Well-posedness of the Cauchy problem for n x n Systems of Conservation Laws

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Abstract. This paper is concerned with the initial value problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + \big[F(u)\big]_x = 0, \qquad u(0,x) = \bar u(x). \tag{$*$}$$

Each characteristic field is assumed to be either linearly degenerate or genuinely nonlinear. We prove that there exists a domain $\mathcal{D} \subset \mathbf{L}^1$, containing all functions with sufficiently small total variation, and a uniformly Lipschitz continuous semigroup $S : \mathcal{D} \times [0,\infty[\, \mapsto \mathcal{D}$ with the following properties. Every trajectory $t \mapsto u(t,\cdot) = S_t \bar u$ of the semigroup is a weak, entropy-admissible solution of $(*)$. Viceversa, if a piecewise Lipschitz, entropic solution $u = u(t,x)$ of $(*)$ exists for $t \in [0,T]$, then it coincides with the semigroup trajectory, i.e. $u(t,\cdot) = S_t \bar u$. For a given domain $\mathcal{D}$, the semigroup S with the above properties is unique.

These results yield the uniqueness, continuous dependence and global stability of weak, entropy-admissible solutions of the Cauchy problem $(*)$, for general $n \times n$ systems of conservation laws, with small initial data.

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1 - Introduction.

Consider the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x). \tag{1.2}$$

For initial data $\bar u \in \mathbf{L}^1$ with small total variation, a well known theorem of Glimm [G] provides the global existence of weak solutions. Aim of the present paper is to show that these solutions are unique and depend continuously on the initial conditions, with a Lipschitz constant in $\mathbf{L}^1$ which is uniform w.r.t. time. More precisely, the following holds.

Theorem 1. *Let $\Omega \subseteq I\!R^n$ be an open set containing the origin, and let $F : \Omega \mapsto I\!R^n$ be a smooth map. Assume that the system (1.1) is strictly hyperbolic and that each characteristic field is either linearly degenerate or genuinely nonlinear. Then there exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R;\ I\!R^n)$, constants η_0, L, and a continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ with the properties:*

(i) Every function $\bar u \in \mathbf{L}^1$ with Tot.Var.$(\bar u) \leq \eta_0$ lies in $\mathcal{D}$.

(ii) For all $\bar u, \bar v \in \mathcal{D}$, $t, s \geq 0$ one has $\big\|S_t\bar u - S_s\bar v\big\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|\bar u - \bar v\|_{\mathbf{L}^1}\big)$.

(iii) If $\bar u \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t,\cdot) = S_t\bar u$ coincides with the solution of (1.1)-(1.2) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

The positively invariant domain $\mathcal{D}$ will have the form

$$\mathcal{D} = cl\,\big\{u \in \mathbf{L}^1(I\!R; I\!R^n);\quad u \text{ is piecewise constant,} \quad V(u) + C\cdot Q(u) < \delta_0\big\}\,, \tag{1.3}$$

for some constants $C, \delta_0 > 0$. Here $V(u)$ and $Q(u)$ denote the total strength of waves and the wave interaction potential of u, while cl denotes closure.

Following [B5], we say that a map S with the properties (i)–(iii) is a *Standard Riemann Semigroup* (SRS). The existence of such a semigroup was proved in [B1, B3] for special classes of $n \times n$ systems with coinciding shock and rarefaction curves, and in [B-C1] for general 2×2 systems. The present theorem, dealing with general $n \times n$ systems, contains all previous results in this direction.

Observe that the statement of Theorem 1 does not explicitly say that the trajectories of the semigroup are actually weak solutions of (1.1). This fact, however, can be deduced as a consequence of (i)–(iii), together with a number of additional properties which are collected below.

Theorem 2. *For a given domain $\mathcal{D}$ of the form (1.3), there can be at most one continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ satisfying the conditions (i)–(iii) listed in Theorem 1. If a SRS does exist, then the following properties also hold:*

(iv) Each trajectory $t \mapsto u(t, \cdot) = S_t \bar{u}$ is a weak, entropy-admissible solution of the corresponding Cauchy problem (1.1)-(1.2).

(v) Let $(u_\nu)_{\nu \geq 1}$ be a sequence of approximate solutions of (1.1)-(1.2) generated by a wave-front tracking algorithm, or by the Glimm scheme with uniformly distributed sampling. Then, as $\nu \to \infty$ we have $\mathbf{L}^1$-$\lim u_\nu(t, \cdot) = S_t \bar{u}$ for every $t \geq 0$.

(vi) Let $u = u(t, x)$ be a piecewise Lipschitz, entropic solution of (1.1)-(1.2) defined on some strip $[0, T] \times I\!R$. Then $u(t, \cdot) = S_t \bar{u}$ for all $t \in [0, T]$.

A proof of Theorem 2 is contained in [B5]. In turn, from the existence of the semigroup, one can prove the uniqueness of the entropy-weak solution of a given Cauchy problem, under a mild assumption on the growth of the total variation [B-LF1]. Another application which is worth mentioning is the error estimate [B-M2], concerning the $\mathbf{L}^1$ distance between the exact solution of a Cauchy problem and an approximate solution generated by the Glimm scheme with uniformly distributed sampling [L].

There are three types of estimates which play a fundamental role in our analysis. These are: the estimates of Glimm on the total strength of waves [G, Sm], the local integral estimates used in the definition of Viscosity Solution [B5], and the decay estimates for positive waves of genuinely nonlinear families [B-C3].

A precise statement of these a-priori bounds requires some notation. Call $A(u) = DF(u)$ the Jacobian matrix of F at u. Smooth solutions of (1.1) thus satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0. \tag{1.4}$$

Let $\lambda_1(u) < \cdots < \lambda_n(u)$ be the eigenvalues of $A(u)$ and choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$\nabla \lambda_i \cdot r_i(u) \ \doteq\ \lim_{h \to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} \ \geq\ 0, \tag{1.5}$$

$$|r_i| \equiv 1, \qquad \langle l_i,\ r_j \rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.6}$$

Following [B-C2, Sch], we now extend the definition of the Glimm functional to a general BV function. Let $u : I\!R \mapsto I\!R^n$ have bounded variation. Then $\mu \doteq D_x u$ is a vector measure, which can be decomposed into a continuous and an atomic part: $\mu = \mu^c + \mu^a$. For $i = 1, \ldots, n$, define the signed measure $\mu_i = \mu_i^c + \mu_i^a$ as follows. The continuous part of μ_i is the Radon measure such that

$$\int \phi \; d\mu_i^c = \int l_i(u) \cdot \phi \; d\mu^c$$

for every scalar continuous function ϕ with compact support. The atomic part of μ_i is the measure concentrated on the countable set $\{x_\alpha;\ \alpha = 1, 2, \ldots\}$ where u has a jump, such that $\mu_i^a\big(\{x_\alpha\}\big)$ is the strength of the i-th wave in the solution of the Riemann problem with data $u(x_\alpha-)$, $u(x_\alpha+)$. Call μ_i^+, μ_i^- the positive and negative parts of the signed measure μ_i, so that

$$\mu_i = \mu_i^+ - \mu_i^-, \qquad\qquad |\mu_i| = \mu_i^+ + \mu_i^-.$$

The *total strength of waves* in u is defined as

$$V(u) \doteq \sum_{i=1}^{n} V_i(u), \qquad\qquad V_i(u) \doteq |\mu_i|(I\!\!R), \tag{1.7}$$

while the *interaction potential* of waves in u is

$$Q(u) \doteq \sum_{i<j} \big(|\mu_j| \times |\mu_i|\big)\Big(\big\{(x,y);\quad x<y\big\}\Big) + \sum_i \big(\mu_i^- \times |\mu_i|\big)\Big(\big\{(x,y);\quad x \neq y\big\}\Big). \tag{1.8}$$

With the above notations, the classical interaction estimates [G, Sm] can be stated as follows.

Proposition 1 (Bounds on wave strengths). *There exists a constant C_1 such that, for every solution $u = u(t,x)$ of (1.1) with small total variation, obtained as limit of wave-front tracking approximations, the following holds. Let $t = \Lambda_j(x)$, $j = 1, 2$ be the equations of two space-like curves in the t-x plane, with $\Lambda_1 \leq \Lambda_2$. Then, calling $u_i(x) \doteq u\big(\Lambda_i(x),\ x\big)$, one has*

$$\begin{aligned} &Q(u_2) \leq Q(u_1), \\ &V(u_2) + C_1 \cdot Q(u_2) \leq V(u_1) + C_1 \cdot Q(u_1) \\ &V_i(u_2) + C_1 \cdot Q(u_2) \leq V_i(u_1) + C_1 \cdot Q(u_1) \qquad i = 1, \ldots, n. \end{aligned} \tag{1.9}$$

It is well known that the estimates (1.9) actually hold not only for exact solutions but also for approximate solutions constructed by various algorithms [B2, G, R]. For convenience, at various stages of this paper we shall work with slightly different definitions of the interaction potential $Q(u)$. Indeed, one may consider two waves of the same family as being always approaching, regardless of their sign, or as being never approaching. This second definition is useful in connection with systems where shock and rarefaction curves coincide. In all cases, we will make sure that the basic interaction estimates (1.9) remain valid.

As proved in [B5], the trajectories of a Standard Riemann Semigroup can be characterized by a set of local integral estimates. Two types of local approximate solutions for (1.1) are considered. One is derived from the self-similar solution of a Riemann problem, the other is obtained by "freezing" the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Let $u : [0,T]\times \mathbb{R} \mapsto \mathbb{R}^n$ be a locally integrable function, and fix any point (τ,ξ) in the domain of u. Assuming $u(\tau,\cdot)\in BV$, consider the limits

$$u^- = \lim_{x\to\xi-} u(\tau,x), \qquad\qquad u^+ = \lim_{x\to\xi+} u(\tau,x).$$

Call $\omega=\omega(t,x)$ the self-similar solution of the Riemann problem

$$u_t + \big[F(u)\big]_x = 0, \qquad\qquad u(0,x) = \begin{cases} u^- & \text{if} \quad x<0,\\ u^+ & \text{if} \quad x>0.\end{cases} \tag{1.10}$$

Let $\hat\lambda$ be an upper bound for all characteristic speeds. For $t>\tau$, define

$$U^\sharp_{(u;\tau,\xi)}(t,x) \doteq \begin{cases} \omega(t-\tau,\ x-\xi) & \text{if} \quad |x-\xi|\le\hat\lambda(t-\tau),\\ u(\tau,x) & \text{if} \quad |x-\xi|>\hat\lambda(t-\tau).\end{cases} \tag{1.11}$$

Observe that the function $t\mapsto U^\sharp_{(u;\tau,\xi)}(t,\cdot)$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, and approaches $u(\tau,\cdot)$ as $t\to\tau+$.

Next, call $\widetilde{A}\doteq DF\big(u(\tau,\xi)\big)$ the Jacobian matrix of F computed at the point $u(\tau,\xi)$. For $t>\tau$, define $U^\flat_{(u;\tau,\xi)}(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$w_t + \widetilde{A}w_x = 0, \qquad\qquad w(\tau,x) = u(\tau,x). \tag{1.12}$$

In the following, by Tot.Var.$\{u(\tau);\ I\}$ we denote the total variation of the function $u(\tau,\cdot)$ over the set I.

