Well-Posedness for a Class of 2×2 Conservation Laws with $\mathbf{L}^\infty$ data

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Abstract. The Cauchy problem for a special class of 2×2 systems of conservation laws with data in $\mathbf{L}^1\cap\mathbf{L}^\infty$ is considered. In the strictly hyperbolic case we prove the existence of a weak solution which depends continuously on the initial data with respect to the $\mathbf{L}^1$-norm. This solution can be characterized in terms of a Kružkov-type entropy condition, which is here introduced.

1 Introduction

For scalar conservation laws of the form

$$u_t + f\big(u, v(x)\big)_x = 0, \tag{1}$$

the existence of solutions and their dependence on initial data and on the flux f can be conveniently studied by looking at the 2×2 system

$$\begin{aligned} u_t + f(u,v)_x &= 0 \\ v_t &= 0. \end{aligned} \tag{2}$$

We shall consider the Cauchy problem for (2) with initial data

$$u(0,\cdot) = \bar{u}, \quad v(0,\cdot) = \bar{v}. \tag{3}$$

We assume that $f \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$, that the system is strictly hyperbolic, and that the data are in $\mathbf{L}^1 \cap \mathbf{L}^\infty$.

Systems of the form (2) also arise in models for porous media, traffic and gas flows, and have been studied by several authors [15, 18, 19]. In particular, a model for polymer flooding of an oil-recovery flow in a porous medium is given by

$$\begin{aligned} s_t + f(s,c)_x &= 0 \\ (cs)_t + \big(cf(s,c)\big)_x &= 0, \end{aligned} \tag{4}$$

where s is the water saturation and c the polymer concentration. This system can be written in the form (2) by a Lagrangian transformation of the independent variables; more precisely it can be reduced to

$$\begin{aligned} (1/s)_t - g(s,c)_x &= 0 \\ c_t &= 0, \end{aligned} \tag{5}$$

where $g = f/s$, see [15]. Notice that the system (5) is not strictly hyperbolic when $f_s = f/s$. In [23] it was proved, in the case when $c(0,x)$ is Lipschitz, that system (4) admits solutions which depend continuously on the initial data in a suitable topology, stronger than the $\mathbf{L}^1$ topology. Indeed, in [14] they show that, in the general case of a non-strictly hyperbolic system, one can not have $\mathbf{L}^1$ continuous dependence on the initial data for the solutions of the Cauchy problem for (4).

In [18, 19] existence results for (2) are obtained by means of Godunov schemes, also in the case where the system is not strictly hyperbolic. In [16] wave-front tracking techniques are used to study existence and uniqueness for a special class of non-strictly hyperbolic systems of this type in the case where v is possibly discontinuous.

In this paper we are mainly concerned with the existence and $\mathbf{L}^1$ continuous dependence for the Cauchy problem for (2) with large data. In the

general case of an $n \times n$ system of strictly hyperbolic conservation laws with each characteristic field either genuinely nonlinear or linearly degenerate, the existence of a global, weak, entropic solution when the data has small total variation is well known [2, 3, 5, 10, 11, 20]. In particular, the semigroup approach is a powerful tool to show continuous dependence on initial data; more precisely [6, 7] show that these systems generate unique Lipschitz continuous flows compatible with the standard (self-similar) solution of Riemann problems. However, in our case we assume neither genuine nonlinearity nor linear degeneracy of the second characteristic family, i.e. f is not supposed to be convex or linear.
Note that (2) is a Temple class system, i.e. the shock and rarefaction curves coincide [22]. For these systems, existence is known even for data in $\mathbf{L}^\infty$, and uniqueness for large $\mathbf{BV}$ data [1, 12, 21]. In [1] a Lipschitz continuous Standard Riemann Semigroup is constructed for data with large total variation. However, in these works the genuine nonlinearity or linear degeneracy of each characteristic field is assumed.
When $\bar{v}$ is fixed, the system (2) is equivalent to the scalar conservation law

$$u_t + f\big(u, \bar{v}(x)\big)_x = 0, \tag{6}$$

with the flux dependent on x. For $\bar{v}$ in $\mathcal{C}^1$ the classical results in [8, 17] show that (6) generates a contractive semigroup. We consider the more general case where $\bar{v}$ may be discontinuous, and we use semigroup techniques based on wave-front tracking to prove the existence and continuous dependence of solutions of (2) for $\mathbf{L}^1 \cap \mathbf{L}^\infty$ data. More precisely, the main result of this paper is the following.

Theorem 1.1 *Assume that $f \in \mathcal{C}^2$ with $f_u > 0$. Given compact intervals $K_1, K_2 \subset \mathbb{R}$, define the domain $\mathcal{D}$ by*

$$\mathcal{D} \doteq \Big\{(\bar{u}, \bar{v}) \in \mathbf{L}^1 \cap \mathbf{L}^\infty\,;\ f\big(\bar{u}(x), \bar{v}(x)\big) \in K_1,\ \bar{v}(x) \in K_2,\ \forall\, x \in \mathbb{R}\Big\}. \tag{7}$$

Then there exists a semigroup

$$\mathbb{S} : [0, \infty) \times \mathcal{D} \to \mathcal{D},$$

satisfying the following conditions

(a) for each $t > 0$, the map $\mathbb{S}_t : \mathcal{D} \to \mathcal{D}$ is continuous with respect to the $\mathbf{L}^1$-norm;

(b) the function $(t, x) \mapsto \mathbb{S}_t(\bar{u}, \bar{v})(x)$ is a weak solution of (2) with initial data $(\bar{u}, \bar{v})$.

Moreover, each trajectory of the semigroup coincides with the unique solution of the corresponding Cauchy problem satisfying a suitable entropy admissibility criterion. The entropy condition that we consider here extends the classical Kružkov condition [17], and yields uniqueness for (2)-(3). We point out that one may expect the semigroup $\mathbb{S}$ to be $\mathbf{L}^1$-contractive as a function of $\bar{u}$, but by the analysis in [1] one may not expect it to be even Lipschitz continuous in $(\bar{u}, \bar{v})$.

The paper is organized as follows. After some preliminary definitions and notations, we define a front-tracking algorithm and show that it yields global approximate solutions of (2) for piecewise constant data. Next we define a semigroup for initial data $(\bar{u}, \bar{v})$ with $\bar{u} \in \mathbf{BV}$ and $\bar{v}$ piecewise constant. By continuity, it is extended to a semigroup whose trajectories are weak solutions of (2) with initial data $(\bar{u}, \bar{v})$ in $\mathbf{L}^1 \cap \mathbf{L}^\infty$. The solutions obtained in this way are showed to depend continuously on the initial data with respect to both $\bar{u}$ and $\bar{v}$, in the $\mathbf{L}^1$-norm.

Finally we introduce the entropy condition and prove that the semigroup trajectories are the unique solutions that satisfy this condition.

2 Preliminaries

The characteristic speeds of (2) are $\mu(u, v) = 0$ and $\lambda(u, v) = f_u(u, v)$. Thus the system is linearly degenerate in the first characteristic field. The integral curves for the first and second characteristic fields are given by $f(u, v) \equiv$ const and $v \equiv$ const, respectively. It is easy to see that the shock and rarefaction curves coincide in each family. We assume strict hyperbolicity, i.e. $f_u(u, v) > 0$, for all u, v. Note that we do not assume genuine nonlinearity in the second characteristic field, i.e. f_{uu} can change sign. We refer to waves corresponding to the first and second field as v-waves and u-waves, respectively. Every v-wave has zero speed, while a u-shock travels with speed given by the Rankine-Hugoniot condition

$$\lambda(u^-, u^+; v) \doteq \frac{f(u^+, v) - f(u^-, v)}{u^+ - u^-},$$

where $v \doteq v^- = v^+$.

A weak solution of (2)-(3) is a function $U = (u, v)$ satisfying

$$v(t, x) = \bar{v}(x), \tag{8}$$

for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$, and

$$\int_{-\infty}^{+\infty} \bar{u}(x)\, \phi(0, x)\, dx + \\ + \int_0^{+\infty} \int_{-\infty}^{+\infty} \Big(u(t, x) \phi_t(t, x) + f\big(u(t, x), \bar{v}(x)\big) \phi_x(t, x) \Big)\, dx dt = 0, \tag{9}$$

for every smooth function ϕ with compact support in $\mathbb{R} \times \mathbb{R}$.

Once $\bar{v}$ is fixed, we shall call a solution of (2) either $U(t,x)$ or $u(t,x)$. Throughout this paper we denote the $\mathbf{L}^1$-norm by $\|\cdot\|$. All the initial data $(\bar{u},\bar{v})$ will belong to the domain $\mathcal{D}$ defined in (7). Finally we denote by C constants depending only on f_u, f_v, f_{uv} and f_{uu}.

