing. More precisely, if the total variation of u remains sufficiently small, the amounts of interaction and cancellation are both controlled in terms of the decrease in Υ. Indeed, using

$$\Delta Q \leq -[\text{Interaction}] + C_0 \cdot V(t-) \cdot [\text{Interaction}] \leq -\frac{1}{2} \cdot [\text{Interaction}],$$

one obtains

$$\Delta\Upsilon \leq -2 \cdot [\text{Cancellation}] - \frac{C_0}{2} \cdot [\text{Interaction}] \tag{2.14}$$

The next two lemmas refer to a piecewise constant solution generated by wave-front tracking. Throughout this paper, by C or c we denote (large or small) strictly positive constants, depending only on the system (1.1) and not on ε or on the particular solution u. In a chain of inequalities, the particular value of these constants may change from one term to the next.

Lemma 1. *For $t \in [\tau, \tau']$, let $x = y(t)$ be the position of a shock front, with size $\sigma(t)$. Then for some constant C the following estimates hold.*

(i) If $\big|\sigma(t)\big| < \big|\sigma(\tau)\big|$, then

$$\Big|\sigma(t) - \sigma(\tau)\Big| \leq C \cdot \Big(\Upsilon(\tau) - \Upsilon(t)\Big). \tag{2.15}$$

(ii) Let $S_{\min} \doteq \min\Big\{\big|\sigma(t)\big|,\ t \in [\tau, \tau']\Big\}$ be the minimum strength of the shock. Then

$$S_{\min} \cdot \Big|\sigma(t) - \sigma(t')\Big| \leq C \cdot \Big[\Upsilon(\tau) - \Upsilon(\tau')\Big], \qquad t, t' \in [\tau, \tau'], \tag{2.16}$$

$$S_{\min} \cdot \Big|\dot{y}(t) - \dot{y}(t')\Big| \leq C \cdot \Big[\Upsilon(\tau) - \Upsilon(\tau')\Big], \qquad t, t' \in [\tau, \tau'], \tag{2.17}$$

Proof. To establish (i), observe that

$$\sigma(t) - \sigma(\tau) \leq \sum_{s \in [\tau, t]} [\![\Delta\sigma(s)]\!]_+. \tag{2.18}$$

Let $\sigma'(s)$ be the size of the wave-front interacting with our shock at time s. In the case where σ' belongs to a different family, or to the same family as σ, we respectively have

$$\begin{aligned} [\![\Delta\sigma(s)]\!]_+ &\leq C \cdot \Big|\sigma'(s) \cdot \sigma(s-)\Big| \leq C \cdot \Big|\Delta Q(s)\Big| \leq C \cdot \Big|\Delta\Upsilon(s)\Big|, \\ [\![\Delta\sigma(s)]\!]_+ &\leq C \cdot [\![\sigma'(s)]\!]_+ \leq C \cdot \Big|\Delta V(s)\Big| \leq C \cdot \Big|\Delta\Upsilon(s)\Big|. \end{aligned} \tag{2.19}$$

Together, (2.18) and (2.19) imply (2.15).

The estimates (2.16)-(2.17) follow respectively from

$$\Big|\Delta\sigma(s)\Big| \leq C\cdot|\sigma'(s)| \leq C\cdot\frac{\Big|\sigma'(s)\cdot\sigma(s-)\Big|}{S_{\min}} \leq C\cdot\frac{\big|\Delta\Upsilon(s)\big|}{S_{\min}}\,,$$

$$\Big|\Delta\dot{y}(s)\Big| \leq C\cdot\Big|\sigma'(s)\Big| \leq C\cdot\frac{\Big|\Delta\Upsilon(s)\Big|}{S_{\min}}\,.$$

Lemma 2. *For some constant $c>0$ the following holds. Assume that at some time τ all shocks have strength $\Big|\sigma_\alpha(\tau)\Big|<\varepsilon^3$. If at a later time $\tau'>\tau$ some shock of strength $>2\varepsilon^3$ has formed, then*

$$\Upsilon(\tau)-\Upsilon(\tau')>c\cdot\varepsilon^6\,. \tag{2.20}$$

Proof. Assume that some shock at time τ' has strength $>2\varepsilon^3$. Let $x=y(t)$ be the polygonal line obtained by following the shock backwards, starting from the terminal time τ'. At a time $s\in[\tau,\tau']$ where two shocks of the same family merge together, we continue backward along the one of larger strength. At every time $s\in[\tau,\tau']$ where an interaction occurs, this choice implies a bound of the form

$$\Big|\sigma(s-)\Big| \geq c\cdot\Big|\sigma(s+)\Big|$$

for some constant $c>0$. In turn, this yields

$$\begin{aligned}\Big|\Delta Q(s)\Big| &\geq 2c\cdot\Big|\Delta\sigma(s)\Big|\cdot\max\Big\{\Big|\sigma(s-)\Big|,\ \Big|\sigma(s+)\Big|\Big\}\\ &\geq c\cdot\Big|\Delta\sigma^2(s)\Big|\,,\end{aligned}$$

possibly with different constants c. Therefore,

$$\Big|\sigma^2(\tau')-\sigma^2(\tau)\Big| \leq C\cdot\Big[Q(\tau)-Q(\tau')\Big]\,.$$

This establishes (2.20).

We conclude this section by stating two lemmas, for future use. The first is concerned with the lower semicontinuity of the interaction functional Q. For a proof, see [1].

Lemma 3. *Consider a sequence of functions $u_\nu:\mathbb{R}\mapsto\mathbb{R}^n$ with sufficiently small total variation. If $u_\nu\to u$ in $\mathbf{L}^1$, then*

$$Q(u)\leq\liminf_{\nu\to\infty}Q(u_\nu)\,. \tag{2.21}$$

The last result is concerned with the behaviour of the measures μ^i in (1.10), w.r.t. pointwise convergence of the corresponding functions u_ν.

Lemma 4. *Let $(u_\nu)_{\nu\geq 1}$ be a sequence of functions with small total variation, converging pointwise to a function u. Call μ^i_ν, μ^i the corresponding measures, defined as in (1.10). Moreover, let $|Du_\nu|$ be the total variation measure of u_ν, and assume the weak convergence $|Du_\nu| \rightharpoonup \tilde\mu$, for some positive Radon measure $\tilde\mu$. Then we have the estimate*

$$\limsup_{\nu\to\infty} \Big|\mu^i_\nu(J) - \mu^i(J)\Big| \leq C\cdot \Big[\tilde\mu(J)\Big]^2, \tag{2.22}$$

valid for every compact interval $J=[a,b]$ and some constant C depending only on the system (1.1).