Proposition 2 (Local integral estimates). *Let $u(t,\cdot)=S_t\bar u$ be any semigroup trajectory. Then, for some constant C_2, at each point (τ,ξ) one has*

$$\frac{1}{\eta}\int_{\xi-\rho+\eta\hat\lambda}^{\xi+\rho-\eta\hat\lambda} \Big|u(\tau+\eta,x)-U^\sharp_{(u;\tau,\xi)}(\tau+\eta,x)\Big|\,dx \ \le C_2\cdot \mathit{Tot.Var.}\big\{u(\tau);\ ]\xi-\rho,\ \xi[\,\cup\,]\xi,\ \xi+\rho[\,\big\}, \tag{1.13}$$

$$\frac{1}{\eta}\int_{\xi-\rho+\eta\hat\lambda\cdot}^{\xi+\rho-\eta\hat\lambda} \Big|u(\tau+\eta,x)-U^\flat_{(u;\tau,\xi)}(\tau+\eta,x)\Big|\,dx \le C_2\cdot\Big(\mathit{Tot.Var.}\big\{u(\tau);\ ]\xi-\rho,\ \xi+\rho[\,\big\}\Big)^2. \tag{1.14}$$

for every $\rho,\eta>0$ sufficiently small. Viceversa, let $u:[0,T]\mapsto\mathbf{L}^1$ be a continuous map taking values inside the domain $\mathcal{D}$ of the semigroup. If the bounds (1.13), (1.14) hold for all $\xi\in\mathbb{R}$ and all but countably many times $\tau\in[0,T]$, then u coincides with a semigroup trajectory.

Following [B5], a continuous function $u : [0,T] \mapsto \mathbf{L}^1$ will be called a *Viscosity Solution* of (1.1) if the inequalities (1.13), (1.14) hold at every (τ, ξ).

To motivate the decay estimates for waves of genuinely nonlinear families, we consider first the scalar case, assuming $F'' > \kappa > 0$. In this case, if $u = u(t,x)$ is a smooth solution of (1.1) defined for $t \geq \tau$, one has

$$(u_x)_t + F'(u)(u_x)_x = -F''(u)u_x^2. \tag{1.15}$$

Integrating (1.15) along characteristics, one obtains the pointwise estimate [O, Sm]

$$u_x(t,x) \leq \frac{1}{\kappa(t-\tau)}. \tag{1.16}$$

In the vector-valued case, introduce the gradient components

$$u_x^i \doteq \langle l_i(u),\ u_x \rangle. \tag{1.17}$$

By (1.4), if $u = u(t,x)$ is a smooth solution of (1.1), one has (see [B1], p.412)

$$(u_x^i)_t + \big[\lambda_i(u)\big](u_x^i)_x = -\big[\nabla\lambda_i \cdot r_i(u)\big](u_x^i)^2 + \sum_{j \neq k} G_{ijk}(u) u_x^j u_x^k. \tag{1.18}$$

Assume that the i-th field is genuinely nonlinear, so that $\nabla\lambda_i \cdot r_i(u) \geq \kappa > 0$, and let $u = u(t,x)$ be a smooth solution defined for $t \geq \tau$. If $G_{ijk} \equiv 0$ for all i,j,k, then the gradient component u_x^i would clearly satisfy an estimate of the form (1.16). In general, (1.16) may fail because of the last term on the right hand side of (1.18). Observe that this summation essentially depends on the (instantaneous) amount of wave interaction. This suggests that the amount by which (1.16) fails, measured by

$$\int_{\{u_x^i > \eta\}} u_x^i(t,x)\ dx \tag{1.19}$$

for any $\eta > 1/\kappa(t-\tau)$, can be estimated in terms of the total amount of interaction taking place during the interval $[\tau, t]$. This quantity, in turn, can be bounded by $Q(\tau) - Q(t)$, i.e. by the decay in the wave interaction potential. To state the result in the most general case of a BV solution, one more piece of notation is needed. Let $u \in BV$, and let μ_i be the measure determined by the i-waves in u, as in (1.7), (1.8). Call m the Lebesgue measure on $\mathbb{R}$ and split $\mu_i = \mu_i^s + \mu_i^{ac}$ according to its singular and absolutely continuous part. For any $\eta > 0$, define

$$V_i^{\eta+}(u) \doteq \mu_i^{s,+}(\mathbb{R}) + \mu_i^{ac}\left(\left\{x;\ \ \frac{d\mu_i^{ac}}{dm}(x) > \eta\right\}\right), \tag{1.20}$$

where $\mu_i^{s,+}$ denotes the positive part of μ_i^s. Observe that (1.20) coincides with (1.19) if u is Lipschitz continuous. With the above notation, one has

Proposition 3 (Decay Estimates). *Assume that the i-th characteristic field is genuinely nonlinear. Then there exist constants $C_3, \kappa > 0$ such that any solution $u(t,\cdot) = S_t\bar u$ of (1.1) satisfies*

$$V_i^{\eta+}\big(u(t,\cdot)\big) \le C_3\left[Q(\tau) - Q(t)\right]\left(1 - \frac{1}{\eta\kappa(t-\tau)}\right)^{-1} \tag{1.21}$$

for every $t > \tau \ge 0$ and $\eta > 1/\kappa(t-\tau)$.

A proof of (1.21) was first derived in [B-C3], for weak solutions obtained as limits of wave-front tracking approximations. In Section 7 of this paper we will show that similar estimates hold for our piecewise Lipschitz approximate solutions as well.

Towards a proof of Theorem 1, the basic strategy for obtaining a Lipschitz semigroup of solutions of (1.1) is to construct suitable approximate solutions, carefully controlling how their distance varies in time. We recall that, for a scalar conservation law, the entropic solutions constitute a contractive semigroup in $\mathbf{L}^1$ [C, K]. Indeed, any two solutions u, u' can be directly compared, showing that the distance

$$\big\|u(t,\cdot) - u'(t,\cdot)\big\|_{\mathbf{L}^1} \tag{1.22}$$

is a non-increasing function of time. This is definitely not true for systems [T2]. In the present case, following [B1], we consider a one-parameter family of (suitably regular) solutions u^θ, $\theta \in [0,1]$, with $u^0 = u$, $u^1 = u'$. For each t, the distance (1.22) is clearly bounded by the $\mathbf{L}^1$-length of the path $\gamma_t : \theta \mapsto u^\theta(t,\cdot)$, defined by

$$\|\gamma_t\|_{\mathbf{L}^1} \doteq \sup\left\{\sum_{j=1}^{\nu}\big\|u^{\theta_j}(t,\cdot) - u^{\theta_{j-1}}(t,\cdot)\big\|_{\mathbf{L}^1},\quad 0 = \theta_0 < \theta_1 < \cdots < \theta_\nu = 1, \qquad \nu \ge 1\right\}.$$

Therefore, if we show that this length satisfies

$$\|\gamma_t\|_{\mathbf{L}^1} \le L \cdot \|\gamma_0\|_{\mathbf{L}^1} \tag{1.23}$$

for every curve γ_0 joining $u(0)$ with $u'(0)$, it will follow that

$$\big\|u(t) - u'(t)\big\|_{\mathbf{L}^1} \le L \cdot \big\|u(0) - u'(0)\big\|_{\mathbf{L}^1} \qquad \forall t \ge 0. \tag{1.24}$$

If the path γ is suitably regular, its length can be computed by integrating the norm of a tangent vector:

$$\|\gamma\|_{\mathbf{L}^1} = \int_0^1 \left\|\frac{d\gamma(\theta)}{d\theta}\right\|_{\mathbf{L}^1} d\theta. \tag{1.25}$$

An estimate of the form (1.24) can thus be obtained by showing that the norm of any tangent vector increases at most by a factor L.

We shall implement the above strategy using paths of piecewise Lipschitz functions. Let $\theta \mapsto u^\theta$ be a one-parameter family of piecewise Lipschitz functions, each u^θ having the same number of jumps, say at the points $y_1^\theta < \ldots < y_N^\theta$. Assume that there exist the functions (fig. 1):

$$v^\theta(x) \doteq \lim_{h\to 0} \frac{u^{\theta+h}(x) - u^\theta(x)}{h} \qquad \text{for a.e. } x \tag{1.26}$$

and the numbers

$$\xi_\alpha^\theta \doteq \lim_{h\to 0} \frac{y_\alpha^{\theta+h} - y_\alpha^\theta}{h} \qquad \alpha = 1, \ldots, N. \tag{1.27}$$

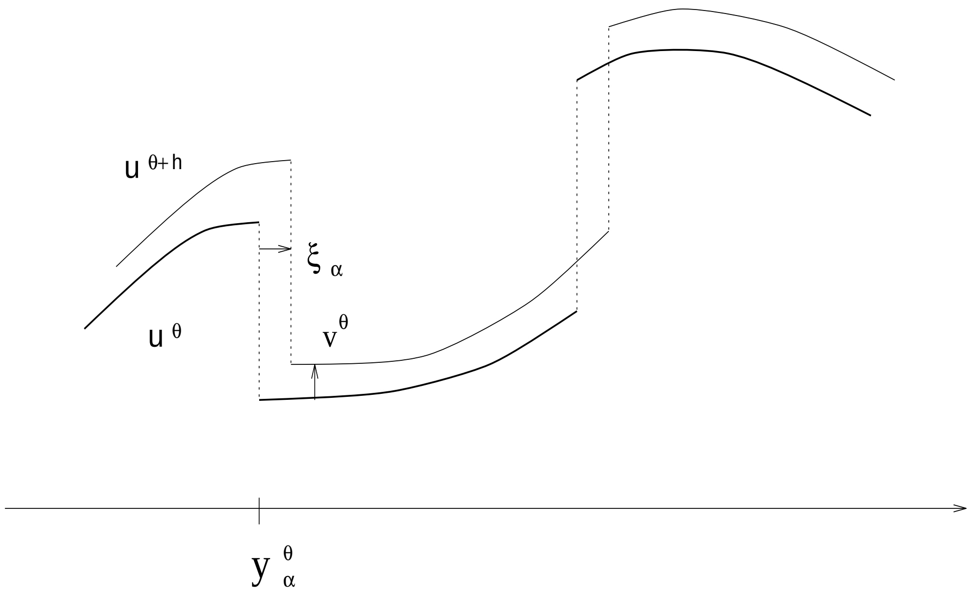

Fig.1

Then, under suitable regularity conditions, the $\mathbf{L}^1$ length of the path $\gamma : \theta \mapsto u^\theta$ is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_0^1 \|v^\theta\|_{\mathbf{L}^1}\, d\theta + \sum_{\alpha=1}^N \int_0^1 \big|u^\theta(y_\alpha+) - u^\theta(y_\alpha-)\big| |\xi_\alpha^\theta|\, d\theta. \tag{1.28}$$

We stress the fact that, in general, the path $\theta \mapsto u^\theta$ is not differentiable w.r.t. the usual differential structure of $\mathbf{L}^1$. Indeed, if the shift rates ξ_α^θ are not equal to zero, as $h \to 0$ the ratio $[u^{\theta+h} - u^\theta]/h$ does not converge to any limit in $\mathbf{L}^1$. In order to correctly measure the length of a path γ, it is essential to work with a class of "generalized tangent vectors" of the form $(v, \xi) \in T_u \doteq \mathbf{L}^1(I\!R;\, I\!R^n) \times I\!R^N$. Observe that the tangent space T_u actually depends on the function u, through the number of points of discontinuity.