2.1 Front tracking

We construct approximate solutions to (2) using a front tracking algorithm. First we define the Riemann solver. Since f_u is strictly positive, for every $v_0 \in \mathbb{R}$ the map $f(\cdot\,, v_0)$ gives a one-to-one correspondence between values of u and values of f. By this correspondence we will use either (u,v) or $\big(f(u,v),v\big)$ to identify a state in the (u,v)-space, whatever is more convenient.
Given $\delta > 0$ consider an equally-spaced δ-grid $\mathcal{F}(\delta) = \{i\delta \mid i \in \mathbb{Z}\}$ along the f-axis, and define a non equally-spaced grid $\mathcal{U}(v_0,\delta)$ along the u-axis by

$$u_0 \in \mathcal{U}(v_0,\delta) \quad \text{iff} \quad f(u_0,v_0) \in \mathcal{F}(\delta).$$

Given the Riemann data $P_l = (f_l, v_l), P_r = (f_r, v_r)$, with $f_l, f_r \in \mathcal{F}(\delta)$, we consider the equation

$$u_t + f^{\delta}(u,v_r)_x = 0, \tag{10}$$

where $f^{\delta}(\cdot\,, v_r)$ is the function which interpolates the curve $u \mapsto f(u,v_r)$ linearly between the points with f-values in $\mathcal{F}(\delta)$. The Riemann problem (P_l, P_r) is solved approximately in the (u,f)-plane as follows. Starting at P_l, follow the horizontal line $f = f_l$ until it meets the curve $u \mapsto f(u,v_r)$ at $P_m = (f_l, v_r)$. Then use the weak entropic solution of (10) with Riemann-initial data (f_l, f_r) as an approximate solution to the Riemann problem (P_m, P_r) for equation (2). This entropic solution of (10) is constructed by taking convex envelopes of $f^{\delta}(\cdot, v_r)$ as in scalar front-tracking [5, 9, 13]. Since we interpolate linearly, the approximate solutions constructed in this way contain only shocks satisfying the Rankine-Hugoniot condition.

Let $\mathcal{PC}(\delta)$ denote the family of pairs of piecewise constant functions $\big(u(x), v(x)\big)$ for which $f\big(u(x), v(x)\big) \in \mathcal{F}(\delta)$ for all $x \in \mathbb{R}$. The Cauchy problem for (2) with $(\bar{u},\bar{v}) \in \mathcal{PC}(\delta)$ is approximately solved in the following way. At $t = 0$ each Riemann problem is solved as indicated above. The fronts are prolonged until the first collision occurs, and the new Riemann problem is solved. The resulting fronts are tracked until the next collision takes place, etc.

Since we do not assume genuine nonlinearity, an interaction may produce a number of outgoing fronts larger than the number of incoming fronts. It is therefore not a priori clear that this algorithm yields globally defined approximate solutions. Also, notice that the total variation of $u(t,\cdot)$ could increase in time; more precisely it is non-increasing across interactions between u-waves, but it can increase across an interaction involving a v-front.

However, the total variation of the function $f(t) \doteq f\big(u(t,\cdot),\bar{v}(\cdot)\big)$ does not increase in time due to the fact that f is constant across a v-discontinuity. This implies that the number of fronts at each fixed time is a-priori bounded. It remains to prove that also the total number of interactions is finite. For this purpose, we introduce a function which decreases by a fixed amount for each collision that produces more than one outgoing u-wave.

Given a time $t > 0$ for which the approximate solution $u(t,x)$ is defined, let $\{x_\alpha(t)\}$ denote the set of discontinuity points of $u(t,\cdot)$, and let $\{y_\beta\}$ denote the points where $v(t,\cdot) = \bar{v}(\cdot)$ is discontinuous. We define the following functions (which depend on u and v)

$$\begin{aligned} R\big(x_\alpha(t)\big) &\doteq \#\big\{\text{jumps in } v \text{ to the right of } x_\alpha(t)\big\} + 1 \\ &= \#\big\{y_\beta\,;\ y_\beta > x_\alpha(t)\big\} + 1, \end{aligned} \tag{11}$$

and

$$W(t) \doteq \sum_\alpha \Big| f\big(U^r_\alpha(t)\big) - f\big(U^l_\alpha(t)\big)\Big| \cdot R\big(x_\alpha(t)\big), \tag{12}$$

where $U^l_\alpha(t) \doteq (u,v)(t, x_\alpha(t)-)$ and $U^r_\alpha(t) \doteq (u,v)(t,x_\alpha(t)+)$.

Lemma 2.1 *The function $W(t)$ is non-increasing (as long as $u(t,\cdot)$ is defined). Moreover, across every interaction with more than one outgoing u-wave it decreases by at least δ.*

Proof. It is clear that W is constant in time intervals where no interactions occur. Assume there is a collision at time τ. Let U_i^-, $i = 0,\dots,N$, be the states separating the incoming u-waves, and let U_j^+, $j = 0,\dots,M$, be the states separating the outgoing u-waves. There are two possible cases depending on whether a v-wave is involved or not. In both cases we have

$$\begin{aligned} \sum_{i=1}^N \big|f(U_i^-) - f(U_{i-1}^-)\big| &\geq \big|f(U_N^-) - f(U_0^-)\big| = \big|f(U_M^+) - f(U_0^+)\big| \\ &= \sum_{j=1}^M \big|f(U_j^+) - f(U_{j-1}^+)\big|. \end{aligned} \tag{13}$$

Here we use that all the outgoing u-jumps have the same sign. In the case where a v-wave is involved we also use that f is constant across the v-discontinuity. Since R decreases by one across v-waves and is constant elsewhere, this shows that W is non-increasing.

Now assume that there are more than one outgoing u-wave. In the first case where only u-waves are involved there are cancellations, and a similar estimate yields

$$\sum_{i=1}^N \big|f(U_i^-) - f(U_{i-1}^-)\big| \geq \sum_{j=1}^M \big|f(U_j^+) - f(U_{j-1}^+)\big| + 2\delta. \tag{14}$$

Thus $W(\tau+)-W(\tau-) \leq -2\delta$. In the second case where a v-wave is present, R decreases by one, and (13) shows that $W(\tau+) - W(\tau-) \leq -\delta$. This concludes the proof of the lemma. □

Since $W(0+)$ is finite, Lemma 2.1 shows that we can have at most $W(0+)/\delta$ interactions where the number of outgoing u-waves is larger than one. This together with hyperbolicity imply that the total number of interactions is finite. It follows that the approximate solution is defined for all positive times.

Notice also that these approximate solutions of (2) are indeed exact weak solutions of

$$u_t + f^\delta\big(u, \bar{v}(x)\big)_x = 0. \tag{15}$$

3 The Semigroup for Piecewise Constant v-data

Using the above construction we now introduce a corresponding approximate semigroup. Throughout this section $\bar{v}$ denotes a fixed piecewise constant function. Given $\delta > 0$, let $\mathcal{U}(\delta)$ denote the set of piecewise constant functions $\bar{u}(x)$ for which $f\big(\bar{u}(x), \bar{v}(x)\big) \in \mathcal{F}(\delta)$ for all x. For $\bar{u} \in \mathcal{U}(\delta)$ let $S_t^{\delta,\bar{v}}\bar{u}$ denote the *approximate* solution given by the above algorithm. It is clear that $S_t^{\delta,\bar{v}}$ is a semigroup mapping $\mathcal{U}(\delta)$ into itself. We use this semigroup to define a semigroup of *exact* weak solutions of (2) in the case of u-data in $\mathbf{L}^1 \cap \mathbf{L}^\infty$. We first treat the case where the u-data has bounded variation. We will need the following lemmas.

Lemma 3.1 *The semigroup $\bar{u} \mapsto S_t^{\delta,\bar{v}}\bar{u}$ satisfies*

$$\big\|S_t^{\delta,\bar{v}}\bar{u}_1 - S_t^{\delta,\bar{v}}\bar{u}_2\big\| \leq \|\bar{u}_1 - \bar{u}_2\|, \tag{16}$$

for any $\bar{u}_1, \bar{u}_2 \in \mathcal{U}(\delta)$ and $t > 0$.

Proof. Let $\overline{w}_1(x) \leq \overline{w}_2(x)$ be two functions in $\mathcal{U}(\delta)$. Since $S_t^{\delta,\bar{v}}\overline{w}_i$, $i = 1, 2$, are weak solutions of (15), it follows that $S_t^{\delta,\bar{v}}\overline{w}_1(x) \leq S_t^{\delta,\bar{v}}\overline{w}_2(x)$ for all $x \in \mathbb{R}$ and $t > 0$. Moreover

$$\int_{-\infty}^{\infty} S_t^{\delta,\bar{v}}\overline{w}_i(x)\,dx = \int_{-\infty}^{\infty} \overline{w}_i(x)\,dx, \qquad i = 1, 2, \quad t > 0. \tag{17}$$

Now, if we take $\overline{w}_1 \doteq \min\{\bar{u}_1, \bar{u}_2\}$ and $\overline{w}_2 \doteq \max\{\bar{u}_1, \bar{u}_2\}$, by monotonicity and (17) we get

$$\big\|S_t^{\delta,\bar{v}}\bar{u}_1 - S_t^{\delta,\bar{v}}\bar{u}_2\big\| \leq \big\|S_t^{\delta,\bar{v}}\overline{w}_1 - S_t^{\delta,\bar{v}}\overline{w}_2\big\| = \|\overline{w}_1 - \overline{w}_2\| = \|\bar{u}_1 - \bar{u}_2\|, \tag{18}$$

and this concludes the proof of the lemma. □

Lemma 3.2 *For $T > 0$ suppose that $u : [0, T] \to \mathcal{U}(\delta)$ is continuous with respect to the $\mathbf{L}^1$-norm. Then*

$$\left\| u(T) - S_T^{\delta,\bar{v}} u(0) \right\| \leq \int_0^T \limsup_{h \to 0} \frac{1}{h} \left\| u(t+h) - S_h^{\delta,\bar{v}} u(t) \right\| dt. \tag{19}$$

Proof. The proof of this lemma is similar to the one given in [4], and we refer to it. □

In order to define the semigroup $S \doteq \lim_{\delta \to 0} S^\delta$, we fix a sequence of grids along the f-axis. Let $\delta_n = 2^{-n}$ and define $\mathcal{F}_n \doteq \mathcal{F}(\delta_n)$, such that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for each n. Also let $\mathcal{U}_n \doteq \mathcal{U}(\delta_n)$. We will later show that the resulting semigroup is unique, and therefore independent of this particular sequence of grids.