Proof. Set $\bar l_i \doteq l_i\big(u(a)\big)$. By the definition (1.10) we have

$$\begin{aligned}\Big|\mu^i_\nu(J) - \mu^i(J)\Big| &= \left|\int_J l_{i,\nu}\cdot Du_\nu - \int_J l_i\cdot Du\right| \\ &\leq \left|\int_J \bar l_i\cdot (Du_\nu - Du)\right| + \int_J |l_{i,\nu}-\bar l_i|\,|Du_\nu| + \int_J |l_i-\bar l_i|\,|Du|\,.\end{aligned} \tag{2.23}$$

From (1.13), the pointwise convergence $u_\nu\to u$ and the weak convergence $|Du_\nu|\rightharpoonup\tilde\mu$ it follows

$$\limsup_{\nu\to\infty}\left(\sup_{x\in J}\Big|u_\nu(x)-u(a)\Big|\right)\leq C\,\tilde\mu(J)\,,$$

$$\sup_{x\in J}\Big|l_i(x)-\bar l_i\Big|\leq C\cdot\tilde\mu(J)\,,\qquad \limsup_{\nu\to\infty}\left(\sup_{x\in J}\Big|l_{i,\nu}(x)-\bar l_i\Big|\right)\leq C\cdot\tilde\mu(J)\,, \tag{2.24}$$

$$|Du|(J)\leq\tilde\mu(J)\,,\qquad \limsup_{\nu\to\infty}|Du_\nu|(J)\leq\tilde\mu(J)\,, \tag{2.25}$$

$$\lim_{\nu\to\infty}\int_J \bar l_i\cdot(Du_\nu-Du)=\lim_{\nu\to\infty}\bar l_i\cdot\Big[\big(u_\nu(b)-u(b)\big)-\big(u_\nu(a)-u(a)\big)\Big]=0\,. \tag{2.26}$$

Using (2.24)-(2.26) in (2.23) we obtain (2.22).

3. - Proof of Proposition 1.

Let $(u_\nu)_{\nu\geq1}$ be a sequence of piecewise constant approximate solutions generated by the wave-front tracking algorithm. Let (τ,ξ) and $\varepsilon>0$ be given. For $i=1,\ldots,n$, call $\mu^i_{\nu,t}$ the purely atomic measure such that $\mu^i_{\nu,t}\Big(\{y_\alpha\}\Big)=\sigma_\alpha$ if and only if $u_\nu(t,\cdot)$ has an i-wave at y_α, of size σ_α. Observe that this is consistent with the previous definitions (1.10)-(1.11), taking $u=u_\nu(t,\cdot)$. Due to the

uniform bounds on the total variation (1.16) and on the propagation speed (1.6), by possibly taking a subsequence we can assume the existence of two non-increasing functions $\widetilde{Q}$, $\widetilde{\Upsilon}$ and n positive measures $\tilde{\mu}^1, \ldots, \tilde{\mu}^n$ such that, as $\nu \to \infty$,

$$\left|\mu^i_{\nu,\tau}\right| \rightharpoonup \tilde{\mu}^i \qquad \text{weakly,} \qquad i = 1, \ldots, n\,, \tag{3.1}$$

$$Q_\nu(t) \to \widetilde{Q}(t), \qquad \Upsilon_\nu(t) \to \widetilde{\Upsilon}(t) \qquad \text{for each} \quad t \geq 0\,, \tag{3.2}$$

where $Q_\nu(t) \doteq Q\Big(u_\nu(t)\Big)$ is the interaction potential as in (2.5), and $\Upsilon_\nu(t) \doteq V\Big(u_\nu(t)\Big) + C_0 \cdot Q\Big(u_\nu(t)\Big)$, as in (2.13).

Now let an arbitrarily small $\varepsilon > 0$ be given. It is not restrictive to assume $\varepsilon << 1$, so that $C\varepsilon < 1$, $\varepsilon/c < 1$ for all the various constants C, c appearing in our future estimates. Choose $\rho > 0$ and then $t^* > \tau$ such that

$$\tilde{\mu}^i\Big([\xi - \rho,\ \xi[\,\cup\,]\xi,\ \xi + \rho]\Big) < \varepsilon^9, \qquad i = 1, \ldots, n\,, \tag{3.3}$$

$$3(t^* - \tau)\hat{\lambda} < \rho\,, \tag{3.4}$$

$$\widetilde{Q}(\tau+) - \widetilde{Q}(t^*) < \varepsilon^{11}, \qquad \widetilde{\Upsilon}(\tau+) - \widetilde{\Upsilon}(t^*) < \varepsilon^{11}\,. \tag{3.5}$$

By (3.3), (3.5), there exist two decreasing sequences $(\delta_\nu)_{\nu\geq 1}$ and $(\tau_\nu)_{\nu\geq 1}$, with $\delta_\nu \to 0$, $\tau_\nu \to \tau$, such that

$$\left|\mu^i_{\nu,\tau_\nu}\right|\Big([\xi - \rho, \xi - \delta_\nu] \cup [\xi + \delta_\nu, \xi + \rho]\Big) < \varepsilon^9, \qquad i = 1, \ldots, n\,, \tag{3.6}$$

$$Q\left(u_\nu(\tau_\nu)\right) - Q\left(u_\nu(t^*)\right) < \varepsilon^{11}, \qquad \Upsilon\left(u_\nu(\tau_\nu)\right) - \Upsilon\left(u_\nu(t^*)\right) < \varepsilon^{11}\,, \tag{3.7}$$

for all ν sufficiently large. Define

$$I_\nu(t) \doteq \Big[\xi - \delta_\nu - (t - \tau)\hat{\lambda},\ \xi + \delta_\nu + (t - \tau)\hat{\lambda}\Big]\,. \tag{3.8}$$

By (3.4), $I_\nu(t) \subset J(t)$ for all $t \in [\tau, t^*]$, where J is the interval at (2.6). From (3.6) we deduce an estimate of the form

$$\begin{aligned} Q\Big(u_\nu(t); J(t)\Big) &\leq Q\Big(u_\nu(\tau_\nu);\ J(\tau_\nu)\Big) \\ &\leq Q\Big(u_\nu(\tau_\nu);\ I_\nu(\tau_\nu)\Big) + V(\tau_\nu) \cdot \sum_{i=1}^n \left|\mu^i_{\nu,\tau_\nu}\right|\Big(J(\tau_\nu) \setminus I_\nu(\tau_\nu)\Big) \\ &\leq Q\Big(u_\nu(\tau_\nu);\ I_\nu(\tau_\nu)\Big) + C\varepsilon^9. \end{aligned} \tag{3.9}$$

To prove Proposition 1, it thus suffices to show that

$$Q\Big(u_\nu(\tau_\nu);\ I_\nu(\tau_\nu)\Big) < \varepsilon/2 \tag{3.10}$$

for all ν sufficiently large.

The interaction potential (3.10) is the sum of a part related to approaching waves of different families and a part related to approaching waves of the same family. We first consider the contribution of waves belonging to different families.