Next, assume that each function $u^\theta(t,\cdot)$ is a solution of the system of conservation laws (1.1). A set of linearized evolution equations for the corresponding tangent vectors (v^θ,ξ^θ) was derived in [B-M1]. To write down these equations, some notation must be introduced. Let $A(u)$ be the Jacobian matrix of f at u and call $\lambda_i(u), l_i(u), r_i(u)$ respectively its eigenvalues and left and right eingenvectors. For $u,u'\in I\!R^n$, define the averaged matrix

$$A(u,u')\doteq\int_0^1 A\big(\theta u+(1-\theta)u'\big)d\theta \tag{1.29}$$

with eigenvalues $\lambda_i(u,u')$, and choose right and left eigenvectors $r_i(u,u')$, $l_i(u,u')$ of $A(u,u')$ according to (1.6). The differential of λ_i at (u,u') is written $D\lambda_i(u,u')$. We thus have

$$D\lambda_i(u,u')\cdot(v,v')=\lim_{\varepsilon\to 0}\varepsilon^{-1}\big[\lambda_i(u+\varepsilon v,u'+\varepsilon v')-\lambda_i(u,u')\big]. \tag{1.30}$$

The same notation will be used for the differentials of the eigenvectors r_i and l_i.

Let $u=u(t,x)$ be a piecewise Lipschitz continuous weak solution of (1.1). At almost every point (t,x), the function u thus satisfies the quasilinear system (1.4) while, along the shock lines $x=y_\alpha(t)$, the Rankine-Hugoniot equations hold:

$$\dot y_\alpha[u^+-u^-]=\big[f(u^+)-f(u^-)\big], \tag{1.31}$$

with $u^+=u(t,y_\alpha+)$, $u^-=u(t,y_\alpha-)$. If the jump at y_α occurs in the k_α-th characteristic family, this implies that u^+-u^- is a right eigenvector of the matrix $A(u^+,u^-)$, with corresponding eigenvalue

$$\dot y_\alpha=\lambda_{k_\alpha}(u^+,u^-). \tag{1.32}$$

The eigenvector condition can also be written as

$$\big\langle l_i(u^+,u^-),\ u^+-u^-\big\rangle=0 \qquad\qquad \forall i\neq k_\alpha. \tag{1.33}$$

A system of linearized evolution equations for the generalized tangent vector (v,ξ) can now be derived from (1.4), (1.32) and (1.33). Namely

$$v_t+A(u)v_x+[DA(u)\cdot v]u_x=0 \tag{1.34}$$

outside the lines of discontinuity, together with the conditions

$$\begin{aligned}&\Big\langle Dl_i(u^+,u^-)\cdot(\xi_\alpha u_x^++v^+,\ \xi_\alpha u_x^-+v^-),\ \ u^+-u^-\Big\rangle\\ &\qquad+\Big\langle l_i(u^+,u^-),\ \ \xi_\alpha u_x^++v^+-\xi_\alpha u_x^--v^-\Big\rangle=0 \qquad\qquad \forall i\neq k_\alpha,\end{aligned} \tag{1.35}$$

$$\dot\xi_\alpha=D\lambda_{k_\alpha}(u^+,u^-)\cdot\big(\xi_\alpha u_x^++v^+,\ \ \xi_\alpha u_x^-+v^-\big), \tag{1.36}$$

on each line $x = y_\alpha(t)$ where u has a shock (or contact discontinuity) in the k_α-th characteristic field. Observe that the equations (1.34) are formally derived from (1.4), replacing u by $u + \varepsilon v$ and differentiating w.r.t. ε. Similarly, the equations (1.35), (1.36) are obtained from (1.33), (1.32) replacing the right and left limits u^+, u^- by $u^+ + \varepsilon v^+ + \varepsilon u_x^+ \xi_\alpha$ and $u^- + \varepsilon v^- + \varepsilon u_x^- \xi_\alpha$ respectively, and differentiating w.r.t. ε. For a rigorous derivation of (1.34)–(1.36) see [B-M1].

According to (1.28), the standard $\mathbf{L}^1$-length of a path γ can be computed by integrating the norm of its tangent vector, defined as

$$\big\|(v,\xi)\big\|_{\mathbf{L}^1} \doteq \|v\|_{\mathbf{L}^1} + \sum_\alpha \big|u(y_\alpha+) - u(y_\alpha-)\big||\xi_\alpha|. \tag{1.37}$$

Example 1. In the scalar case, it is well known that the conservation law (1.1) generates a contractive semigroup in $\mathbf{L}^1$. In particular, the $\mathbf{L}^1$-length of a path of solutions does not increase in time, and the same holds for the norm (1.37) of a tangent vector. It is an instructive exercise to carry out the computations in this simple case. Let F be convex and let $u = u(t,x)$ be a piecewise Lipschitz solution of (1.1), with jumps at the points $x = y_\alpha(t)$, $\alpha = 1, \ldots, N$. Calling $\lambda(u) \doteq DF(u)$ the characteristic speed, (1.4) takes the form

$$u_t + \lambda(u)u_x = 0. \tag{1.38}$$

At points of jump, the Rankine-Hugoniot and the entropy conditions yield

$$\dot y_\alpha = \frac{1}{|\sigma_\alpha|}\int_{\sigma_\alpha}^0 \lambda\big(u(y_\alpha-)+s\big)\,ds = \frac{F\big(u(y_\alpha+)\big) - F\big(u(y_\alpha-)\big)}{u(y_\alpha+) - u(y_\alpha-)}, \tag{1.39}$$

$$\sigma_\alpha = u(y_\alpha+) - u(y_\alpha-) < 0. \tag{1.40}$$

The linearized evolution equations (1.34)–(1.36) for a generalized tangent vector (v,ξ) take the form

$$v_t + \big[\lambda(u)v\big]_x = 0, \tag{1.41}$$

$$\dot\xi_\alpha = D\lambda(u^-,u^+)\cdot\big(v^- + \xi_\alpha u_x^-,\ v^+ + \xi_\alpha u_x^+\big) \qquad \alpha = 1,\ldots,N. \tag{1.42}$$

Here and in the following, we use the shorter notation $u^\pm \doteq u(y_\alpha\pm)$, and similarly for $v^\pm, u_x^\pm$. Observing that

$$\dot\sigma_\alpha = \big(\lambda(u^-) - \dot y_\alpha\big)u_x^- + \big(\dot y_\alpha - \lambda(u^+)\big)u_x^+, \qquad \sigma_\alpha = u^+ - u^- < 0, \tag{1.43}$$

$$\begin{aligned} D\lambda(u^-,u^+)\cdot(\eta^-,\eta^+) &\doteq \frac{d}{d\varepsilon}\left[\frac{F(u^-+\varepsilon\eta^-) - F(u^+ + \varepsilon\eta^+)}{(u^-+\varepsilon\eta^-) - (u^+ + \varepsilon\eta^+)}\right]_{\varepsilon=0} \\ &= \frac{\lambda(u^-)\eta^- - \lambda(u^+)\eta^+}{u^- - u^+} - \frac{f(u^-) - f(u^+)}{u^- - u^+}\cdot\frac{\eta^- - \eta^+}{u^- - u^+} \\ &= \big(\lambda(u^-) - \dot y\big)\frac{\eta^-}{|\sigma_\alpha|} + \big(\dot y_\alpha - \lambda(u^+)\big)\frac{\eta^+}{|\sigma_\alpha|}, \end{aligned} \tag{1.44}$$

from (1.41), (1.42) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left\{\int_{-\infty}^{\infty}\big|v(t,x)\big|\,dx+\sum_{\alpha=1}^{N}|\xi_\alpha||\sigma_\alpha|\right\}\\
&=-\left\{\sum_\alpha\big(\lambda(u^-)-\dot y_\alpha\big)|v^-|+\sum_\alpha\big(\dot y_\alpha-\lambda(u^+)\big)|v^+|\right\}\\
&\quad+\sum_\alpha D\lambda(u^-,u^+)\cdot\big(v^-+\xi_\alpha u_x^-,\ v^++\xi_\alpha u_x^+\big)(\text{sign }\xi_\alpha)|\sigma_\alpha|\\
&+\sum_\alpha|\xi_\alpha|\Big[\big(\lambda(u^-)-\dot y_\alpha\big)u_x^- -\big(\dot y_\alpha-\lambda(u^+)\big)\Big]\\
&\le 0.
\end{aligned}
\tag{1.45}
$$

In the case of systems, on the other hand, the norm (1.37) may well increase along solutions of (1.34)–(1.36). In [B4], however, the following was proved.

Proposition 4. *There exists a set $\mathcal{U}$, containing all piecewise Lipschitz functions with sufficiently small total variation, and a family of weighted norms $\|\cdot\|_u$, defined for $u\in\mathcal{U}$, with the following properties.*

(i) If $u=u(t,x)$ is a piecewise Lipschitz continuous solution of (1.1) with $u(t)\in\mathcal{U}$, and if the pair $\big(v(t),\xi(t)\big)$ is any solution of the corresponding linearized system, then the norm $\big\|\big(v(t),\xi(t)\big)\big\|_{u(t)}$ is a non-increasing function of time, even at times where two shocks interact.

(ii) There exists a constant L such that, for all $u\in\mathcal{U}$ and $(v,\xi)\in T_u$, one has

$$
\big\|(v,\xi)\big\|_{\mathbf{L}^1}\le\big\|(v,\xi)\big\|_u\le L\cdot\big\|(v,\xi)\big\|_{\mathbf{L}^1}. \tag{1.46}
$$

In turn, the Riemann metric $\|\cdot\|_u$ determines a weighted distance $d_\star$ on $\mathcal{U}$. Roughly speaking, $d_\star(u,u')$ is the infimum of the weighted length of paths joining u with u'. A more careful construction goes as follows.

Definition 1. We say that a continuous map $\gamma:\theta\mapsto u^\theta\doteq\gamma(\theta)$ from an open interval J into $\mathbf{L}^1_{loc}$ is a *Regular Path* (RP) if the following holds. For $\theta\in J$, all functions u^θ are piecewise Lipschitz continuous, with the same number of jumps, say at $x_1^\theta<\cdots<x_N^\theta$, and the same Lipschitz constant outside these points of jump. They all coincide outside some fixed interval $[-M,M]$. Moreover, the function $\theta\mapsto u_x^\theta$ is continuous from J into $\mathbf{L}^1$. The map $\theta\mapsto u^\theta$ admits a generalized tangent vector $D\gamma(\theta)=(v^\theta,\xi^\theta)\in T_{\gamma(\theta)}=\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)\times\mathbb{R}^N$, continuously depending on θ.

Definition 2. A continuous map $\gamma:[a,b]\mapsto\mathbf{L}^1$ is a *Piecewise Regular Path* (PRP) if there exist finitely many values $a=\theta_0<\theta_1<\cdots<\theta_N=b$ such that the restriction of γ to each open subinterval $J_\ell\doteq\,]\theta_{\ell-1},\ \theta_\ell[$ is a regular path.