Now, given $\bar{u} \in \mathbf{BV}$, choose a sequence $\{\bar{u}_n\}$ with $\bar{u}_n \in \mathcal{U}_n$ for each $n \geq 1$, such that $\bar{u}_n \to \bar{u}$ in $\mathbf{L}^1$ and with Tot.Var.$(\bar{u}_n) \leq M_0$ for every n, for a suitable constant M_0.

We prove that for all $T > 0$ the sequence $S_T^{\delta_n,\bar{v}} \bar{u}_n$ is a Cauchy sequence in $\mathbf{L}^1$. Take $m \leq n$. Since $\mathcal{U}_m \subset \mathcal{U}_n$ we can apply the approximate semigroup $S_t^{\delta_n,\bar{v}}$ to any function $u \in \mathcal{U}_m$. In particular, by applying Lemma 3.1 and Lemma 3.2 with $u(t) = S_t^{\delta_m,\bar{v}} \bar{u}_m$, it follows that

$$\begin{aligned} &\left\| S_T^{\delta_n,\bar{v}} \bar{u}_n - S_T^{\delta_m,\bar{v}} \bar{u}_m \right\| \\ &\qquad \leq \left\| S_T^{\delta_n,\bar{v}} \bar{u}_n - S_T^{\delta_n,\bar{v}} \bar{u}_m \right\| + \left\| S_T^{\delta_n,\bar{v}} \bar{u}_m - S_T^{\delta_m,\bar{v}} \bar{u}_m \right\| \\ &\qquad \leq \left\| \bar{u}_n - \bar{u}_m \right\| + \int_0^T \limsup_{h \to 0} \frac{1}{h} \left\| S_h^{\delta_m,\bar{v}} u(t) - S_h^{\delta_n,\bar{v}} u(t) \right\| dt. \end{aligned} \tag{20}$$

We want to estimate $\left\| S_h^{\delta_m,\bar{v}} u(t) - S_h^{\delta_n,\bar{v}} u(t) \right\|$ at any time t where no interactions occur in $u(t)$. Let $\{x_\alpha\}$ and $\{y_\beta\}$ be the sets of positions where $u(t)$ has a discontinuity across a u-wave or a v-wave, respectively. Call $u_\alpha^\pm = u\big(t, x_\alpha(t) \pm\big)$ and $u_\beta^\pm = u\big(t, y_\beta(t) \pm\big)$. For the Riemann problems (u_α^-, u_α^+) we have that

- the δ_m-approximate solution of the Riemann problem is given by a single u-wave front connecting the states u_α^-, u_α^+ and moving with speed $\lambda_\alpha \doteq \lambda\big(u_\alpha^-, u_\alpha^+; \bar{v}(x_\alpha)\big)$.
- the δ_n-approximate solution of the Riemann problem is given by possibly several u-waves connecting the states $u_\alpha^- \doteq u_\alpha^0, \ldots, u_\alpha^k, \ldots, u_\alpha^N \doteq u_\alpha^+$, where u_α^k are points in $\mathcal{U}\big(\bar{v}(x_\alpha), \delta_n\big)$ between u_α^- and u_α^+. The wave-front connecting the states u_α^{k-1} and u_α^k travels with shock-speed $\lambda_\alpha^k \doteq \lambda\big(u_\alpha^{k-1}, u_\alpha^k; \bar{v}(x_\alpha)\big)$, for $k = 1, \ldots, N$.

On the other hand, since $\mathcal{F}_m \subset \mathcal{F}_n$, the δ_n- and δ_m-approximate solutions of a Riemann problem (u_β^-, u_β^+) across a v-discontinuity coincide. Hence, for

$\rho, h > 0$ sufficiently small we have

$$\left\| S_h^{\delta_m,\bar v} u(t) - S_h^{\delta_n,\bar v} u(t) \right\| = \sum_\alpha \int_{x_\alpha-\rho}^{x_\alpha+\rho} \left| S_h^{\delta_m,\bar v} u(t,x) - S_h^{\delta_n,\bar v} u(t,x) \right| dx. \quad (21)$$

In the following computations we simplify the notation by writing $f(\cdot)$ for $f\big(\cdot\,,\bar v(x_\alpha)\big)$. Consider the α-th term in this sum. Note also that since $m \le n$ we have $\lambda_\alpha \in \big[\lambda_\alpha^{k_\alpha}, \lambda_\alpha^{k_\alpha+1}\big]$ for some k_α. We assume that $u_\alpha^0 < u_\alpha^N$ (the case $u_\alpha^0 > u_\alpha^N$ being similar), such that the α-th term in the above sum is given by

$$\begin{aligned} h\Big[\sum_{k=1}^{k_\alpha}(\lambda_\alpha - \lambda_\alpha^k)(u_\alpha^k - u_\alpha^{k-1}) + \sum_{k=k_\alpha+1}^{N}(\lambda_\alpha^k - \lambda_\alpha)(u_\alpha^k - u_\alpha^{k-1})\Big] \\ = h\Big[\lambda_\alpha(u_\alpha^{k_\alpha} - u_\alpha^0) - \big(f(u_\alpha^{k_\alpha}) - f(u_\alpha^0)\big)\Big] + \\ + h\Big[\big((f(u_\alpha^N) - f(u_\alpha^{k_\alpha})\big) - \lambda_\alpha(u_\alpha^N - u_\alpha^{k_\alpha})\Big]. \quad (22) \end{aligned}$$

The last two terms are both equal to $h(\lambda_\alpha - \hat\lambda_\alpha)(u_\alpha^{k_\alpha} - u_\alpha^0)$, where $\hat\lambda_\alpha \doteq \lambda\big(u_\alpha^0, u_\alpha^{k_\alpha}; \bar v(x_\alpha)\big)$.
By assumption we have that $f(u_\alpha^0), f(u_\alpha^N) \in \mathcal{F}_m$ while $f(u_\alpha^{k_\alpha}) \in \mathcal{F}_n \setminus \mathcal{F}_m$. Since $n \ge m$ we can find points $u', u'' \in [u_\alpha^0, u_\alpha^N] \cap \mathcal{U}\big(\bar v(x_\alpha), \delta_m\big)$ such that $u' < u_\alpha^{k_\alpha} < u''$ and $f(u'') - f(u') = 2^{-m}$. Let $\lambda' \doteq \lambda\big(u', u_\alpha^{k_\alpha}; \bar v(x_\alpha)\big)$ and $\lambda'' \doteq \lambda\big(u_\alpha^{k_\alpha}, u''; \bar v(x_\alpha)\big)$. Notice that the points $\big(u', f(u')\big)$ and $\big(u'', f(u'')\big)$ lie above the straight line through the points $\big(u_\alpha^0, f(u_\alpha^0)\big)$ and $\big(u_\alpha^N, f(u_\alpha^N)\big)$ (see Figure 1). Hence $\lambda' \le \hat\lambda_\alpha < \lambda_\alpha \le \lambda''$, such that

$$\lambda_\alpha - \hat\lambda_\alpha \le \lambda'' - \lambda' \le C(u'' - u') \le C \cdot 2^{-m}. \quad (23)$$

By (21)-(23) it follows that

$$\begin{aligned} \big\| S_h^{\delta_m,\bar v} u(t) &- S_h^{\delta_n,\bar v} u(t) \big\| \\ &= 2h \cdot \sum_\alpha |\lambda_\alpha - \hat\lambda_\alpha|\,|u_\alpha^{k_\alpha} - u_\alpha^0| \le hC2^{-m} \cdot \sum_\alpha |u_\alpha^+ - u_\alpha^-| \\ &\le hC2^{-m} \cdot \text{Tot.Var.}\big(u(t)\big). \quad (24) \end{aligned}$$

As noticed above, the total variation of $u(t) = S_t^{\delta_m,\bar v}\bar u_m$ could increase in time. However, the total variation of the function $f(t) \doteq f\big(u(t,\cdot), \bar v(\cdot)\big)$ does not increase in time. This implies that

$$\begin{aligned} \text{Tot.Var.}\big(u(t)\big) &\le C\big[\text{Tot.Var.}\big(f(t)\big) + \text{Tot.Var.}(\bar v)\big] \\ &\le C\big[\text{Tot.Var.}\big(f(0)\big) + \text{Tot.Var.}(\bar v)\big], \end{aligned}$$

which, by assumption, is bounded by some constant C'. We thus have

$$\left\| S_h^{\delta_m,\bar v} u(t) - S_h^{\delta_n,\bar v} u(t) \right\| \le hCC'2^{-m}. \quad (25)$$