Fix any two indices $i < j$. Consider the triangular domain

$$\mathcal{D}_\nu^{ij} \doteq \left\{ (t,x); \quad t \geq \tau_\nu, \quad x \in I_\nu^{ij}(t) \right\}, \tag{3.11}$$

where

$$I_\nu^{ij}(t) \doteq \Big[\xi - \delta_\nu + (t-\tau_\nu)\lambda_j^{\min},\ \xi + \delta_\nu + (t-\tau_\nu)\lambda_i^{\max} \Big], \tag{3.12}$$

$$\lambda_j^{\min} \doteq \inf_u \lambda_j(u), \qquad\qquad \lambda_i^{\max} \doteq \sup_u \lambda_i(u)\,. \tag{3.13}$$

Because of strict hyperbolicity, by possibly letting u vary in a smaller domain, we can assume that $\lambda_i^{\max} < \lambda_j^{\min}$. Consider the interaction potential of couples of i,j-waves contained in $I_\nu^{ij}(t)$, i.e.

$$Q_\nu^{ij}(t) \doteq \sum_{(\alpha,\beta)\in\mathcal{A}_\nu^{ij}} |\sigma_\alpha \sigma_\beta|\,. \tag{3.14}$$

Here $\mathcal{A}_\nu^{ij}$ is the set of all couples (α,β) such that $u_\nu(t)$ contains an i-wave at y_β and a j-wave at y_α, with $y_\alpha < y_\beta$ and $y_\alpha, y_\beta \in I_\nu^{ij}(t)$.

From the definition of the triangular set D_ν^{ij} it follows that no j-wave can exit through the left boundary. Similarly, no i-wave can exit through the right boundary. Therefore, the quantity Q_ν^{ij} can decrease only at a time t where one of the following cases occurs:

- two i-waves (or two j-waves) collide with each other,
- an i-wave interacts with a j-wave,
- an i-wave (or a j-wave) collides with some other h-wave, with $h \neq i,j$.

At every such interaction time, the decrease in Q_ν^{ij} is bounded by the decrease of Υ_ν. More precisely, for some constant $C > 0$ we have

$$[\![\Delta Q_\nu^{ij}(t)]\!]_- \leq C \cdot \Big|\Delta \Upsilon_\nu(t)\Big|\,. \tag{3.15}$$

We now observe that, if ν is sufficiently large, then the interval $I_\nu^{ij}(t^*)$ in (3.12) is empty, because $\delta_\nu \to 0$ and $\lambda_i^{\max} < \lambda_j^{\min}$. Hence $Q_\nu^{ij}(t^*) = 0$. By (3.15), (2.13) and (3.7) we thus obtain

$$Q_\nu^{ij}(\tau_\nu) = Q_\nu^{ij}(\tau_\nu) - Q_\nu^{ij}(t^*)$$

$$
\begin{aligned}
&\leq \sum_{t\in[\tau_\nu,t^*]} [\![\Delta Q^{ij}_\nu(t)]\!]_- \\
&\leq C\cdot \sum_{t\in[\tau_\nu,t^*]} \big|\Delta\Upsilon_\nu(t)\big| \\
&= C\cdot \Big(\Upsilon\big(u_\nu(\tau_\nu)\big)-\Upsilon\big(u_\nu(t^*)\big)\Big) \\
&< \varepsilon^{10}
\end{aligned}
\tag{3.16}
$$

for every couple of indices $i<j$. This provides a bound on the interaction potential due to waves of different families.

We claim that, at time τ_ν, the waves in $u_\nu(\tau_\nu)$ contained in the interval $I_\nu(\tau_\nu)$ are almost separated, i.e. there exist n subintervals I^i_ν such that

$$
\bigcup_{i=1}^n I^i_\nu = I_\nu(\tau_\nu), \qquad\qquad \sum_{j\neq i}\Big|\mu^j_{\nu,\tau_\nu}\Big|\,(I^i_\nu)\leq \varepsilon^4\,. \tag{3.17}
$$

Indeed, to construct the intervals I^i_ν we set $I_\nu(\tau_\nu)\doteq[a_\nu,b_\nu]$ and choose points $a_\nu=p^0_\nu<p^1_\nu<\ldots<p^n_\nu=b_\nu$ such that

$$
\begin{aligned}
&\Big|\mu^j_{\nu,\tau_\nu}\Big|\Big([p^i_\nu,b_\nu]\Big)<\varepsilon^4/2n \qquad \text{for all } j<i\,,\\
&\Big|\mu^j_{\nu,\tau_\nu}\Big|\Big([a_\nu,p^i_\nu]\Big)<\varepsilon^4/2n \qquad \text{for all } j\geq i.
\end{aligned}
$$

Such points p^i_ν exist because of (3.16). Setting $I^i_\nu\doteq[p^{i-1}_\nu,\ p^i_\nu]$ we obtain (3.17).

Next, we estimate the interaction potential $Q^{ii}_\nu(\tau_\nu)$ corresponding to all pairs of i-waves in $u_\nu(\tau_\nu)$, both contained inside $I_\nu(\tau_\nu)$. Calling $\widehat{Q}^i_\nu$ the interaction potential of pairs of i-waves in $u_\nu(\tau_\nu)$ both contained in the subinterval I^i_ν, by (3.17) we have

$$
\begin{aligned}
Q^{ii}_\nu(\tau_\nu) &\leq \widehat{Q}^i_\nu + \Big|\mu^i_{\nu,\tau_\nu}\Big|\Big(I_\nu(\tau_\nu)\Big)\cdot\Big|\mu^i_{\nu,\tau_\nu}\Big|\Big(I_\nu(\tau_\nu)\setminus I^i_\nu\Big)\\
&\leq \widehat{Q}^i_\nu+\varepsilon^3.
\end{aligned}
\tag{3.18}
$$

Therefore, it will suffice to derive an estimate on $\widehat{Q}^i_\nu$. In the following, we write μ^{i+},μ^{i-} for the positive and negative parts of μ^i, as in (1.9). We shall consider three cases.

1. $\Big(\mu^{i-}_{\nu,\tau_\nu}\Big)(I^i_\nu)<\varepsilon^2$. In this case we simply have

$$
\widehat{Q}^i_\nu\leq\Big|\mu^i_{\nu,\tau_\nu}\Big|\,(I^i_\nu)\cdot\mu^{i-}_{\nu,\tau_\nu}(I^i_\nu)\leq C\,\varepsilon^2\,. \tag{3.19}
$$

2. The function $u_\nu(\tau_\nu)$ contains at least one i-shock of strength $|\bar\sigma|\geq\varepsilon^3$, located at some point $\bar y\in I^i_\nu$.

In this case, we claim that the total amount of all other i-waves contained in I^i_ν satisfies

$$\left|\mu^i_{\nu,\tau_\nu}\right|\Big(I^i_\nu \setminus \{\bar y\}\Big) < \varepsilon^4, \tag{3.20}$$

hence

$$\widehat{Q}^i_\nu \leq C\,\varepsilon^7\,. \tag{3.21}$$

To prove (3.20), we will show that in the opposite case the large shock would attract an amount $> c\varepsilon^4$ of i-waves, for some constant $c > 0$. This would produce an amount of interaction and cancellation $> \varepsilon^3 \cdot c\varepsilon^4$ within the interval $[\tau_\nu, t^*]$, in contrast with (3.7).