Given any two piecewise Lipschitz continuous functions $u, u' \in \mathcal{U}$, call $\Sigma_{u,u'}$ the family of all regular paths $\gamma : [0,1] \mapsto \mathcal{U}$ with $\gamma(0) = u$, $\gamma(1) = u'$. The weighted length of a path $\gamma \in \Sigma_{u,u'}$ is then defined as

$$\|\gamma\|_\star \doteq \int_0^1 \big\|D\gamma(\theta)\big\|_{\gamma(\theta)} d\theta, \tag{1.47}$$

while the Riemannian distance between u and u' is given by

$$d_\star(u, u') \doteq \inf \big\{ \|\gamma\|_\star; \quad \gamma \in \Sigma_{u,u'} \big\}. \tag{1.48}$$

As proved in [B-B], the weighted length (1.47) is lower semicontinuous: if $(\gamma_\nu)_{\nu \geq 0}$ is a sequence of piecewise regular paths such that

$$\lim_{\nu \to \infty} \sup_{\theta \in [0,1]} \big\|\gamma_\nu(\theta) - \gamma_0(\theta)\big\|_{\mathbf{L}^1} = 0,$$

then

$$\|\gamma_0\|_\star \leq \liminf_{\nu \to \infty} \|\gamma_\nu\|_\star.$$

Because of (ii) in Proposition 4, the distance $d_\star$ is uniformly equivalent to the standard $\mathbf{L}^1$ distance. Hence, it can be extended by continuity to the $\mathbf{L}^1$ closure of $\mathcal{U}$. Moreover, by (i), the length of every regular path does not increase in time along the flow of (1.1). This suggests that the flow generated by (1.1) should be globally contractive w.r.t. the weighted distance $d_\star$, and hence uniformly Lipschitz continuous w.r.t. the usual $\mathbf{L}^1$ distance.

Unfortunately, a rigorous proof of this fact runs into a major difficulty. Indeed, the estimates on the norm of a tangent vector, and on the length of a path $\theta \mapsto \gamma_t(\theta)$, are valid assuming that all solutions remain piecewise Lipschitz throughout a given interval $[0, T]$. This is not the case in general. Indeed, a piecewise Lipschitz solution may lose its regularity in two ways (fig. 2):

(i) The number of shock fronts may become infinite in finite time, due to repeated shock interactions.

(ii) The Lipschitz constant outside the shocks may become infinite, due to the genuine nonlinearity of some characteristic fields.

We recall that, in the special case where all characteristic fields are linearly degenerate, solutions which are initially smooth remain smooth for all times. In this case, as soon as Proposition 4 has been established, the construction of the semigroup is straightforward [B1]. The purpose of the present paper is to show that this construction can still be accomplished, for general $n \times n$ genuinely nonlinear systems, with the aid of three technical tools:

(1) A restarting procedure, which replaces a path $\gamma_\tau : \theta \mapsto u^\theta(\tau)$ with a new path $\gamma_{\tau+}$. This is used when some of the functions u^θ are about to lose their regularity and cannot be prolonged further in time.

(2) A slight modification of the Rankine-Hugoniot equations, which forces shock curves to coincide with rarefaction curves, for small amplitudes.

(3) A cyclical concatenation of flows generated by quasilinear systems where $n-1$ characteristic fields are linearly degenerate and only one is genuinely nonlinear. In the limit, this yields the flow generated by a general system, with an arbitrary number of genuinely nonlinear fields.

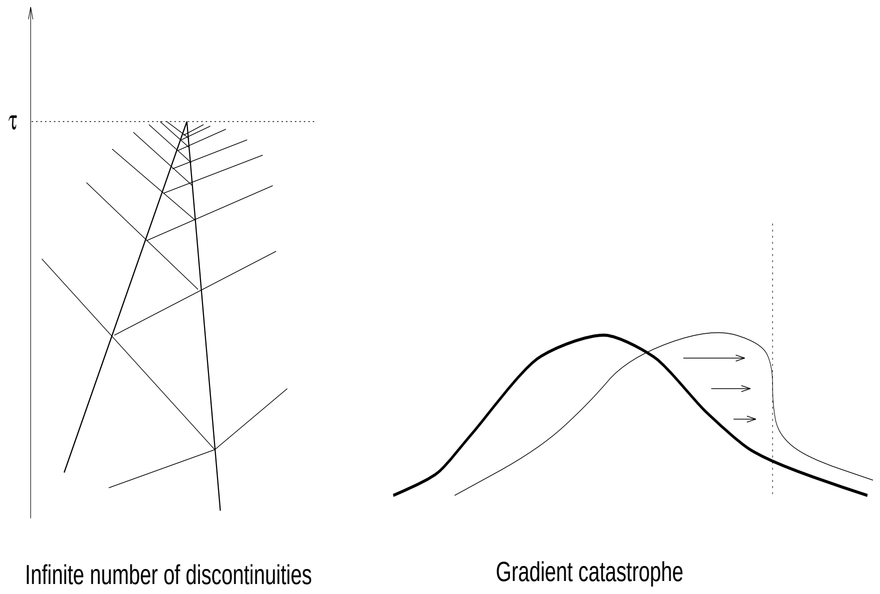

Fig.2

Restarting procedures, applied to approximate solutions, are well known in the literature. A classical example is the Glimm scheme. Another one, consisting of periodic mollifications, occurs in [B3]. We remark that, in our case, the new path $\gamma_{\tau+}$ should satisfy

$$\left\|\gamma_{\tau+}(\theta)-\gamma_\tau(\theta)\right\|_{\mathbf{L}^1}\leq\varepsilon_0 \qquad \forall\theta, \tag{1.49}$$

$$\|\gamma_{\tau+}\|_\star\leq\|\gamma_\tau\|_\star+\varepsilon_0 \tag{1.50}$$

for some $\varepsilon_0>0$ suitably small. In other words, the new path must be close to the old one, and its length should be almost the same. In addition, the solutions u^θ should be well defined and remain regular on some interval $[\tau,\tau+\delta]$, with $\delta>0$ independent of ε_0. Working with exact solutions, it is apparently not possible to meet all these requirements. This is why the approximations **(2)-(3)** are used.

The idea of interpolating between shock and rarefaction curves was introduced in [B-C1]. In an ε-approximate solution, shocks of strength $|\sigma|\geq 4\varepsilon$ satisfy the Rankine-Hugoniot equations exactly. On the other hand, shocks of strength $\leq 3\varepsilon$ connect a right and a left state lying on the same rarefaction curve. Observing that any solution contains at most finitely many shocks of strength $> 3\varepsilon$, the advantage of this approximation is clear. Indeed, outside a finite number of lines in the t-x plane (which we regard as "free boundaries"), we are now dealing with a system where shock and rarefaction curves coincide.

The cyclical concatenation of flows, a particular kind of flux-splitting method, is the main new technique introduced in the present paper. It provides the key for extending the result in [B-C1] to the general $n \times n$ case. The construction goes as follows. Given the system (1.4), fix a state u^* and let $\lambda_1^* < \cdots < \lambda_n^*$ be the eigenvalues of $A(u^*)$. For $h = 1, \ldots, n$, call $A_h(u)$ the matrix with the same eigenvectors $r_1(u), \ldots, r_n(u)$ as $A(u)$, but whose eigenvalues are

$$\lambda_1^*, \ \cdots \ , \ \lambda_{h-1}^*, \ \lambda_h^* + n\big(\lambda_h(u) - \lambda_h^*\big), \ \ldots \ , \ \lambda_n^*. \tag{1.51}$$

Clearly, for each h, the system

$$u_t + A_h(u)u_x = 0 \tag{1.52}$$

has $n-1$ linearly degenerate fields. Call $(t, \bar{u}) \mapsto S_t^h \bar{u}$ the corresponding flow. Given a time step $\Delta t > 0$, we now concatenate the flows of the semigroups $S^1, \ldots, S^n$ cyclically, on subintervals of length $\Delta t/n$. Letting $\Delta t \to 0$, in the limit we obtain the flow determined by (1.4).

2 - Outline of the proof

We collect here the basic steps in the proof of Theorem 1. Technical details will be worked out in the remaining sections.

By possibly performing a linear rescaling of time, it is not restrictive to assume that all wave speeds are < 1 in absolute value. Moreover, throughout the main construction we shall assume that all characteristic fields are genuinely nonlinear. When one or more linearly degenerate fields are present, the minor modifications needed in the proof will be described in the last section of the paper.

We begin by defining, for a given $\varepsilon > 0$, a set of approximate Rankine-Hugoniot conditions. These coincide with the usual ones for shocks of strength $|\sigma| \geq 4\varepsilon$. For a given state $u \in I\!R^n$ and $i = 1, \ldots, n$, denote by

$$\sigma \mapsto S_i(\sigma)(u), \qquad \sigma \mapsto R_i(\sigma)(u),$$

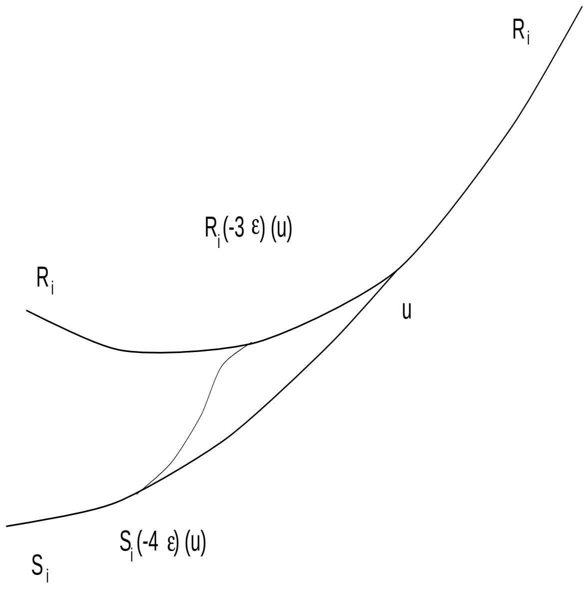

Fig.3

the usual i-shock and i-rarefaction curves through u, parametrized by arclength. As customary, the orientation is chosen so that the i-th characteristic speed is increasing along the curves S_i, R_i. Consider a smooth, non-increasing map $\varphi : I\!R \mapsto [0, 1]$ such that

$$\begin{cases} \varphi(\sigma) = 1 & \text{if} \quad \sigma < -4 \\ \varphi(\sigma) = 0 & \text{if} \quad \sigma > -3 \end{cases} \qquad \dot\varphi \in [-2, 0]. \tag{2.1}$$

Define the interpolated curves (fig. 3)

$$\Psi_i^\varepsilon(\sigma)(u) \doteq \varphi(\sigma/\varepsilon) \cdot S_i(\sigma)(u^-) + \big(1 - \varphi(\sigma/\varepsilon)\big) \cdot R_i(\sigma)(u^-). \tag{2.2}$$