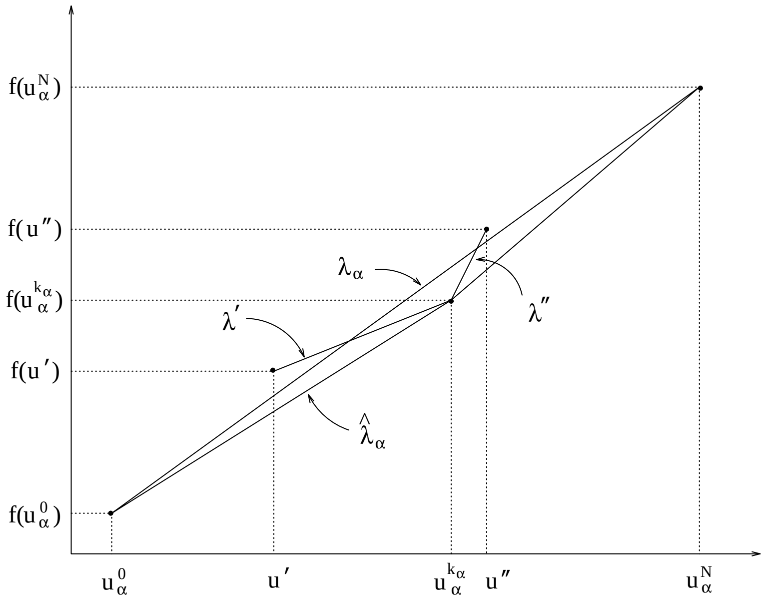

Figure 1

Using this estimate in (20) gives

$$\left\| S_T^{\delta_n,\bar v}\bar u_n - S_T^{\delta_m,\bar v}\bar u_m \right\| \le \|\bar u_n - \bar u_m\| + CC'T2^{-m}. \tag{26}$$

Since the sequence $\{\bar u_n\}$ converges to $\bar u$, we see that $\{S_T^{\delta_n,\bar v}\bar u_n\}$ is a Cauchy sequence in $\mathbf{L}^1$. The limit is denoted by $S_T^{\bar v}\bar u$ and we note that the map $x \mapsto \big(S_t^{\bar v}\bar u(x), \bar v(x)\big)$ is still in $\mathcal{D}$. Note that this definition does not depend on the choice of the sequence $\bar u_n \in \mathcal{U}_n$. Also, the semigroup is contractive, as follows by passing to the limit in (16).

This shows the continuity of $S_t^{\bar v}\bar u$ with respect to $\bar u \in \mathbf{BV}$. We can thus extend the definition of $S_t^{\bar v}\bar u$ to the case $\bar u \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ by letting

$$S_t^{\bar v}\bar u \doteq \mathbf{L}^1\text{–}\lim_{n\to\infty} S_t^{\bar v}\bar u_n, \tag{27}$$

where $\{\bar u_n\}$ is any sequence of functions with bounded variation, converging to $\bar u$. By the $\mathbf{L}^1$-contractivity of the semigroup S on $\mathbf{BV}$, this limit is well-defined, and the resulting semigroup is also contractive.

4 The Semigroup for general v-data

In this section we extend the semigroup to the case where also the v-component of the initial data lies in $\mathbf{L}^1 \cap \mathbf{L}^\infty$, and we establish part (a) of Theorem 1.1. First, we notice that the approximate semigroups are jointly continuous with respect to (u, v). More precisely we have the following result.

Lemma 4.1 *Let $\delta > 0$ be fixed. Let $\{(\bar{u}_n, \bar{v}_n)\}$ be a sequence of $\mathcal{PC}(\delta)$-functions of bounded variation converging in $\mathbf{L}^1$ to $(\bar{u}, \bar{v}) \in \mathcal{PC}(\delta)$ as $n \to \infty$. Then for every $t > 0$ one has*

$$S_t^{\delta, \bar{v}_n} \bar{u}_n \to S_t^{\delta, \bar{v}} \bar{u}, \quad \textit{in } \mathbf{L}^1 \quad \textit{as} \quad n \to \infty. \tag{28}$$

Proof. Referring to Section 5, for any $(\bar{u}, \bar{v}) \in \mathcal{PC}(\delta)$, the function $(t, x) \mapsto S_t^{\delta, \bar{v}} \bar{u}(x)$ is an entropy admissible solution of (15). Moreover it is clear that $\mathbf{L}^1_{loc}$-limits of sequences of entropy admissible solutions are entropy admissible solutions. By Helly's theorem, passing to subsequences if necessary, we can assume that $S_t^{\delta, \bar{v}_n} \bar{u}_n$ converges in $\mathbf{L}^1_{loc}(\mathbb{R}^+ \times \mathbb{R}; \mathbb{R})$ to a function $w(t, x)$, which is again an entropy admissible solution of (15). Since $w(0, x) = \bar{u}(x)$, by the uniqueness result stated in Section 5, it follows that $w(t, x)$ coincides with $S_t^{\delta, \bar{v}} \bar{u}$. This completes the proof of the lemma. □

The proof of Theorem 1.1 is based on the following fundamental estimate.

Proposition 4.2 *Let $\hat{f}$ be a piecewise constant function with values in $\mathcal{F}_{n_0}$, and let $U \doteq (u, v)$, $U^* \doteq (u^*, v^*)$ be two pairs of piecewise constant initial data satisfying the relation*

$$f(U(x)) = f(U^*(x)) \doteq \hat{f}(x), \qquad \forall\, x \in \mathbb{R}. \tag{29}$$

Then for each $T > 0$ and $n \geq n_0$ one has

$$\left\| S_T^{\delta_n, v} u - S_T^{\delta_n, v^*} u^* \right\| \leq C_1 \cdot \left(1 + \text{Tot.Var. } (\hat{f})\right) \|v - v^*\|, \tag{30}$$

where C_1 is a constant independent of n. In particular, by passing to the limit $n \to \infty$ in (30), we obtain

$$\left\| S_T^{v} u - S_T^{v^*} u^* \right\| \leq C_1 \cdot \left(1 + \text{Tot.Var. } (\hat{f})\right) \|v - v^*\|. \tag{31}$$

Proof. Fix $n \geq n_0$ and let $\delta = \delta_n$. To estimate the distance between two solutions, we follow the approach in [6], estimating the length of a path joining u with u^*. We recall some definitions. A pseudopolygonal is a continuous curve $\Gamma : [a, b] \mapsto \mathbf{L}^1$ for which there is a finite partition $\{(\theta_i, \theta_{i+1})\}$ such that Γ on each interval is given by

$$\Gamma(\theta) = \sum_\alpha \omega_\alpha \chi_{(x^\theta_{\alpha-1}, x^\theta_\alpha]}, \qquad x^\theta_\alpha = x_\alpha + \xi_\alpha \theta,$$

where χ_A denotes the characteristic function of the set A. Here the states ω_α are fixed and the positions of the jumps x^θ_α shift at constant rates ξ_α as θ varies. For a full description of this technique see [6].

Consider the continuous curve $\Gamma : \theta \mapsto U^\theta \doteq (u^\theta, v^\theta)$ in $\mathbf{L}^1$ given by

$$U^\theta \doteq U^* \cdot \chi_{(-\infty, \theta]} + U \cdot \chi_{(\theta, \infty)}. \tag{32}$$

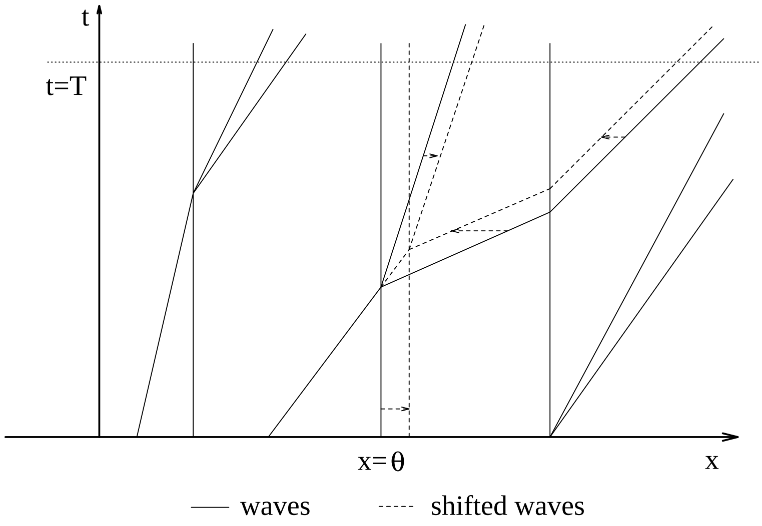

Figure 2: wave-front configuration

The curve Γ is a pseudopolygonal connecting U and U^* and which takes values in $\mathcal{PC}(\delta)$. For each fixed θ we consider the corresponding solution obtained by performing wave-front tracking on U^θ and define $\gamma_t(\theta) \doteq S_t^{\delta,v^\theta} u^\theta$. By the way the front-tracking algorithm is defined we have that for every $t > 0$ the map $\theta \mapsto \gamma_t(\theta)$ is a pseudopolygonal connecting $S_t^{\delta,v} u$ and $S_t^{\delta,v^*} u^*$, such that $\big(\gamma_t(\theta)(x), v^\theta(x)\big) \in \mathcal{F}(\delta)$. Indeed, the continuity of the map $\theta \mapsto \gamma_t(\theta)$ was proved in Lemma 4.1.