A more detailed proof goes as follows. Let $y(t)$ be the position of the shock at time $t \in [\tau_\nu, t^*]$, and let $\sigma(t) < 0$ be its size. By Lemma 1, the bounds (3.5)-(3.6) on the amounts of interaction and cancellation imply that

$$\Big|\sigma(t) - \bar\sigma\Big| < C\cdot\varepsilon^6, \qquad \Big|\dot y(t) - \dot y(\tau_\nu)\Big| \leq C\cdot\varepsilon^6, \qquad\qquad t\in[\tau_\nu,\ t^*]\,. \tag{3.22}$$

Consider the triangular domain

$$\Gamma \doteq \left\{(t,x);\quad t\in[\tau_\nu,t^*],\quad x\in\Big[y^-(t),\ y^+(t)\Big]\right\},$$

where

$$y^-(t) = y(t^*) - (t^*-t)\Big[\dot y(t^*) + \varepsilon^5\Big], \qquad\qquad y^+(t) = y(t^*) - (t^*-t)\Big[\dot y(t^*) - \varepsilon^5\Big].$$

We now introduce a functional which roughly measures the speed at which i-waves approach the shock:

$$\Psi(t) \doteq \sum_{y_\alpha(t)\in]y(t),y^+(t)]} |\sigma_\alpha|\cdot[\![\dot y^+ - \dot y_\alpha(t)]\!]_+ + \sum_{y_\alpha(t)\in[y^-(t),y(t)[} |\sigma_\alpha|\cdot[\![\dot y_\alpha(t) - \dot y^-]\!]_+ \tag{3.23}$$

The above summations refer to i-waves in $u_\nu(t)$, located at y_α with strength $|\sigma_\alpha|$. By the above definition, it is clear that Ψ does not change whenever an i-wave leaves Γ, crossing one of the lines y^-, y^+. Therefore,

$$[\![\Delta\Psi(t)]\!]_- \leq C\Big|\Delta\Upsilon_\nu(t)\Big|\cdot\varepsilon^3\,. \tag{3.24}$$

Assume that the total amount of i-waves inside I^i_ν is $> \varepsilon^4$. Since, for ν large,

$$I^i_\nu \subset I_\nu(\tau_\nu) \subset \Big[y^-(\tau_\nu),\ y^+(\tau_\nu)\Big],$$

the assumption of genuine nonlinearity implies

$$\Psi(\tau_\nu) \geq c\varepsilon^7 \tag{3.25}$$

for some constant $c > 0$. Observing that $\Psi(t^*) = 0$ because $y^-(t^*) = y^+(t^*)$, from (3.24), (3.25) it follows

$$\sum_{t\in[\tau_\nu,t^*]} \Big|\Delta\Upsilon(t)\Big| \geq c\varepsilon^{10},$$

a contradiction with (3.7). This establishes (3.20), and hence (3.21).

3. We are now left with the case where $\mu^{i-}_{\nu,\tau_\nu}(I^i_\nu) \geq \varepsilon^2$, and all the i-waves inside I^i_ν have strength $|\sigma| < \varepsilon^3$. Our aim is to show that this case cannot happen, because it would cause a large amount of interaction and cancellation, in contrast with (3.7). Recalling that $I^i_\nu = [p^{i-1}_\nu, p^i_\nu]$ in (3.17), consider the domain

$$\mathcal{D}^i_\nu \doteq \Big\{(t,x);\quad t\in[\tau_\nu,t^*],\quad x\in I^i_\nu(t)\Big\}, \tag{3.26}$$

where

$$I^i_\nu(t) = \Big[a^i_\nu(t),\ b^i_\nu(t)\Big] \doteq \Big[p^{i-1}_\nu + (t-\tau_\nu)\lambda^{\min}_i,\quad p^i_\nu + (t-\tau_\nu)\lambda^{\max}_i\Big]. \tag{3.27}$$

Consider all shock lines $x = y_\alpha(t)$ of the solution u_ν, which are defined on the whole time interval $[\tau_\nu, t^*]$ (fig.1).

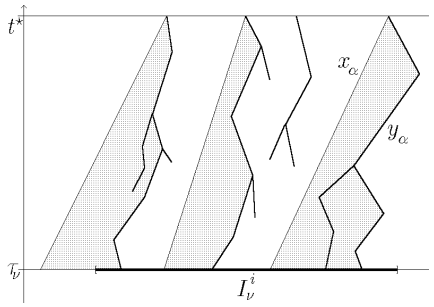

Figure 1.

By the assumption (3.7) on the total amount of interaction and cancellation, for every $t \in [\tau_\nu, t^*]$ the total strength of all shocks in $u_\nu(t)$ located on these lines is

$$\sum_\alpha \Big|\sigma_\alpha(t)\Big| > \sum_\alpha \Big|\sigma_\alpha(\tau_\nu)\Big| - C\,\varepsilon^{11} > \varepsilon^2 - \varepsilon^{10}. \tag{3.28}$$

Moreover, since the initial strength of any i-shock is $< \varepsilon^3$, by (3.7) and Lemma 2 it follows that all shocks remain small:

$$\Big|\sigma_\alpha(t)\Big| < 2\varepsilon^3 \qquad \text{for all } \alpha,\quad t\in[\tau_\nu,t^*]. \tag{3.29}$$

Call $y_1(t^*) < \dots y_N(t^*)$ the positions of those i-shocks in $u(t^*)$ which can be continued backward on the whole interval $[\tau_\nu, t^*]$, reaching a point inside I^i_ν at time τ_ν. For each $\alpha = 1, \dots, N$, let $x = y_\alpha(t)$ be the location of the maximal among these backward continuations. Moreover, call $x = x_\alpha(t)$ be the minimal backward characteristic through the point $\Big(t^*, y_\alpha(t^*)\Big)$ (fig.1). By construction, the intervals $\Big[x_\alpha(t),\ y_\alpha(t)\Big]$ are mutually disjoint, for every $t \in [\tau_\nu, t^*]$. For $\alpha = 2, \dots, N$, these intervals are also contained inside I^i_ν, and hence inside $I_\nu(\tau_\nu)$, defined at (3.8). Therefore

$$\sum_{\alpha=2}^{N} y_\alpha(\tau_\nu) - x_\alpha(\tau_\nu) \leq 2\delta_\nu + 2(\tau_\nu - \tau)\hat\lambda < \varepsilon^9 \tag{3.30}$$

for all ν sufficiently large. Using the genuine nonlinearity and the fact that in our algorithm all rarefaction fronts travel with their right-characteristic speed, we compute

$$\begin{aligned} y_\alpha(\tau_\nu) - x_\alpha(\tau_\nu) &= \big(y_\alpha(t^*) - x_\alpha(t^*)\big) - \int_{\tau_\nu}^{t^*} \big(\dot y_\alpha(s) - \dot x_\alpha(s)\big)\, ds \\ &\geq c \cdot \int_{\tau_\nu}^{t^*} \mu^{i-}_{\nu,s}\Big(\big]x_\alpha(s), y_\alpha(s)\big]\Big)\, ds \\ &\quad - C \cdot \int_{\tau_\nu}^{t^*} \mu^{i+}_{\nu,s}\Big(\big]x_\alpha(s), y_\alpha(s)\big]\Big)\, ds \\ &\quad - C \cdot \int_{\tau_\nu}^{t^*} \sum_{j\neq i} \Big|\mu^j_{\nu,s}\Big|\Big(\big]x_\alpha(s), y_\alpha(s)\big]\Big)\, ds, \end{aligned} \tag{3.31}$$

for some constants $C, c > 0$. The three terms on the right hand side of (3.31) will now be estimated one at a time. Call $\mathcal{T}_\alpha$ the set of times at which an interaction takes place inside the domain

$$\mathcal{D}_\alpha \doteq \Big\{(t,x);\qquad t \in [\tau_\nu,\ t^*],\quad x \in \Big]x_\alpha(t),\ y_\alpha(t)\Big]\Big\}. \tag{3.32}$$