Definition 3. We say that the jump (u^-, u^+), located at $x = y(t)$ and travelling with speed $\dot{y}$, satisfies the *ε-Rankine-Hugoniot conditions* (ε-RH) if, for some $\sigma < 0$ and some $i \in \{1, \ldots, n\}$, we have

$$u^+ = \Psi_i^\varepsilon(\sigma)(u^-), \tag{2.3}$$

$$\dot{y} = \lambda_i^\varepsilon(u^-, u^+) \doteq \varphi(\sigma/\varepsilon)\lambda_i^s + \big(1 - \varphi(\sigma/\varepsilon)\big)\lambda_i^r. \tag{2.4}$$

Here λ_i^s is the speed of a true shock connecting u^- with $S_i(\sigma)(u^-)$, while

$$\lambda_i^r \doteq \frac{1}{|\sigma|} \int_\sigma^0 \lambda_i\big(R_i(s)(u^-)\big)\, ds. \tag{2.5}$$

The modified Rankine-Hugoniot conditions introduced above, in turn, determine a new way of approximately solving a Riemann problem. More precisely, consider the initial data

$$u(0, x) = \begin{cases} u^- & \text{if} \quad x < 0, \\ u^+ & \text{if} \quad x > 0. \end{cases} \tag{2.6}$$

We seek a self-similar, piecewise Lipschitz function $\omega = \omega(t, x)$ which satisfies the quasilinear system (1.4) almost everywhere, and the ε-approximate Rankine-Hugoniot conditions along each shock line. The solution to this problem is provided by the following

ε-Riemann Solver: Using the implicit function theorem, determine wave sizes $\sigma_1, \ldots, \sigma_n$ such that

$$u^+ = \Psi_n^\varepsilon(\sigma_n) \circ \cdots \circ \Psi_1^\varepsilon(\sigma_1)(u^-). \tag{2.7}$$

Let $\omega_0 = u^-$, $\omega_1 = \Psi_1^\varepsilon(\sigma_1)(\omega_0)$, ... , $\omega_n = \Psi_n^\varepsilon(\sigma_n)(\omega_{n-1}) = u^+$ be the corresponding intermediate states. If $\sigma_i \geq 0$, the states ω_{i-1} and ω_i are connected by a centered rarefaction wave of the i-th family, as usual. If $\sigma_i < 0$, these two states are connected by a single jump, travelling with the speed $\dot{y} = \lambda_i^\varepsilon(\omega_{i-1}, \omega_i)$ defined at (2.4).

Definition 4. A continuous map $u : [0, T[\mapsto BV$ is a *Viscosity ε-solution* of (1.1) if at each point (τ, ξ) the inequalities (1.13), (1.14) hold for all $\rho, \eta > 0$ sufficiently small, with $U^\flat$ the solution of (1.12), and

$$U^\sharp_{(u;\tau,\xi)}(t, x) \doteq \begin{cases} \omega^\varepsilon(t - \tau,\ x - \xi) & \text{if} \quad |x - \xi| \leq t - \tau, \\ u(\tau, x) & \text{if} \quad |x - \xi| > t - \tau. \end{cases}$$

Here ω^ε is the ε-solution of the Riemann problem with data $u(\tau, \xi-), u(\tau, \xi+)$. Observe that, by our initial assumption, we can take here $\hat{\lambda} = 1$ as an upper bound for all wave speeds.

For the most part, our work will be devoted to a proof of

Theorem 3. *With the same assumptions of Theorem 1, for some constants $L, \eta_0 > 0$ the following holds. For each $\varepsilon > 0$ there exists a closed domain $\mathcal{D}^\varepsilon \subset \mathbf{L}^1(I\!R;\ I\!R^n)$ and a continuous semigroup $S^\varepsilon : \mathcal{D}^\varepsilon \times [0, \infty[\mapsto \mathcal{D}^\varepsilon$ with the properties:*

(i) *Every function $\bar u \in \mathbf{L}^1$ with Tot.Var.$(\bar u) \leq \eta_0$ lies in $\mathcal{D}^\varepsilon$.*

(ii) *For all $\bar u, \bar v \in \mathcal{D}^\varepsilon$, $t, s \geq 0$ one has $\big\| S^\varepsilon_t \bar u - S^\varepsilon_s \bar v \big\|_{\mathbf{L}^1} \leq L \cdot \big(|t - s| + \|\bar u - \bar v\|_{\mathbf{L}^1} \big)$.*

(iii) *If $\bar u \in \mathcal{D}^\varepsilon$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t, \cdot) = S^\varepsilon_t \bar u$ coincides with the solution of (1.1)-(1.2) obtained by piecing together the solutions of the corresponding Riemann problems determined by the ε-Riemann Solver.*

As soon as Theorem 3 is established, letting $\varepsilon \to 0$ it will be an easy matter to show that the semigroups S^ε converge to a unique semigroup S, having all the properties stated in Theorem 1. Throughout the following, we thus fix some $\varepsilon > 0$ and concentrate on the construction of the semigroup S^ε, with constants $L, \eta_0 > 0$ independent of ε. This requires several steps.

STEP 1. We begin by showing that a Lipschitz semigroup exists, whose domain contains all suitably small perturbations of a Riemann data.

Proposition 5. *There exists a neighborhood of the origin $\Omega_0 \subset I\!R^n$ and positive constants $L, \eta = \eta(\varepsilon) > 0$ for which the following holds. For every $u^-, u^+ \in \Omega_0$, there exists a closed domain $\mathcal{D} = \mathcal{D}_{(u^-, u^+)}$ of the form*

$$\mathcal{D} \doteq \left\{ u \in BV;\quad \int_{-\infty}^0 \big| u(x) - u^- \big|\, dx + \int_0^\infty \big| u(x) - u^+ \big|\, dx < \infty,\quad Q(u) \leq \eta \right\}, \tag{2.8}$$

and a Lipschitz semigroup S on $\mathcal{D}$ satisfying (ii) and (iii) in Theorem 3 (with $\mathcal{D}^\varepsilon$ replaced by $\mathcal{D}$). Such a semigroup is unique, up to the domain. Its trajectories are Viscosity ε-solutions of (1.1).

It should be noted that Proposition 5 is much weaker than Theorem 3. Indeed, in the theorem the constant η_0 is independent of ε. This is essential, since we eventually need to consider the limit as $\varepsilon \to 0$. In Proposition 5, however, we allow ourselves to choose η small depending on ε. In particular, by taking $\eta << \varepsilon^2$, we can assume that every function $u \in \mathcal{D}_{(u^-, u^+)}$ has the same number of large shocks of strength $|\sigma| > \varepsilon$ (i.e. the same "qualitative structure") as the solution of the Riemann problem with data u^-, u^+.

The construction of the semigroup S relies on the ideas described in the Introduction. For sake of clarity, we first describe the constructive procedure in the special case where the solution of the Riemann problem (2.6) does not contain any shock of strength $\geq 2\varepsilon$. In this case, by choosing $\eta > 0$ sufficiently small, we can assume that all functions u in the set $\mathcal{D} = \mathcal{D}_{(u^-,u^+)}$ in (2.8) contain shocks only of strength $< 3\varepsilon$. Because of the definitions (2.1)-(2.2), inside the positively invariant domain $\mathcal{D}$ the time evolution is thus determined by a system where shock and rarefaction curves coincide.

On a given interval $[0,T]$, our approximations will be piecewise Lipschitz solutions of a quasilinear system, except for a finite number of times $0 = t_0 < t_1 < \ldots < t_\nu < T$ where a restarting procedure is used. Namely, to prevent a loss of regularity, at suitable times t_ℓ we shall replace $u(t_\ell-,\cdot)$ with a "nicer" function $u(t_\ell+,\cdot)$.

First of all, we choose a time step $\Delta t = \delta_1 > 0$ and partition the time axis into intervals of the form

$$I_{m,h} \doteq \big[\tau_{m,h-1},\ \tau_{m,h}\big[\ \doteq \ \left[\Big(m + \frac{h-1}{n}\Big)\delta_1,\ \ \Big(m + \frac{h}{n}\Big)\delta_1\right[\,, \tag{2.9}$$

with $m \in \mathbb{N}$, $h \in \{1,\ldots,n\}$. Moreover, we fix a constant state u^* and call $\lambda_1^* < \ldots < \lambda_n^*$ the eigenvalues of the matrix $A(u^*)$. For $h = 1,\ldots,n$, let $A_h(u)$ be the matrix with the same eigenvectors as $A(u)$, but whose eigenvalues are

$$\lambda_1^*,\ \ldots\ ,\lambda_{h-1}^*,\ \lambda_h^* + n\big(\lambda_h(u) - \lambda_h^*\big),\ \ldots\ ,\lambda_n^*. \tag{2.10}$$

In other words,

$$A_h(u)\mathbf{v} = n\big(\lambda_h(u) - \lambda_h^*\big)\big(l_h(u)\cdot\mathbf{v}\big)r_h(u) + \sum_{i=1}^n \lambda_i^*\big(l_i(u)\cdot\mathbf{v}\big)r_i(u) \qquad\qquad \mathbf{v} \in \mathbb{R}^n.$$

On each subinterval $I_{m,h}$, our approximation u will have the following properties. Each $u(t,\cdot)$ is piecewise Lipschitz continuous and satisfies

$$u_t + A_h(u)u_x = 0 \tag{2.11}$$

a. e. outside the jumps. At every point $x_\alpha(t)$ where $u(t,\cdot)$ is discontinuous, the jump occurs always in the h-characteristic family and the following approximate Rankine-Hugoniot conditions hold:

$$u(t,x_\alpha+) = R_h(\sigma_\alpha)\big(u(t,x_\alpha-)\big), \tag{2.12}$$

$$\dot{x}_\alpha = \frac{1}{|\sigma_\alpha|}\int_{\sigma_\alpha}^0 \lambda_h\Big(R_h(s)\big(u(t,x_\alpha-)\big)\Big)\,ds, \tag{2.13}$$

for some $\sigma_\alpha < 0$. As usual, $\sigma \mapsto R_h(\sigma)(u)$ describes the h-rarefaction curve through the state u.

For a given piecewise Lipschitz initial data, the local existence and uniqueness of a piecewise Lipschitz solution to (2.11)–(2.13) follows from the standard theory of quasilinear hyperbolic equations. Let us briefly examine for how long this solution can retain its piecewise Lipschitz regularity. To fix the ideas, assume that, at some time $\tau \in I_{m,h}$, the gradient components $u_x^i = l_i(u)\cdot u_x$

of $u(\tau,\cdot)$ satisfy the bounds (for a. e. $x \in I\!R$)

$$|u_x^i| \leq M \qquad\qquad i = 1,\ldots,n, \tag{2.14}$$

$$u_x^h \geq -1. \tag{2.15}$$

(i) Whenever two or more h-shocks interact, they simply join together forming a single h-shock, without generating outgoing waves of any other family. Indeed, by construction all shocks in u belong the the h-family. Moreover, according to our modified dynamics, shock and rarefaction curves coincide. As a consequence, the number of shock fronts can only decrease in time.