Note that as θ increases, at time $t = 0$ there is always one v-discontinuity located at $x = \theta$ shifting with speed $\xi = 1$. As t increases, the u-waves from the left of $x = \theta$ will interact with this discontinuity and will be shifted. Thus at times $t > 0$ there can be more than one wave shifting: the moving v-discontinuity and also some u-fronts located to the right of $x = \theta$ (see Figure 2).

Let $T > 0$ be given. We have that

$$\left\| S_T^{\delta,v} u - S_T^{\delta,v^*} u^* \right\| \leq \mathbf{L}^1\text{-length of } \gamma_T = \int_{-\infty}^{+\infty} \Xi(\theta, T)\, d\theta, \tag{33}$$

where

$$\Xi(\theta, T) \doteq \sum_\alpha |\Delta_\alpha u^\theta| \cdot |\xi_\alpha|.$$

Here the sum is over all the discontinuities of $\gamma_T(\theta)$ located at x_α with corresponding u-jumps $|\Delta_\alpha u^\theta|$ and shifts $|\xi_\alpha|$. Note that $\Xi(\theta, T)$ does not contain any terms corresponding to discontinuities to the left of the line $x = \theta$ since none of these are shifted.

For almost every θ the wave-front configuration in $\gamma_t(\theta)$ remains the same for $t \in [0, T]$ and θ-values close to θ. Fix θ to be one of these values. We can also assume that no interactions occur in $u^\theta(T, \cdot)$.

It is clear that, for θ fixed, $\Xi(\theta, \cdot)$ remains constant in any time interval where no collisions occur. However, $\Xi(\theta, \cdot)$ may change across interactions. Now, if the interaction occurs in the region $x > \theta$, then only u-waves and possibly a non-shifted v-wave are involved. By $\mathbf{L}^1$-contractivity of the map $u \mapsto S_t^{\delta,v} u$ it follows that $\Xi(\theta, \cdot)$ decreases across these collisions as time increases.

If the collision occurs on the line $x = \theta$ an additional analysis is needed. Denote the times of interaction along the line $x = \theta$ by $0 < t_1 < \cdots < t_p$, and consider one of these where a single u-wave interacts with the shifting v-wave from the left. Note that since only the v-wave is shifting we may assume that all the collisions along $x = \theta$ involve a *single* u-wave only.

Fix a t_β and let $(u^l, v^-), (u^r, v^-)$ be the left and right states of the incoming u-wave, and let (u_i^+, v^+), $i = 0, \ldots, M$, be the states separating the outgoing u-waves. Note that $v^- = v^*(\theta)$ and $v^+ = v(\theta)$. Let the incoming wave have speed λ^- while the outgoing waves have speeds λ_i^+ and shifts ξ_i^+, $i = 1, \ldots, M$.

Since the v-discontinuity shifts with speed $\xi = 1$ we have

$$\xi_i^+ = 1 - \frac{\lambda_i^+}{\lambda^-}. \tag{34}$$

Put $\Delta_i u^+ \doteq u_i^+ - u_{i-1}^+$ and $f_i^+ \doteq f(u_i^+, v^+)$. Assume that $\lambda_i^+ \leq \lambda^-$ for $i = 1, \ldots, k$, and $\lambda_i^+ > \lambda^-$ for $i = k+1, \ldots, M$. The u-jumps across the shifted v-wave before and after the interactions are denoted by σ^- and σ^+, respectively. Since all the jumps $\Delta_i u^+$ have the same sign, we can assume also that $\Delta_i u^+ > 0$ for all i, a similar analysis holding if $\Delta_i u^+ < 0$ for all i. It follows that the sign of $\Delta_i u^+ \cdot \xi_i^+$ is positive for $i = 1, \ldots, k$, and negative for $i = k+1, \ldots, M$. If the interaction occurs at time t_β we thus have

$$\begin{aligned}
&\Xi(\theta, t_\beta+) - \Xi(\theta, t_\beta-) \\
&\quad = |\sigma^+| - |\sigma^-| + \sum_{i=1}^{M} \left|1 - \frac{\lambda_i^+}{\lambda^-}\right| |\Delta_i u^+| \\
&\quad \leq \left|(u_0^+ - u^l) - (u_M^+ - u^r)\right| + \\
&\qquad + \sum_{i=1}^{k} \Delta_i u^+ \left(1 - \frac{\lambda_i^+}{\lambda^-}\right) - \sum_{i=k+1}^{M} \Delta_i u^+ \left(1 - \frac{\lambda_i^+}{\lambda^-}\right) \\
&\quad = \left|(u_M^+ - u_0^+) - (u^r - u^l)\right| + \\
&\qquad + \left((u_k^+ - u_0^+) - \frac{f_k^+ - f_0^+}{\lambda^-}\right) - \left((u_M^+ - u_k^+) - \frac{f_M^+ - f_k^+}{\lambda^-}\right).
\end{aligned} \tag{35}$$

Now consider the point u^m defined through the relation $f(u^m, v^-) = f_k^+$, and define $\lambda^* \doteq \lambda(u^l, u^m; v^-)$ and $\lambda^{**} \doteq \lambda(u^m, u^r; v^-)$. Then $\lambda^* \geq \lambda^- \geq \lambda^{**}$

such that

$$
\begin{aligned}
&\Xi(\theta, t_\beta+) - \Xi(\theta, t_\beta-) \\
&\quad \leq \big|(u_M^+ - u_0^+) - (u^r - u^l)\big| + \\
&\qquad + \left((u_k^+ - u_0^+) - \frac{f_k^+ - f_0^+}{\lambda^*}\right) - \left((u_M^+ - u_k^+) - \frac{f_M^+ - f_k^+}{\lambda^{**}}\right) \\
&\quad = \big|(u_M^+ - u_0^+) - (u^r - u^l)\big| + \\
&\qquad + \big((u_k^+ - u_0^+) - (u^m - u^l)\big) - \big((u_M^+ - u_k^+) - (u^r - u^m)\big).
\end{aligned} \tag{36}
$$

To estimate the terms in this expression we recall that the change of coordinates $(u, v) \leftrightarrow (f, v)$ implies that $u = G(f, v)$ with G of class $\mathcal{C}^2$. For every $f_1, f_2 \in K_1$ and $v \in K_2$, we can thus write

$$
G(f_2, v) - G(f_1, v) = \int_0^1 \partial_1 G\big(sf_2 + (1-s)f_1, v\big)(f_2 - f_1)\, ds. \tag{37}
$$

By (37) it follows that

$$
\begin{aligned}
&(u_k^+ - u_0^+) - (u^m - u^l) \\
&\quad = \big(G(f_k^+, v^+) - G(f_0^+, v^+)\big) - \big(G(f_k^+, v^-) - G(f_0^+, v^-)\big) \\
&\quad = (f_k^+ - f_0^+) \cdot \\
&\qquad \cdot \int_0^1 \Big(\partial_1 G\big(sf_k^+ + (1-s)f_0^+, v^+\big) - \partial_1 G\big(sf_k^+ + (1-s)f_0^+, v^-\big)\Big)\, ds \\
&\quad = (f_k^+ - f_0^+)(v^+ - v^-) \cdot \\
&\qquad \cdot \int_0^1 \int_0^1 \partial_2 \partial_1 G\big(sf_k^+ + (1-s)f_0^+, \tau v^+ + (1-\tau)v^-\big)\, ds d\tau \\
&\quad \leq C|v^+ - v^-|\,(f_k^+ - f_0^+),
\end{aligned} \tag{38}
$$

for some constant C depending on the max of $\partial_2\partial_1 G$. Similar computations show that

$$
(u_M^+ - u_k^+) - (u^r - u^m) \leq C|v^+ - v^-|\,(f_M^+ - f_k^+), \tag{39}
$$

and that

$$
\big|(u_M^+ - u_0^+) - (u^r - u^l)\big| \leq C|v^+ - v^-|\,(f_M^+ - f_0^+). \tag{40}
$$

By (36)-(40) we get

$$
\begin{aligned}
\Xi(\theta, t_\beta+) \;-\; &\Xi(\theta, t_\beta-) \\
&\leq C\big|v^+ - v^-\big|\Big(\big|f_M^+ - f_0^+\big| + \big|f_k^+ - f_0^+\big| + \big|f_M^+ - f_k^+\big|\Big) \\
&= 2C\big|v^+ - v^-\big|\,\big|f_M^+ - f_0^+\big| = 2C\big|v^+ - v^-\big|\,\big|\Delta f(t_\beta)\big|,
\end{aligned} \tag{41}
$$

where $\Delta f(t_\beta) \doteq f_M^+ - f_0^+$. This estimate holds across each interaction along the line $x = \theta$ corresponding to a time $t_\beta > 0$. At $t = 0+$ only the v-wave is shifted with corresponding u-jump $\big|u(x=\theta)-u^*(x=\theta)\big|$ which by strict hyperbolicity is bounded by $C|v^+-v^-|$. For $T>0$ we thus obtain the following estimate

$$\begin{aligned}\Xi(\theta,T) &= \Xi(\theta,0+) + \sum_{0<t_\beta<T}\big(\Xi(\theta,t_\beta+)-\Xi(\theta,t_\beta-)\big)\\ &\le C\big|v^*(\theta)-v(\theta)\big|\bigg(1+\sum_{0<t_\beta<T}\big|\Delta f(t_\beta)\big|\bigg)\\ &\le C\big|v^*(\theta)-v(\theta)\big|\bigg(1+\sum_{\beta=1}^{p}\big|\Delta f(t_\beta)\big|\bigg).\end{aligned} \tag{42}$$

Notice that for $t\ge t_p$ only v-waves are present to the left of $x=\theta$.