Since no i-shock can exit from $\mathcal{D}_\alpha$, for $s \in \,]\tau_\nu, t_*[$ one has

$$\mu^{i-}_{\nu,s}\Big(\big]x_\alpha(s), y_\alpha(s)\big]\Big) \geq \mu^{i-}_{\nu,\tau_\nu}\Big(\big]x_\alpha(\tau_\nu), y_\alpha(\tau_\nu)\big]\Big) - C \cdot \sum_{t\in\mathcal{T}_\alpha} \big|\Delta\Upsilon_\nu(t)\big| \tag{3.33}$$

Similarly, rarefaction waves cannot exit from $\mathcal{D}_\alpha$. Observing that all positive waves contained inside $\mathcal{D}_\alpha$ are cancelled within time t^*, we obtain

$$\mu^{i+}_{\nu,\tau_\nu}\Big(\big]x_\alpha(\tau_\nu), y_\alpha(\tau_\nu)\big]\Big) \leq C \cdot \sum_{t\in\mathcal{T}_\alpha} \big|\Delta\Upsilon_\nu(t)\big|\ . \tag{3.34}$$

Concerning the third integral on the right hand side of (3.31), observing that $\mathcal{D}_\alpha \subset \mathcal{D}^{ii}_\nu$ for all $\alpha = 2, \dots, N$, we have

$$\sum_{\alpha=2}^{N} \Big|\mu^j_{\nu,s}\Big|\Big(\big]x_\alpha(s), y_\alpha(s)\big]\Big) \leq \sum_{j\neq i} \Big|\mu^j_{\nu,\tau_\nu}\Big|\,(I^i_\nu) + C \cdot \Big[\Upsilon_\nu(\tau_\nu) - \Upsilon_\nu(t^*)\Big] \tag{3.35}$$

Using (3.33)–(3.35) inside (3.31) we obtain

$$\sum_{\alpha=2}^{N} \big(y_\alpha(\tau_\nu) - x_\alpha(\tau_\nu)\big) \geq c \sum_{\alpha=2}^{N} \Big(\mu^{i-}_{\nu,\tau_\nu}\Big) \Big(\big]x_\alpha(\tau_\nu), y_\alpha(\tau_\nu)\big]\Big) - C \sum_{j\neq i} \Big|\mu^{j}_{\nu,\tau_\nu}\Big| \,(I^h_\nu) \\ - C \cdot \big(\Upsilon_\nu(\tau_\nu) - \Upsilon_\nu(t^*)\big) \tag{3.36}$$

Thanks to the estimates (3.30), (3.17) and (3.7), from (3.36) it follows

$$\sum_{\alpha=2}^{N} \mu^{i-}_{\nu,\tau_\nu} \Big(\big]x_\alpha(\tau_\nu), y_\alpha(\tau_\nu)\big]\Big) \leq C\varepsilon^3 . \tag{3.37}$$

This provides an estimate concerning the amount of negative waves inside all intervals $]x_\alpha,\ y_\alpha]$ with $\alpha \geq 2$. Concerning the first interval, observe that the final time t^* all i-waves contained inside D_1 collapse to the single shock located at $y_1(t^*)$. By assumption, at the initial time τ_ν every shock has strength $|\sigma| < \varepsilon^3$. Since the total amount of interaction and cancellation is $< \varepsilon^{11}$, this implies

$$\mu^{i-}_{\nu,\tau_\nu}\Big(\big]x_1(\tau_\nu),\ y_1(\tau_\nu)\big]\Big) < C\varepsilon^3 . \tag{3.38}$$

Together, (3.37) and (3.38) show that this third case cannot occur, completing the proof of Proposition 1.

4. - Proof of Theorem 1.

Let $(u_\nu)_{\nu\geq 1}$ be a sequence of piecewise constant approximate solutions generated by the wave-front tracking algorithm described in Section 2. For each ν, consider the measure $\mu_{\nu,t}$ given by the distributional derivative of $u_\nu(t,\cdot)$, with components $\mu^i_{\nu,t}$ defined as in Section 1.

1. In the first part of the proof, we analyse the behavior of rarefaction fronts in one particular approximate solution u_ν.

Fix a point $(\tau,\xi) \in]0,+\infty[\times I\!R$ and some $i \in \{1,\ldots,n\}$, assuming that the i-th family is genuinely nonlinear. By $t \mapsto y^i_\nu(t;\tau,\xi)$ we denote the *minimal backward i-characteristic* through (τ,ξ), that is, the minimal solution to the Cauchy problem

$$\begin{cases} \dot x(t) = \lambda_i\big(u_\nu(t,x)\big) & t \in [0,\tau]\,, \\ x(\tau) = \xi\,. \end{cases} \tag{4.1}$$

Since the above differential equation has discontinuous right hand side, solutions are here interpreted in the generalized sense, according to Filippov [10]. For the basic theory of generalized

characteristics we refer to [9]. Observe that, for our piecewise constant approximate solution u_ν, the i-characteristics run into an i-shock from both sides. On the other hand, i-characteristics move away from an i-rarefaction front from the left, and run parallel to it on the right.

Now fix a time $\tau > 0$ and let $I = [\xi', \xi'']$ be any compact interval. For $t \in [0, \tau]$ define

$$I(t) \doteq \Big[y^i_\nu \left(t; \tau, \xi'\right),\ y^i_\nu \left(t; \tau, \xi''\right) \Big] \doteq \Big[a(t),\ b(t) \Big] . \tag{4.2}$$

Call $\mathcal{T}(I, \tau)$ the set of all times at which an interaction occurs within the region

$$\Gamma \doteq \Big\{ (t, x) \in I\!R^2 \ ;\ \ t \in [0, \tau] \text{ and } x \in I(t) \Big\} . \tag{4.3}$$