(ii) For $i \neq h$, the i-th eigenvalue of the matrix $A_h(u)$ is constantly equal to λ_i^*. Hence, for the system (2.11), the i-th family is linearly degenerate. As a consequence, all gradient components u_x^i, $i \neq h$, remain uniformly bounded. To estimate the growth in the (genuinely nonlinear) component u_x^h, let $x = x(t)$ be an h-characteristic line, so that

$$\dot{x}(t) = \lambda_h^* + n\Big(\lambda_h\big(u(t,x)\big) - \lambda_h^*\Big).$$

Differentiating along this line, from (2.11) we obtain an equation of the form

$$\frac{d}{dt}u_x^h\big(t,x(t)\big) = -n\big[\nabla\lambda_h \cdot r_h(u)\big]\cdot(u_x^h)^2 + \sum_{i\neq j} G_{hij}(u)u_x^i u_x^j, \tag{2.16}$$

where the functions G_{hij} are uniformly bounded. Due to the squared term on the right-hand side, the component u_x^h may well approach $-\infty$ in finite time. However, by (2.16) and (2.15), this component can be bounded from below in terms of the solution to the O. D. E.

$$\dot{Z} = -aZ^2 + bZ - c, \qquad\qquad Z(0) = -1, \tag{2.17}$$

where the positive constants a, b, c depend only on the coefficients G_{hij} and on the uniform bounds already available on the other components u_x^i, $i \neq h$. This provides a lower bound on the time when the solution u of (2.11)-(2.13) may lose its piecewise Lipschitz regularity.

Since the regularity of solutions is not preserved globally in time, the construction of piecewise Lipschitz approximate solutions on a given interval $[0,T]$ must involve some restarting procedures. These restartings are of two types.

(1) At time $t = 0$ we approximate the initial data $\bar{u}$ with a piecewise Lipschitz function $u(0+,\cdot)$ containing only 1-shocks. At each time $t = \tau_{m,h}$ at (2.9), we replace the function $u(t-,\cdot)$ (which contains only h-shocks) with a new function $u(t+,\cdot)$ containing only $(h+1)$-shocks [only 1-shocks in the case $h=n$].

(2) At a time $t \in I_{m,h}$ when $\inf_x u_x^h(t,x)$ is getting close to $-\infty$, we replace $u(t-,\cdot)$ with a new function $u(t+,\cdot)$, still containing only h-shocks, which satisfies (2.15). This can always be accomplished by inserting several small downward jumps in regions where u_x^h is large and negative (fig. 4).

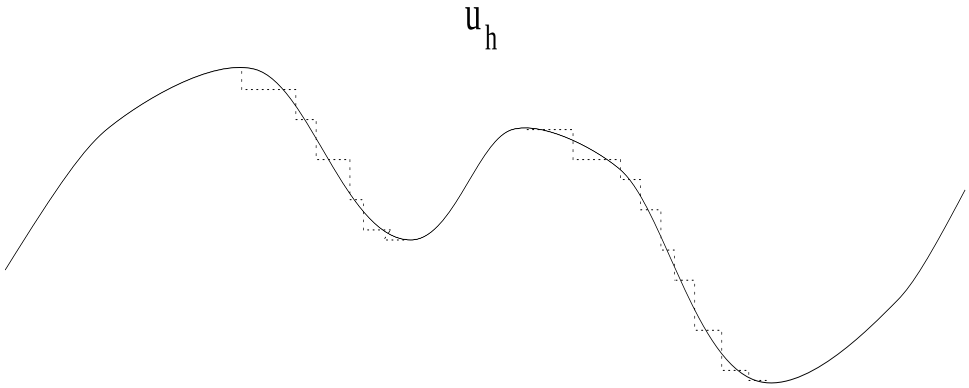

Fig.4

In all cases, at each restarting time t, we require that the distance $\big\|u(t+,\cdot)-u(t-,\cdot)\big\|_{\mathbf{L}^1}$ be suitably small. Moreover, we make sure that the total strength of waves and the interaction potential are changed very little by the restarting procedure.

By the analysis in **(i)-(ii)**, the construction of an approximate solution on a given interval $[0,T]$ can be accomplished with a finite number of restartings. Indeed, on each subinterval $I_{m,h}$ we can derive an a-priori bound of the form (2.14) on every linearly degenerate component u_x^i, $i \neq h$. Hence, by (2.15) and (2.16), the difference $t_{\ell+1} - t_\ell$ between two restarting times is bounded from below by the length of the interval where the solution of (2.17) is defined.

We can now repeat the above construction with different values of the time step δ_1, letting $\delta_1 \to 0$, and obtain a sequence of approximate solutions $(u_\nu)_{\nu \geq 1}$. By possibly taking a subsequence, a compactness argument yields some function $u = \lim u_\nu$. We claim that this limit provides an ε-solution to (1.1)-(1.2).

Intuitively, this is seen as follows. Consider any wave, say of the k-th family. In an exact solution, this wave should travel with speed $\lambda_k(u)$. On the other hand, in an approximate solution with time step $\Delta t = \delta_1$, by (2.10) such a wave will travel with speed

$$\dot{x} = \begin{cases} \lambda_k^* & \text{if} \qquad t \notin \bigcup_{m \geq 1} I_{m,k}, \\ \lambda_k^* + n\big(\lambda_k(u) - \lambda_k^*\big) & \text{if} \qquad t \in \bigcup_{m \geq 1} I_{m,k}. \end{cases}$$

Therefore, on any interval of the form

$$\big[m\delta_1,\ (m+1)\delta_1\big[\ =\ I_{m,1}\cup\cdots\cup I_{m,n},$$

the average speed of a k-wave coincides with $\lambda_k(u)$. Letting $\delta_1 \to 0$ we thus obtain an exact solution.

Of course, one could construct approximate solutions by a less sophysticated technique, such as the Glimm scheme or wave-front tracking. The use of piecewise Lipschitz approximations with successive restartings, however, offers a major advantage compared with all other methods. Namely, the distance between any two approximate solutions can now be carefully estimated. This is done as follows (fig. 5).

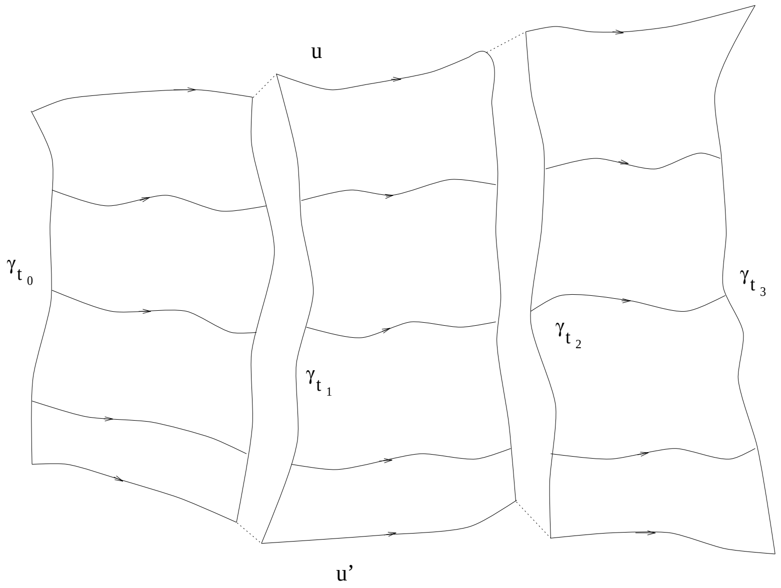

Fig.5

For a fixed $\delta_1 > 0$, let u, u' be any two approximate solutions, constructed on the interval $[0,T]$ according to the above procedure. Consider any piecewise regular path $\gamma_0 : \theta \mapsto u^\theta(0,\cdot)$ joining $u(0,\cdot)$ with $u'(0,\cdot)$. We can then construct a one-parameter family of approximate solutions u^θ within the class of piecewise Lipschitz functions, implementing a restarting procedure at suitable times t_ℓ and carefully controlling the length of the path $\gamma_t : \theta \mapsto u^\theta(t,\cdot)$.

At times where all functions u^θ solve the same quasilinear system (2.11) with jump conditions (2.12)-(2.13), the weighted length of the generalized tangent vector $\|du^\theta/d\theta\|_{u^\theta}$ does not increase. Moreover, a careful construction guarantees that, at each time t where a restarting procedure is applied (simultaneously to all functions $u^\theta(t-,\cdot)$), the weighted length of the path γ_t changes only by a very small amount. In the end, we obtain a bound on the length of the path γ_T, and hence on the distance $\|u(T,\cdot) - u'(T,\cdot)\|_{\mathbf{L}^1}$, based on the length of the path γ_0. Letting $\delta_1 \to 0$, this yields the Lipschitz estimate (ii) in Theorem 3.

It is worth observing that, if (1.1) is a special system where all shock and rarefaction curves coincide [T1], then the above construction already provides a complete proof of Theorem 1. For general systems, however, we still have to describe a proof of Proposition 5, in the case where the solution of the Riemann problem (u^-, u^+) contains one or more shocks with strength $|\sigma_i| \geq 2\varepsilon$.

To fix the ideas, assume that the ε-solution of the Riemann problem (1.1)-(2.6) has ν shocks of strength $\geq 2\varepsilon$, say in the families $j_1, \ldots, j_\nu \in \{1, \ldots, n\}$, plus possibly other shocks of smaller strength. Then, if we choose $\eta << \varepsilon^2$ sufficiently small, every function $u \in \mathcal{D}_{(u^-,u^+)}$ will contain exactly one shock of strength $> \varepsilon$ for each of the families $j_1, \ldots, j_\nu$, plus possibly other shocks (of different families) all with strength $< 3\varepsilon$. For any fixed $\delta_1, \delta_2 > 0$, a piecewise Lipschitz continuous approximation $u = u(t,x)$ is constructed by choosing a time step $\Delta t = \delta_1$ and inserting ν "boundary layers" of width δ_2 around the large shocks, say located at $y_{j_1}(t) < \cdots < y_{j_\nu}(t)$. For $t \in I_{m,h}$ as in (2.9), the function u will have the following properties.

- At each point y_i, $i \in \{j_1, \ldots, j_\nu\}$, the ε-Rankine-Hugoniot conditions (2.3)-(2.4) hold.

- Inside each boundary layer

$$B_i \doteq \big[y_i(t) - \delta_2,\ y_i(t) + \delta_2\big] \qquad i \in \{j_1, \ldots, j_\nu\}, \tag{2.18}$$

  the evolution of u is determined by a quasilinear system where all characteristic speeds are constant (hence all fields are linearly degenerate).

- Outside the boundary layers B_i, the function u is a solution of the quasilinear system (2.11), with jumps occurring only in the h-characteristic field. All these jumps have strength $|\sigma| < 3\varepsilon$ and satisfy the ε-Rankine-Hugoniot equations (2.12)-(2.13).

The reason for inserting the boundary layers is the following. Within a time interval $I_{m,h}$, if a small h-shock were to hit one of the large shocks, the interaction could produce several outgoing shocks of various families. By subsequent interactions, the total number of shocks could thus approach $+\infty$ within a very short time, and the algorithm would break down. To avoid this, we adopt an additional restarting procedure:

(3) At a time t when an h-shock has penetrated inside one of the boundary layers B_i, this shock is replaced by a steep Lipschitz continuous compression wave, before hitting the big shock at $y_i(t)$.