We finally relate the sum in (42) to the variation of $\hat f$ along the half line $x\le\theta$. For $t>0$ let $u^*(t,x)\doteq S_t^{\delta,v^*}u^*(x)$ and consider the variation $\Lambda(t)$ of $f\big(u^*(t,\cdot),v^*(\cdot)\big)$ on the interval $(-\infty,\theta]$. It is clear that $\Lambda(t)$ is constant in every time interval where no interactions occur in $u^*(t,x)$ in the region $t>0$, $x\le\theta$. Suppose there is an interaction at time $t>0$ for $x<\theta$. If the interaction involves more than one u-wave a cancellation may take place and since f is constant across v-waves, we have that

$$\Lambda(t+)\le\Lambda(t-).$$

Thus $\Lambda(t)$ is non-increasing across these interactions. As time increases the u-waves in the region $x<\theta$ will eventually cross the shifting v-discontinuity at $x=\theta$. Again fix one of these interaction times t_β. We see that the term in $\Lambda(t_\beta-)$ corresponding to the incoming u-wave is exactly equal to the term $\big|\Delta f(t_\beta)\big|$ in the above sum. We thus have

$$\Lambda(t_\beta+)-\Lambda(t_\beta-)=-\big|\Delta f(t_\beta)\big|, \tag{43}$$

and

$$\begin{aligned}\sum_{\beta=1}^{p}\big|\Delta f(t_\beta)\big| &= -\sum_{\beta=1}^{p}\big(\Lambda(t_\beta+)-\Lambda(t_\beta-)\big)\le\Lambda(0)-\Lambda(t_p+)=\Lambda(0)\\ &= \text{Tot.Var.}\Big(\hat f\big|_{(-\infty,\theta]}\Big)\le\text{Tot.Var.}(\hat f).\end{aligned} \tag{44}$$

This and (42) imply that

$$\Xi(\theta,T)\le C\big|v^*(\theta)-v(\theta)\big|\,\big(1+\text{Tot.Var.}(\hat f)\big), \tag{45}$$

which together with (33) yields the conclusion of the proposition. □

Using this proposition we can now prove part (a) of Theorem 1.1, while part (b) is postponed to the next section. As a preliminary, we state the following lemma.

Lemma 4.3 *Let $(\bar u, \bar v) \in \mathcal{D}$ be given. Then for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ and a piecewise constant function $\hat f$ with values in $\mathcal{F}_{n_0}$ such that for every pair of piecewise constant functions v, v^* the following holds*

$$\begin{aligned} &\left\| S_t^v \bar u - S_t^{v^*} \bar u \right\| \\ &\quad \le C \Big\{ \varepsilon + \|v - \bar v\| + \|v^* - \bar v\| + \big(1 + \text{Tot.Var.}(\hat f)\big) \, \|v - v^*\| \Big\}. \end{aligned} \tag{46}$$

Proof. Let $\bar f(x) \doteq f\big(\bar u(x), \bar v(x)\big)$. Fix $\varepsilon > 0$ and choose n_0 such that there exists a piecewise constant function $\hat f$ with values in $\mathcal{F}_{n_0}$ and with

$$\left\| \hat f - \bar f \right\| < \varepsilon. \tag{47}$$

Take two piecewise constant functions v, v^* and define u, u^* through the relation

$$f\big(u(x), v(x)\big) = f\big(u^*(x), v^*(x)\big) = \hat f(x), \qquad \forall\, x \in \mathbb{R}.$$

By hyperbolicity, one has

$$\|u - \bar u\| \le C\Big(\left\|\hat f - \bar f\right\| + \|v - \bar v\|\Big) \le C\big(\varepsilon + \|v - \bar v\|\big), \tag{48}$$

and similarly

$$\|u^* - \bar u\| \le C\big(\varepsilon + \|v^* - \bar v\|\big). \tag{49}$$

By Proposition 4.2, (48), (49) and the $\mathbf{L}^1$-contractivity of S_t^v, it follows that

$$\begin{aligned} &\left\| S_t^v \bar u - S_t^{v^*} \bar u \right\| \\ &\quad \le \left\| S_t^v \bar u - S_t^v u \right\| + \left\| S_t^v u - S_t^{v^*} u^* \right\| + \left\| S_t^{v^*} u^* - S_t^{v^*} \bar u \right\| \\ &\quad \le \|\bar u - u\| + \left\| S_t^v u - S_t^{v^*} u^* \right\| + \|u^* - \bar u\| \\ &\quad \le C \Big\{ \varepsilon + \|v - \bar v\| + \|v^* - \bar v\| + \big(1 + \text{Tot.Var.}(\hat f)\big) \, \|v - v^*\| \Big\}, \end{aligned} \tag{50}$$

which completes the proof. □

In particular, take a sequence of piecewise constant functions $\{v_n\}$ converging to $\bar v$ in $\mathbf{L}^1$. Given $\varepsilon > 0$, by Lemma 4.3 there exists $\hat f$ such that for every n and m one has

$$\begin{aligned} &\left\| S_t^{v_n} \bar u - S_t^{v_m} \bar u \right\| \\ &\quad \le C \Big\{ \varepsilon + \|v_n - \bar v\| + \|v_m - \bar v\| + \big(1 + \text{Tot.Var.}(\hat f)\big) \, \|v_n - v_m\| \Big\}. \end{aligned} \tag{51}$$

Since $v_n \to \bar v$ this shows that the sequence $\big\{ S_t^{v_n} \bar u \big\}$ is Cauchy. Hence we can extend the semigroup to the case when $(\bar u, \bar v) \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ by letting

$$S_t^{\bar v} \bar u \doteq \mathbf{L}^1 - \lim_{n \to \infty} S_t^{v_n} \bar u. \tag{52}$$

By the previous analysis this limit is well-defined. Moreover, the map $\bar{u} \mapsto S_t^{\bar{v}}\bar{u}$ is again a contraction. Finally we define

$$\mathbb{S}_t(\bar{u},\bar{v}) \doteq \big(S_t^{\bar{v}}\bar{u},\bar{v}\big). \tag{53}$$

We claim that the semigroup $\mathbb{S}$ is jointly continuous in (u,v). Indeed, given another point $(\tilde{u},\tilde{v}) \in \mathcal{D}$, choose sequences $\{v_n\}$, $\{\tilde{v}_n\}$ of piecewise constant functions such that $v_n \to \bar{v}$ and $\tilde{v}_n \to \tilde{v}$ in $\mathbf{L}^1$ as $n \to \infty$. By Lemma 4.3 we obtain

$$\begin{aligned}
&\big\|S_t^{v_n}\bar{u} - S_t^{\tilde{v}_n}\tilde{u}\big\| \\
&\qquad \le \big\|S_t^{v_n}\bar{u} - S_t^{\tilde{v}_n}\bar{u}\big\| + \big\|S_t^{\tilde{v}_n}\bar{u} - S_t^{\tilde{v}_n}\tilde{u}\big\| \\
&\qquad \le C\big\{\varepsilon + \|v_n - \bar{v}\| + \|\tilde{v}_n - \bar{v}\| + \big(1 + \text{Tot.Var.}(\hat{f})\big)\,\|v_n - \tilde{v}_n\|\big\} + \\
&\qquad\qquad + \|\bar{u} - \tilde{u}\|.
\end{aligned} \tag{54}$$

By passing to the limit $n \to \infty$ in this last estimate, we finally get

$$\big\|S_t^{\bar{v}}\bar{u} - S_t^{\tilde{v}}\tilde{u}\big\| \le \|\bar{u} - \tilde{u}\| + C\Big\{\varepsilon + \big(2 + \text{Tot.Var.}(\hat{f})\big)\,\|\bar{v} - \tilde{v}\|\Big\}. \tag{55}$$

This shows that for every $(\tilde{u},\tilde{v}) \in \mathcal{D}$ with

$$\|\bar{u} - \tilde{u}\| \le \varepsilon, \qquad \|\bar{v} - \tilde{v}\| \le \frac{\varepsilon}{C\big(2 + \text{Tot.Var.}(\hat{f})\big)},$$

one has

$$\big\|S_t^{\bar{v}}\bar{u} - S_t^{\tilde{v}}\tilde{u}\big\| \le C\varepsilon. \tag{56}$$

This completes the proof of part (a) of Theorem 1.1.

5 Entropy Conditions and Uniqueness

In this section we formulate an entropy condition for the system (2) which yields uniqueness of entropy admissible solutions in the same way as the Kružkov condition for scalar equations [17]. Since the second equation is solved uniquely by the function $v(t,x) = \bar{v}(x)$, this amounts to give a condition for the scalar conservation law

$$u_t + F(u,x)_x = 0, \tag{57}$$

where the flux function $F(u,x) \doteq f\big(u,\bar{v}(x)\big)$ depends explicitly on x. Kružkov's original formulation is not meaningful if this dependence is discontinuous. In the case where F does not depend on x we have that the constants are stationary solutions of (57). In our situation we see that the stationary solutions are those functions $u(x)$ satisfying $F\big(u(x),x\big) \equiv \text{const.}$

These solutions play the role of the constants in the Kružkov formulation. This motivates the following construction.