We seek an estimate on the amount of positive i-waves in the approximate solution $u_\nu(\tau, \cdot)$ contained in the interval I. For a fixed t, the sizes and the locations of the j-waves in $u_\nu(t, \cdot)$ are denoted respectively by $\sigma^\ell_j(t)$, $x^\ell_j(t)$, $\ell = 1, \dots \widetilde{N}_j$. For notational convenience, we assume that the indices ℓ are assigned so that

$$x^\ell_j(t) \in I(t) \qquad \ell = 1, \dots, N_j , \qquad\qquad x^\ell_j(t) \notin I(t) \qquad \ell = N_j + 1, \dots, \widetilde{N}_j , \tag{4.4}$$

for some $N_j \leq \widetilde{N}_j$. Define

$$m(t) \doteq \text{meas} \Big(I(t) \Big) = b(t) - a(t) . \tag{4.5}$$

and call

$$M(t) \doteq \sum_{\ell=1}^{N_i} \sigma^\ell_i(t) = \mu^i_{\nu,t} \Big(I(t) \Big) \tag{4.6}$$

the sum of the (signed) i-waves in $u_\nu(t, \cdot)$ contained in $I(t)$. The sum of the cubes of the strengths of the i-shocks inside $I(t)$ will be denoted by

$$B(t) \doteq \sum_{\ell=1}^{N_i} [\![\sigma^\ell_i(t)]\!]^{\,3}_{-} . \tag{4.7}$$

Finally, the total strength of all the j-waves, with $j \neq i$, contained in $I(t)$ is written

$$A(t) \doteq \sum_{j \neq i} \sum_{\ell=1}^{N_j} \Big| \sigma^\ell_j(t) \Big| . \tag{4.8}$$

Observe that, at the endpoints $x = a(t), b(t)$, the function $u_\nu(t, \cdot)$ either is continuous or else it has a rarefaction front. Since all rarefaction fronts have strength $\leq C/\nu$, the assumption of genuine nonlinearity (1.8) yields

$$\dot{m}(t) \geq \kappa_i \cdot M(t) - C \Big(B(t) + A(t) + \frac{1}{\nu} \Big) , \tag{4.9}$$

for some constant C depending only on the system (1.1), and all but finitely many $t \in [0, \tau]$. The changes $\Delta M(t)$ and $\Delta B(t)$ of the functions M and B at an interaction time $t \in \mathcal{T}(I, \tau)$ can be bounded by

$$\left|\Delta M(t)\right| \leq C \left|\Delta Q\left(u_\nu(t)\right)\right|, \qquad \left|\Delta B(t)\right| \leq C \left|\Delta Q\left(u_\nu(t)\right)\right|. \tag{4.10}$$

Two additional cases need to be considered, when an i-wave enters or exits across the right boundary of Γ (fig.2).

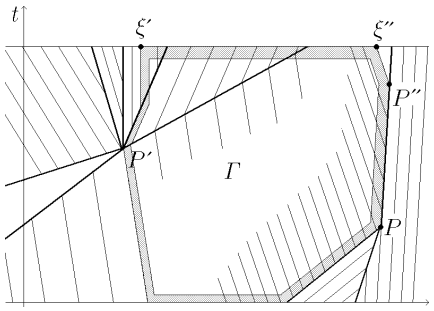

Figure 2.

In P the collision between an i-rarefaction and an i-shock leads to the formation of an i-rarefaction. In P' two j-waves interact forming three i-rarefactions. In P'' an i-rarefaction front leaves Γ.

At a point P where an i-rarefaction interacts with a small i-shock and an i-rarefaction front emerges, we have

$$\Delta M(t) \leq 0, \qquad \Delta B(t) = 0. \tag{4.11}$$

At a point P'' where an i-rarefaction front leaves Γ and no interaction occurs, the estimates (4.11) still hold.

To estimate the contribution of the term $A(t)$ in (4.9), we introduce the function

$$\Phi_i(t) \doteq \sum_{j \neq i} \sum_{\ell=1}^{\widetilde{N}_j} \phi_j\left(t, x_j^\ell(t)\right) \cdot \left|\sigma_j^\ell(t)\right| \tag{4.12}$$

where

$$\phi_j(t,x) \doteq \begin{cases} 1 & \text{if } x < a(t) \\ \dfrac{b(t)-x}{m(t)} & \text{if } x \in [a(t), b(t)] \\ 0 & \text{if } x > b(t) \end{cases} \qquad \text{or} \qquad \phi_j(t,x) \doteq \begin{cases} 0 & \text{if } x < a(t) \\ \dfrac{x-a(t)}{m(t)} & \text{if } x \in [a(t), b(t)] \\ 1 & \text{if } x > b(t) \end{cases}$$

in the cases $j < i$ or $j > i$, respectively. Throughout this computation we are assuming that $m(t) > 0$. Observe that Φ_i is piecewise Lipschitz with a finite number of discontinuities occuring at interaction times, where

$$\left|\Delta\Phi_i(t)\right| \leq C \cdot \left|\Delta Q\left(u_\nu(t)\right)\right| . \tag{4.13}$$

Outside the interaction times, the function Φ_i is non-decreasing. Indeed

$$\begin{aligned}
\dot{\Phi}_i(t) &= \sum_{j\neq i}\sum_{\ell=1}^{N_j}\left|\sigma_j^\ell(t)\right|\cdot\frac{d}{dt}\phi_j\Big(t, x_j^\ell(t)\Big) \\
&= \sum_{j<i}\sum_{\ell=1}^{N_j}\left|\sigma_j^\ell(t)\right|\cdot\left(\frac{\dot{b}-\dot{x}_j^\ell}{m}-\frac{(b-x_j^\ell)\dot{m}}{m^2}\right)+\sum_{j>i}\sum_{\ell=1}^{N_j}\left|\sigma_j^\ell(t)\right|\cdot\left(\frac{\dot{x}_j^\ell-\dot{a}}{m}-\frac{(x_j^\ell-a)\dot{m}}{m^2}\right) \\
&\geq \sum_{j\neq i}\sum_{\ell=1}^{N_j}\left|\sigma_j^\ell(t)\right|\cdot\frac{c_0}{m(t)}
\end{aligned} \tag{4.14}$$

for some constant $c_0 > 0$. This yields the bound

$$A(t) \leq C \cdot m(t) \cdot \dot{\Phi}_i(t) \tag{4.15}$$

valid for all but finitely many times t. Inserting the estimate (4.15) in (4.9) we obtain

$$\dot{m}(t) \geq \kappa_i \cdot M(t) - C \cdot \left(m(t)\cdot\dot{\Phi}_i(t) + B(t) + \frac{1}{\nu}\right) .$$

Using (4.10) and (4.11) to estimate $M(t), B(t)$ in terms of $M(\tau), B(\tau)$, this yields

$$\dot{m}(t) + C\cdot m(t)\cdot\dot{\Phi}_i(t) \geq \kappa_i\cdot M(\tau) - C\cdot\left(B(\tau)+\frac{1}{\nu}+\sum_{t'\in\mathcal{T}(I,\tau)}\left|\Delta Q\left(u_\nu(t')\right)\right|\right) , \tag{4.16}$$

possibly with a different constant C. We now observe that m is a continuous, piecewise linear function of t. Moreover, by (4.13) and (4.14), the total variation of Φ_i is bounded, say

$$\text{Tot.Var.}(\Phi_i) \leq K , \tag{4.17}$$

for some constant K depending only on the total variation of u_ν and on the system (1.1). From the bounds (4.16)-(4.17) we now obtain

$$\begin{aligned}
m(\tau) &\geq e^{-CK\tau}\cdot\left[\kappa_i M(\tau) - C\left(B(\tau)+\frac{1}{\nu}+\sum_{t'\in\mathcal{T}(I,\tau)}\left|\Delta Q\left(u_\nu(t')\right)\right|\right)\right] \\
&= C_3\tau\left(\kappa_i' M(\tau) - B(\tau) - \frac{1}{\nu} - \sum_{t'\in\mathcal{T}(I,\tau)}\left|\Delta Q\Big(u_\nu(t')\Big)\right|\right) ,
\end{aligned} \tag{4.18}$$

for suitable constants $C_3, \kappa_i' > 0$, depending only on the system (1.1) and on the total variation of the initial data.