By letting all wave speeds be constant inside B_i, we make sure that this steep compression wave (of the h-family) does not "break", reforming an h-shock almost immediately. Since all wave speeds are bounded, it takes a positive amount of time for an h-wave to penetrate the boundary layer B_i of width $\delta_2 > 0$ and hit the shock at $y_i(t)$. Therefore, the length of the interval $[t_\ell,\ t_{\ell+1}]$ between two consecutive restarting times remains uniformly positive. In a finite number of steps, our approximate solution can thus be constructed on the whole interval $[0, T]$.

A detailed construction of approximate solutions in this more general case will be given in Section 3, together with the estimates on the weighted length of a generalized tangent vector. The restarting procedures are carefully analyzed in Section 4. The proof of Proposition 5 is then completed in Section 5, showing that, as $\delta_1, \delta_2 \to 0$, our approximate solutions converge to a unique limit, continuously depending on the initial data.

STEP 2. For a given $\varepsilon > 0$, we now construct a continuous semigroup S^ε on a domain $\mathcal{D}^\varepsilon$, satisfying (i) and (iii) in Theorem 3.

Observe that, in Step 1, we obtained a family of semigroups with the properties (ii) and (iii) but not (i), because the domains $\mathcal{D}_{(u^-,u^+)}$ would shrink to a point as $\varepsilon \to 0$. On the other hand, the domains $\mathcal{D}^\varepsilon$ which we now consider remain uniformly large, satisfying (i) for some $\eta_0 > 0$ independent of ε. The trajectories of the semigroup S^ε are constructed by a wave-front tracking algorithm. This algorithm is essentially the same as described in [B-J, B6], except for the use of the ε-approximate Riemann solver (2.7).

The basic idea in the wave-front tracking method is well known [D, DP2, B2, R]. Start with a piecewise constant function $u(0,\cdot)$ close to the initial data $\bar{u}$. At each point of jump, one approximately solves the corresponding Riemann problem within the class of piecewise constant functions. This yields an approximate solution defined up to the first time t_1 where one or more wave-front interactions take place. The new Riemann problem is then solved again within the class of piecewise constant functions, prolonging the solution up to some time t_2 where the second set of interactions takes place, etc. In order to keep finite the total number of wave-fronts, we shall use two distinct procedures for approximately solving a Riemann problem: an accurate method, which possibly introduces several new fronts, and a simplified method, which avoids the introduction of new wave-fronts. The algorithm involves two (strictly positive) parameters: $\bar{\sigma}$, bounding the maximum size of rarefaction fronts, and $\bar{\rho}$, determining which Riemann problems will be accurately solved.

For a given Riemann problem with data u^-, u^+, these two solution methods are described below. By $[\![s]\!]$ we denote here the integer part of s.

Accurate ε-Riemann solver: Let $\omega_0 = u^-, \omega_1, \ldots, \omega_n = u^+$ be the states present in the ε-

approximate solution of the Riemann problem, as in (2.7). The piecewise constant approximation u is then obtained by replacing each rarefaction wave with a rarefaction fan. More precisely, if $\omega_i = \Psi_i^\varepsilon(\sigma_i)(\omega_{k-1})$ with $\sigma_i > 0$, we divide this jump into $p_i = 1 + [\![\sigma_i/\bar\sigma]\!]$ equal parts, inserting the intermediate states

$$\omega_{i,0} = \omega_{i-1},\ \omega_{i,1},\ \ldots\ ,\ \omega_{i,p_i} = \omega_i.$$

Each small jump $(\omega_{i,\ell-1}, \omega_{i,\ell})$ travels with speed $\lambda_i(\omega_{i,\ell})$, i.e. with the characteristic speed of its right state.

Simplified ε-Riemann Solver: Assume that the Riemann problem is determined by the interaction of two waves of distinct families, say $i < j$, with sizes σ_i, σ_j. Call u^l, u^m, u^r the left, middle and right states before the interaction. Clearly, $u^m = \Psi_j^\varepsilon(\sigma_j)(u^l)$, $u^r = \Psi_i^\varepsilon(\sigma_i)(u^m)$. We then solve the Riemann problem in terms of two outgoing wave-fronts of the same families, still with sizes σ_i, σ_j. The solution will thus involve a middle state $\tilde u^m = \Psi_i^\varepsilon(\sigma_i)(u^l)$ and a new right state $\tilde u^r = \Psi_j^\varepsilon(\sigma_j)(\tilde u^m)$. In general, $\tilde u^r \neq u^r$. The jump $(\tilde u^r, u^r)$ is then propagated along a *non-physical* wave-front, travelling with a fixed speed $\hat\lambda$, larger than all characteristic speeds.
In the case where both incoming wave-fronts belong to the same i-th family and have sizes σ_i, σ_i', the Riemann problem is solved by a single outgoing i-wave of size $\sigma_i + \sigma_i'$, together with a non-physical wave-front connecting the states $\Psi_i^\varepsilon(\sigma_i + \sigma_i')(u^l)$ and u^r, travelling with speed $\hat\lambda$.
Finally, in the case where a non-physical front hits an i-wave of size σ_i, the Riemann problem is solved in terms of an outgoing i-wave of the same size σ_i, and a non-physical front always travelling with speed $\hat\lambda$.

To complete the description of the algorithm, it remains to specify which Riemann solver is used at any given interaction:

- The accurate method is used at time $t = 0$, and at every interaction where the product of the strengths of the incoming waves is $\geq \bar\rho$.
- The simplified method is used at every interaction involving a non-physical wave-front, and also at interactions where the product of the strengths of the incoming waves is $< \bar\rho$.

In the above, we tacitly assumed that only two wave-fronts interact at any given point. This can always be achieved by an arbitrarily small change in the speed of one of the interacting fronts.

Given any initial data $\bar u$ with sufficiently small total variation, consider a sequence of piecewise constant functions such that $\bar u_\nu \to \bar u$ in $\mathbf{L}^1$. Calling N_ν the number of jumps in $\bar u_\nu$, choose the parameter values

$$\bar\sigma_\nu \doteq \frac{1}{\nu}, \qquad\qquad \bar\rho_\nu \doteq e^{-(N_\nu+\nu)}.$$

For each $\nu \geq 1$, using the above algorithm we now construct a piecewise constant approximate solution u_ν, with $u_\nu(0,\cdot) = \bar{u}_\nu$. Each u_ν is defined for all $t \in [0,\infty[$ and has a finite number of lines of discontinuity in the t-x plane. By a straightforward adaptation of the estimates in [B-J, B6] one checks that, as $\nu \to \infty$,

(i) The total variation of $u_\nu(t,\cdot)$ remains uniformly small,

(ii) The maximum strength of rarefaction fronts in u_ν approaches zero,

(iii) The total strength of all non-physical waves approaches zero.

By (i), Helly's theorem guarantees the existence of a subsequence which converges to some function u in $\mathbf{L}^1_{loc}$. By (ii) and (iii), u is an ε-solution of (1.1). Moreover, relying on the lengthy construction performed in Step 1, we can show that this limit function u is unique and depends continuously on the initial data.

Proposition 6. *Let $\bar{u} \in \mathbf{L}^1$ have sufficiently small total variation. Then any sequence of approximations u_ν generated by the above wave-front tracking algorithm converges to a unique limit $u = u(t,x)$, continuously depending on the initial data $\bar{u}$. The map*

$$(t,\bar{u}) \mapsto u(t,\cdot) \doteq S_t^\varepsilon \bar{u}$$

is a continuous semigroup. Every trajectory is a Viscosity ε-solution of the corresponding Cauchy problem (1.1), (1.2).

The key step in the proof of Proposition 6 is to show that, locally in the t-x plane, the limit solution u coincides with a trajectory of one of the Lipschitz semigroups constructed in Step 1. More precisely, fix any point $(\tau,\bar{x})$ and define the one-sided limits $u^- = u(\tau,\bar{x}-)$, $u^+ = u(\tau,\bar{x}+)$. Choosing $\rho > 0$ small enough, the truncated function

$$\tilde{u}(x) = \begin{cases} u(\tau,x) & \text{if} \quad x \in [\bar{x}-\rho,\ \bar{x}+\rho], \\ u^- & \text{if} \quad x < \bar{x}-\rho, \\ u^+ & \text{if} \quad x > \bar{x}+\rho, \end{cases} \tag{2.19}$$

lies in the domain $\mathcal{D}_{(u^-,u^+)}$ of one of the semigroups S constructed in Step 1. One then has

$$\big(S_{t-\tau}\tilde{u}\big)(x) = u(t,x) \qquad\qquad \forall (t,x) \in \Gamma, \tag{2.20}$$

where Γ is the domain of dependency

$$\Gamma \doteq \big\{(t,x); \quad t \geq \tau, \quad |x-\bar{x}| \leq \rho - (t-\tau)\big\}. \tag{2.21}$$

From the uniqueness of the semigroup S, proved as in [B4], it thus follows the uniqueness of the limit function u.

STEP 3. All the remaining analysis aims at establishing the uniform Lipschitz continuity of the semigroup S^ε, with a Lipschitz constant independent of ε, thus proving (ii) in Theorem 3. This is not an easy task: at this stage we can compare different solutions, and show the Lipschitz continuity on the initial data, only within a narrow class of functions with the same wave-front structure. Roughly speaking, by "wave-front structure" or "configuration" we refer here to the number of shocks of strength $> \varepsilon$, and to the order in which they interact. As shown in Step 1, we can construct suitable approximations by choosing artificial wave speeds λ_i^* and inserting the buffer zones (2.18) around each big shock. The distance between two approximations can then be estimated, but only if these approximations are obtained by the same choice of the λ_i^*, and by the insertion of the same number of boundary layers. Of course, this cannot be the case for solutions with different wave-front configurations (fig. 6).

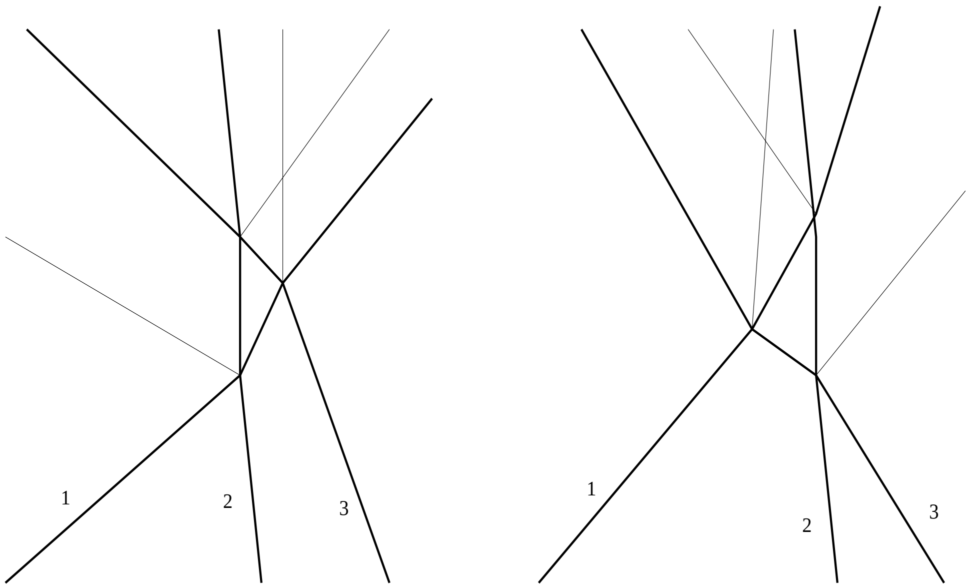

Fig.6

Our basic approach is the following. Consider a regular path of initial data $\gamma_0 : \theta \mapsto \bar{u}^\theta$, $\theta \in [0,1]$. For each $t > 0$, define the corresponding path

$$\gamma_t : \theta \mapsto S_t^\varepsilon \bar{u}^\theta. \tag{2.22}$$

First, we prove that the weighted length of γ does not increase in time if all solutions u^θ have the same wave-front structure. By the continuity of the semigroup S^ε, proved in Step 2, the result can be extended to the case where the configuration changes only at finitely many values of the parameter θ. Finally, we show that any path γ of solutions can be uniformly approximated by a path $\tilde{\gamma}$ such that the wave-front structure of the function $\tilde{u}^\theta = \tilde{\gamma}(\theta)$ changes only at finitely many parameter values.