For any fixed $x \in \mathbb{R}$ the function $u \mapsto F(u,x)$ is one to one, hence for every constant $\zeta \in \mathbb{R}$ we can define the function $u^\zeta(x)$ through the relation

$$F\big(u^\zeta(x), x\big) = \zeta, \qquad \forall\, x \in \mathbb{R}. \tag{58}$$

We introduce the following entropy-entropy flux pair $\big(\eta^\zeta(u,x), q^\zeta(u,x)\big)$ (which depends also on the variable x) given by

$$\begin{aligned} \eta^\zeta(u,x) &\doteq \big|u - u^\zeta(x)\big| \\ q^\zeta(u,x) &\doteq \big|F(u,x) - \zeta\big|. \end{aligned} \tag{59}$$

Fix $(\bar{u}, \bar{v})$. We say that a continuous function $t \mapsto u(t,\cdot)$ from $[0,+\infty)$ to $\mathbf{L}^1_{loc}$ is an *entropy admissible solution* of system (2)-(3) if for every $\zeta \in \mathbb{R}$ the following holds

$$\eta^\zeta\big(u(t,x),x\big)_t + q^\zeta\big(u(t,x),x\big)_x \leq 0, \tag{60}$$

in distributional sense. That is, for any nonnegative $\mathcal{C}^1$ function ϕ with compact support in $(0,+\infty)\times\mathbb{R}$ we have

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \Big\{ \big|u(t,x) - u^\zeta(x)\big|\phi_t(t,x) + \\ + \big|F\big(u(t,x),x\big) - \zeta\big|\phi_x(t,x)\Big\}\, dxdt \geq 0. \end{aligned} \tag{61}$$

Notice that since $F(\cdot,x)$ is increasing, $\mathrm{sign}\big(F(u,x)-\zeta\big) = \mathrm{sign}\big(u - u^\zeta(x)\big)$, and we have

$$q^\zeta(u,x) = \mathrm{sign}\big(u - u^\zeta(x)\big)\big[F(u,x) - \zeta\big].$$

In particular, when f does not depend on v we recover the well-known Kružkov entropy condition [17].
Moreover by taking $\zeta = \pm \sup_{(t,x)} \big|f\big(u(t,x), \bar{v}(x)\big)\big|$ we see that also in our case a bounded entropy admissible solution is a weak solution.

The following lemma yields an alternative characterization of entropy admissibility in the case of a piecewise $\mathcal{C}^1$ solution.

Lemma 5.1 *Assume that $\big(u(t,x), \bar{v}(x)\big)$ is a piecewise $\mathcal{C}^1$ weak solution of (2) having discontinuities only along a finite number of piecewise Lipschitz continuous curves, say $x_\alpha = x_\alpha(t)$. Denote by $u_\alpha^\pm \doteq u\big(t, x_\alpha(t)\pm\big)$ and $v_\alpha^\pm \doteq \bar{v}\big(x_\alpha(t)\pm\big)$.*
Then $u(t,x)$ is an entropy admissible solution if and only if at each discontinuity point $\big(t, x_\alpha(t)\big)$

- *either $\dot{x}_\alpha(t) = 0$ and*

$$f\big(u_\alpha^-, v_\alpha^-\big) = f\big(u_\alpha^+, v_\alpha^+\big); \tag{62}$$

- *or $\dot x_\alpha(t) > 0$. In this case $v_\alpha^+ = v_\alpha^- \doteq v_\alpha$ and for any $s \in [0,1]$, if $u_\alpha^- < u_\alpha^+$ one has*

$$f\big(su_\alpha^+ + (1-s)u_\alpha^-, v_\alpha\big) \geq sf\big(u_\alpha^+, v_\alpha\big) + (1-s)f\big(u_\alpha^-, v_\alpha\big), \tag{63}$$

whereas if $u_\alpha^- > u_\alpha^+$ one has

$$f\big(su_\alpha^+ + (1-s)u_\alpha^-, v_\alpha\big) \leq sf\big(u_\alpha^+, v_\alpha\big) + (1-s)f\big(u_\alpha^-, v_\alpha\big). \tag{64}$$

Proof. Fix $\zeta \in \mathbb{R}$ and take a nonnegative $\mathcal{C}^1$ function ϕ with compact support in $[0,T] \times [a,b]$. By the divergence theorem we obtain that

$$\begin{aligned} 0 &\leq \int_0^\infty \int_{-\infty}^\infty \Big\{\eta^\zeta\big(u(t,x),x\big)\phi_t(t,x) + q^\zeta\big(u(t,x),x\big)\phi_x(t,x)\Big\}\,dxdt \\ &= \int_0^\infty \int_{-\infty}^\infty \Big\{\eta^\zeta\big(u(t,x),x\big)_t + q^\zeta\big(u(t,x),x\big)_x\Big\}\phi(t,x)\,dxdt + \\ &\qquad + \sum_\alpha \int_0^T \Big\{\dot x_\alpha(t)\Delta\eta^\zeta\big(x_\alpha(t)\big) - \Delta q^\zeta\big(x_\alpha(t)\big)\Big\}\phi\big(t,x_\alpha(t)\big)\,dt, \end{aligned} \tag{65}$$

where $\Delta\eta^\zeta\big(x_\alpha(t)\big) \doteq \eta^\zeta\big(u_\alpha^+, x_\alpha(t)+\big) - \eta^\zeta\big(u_\alpha^-, x_\alpha(t)-\big)$ and similarly for $\Delta q^\zeta\big(x_\alpha(t)\big)$. Since $(u,\bar v)$ is piecewise $\mathcal{C}^1$ the first integrand on the right-hand side of (65) is zero a.e., thus the last inequality holds if and only if

$$\dot x_\alpha(t)\Delta\eta^\zeta\big(x_\alpha(t)\big) - \Delta q^\zeta\big(x_\alpha(t)\big) \geq 0, \tag{66}$$

for all $t > 0$, α and ζ. Now, if $\dot x_\alpha(t) = 0$ then (66) is equivalent to

$$0 \geq \big|f(u_\alpha^+, v_\alpha^+) - \zeta\big| - \big|f(u_\alpha^-, v_\alpha^-) - \zeta\big|. \tag{67}$$

Since $(u,\bar v)$ has a v-discontinuity located at x_α, by the Rankine-Hugoniot conditions it follows that (67) is equivalent to (62).

If, instead, $\dot x_\alpha(t) > 0$ then the solution has a u-discontinuity at $x_\alpha(t)$, hence $v_\alpha^+ = v_\alpha^- \doteq v_\alpha$. In this case $u^\zeta\big(x_\alpha(t)+\big) = u^\zeta\big(x_\alpha(t)-\big) \doteq h$, so that $\zeta = f(h, v_\alpha)$. Recalling that $\dot x_\alpha\big(u_\alpha^+ - u_\alpha^-\big) = f(u_\alpha^+, v_\alpha) - f(u_\alpha^-, v_\alpha)$, from (66) one gets

$$\begin{aligned} &\big(f(u_\alpha^+, v_\alpha) - f(u_\alpha^-, v_\alpha)\big)\big(u_\alpha^+ + u_\alpha^- - 2h\big) \\ &\qquad \geq \big(u_\alpha^+ - u_\alpha^-\big)\big(f(u_\alpha^+, v_\alpha) + f(u_\alpha^-, v_\alpha) - 2f(h, v_\alpha)\big). \end{aligned} \tag{68}$$

If we choose $h = su_\alpha^+ + (1-s)u_\alpha^-$, we recover conditions (63) and (64). $\square$

In particular, the approximate solutions constructed by the wave-front tracking algorithm are entropy admissible solutions of (15).

Now we state the main theorem of this section.

Theorem 5.2 *Let $\bar{v} \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ be fixed and define $F(u,x) \doteq f\big(u, \bar{v}(x)\big)$. Let $u(t,x)$ and $w(t,x)$ be two bounded entropy admissible weak solutions of*

$$u_t + F(u,x)_x = 0.$$

Let $M, L > 0$ be constants such that

$$\big|u(t,x)\big| \le M, \qquad \big|w(t,x)\big| \le M, \qquad\qquad \forall\, t, x,$$

$$\big|F(u',x) - F(u'',x)\big| \le L|u' - u''|, \qquad\qquad \forall\, u', u'' \in [-M, M], \quad \forall\, x \in \mathbb{R}.$$

Then for every $R > 0$ and $\tau' \ge \tau \ge 0$, one has

$$\int_{|x|\le R} \big|u(\tau,x) - w(\tau,x)\big|\, dx \ge \int_{|x|\le R - L(\tau'-\tau)} \big|u(\tau',x) - w(\tau',x)\big|\, dx. \tag{69}$$

Proof. The proof is similar to the one in [5, 17]. For any $\xi, \zeta \in \mathbb{R}$ and for any nonnegative $\mathcal{C}^1$ function $\phi = \phi(s,x,t,y)$ with compact support in $s \ge 0$, $t \ge 0$, we have

$$\int_0^\infty \int_{-\infty}^\infty \Big\{ \big|u(t,x) - u^\xi(x)\big|\phi_t + \big|F\big(u(t,x),x\big) - \xi\big|\phi_x \Big\}\, dxdt \ge 0, \tag{70}$$

$$\int_0^\infty \int_{-\infty}^\infty \Big\{ \big|w(s,y) - u^\zeta(y)\big|\phi_s + \big|F\big(w(s,y),y\big) - \zeta\big|\phi_y \Big\}\, dyds \ge 0. \tag{71}$$