2. We shall need a more general version of (4.18), valid for the union of finitely many compact intervals. Fix $\tau > 0$ and let $I_1, \ldots, I_{\bar k}$ be disjoint compact intervals. For each $k = 1, \ldots, \bar k$ and $t \in [0, \tau]$, define the intervals $I_k(t) = \Big[a_k(t),\ b_k(t)\Big]$ as in (4.2). Let M_k, B_k be as in (4.6), (4.7), with $I(t)$ replaced by $I_k(t)$. Define

$$\tau_k \doteq \max\Big\{t \in [0,\tau];\quad a_k(t) = b_k(t)\Big\}$$

and call $\mathcal{T}_k$ the set of times when an interaction occurs within the region

$$\Gamma_k \doteq \Big\{(t,x);\quad \tau_k < t \leq \tau,\quad x \in I_k(t)\Big\}.$$

Observe that each point (t,x) can lie at most inside two regions Γ_k. Applying the estimate (4.18) to each interval I_k and summing over k we thus obtain

$$\begin{aligned}\text{meas}\left(\bigcup_{k=1}^{\bar k} I_k(\tau)\right) &\geq C_3\tau\cdot\sum_{k=1}^{\bar k}\left(\kappa_i' M_k(\tau) - B_k(\tau) - \frac{1}{\nu} - \sum_{t'\in\mathcal{T}_k}\Big|\Delta Q\Big(u_\nu(t')\Big)\Big|\right)\\ &\geq C_3\tau\left(\kappa_i'\cdot\mu^i_{\nu,\tau}\left(\bigcup_{k=1}^{\bar k} I_k(\tau)\right) - \frac{\bar k}{\nu} - 2\Big[Q\big(u_\nu(0)\big) - Q\big(u_\nu(\tau)\big)\Big] - \sum_{k=1}^{\bar k} B_k(\tau)\right).\end{aligned} \tag{4.19}$$

3. We now consider an exact solution $u = \lim u_\nu$ of (1.1), and establish the bound

$$\mu^i(\tau)\Big(\{p_1,\ldots,p_r\}\Big) \leq C_4\cdot\Big[Q\big(u(0)\big) - Q\big(u(\tau)\big)\Big], \tag{4.20}$$

for some constant C_4 and every finite set of points $p_1, \ldots, p_r$. This estimate should be intuitively clear: assume that

$$\mu^i_\tau\Big(\{p_k\}\Big) \doteq \sigma^i_k > 0 \qquad\qquad k = 1,\ldots,r\,. \tag{4.21}$$

This means that, for the solution $u = u(t,x)$, the point (τ, p_k) is the center of an i-rarefaction wave with strength σ^i_k. Such a wave can only be produced by an interaction occurring exactly at time τ. Summing over k, we thus expect

$$\sum_{k=1}^{\bar k}\sigma^i_k \leq C_4\Big|Q(\tau) - Q(\tau-)\Big|.$$

A rigorous proof of (4.20) will now be given, based on Proposition 1. It is clearly not restrictive to assume that (4.21) holds. As in (1.6), $\hat\lambda$ will denote an upper bound for all characteristic speeds. Define

$$I_k(t) \doteq \Big[p_k - 2(t-\tau)\hat\lambda,\ \ p_k + 2(t-\tau)\hat\lambda\Big]\,.$$

By Proposition 1, we can find a strictly decreasing sequence $\tau_\nu \to \tau$ such that, calling $I_k(\tau_\nu) = [a_{k,\nu},\ b_{k,\nu}]$, for $k = 1, \ldots, \bar k$ one has

$$\lim_{\nu\to\infty} Q\Big(u_\nu(\tau_\nu);\ I_k(\tau_\nu)\Big) = 0\,, \tag{4.22}$$

$$\lim_{\nu\to\infty} u_\nu(\tau_\nu, a_{k,\nu}) = u(\tau, p_k-)\,, \qquad\qquad \lim_{\nu\to\infty} u_\nu(\tau_\nu, b_{k,\nu}) = u(\tau, p_k+)\,. \tag{4.23}$$

Let the quantities M_k^ν, B_k^ν be defined as in (4.6)-(4.7), with t replaced by τ_ν and $I(t)$ replaced by $I_k(\tau_\nu)$. Using (4.19) with τ replaced by τ_ν we thus obtain

$$\begin{aligned}\text{meas}\left(\bigcup_{k=1}^{\bar k} I_k(\tau_\nu)\right) &= 4\bar k\hat\lambda(\tau_\nu - \tau)\\ &\geq C_3\tau\left(\kappa_i'\sum_{k=1}^{\bar k} M_k^\nu(\tau_\nu) - \frac{\bar k}{\nu} - 2\Big[Q\left(u_\nu(0)\right) - Q\left(u_\nu(\tau_\nu)\right)\Big] - \sum_{k=1}^{\bar k} B_k^\nu(\tau_\nu)\right).\end{aligned} \tag{4.24}$$

As $\nu\to\infty$, (4.22)-(4.23) imply

$$M_k^\nu(\tau_\nu) \doteq \mu^i_{\nu,\tau_\nu}\Big([a_{k,\nu},\ b_{k,\nu}]\Big) \to \sigma_k^i\,, \qquad\qquad B_k^\nu \to 0\,. \tag{4.25}$$

Letting $\nu\to\infty$ in (4.24) and using (4.25) we obtain

$$\liminf_{\nu\to\infty} 2\cdot\Big[Q\left(u_\nu(0)\right) - Q\left(u_\nu(\tau_\nu)\right)\Big] \geq \kappa_i'\cdot\sum_{k=1}^{\bar k}\sigma_k^i\,. \tag{4.26}$$

Observe that $u_\nu(\tau_\nu)\to u(\tau)$ in $\mathbf{L}^1$. The lower semicontinuity of the functional Q, stated in Lemma 3, together with the assumption (2.2) now yields

$$2\left[Q\Big(u(0)\Big) - Q\Big(u(\tau)\Big)\right] \geq \kappa_i'\sum_k\sigma_k^i = \kappa_i'\cdot\mu_\tau^i\Big(\{p_1,\ldots,p_r\}\Big)\,. \tag{4.27}$$

This gives (4.20), with $C_4 = 2/\kappa_i'$.

4. We now show that (1.20) is true for all Borel sets if and only if it is true for every finite union of compact intervals:

$$J = \bigcup_{h=1}^{\bar h}[a_h,\ b_h]. \tag{4.28}$$

Indeed, assume that (1.20) fails for some Borel set $\widetilde{J}$, say

$$\mu^i_\tau\left(\widetilde{J}\right) > \frac{\mathrm{meas}(\widetilde{J})+\varepsilon}{\kappa\tau} + C_2\cdot\Big[Q(0)-Q(\tau)\Big] \tag{4.29}$$

for some $\varepsilon>0$. Choose an open set $J'\supset\widetilde{J}$ with $\mathrm{meas}(J')<\mathrm{meas}(\widetilde{J})+\varepsilon$. Write $J'=\bigcup_h\,]a'_h,\ b'_h[$ as a disjoint union of countably many open intervals. By choosing

$$J''\doteq\bigcup_{h=1}^{\bar h}\,]a'_h,\ b'_h[$$

with $\bar h$ sufficiently large, from (4.29) we thus obtain

$$\mu^i_\tau(J'') > \frac{\mathrm{meas}(J'')}{\kappa\tau} + C_2\cdot\Big[Q(0)-Q(\tau)\Big]\,. \tag{4.30}$$

Defining $[a_h,\ b_h]\doteq[a'_h+\rho,\ b'_h-\rho]$ and choosing $\rho>0$ sufficiently small, by (4.30) the corresponding set J in (4.28) satisfies

$$\mu^i_\tau(J) > \frac{\mathrm{meas}(J)}{\kappa\tau} + C_2\cdot\Big[Q(0)-Q(\tau)\Big]\,.$$

This proves our claim.

5. In the remainder of the proof, we shall thus assume that J is a compact set of the form (4.28). Call $\Big|Du_\nu(\tau)\Big|$ the total variation measure of $u_\nu(\tau,\cdot)$. By possibly taking a subsequence, we can assume the weak convergence

$$\Big|Du_\nu(\tau)\Big| \rightharpoonup \tilde\mu \tag{4.31}$$

for some positive Radon measure $\tilde\mu$. Let $\varepsilon>0$ be given. The set J can be decomposed as a finite union of disjoint open intervals and points:

$$J=\left(\bigcup_{\ell=1}^m\,]\alpha'_\ell,\ \beta'_\ell[\right)\cup\{p_1,\ldots,p_r\}\,, \tag{4.32}$$

with

$$\tilde\mu\Big(]\alpha'_\ell,\ \beta'_\ell[\Big)<\varepsilon\qquad\qquad \ell=1,\ldots,m\,. \tag{4.33}$$

Choose compact subintervals $[\alpha_\ell,\beta_\ell]\subset]\alpha'_\ell,\beta'_\ell[$ such that

$$\sum_{\ell=1}^m\mu^i_\tau\Big(]\alpha'_\ell,\beta'_\ell[\Big)<\sum_{\ell=1}^m\mu^i_\tau\Big([\alpha_\ell,\beta_\ell]\Big)+\varepsilon\,. \tag{4.34}$$

Applying (4.19) to the union of the intervals $[\alpha_\ell,\beta_\ell]$, for every $\nu\geq1$ we obtain

$$\begin{aligned}\mathrm{meas}(J)&\geq\sum_{\ell=1}^m(\beta_\ell-\alpha_\ell)\\&\geq C_3\tau\left(\kappa'_i\cdot\mu^i_{\nu,\tau}\Big(\bigcup_\ell[\alpha_\ell,\beta_\ell]\Big)-\frac{m}{\nu}-2\Big[Q\Big(u_\nu(0)\Big)-Q\Big(u_\nu(\tau)\Big)\Big]-\sum_\ell B^\nu_\ell(\tau)\right)\,.\end{aligned} \tag{4.35}$$

Recall that $B^\nu_\ell(\tau)$ is the sum of cubes of the strengths of i-shocks in $u_\nu(\tau,\cdot)$ contained in the interval $[\alpha_\ell, \beta_\ell]$. By (4.33), as $\nu \to \infty$, the maximum strength of these shocks is $O(\varepsilon)$. Hence

$$\limsup_{\nu\to\infty} \sum_\ell B^\nu_\ell(\tau) \leq C\varepsilon^2 . \tag{4.36}$$

Lemma 3 and the assumption (2.2) now imply

$$\limsup_{\nu\to\infty} \Big[Q\big(u_\nu(0)\big) - Q\Big(u_\nu(\tau)\Big)\Big] \leq Q\Big(u(0)\Big) - Q\Big(u(\tau)\Big) . \tag{4.37}$$

Moreover, by Lemma 4,

$$\limsup_{\nu\to\infty} \Big|\mu^i_\tau\Big([\alpha_\ell,\beta_\ell]\Big) - \mu^i_{\nu,\tau}\Big([\alpha_\ell,\beta_\ell]\Big)\Big| \leq C\cdot\Big[\tilde\mu\Big([\alpha_\ell,\beta_\ell]\Big)\Big]^2 \leq C\cdot\varepsilon\cdot\tilde\mu\Big([\alpha_\ell,\beta_\ell]\Big) . \tag{4.38}$$

Letting $\nu\to\infty$ in (4.35) and using (4.36)–(4.38) we thus obtain

$$\mathrm{meas}(J) \geq C_3\tau\left(\kappa'_i\cdot\sum_\ell\Big[\mu^i_\tau\Big([\alpha_\ell,\beta_\ell]\Big) - C\varepsilon\cdot\tilde\mu\Big([\alpha_\ell,\beta_\ell]\Big)\Big] - 2\Big[Q\Big(u(0)\Big) - Q\Big(u(\tau)\Big)\Big] - C\varepsilon^2\right) .$$

Hence

$$\mu^i_\tau\Big(\bigcup_\ell [\alpha_\ell,\ \beta_\ell]\Big) \leq C\varepsilon\tilde\mu(I\!R) + \frac{1}{\kappa'_i}\left\{\frac{\mathrm{meas}(J)}{C_3\tau} + 2\Big[Q\Big(u(0)\Big) - Q\Big(u(\tau)\Big)\Big] + C\varepsilon^2\right\} . \tag{4.39}$$

Putting together (4.27) and (4.39) and recalling (4.32), (4.34), we finally obtain

$$\begin{aligned}\mu^i_\tau(J) &\leq \mu^i_\tau\Big(\bigcup_\ell[\alpha_\ell,\beta_\ell]\Big) + \mu^i_\tau\Big(\{p_1,\ldots,p_r\}\Big) + \varepsilon\\ &\leq \frac{1}{\kappa'_i}\left\{\frac{\mathrm{meas}(J)}{C_3\tau} + 4\Big[Q\ \big(u(0)\big) - Q\ \big(u(\tau)\big)\Big] + \varepsilon\Big(C\varepsilon + C\kappa'_i\tilde\mu(I\!R) + 1\Big)\right\} .\end{aligned} \tag{4.40}$$

Since $\varepsilon > 0$ was arbitrary, (4.40) implies (1.20), for suitable constants C_1, κ.

Aknowledgement: The second author thanks the 3rd Faculty of Sciences in Varese for having partially supported this work.

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