Define the functions $\tilde{w}(s,y;x)$ and $\tilde{u}(t,x;y)$ through the relations

$$F\big(\tilde{w}(s,y;x),x\big) = F\big(w(s,y),y\big), \qquad F\big(\tilde{u}(t,x;y),y\big) = F\big(u(t,x),x\big). \tag{72}$$

Put $\xi = F\big(w(s,y),y\big)$ in (70) and $\zeta = F\big(u(t,x),x\big)$ in (71). Integrating the first equation w.r.t. s, y, the second w.r.t. t, x and adding the results one gets

$$\begin{aligned} \iiiint \Big\{ &\big|u(t,x) - \tilde{w}(s,y;x)\big|\phi_t + \big|w(s,y) - \tilde{u}(t,x;y)\big|\phi_s + \\ &+ \big|F\big(u(t,x),x\big) - F\big(w(s,y),y\big)\big|\,(\phi_x + \phi_y)\Big\}\, dxdydtds \ge 0. \end{aligned} \tag{73}$$

Now, take a sequence of $\mathcal{C}^\infty$ functions $\{\delta_\nu\}_{\nu\ge1}$ approximating the Dirac delta at the origin. More precisely take $\delta_\nu : \mathbb{R} \mapsto [0, +\infty)$ such that

$$\int_{-\infty}^\infty \delta_\nu(z)\, dz = 1, \qquad \mathrm{supp}(\delta_\nu) \subset [-1/\nu, 1/\nu],$$

and define

$$\alpha_\nu(x) \doteq \int_{-\infty}^x \delta_\nu(z)\, dz.$$

Fix $R>0$, $\tau'>\tau>0$ and define ψ, a nonnegative smooth approximation of the characteristic function of the set $\Omega \doteq \big\{(T,X);\ \tau \leq T \leq \tau',\ |X| \leq R - L(T-\tau)\big\}$, as follows

$$\psi(T,X) \doteq \big[\alpha_\nu(T-\tau) - \alpha_\nu(T-\tau')\big] \cdot \big[1 - \alpha_\nu\big(|X| - R + L(T-\tau)\big)\big]. \tag{74}$$

For any $h,k \in \mathbb{N}$ choose

$$\phi(t,x,s,y) \doteq \psi\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_h\left(\frac{t-s}{2}\right) \delta_k\left(\frac{x-y}{2}\right). \tag{75}$$

It is easy to see that

$$(\phi_x + \phi_y)(t,x,s,y) = \psi_X\left(\frac{t+s}{2}, \frac{x+y}{2}\right) \delta_h\left(\frac{t-s}{2}\right) \delta_k\left(\frac{x-y}{2}\right).$$

Notice that $\tilde{w}(s,y;y) \equiv w(s,y)$ and $\tilde{u}(t,x;x) \equiv u(t,x)$. Inserting this function ϕ in (73) and using the coordinates $\big(t, X = (x+y)/2, s, Y = (x-y)/2\big)$, taking the limit $k \to \infty$ one has

$$\begin{aligned} \iiint \Big\{ \big|u(t,X) - w(s,X)\big| (\varphi_t + \varphi_s) + \\ + \big|F\big(u(t,X),X\big) - F\big(w(s,X),X\big)\big|\, \varphi_X \Big\}\, dt\,ds\,dX \geq 0, \end{aligned} \tag{76}$$

where now

$$\varphi(t,s,X) \doteq \psi\left(\frac{t+s}{2}, X\right) \delta_h\left(\frac{t-s}{2}\right). \tag{77}$$

In a similar way, by using coordinates $\big(T = (t+s)/2, S = (t-s)/2, X\big)$ and letting $h \to \infty$ it follows that

$$\begin{aligned} \iint \Big\{ \big|u(T,X) - w(T,X)\big|\, \psi_T + \\ + \big|F\big(u(T,X),X\big) - F\big(w(T,X),X\big)\big|\, \psi_X \Big\}\, dT\,dX \geq 0. \end{aligned} \tag{78}$$

With our particular choice of ψ, for ν large enough we obtain

$$\begin{aligned} &\iint \big|u(t,x) - w(t,x)\big| \cdot \\ &\quad \cdot \big[\delta_\nu(t-\tau) - \delta_\nu(t-\tau')\big] \cdot \big[1 - \alpha_\nu\big(|x| - R + L(t-\tau)\big)\big]\, dx\,dt \\ &\geq \iint \left\{ \frac{x}{|x|} \big|F\big(u(t,x),x\big) - F\big(w(t,x),x\big)\big| + L\big|u(t,x) - w(t,x)\big| \right\} \cdot \\ &\quad \cdot \big[\alpha_\nu(t-\tau) - \alpha_\nu(t-\tau')\big]\, \delta_\nu\big(|x| - R + L(t-\tau)\big)\, dx\,dt. \end{aligned} \tag{79}$$

The right-hand side of (79) is easily seen to be positive, hence it follows that

$$\iint \big|u(t,x) - w(t,x)\big|\, \big[\delta_\nu(t-\tau) - \delta_\nu(t-\tau')\big]\, \big[1 - \alpha_\nu\big(|x| - R + L(t-\tau)\big)\big]\, dx\,dt \geq 0. \tag{80}$$

Since the maps $t \mapsto u(t,\cdot)$ and $t \mapsto w(t,\cdot)$ are continuous, by letting $\nu \to \infty$ we obtain (69) in the case $\tau' > \tau > 0$. Finally by continuity, the assertion is also true for $\tau' = \tau$ or $\tau = 0$. $\square$

As an immediate consequence of Theorem 5.2 we have $\mathbf{L}^1$-contractivity for entropy admissible solutions of (2) when $\bar{v}$ is fixed.

Corollary 5.3 *If u, w and $\bar{v}$ are as in the above theorem, then for every $t > 0$ we have*

$$\int_{-\infty}^{\infty} \big|u(t,x) - w(t,x)\big|\, dx \le \int_{-\infty}^{\infty} \big|u(0,x) - w(0,x)\big|\, dx. \tag{81}$$

In particular, bounded entropy admissible solutions to the Cauchy problem for system (2) are unique.

Finally we prove part (b) of Theorem 1.1, i.e. that the semigroup trajectories are weak solutions. Let us prove it first for $(\bar{u}, \bar{v}) \in \mathbf{BV}$. Take piecewise constant functions (u_n, v_n) converging to $(\bar{u}, \bar{v})$, and such that $u_n(t,x) \doteq S_t^{\delta_n, v_n} u_n$ converges to $u(t,x) \doteq S_t^{\bar{v}} \bar{u}$ in the $\mathbf{L}^1$-norm. As noticed above $u_n(t,x)$ is actually the entropy admissible solution for (15) with initial data (u_n, v_n), hence

$$\begin{aligned} \int_0^{+\infty}\int_{-\infty}^{+\infty} \Big\{ &\big|u_n(t,x) - u_n^{\zeta}(x)\big|\phi_t(t,x) + \\ &+ \big|f^{\delta_n}\big(u_n(t,x), v_n(x)\big) - \zeta\big|\, \phi_x(t,x)\Big\}\, dx\,dt \ge 0, \end{aligned} \tag{82}$$

for every $\zeta \in \mathbb{R}$ and every smooth function ϕ with compact support, where $u_n^{\zeta}(x)$ is defined through the relation

$$f^{\delta_n}\big(u_n^{\zeta}(x), v_n(x)\big) = \zeta, \qquad\qquad \forall\, x \in \mathbb{R}.$$

We have also

$$\begin{aligned} \big|u^{\zeta}(x) - u_n^{\zeta}(x)\big| &\le C\big|f^{\delta_n}\big(u_n^{\zeta}(x), v_n(x)\big) - f^{\delta_n}\big(u^{\zeta}(x), v_n(x)\big)\big| \\ &\le C\Big\{\big|f\big(u^{\zeta}(x), v(x)\big) - f^{\delta_n}\big(u^{\zeta}(x), v(x)\big)\big| + \\ &\qquad + \big|f^{\delta_n}\big(u^{\zeta}(x), v(x)\big) - f^{\delta_n}\big(u^{\zeta}(x), v_n(x)\big)\big|\Big\}. \end{aligned} \tag{83}$$

Since f^{δ_n} converges to f uniformly on K as $n \to \infty$, one has that u_n^{ζ} converges to u^{ζ} in $\mathbf{L}^1$. Thus for every nonnegative smooth function ϕ with compact support it follows that

$$\begin{aligned} &\iint \Big\{\big|u(t,x) - u^{\zeta}(x)\big|\phi_t + \big|f\big(u(t,x), v(x)\big) - \zeta\big|\phi_x\Big\}\, dx\,dt \\ &= \lim_{n\to\infty} \iint \Big\{\big|u_n(t,x) - u_n^{\zeta}(x)\big|\phi_t + \big|f^{\delta_n}\big(u_n(t,x), v_n(x)\big) - \zeta\big|\phi_x\Big\} dx\,dt \ge 0. \end{aligned} \tag{84}$$

This shows that $u(t,x) = S_t^{\bar v}\bar u$ is an entropy admissible solution of (2). The corresponding result for $(\bar u, \bar v) \in \mathbf{L}^1 \cap \mathbf{L}^\infty$ now follows by approximation. This concludes the proof of Theorem 1.1.
Notice that this implies also the uniqueness of the semigroup $\mathbb{S}$.

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