To implement the above program, we introduce a set of conditions providing the structural stability of an ε-solution u. They will imply that any ε-solution u' sufficiently close to u has the same number of large shocks as u, interacting in the same order. In particular, u' can be obtained as limit of the same type of piecewise Lipschitz approximations used in the construction of u.

As a preliminary, given an ε-solution u and a point (t,x) with $t>0$, define

$$u^\eta(t',x') \doteq u(t+\eta t',\ x+\eta x'), \qquad \omega_{(t,x)}(t',x') \doteq \lim_{\eta\to 0+} u^\eta(t',x'). \tag{2.23}$$

By the analysis in [DP1, B-LF2], the self-similar limit in (2.23) is well defined in $\mathbf{L}^1_{loc}$. On the upper half plane where $t'>0$, $\omega_{(t,x)}$ coincides with the solution of the standard Riemann problem with data $u(t,x-)$, $u(t,x+)$. On the lower half plane where $t'<0$, the function $\omega_{(t,x)}$ may contain a set of incoming waves, including shocks and centered compression waves of various families. We shall distinguish three cases.

CASE 1: The strength of all incoming waves is $<\varepsilon^2$.

CASE 2: There exists an incoming shock of strength $>\varepsilon^3$ while the total strength of all other incoming waves is $<\varepsilon^8$.

CASE 3: There exist two incoming shocks, both of strength $>\varepsilon^9$, while the total strength of all other incoming waves is $<\varepsilon^{20}$.

Observe that the above cases are not mutually exclusive, nor do they cover all possibilities. A suitable definition of structural stability can now be introduced.

Definition 5. An ε-solution u of (1.1) is *Structurally Stable at the point* (t,x) if the corresponding function $\omega_{(t,x)}$ in (2.23) satisfies one of the Cases 1, 2, 3. We say that u is *Structurally Stable on* $[t^*,T]$ iff u is stable at each point (t,x) with $t^*\le t\le T$, $x\in\mathbb{R}$.

There are two typical examples where structural stability fails:

- Three shocks, each with strength $>\varepsilon$, interact at a single point (t,x).
- The point (t,x) is the center of a compression wave, of strength $>\varepsilon$.

Observe that, in both cases, an arbitrarily small perturbation may destroy the wave-front configuration of the solution u. The next proposition states that this does not happen for a structurally stable solution $\bar{u}$.

In the following, by $V\big(u(t);\ J\big)$ we denote the total strength of waves in $u(t)$ inside the set J.

Proposition 7. *Let $\tilde u$ be an ε-solution of (1.1), structurally stable at the point $(\tau, \bar x)$. Then there exists $r^* > 0$ such that, for every fixed $r \in]0, r^*]$, on the interval $J \doteq [\bar x - 7r^* - r,\ \bar x + 7r^* + r]$ the following holds. For every ε-solution u sufficiently close to $\tilde u$ in the $\mathbf{L}^1$ norm, one has:*

(i) In Case 1, $u(\tau - r)$ satisfies

$$V\big(u(\tau - r);\ J\big) \le C\varepsilon^2. \tag{2.24}$$

(ii) In Case 2, $u(\tau - r)$ has a shock of strength $> \varepsilon^3/2$, located at some point $y \in J$. Moreover,

$$V\big(u(\tau - r);\ J \setminus \{y\}\big) \le C\varepsilon^8. \tag{2.25}$$

(iii) In Case 3, $u(\tau - r)$ has two approaching shocks, both of strength $> \varepsilon^9/2$, located at points $y, y' \in J$. Moreover,

$$V\big(u(\tau - r);\ J \setminus \{y, y'\}\big) \le C\varepsilon^{20}. \tag{2.26}$$

As usual, by C we denote a constant depending only on the system, and not on $u, \tilde u, r, r^*, \varepsilon \ldots$ On the other hand, it is understood that the conclusion of Proposition 7 holds for every solution u satisfying

$$\big\|u(t) - \tilde u(t)\big\|_{\mathbf{L}^1} < \varepsilon_* \qquad \forall t,$$

with $\varepsilon_* > 0$ possibly depending on r. For exact solutions of the system (1.1), or approximations obtained by wave-front tracking, a more general result in this direction is established in [B-LF2], relying on the decay estimates (1.21). All proofs remain valid for ε-solutions, without any change. In Section 7 we will show that the same conclusions (i)–(iii) also hold if u is a sufficiently accurate piecewise Lipschitz approximation, constructed according to our algorithm.

In the above setting, a trapezoid of the form

$$\Gamma \doteq \Big\{(t,x); \qquad t \in [\tau - r',\ \tau + r''],\ \ |x - \bar x| \le 7r^* - (t - \tau)\Big\} \tag{2.27}$$

with $r', r'' \in]0, r^*]$ will be called a *Stabilizing Block* for the solution $\tilde u$ around the point $(\tau, \bar x)$. Observe that, by Proposition 7, every solution u suitably close to $\tilde u$ in the $\mathbf{L}^1$ norm has the same wave-front configuration as $\tilde u$ inside the trapezoid Γ. This motivates our definition. We say that the stabilizing block Γ is of type 1, 2 or 3 according to the three cases considered in Proposition 7.

A straightforward compactness argument yields the following covering lemma.

Lemma 1. *Let $M, t_* > 0$ be given and let $\tilde u$ be structurally stable on $[t_*, T]$. Then there exist times $0 < t_0 < \tau_1 < t_1 < \tau_2 < \cdots < \tau_N < t_N = T$, points x_{ij} and values $r_{ij} > 0$, $j = 1, \ldots, N_i$ with* *the following properties. For each* $i = 1, \ldots, N$*, the intervals* $[x_{ij} - r_{ij},\ x_{ij} + r_{ij}]$ *cover* $[-M, M]$*. Moreover,*

$$[t_*, T] \subset [t_0, t_N], \qquad \max\left\{t_i - \tau_i,\ \tau_i - t_{i-1}\right\} \leq \min_j r_{ij} \qquad i = 1, \ldots N. \tag{2.28}$$

Each trapezoid

$$\Gamma_{ij} \doteq \left\{(t,x); \qquad t \in [t_{i-1},\ t_i], \quad |x - x_{ij}| \leq 7r_{ij} - (t - \tau_i)\right\} \tag{2.29}$$

is a stabilizing block for $\tilde{u}$ *around the point* (τ_i, x_{ij})*.*

Example 2. Consider an ε-solution $u = u(t,x)$ with the following wave-front structure (fig. 7). Initially, u contains only one large shock, say at a point y_1, plus other small waves. As time progresses, a second shock is formed, say located at y_2. The strength of this shock increases continuously from 0 to some value $> \varepsilon$. At some time τ_2, the two shocks interact, generating three outgoing shocks.

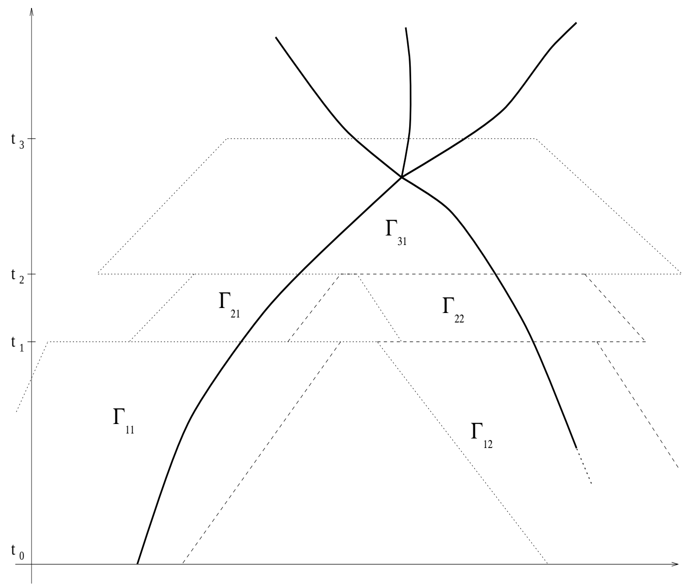

Fig.7

In this case, u is structurally stable. A covering in terms of stabilizing blocks is illustrated in fig. 7. The block Γ_{12} is of type 1, $\Gamma_{11}, \Gamma_{21}, \Gamma_{22}$ are of type 2, while Γ_{31} is of type 3.

We can now take the main step toward establishing the Lipschitz continuity of the semigroup S^ε. More precisely, we will show that S^ε is contractive w.r.t. a weighted distance defined as follows. Let u be a piecewise Lipschitz function having jumps at the points $x_1 < \cdots < x_N$. Assume that the ε-solution of the Riemann problem determined by the jump at x_α consists of a single shock in the k_α-th characteristic family, of strength $|\sigma_\alpha|$. For any $v \in \mathbf{L}^1$, define the components

$$v_i(x) \doteq \big\langle l_i(u(x)),\ v(x)\big\rangle, \qquad u_x^i(x) \doteq \big\langle l_i(u(x)),\ u_x(x)\big\rangle \qquad i = 1, \ldots, n. \tag{2.30}$$

The weighted norm of a generalized tangent vector $(v, \xi) \in T_u = \mathbf{L}^1 \times \mathbb{R}^N$ is then defined as

$$\big\|(v,\xi)\big\|_u \doteq \sum_{i=1}^n \int_{-\infty}^{\infty} \big|v_i(x)\big| W_i^u(x)\, dx + \sum_{\alpha=1}^N \big|\xi_\alpha\big| |\sigma_\alpha| W_{k_\alpha}^u(x_\alpha). \tag{2.31}$$

Here $W_i^u(x)$ is the weight assigned to an i-wave located at x. It has the form

$$W_i^u(x) \doteq 1 + \kappa_1 R_i^u(x) + \kappa_1\kappa_2 Q(u), \tag{2.32}$$

where

$$R_i^u(x) \doteq \left[\sum_{j\le i}\int_x^\infty + \sum_{j\ge i}\int_{-\infty}^x\right] \big|u_x^j(y)\big|dy + \left[\sum_{\substack{k_\alpha \le i\\ x_\alpha > x}} + \sum_{\substack{k_\alpha \ge i\\ x_\alpha < x}}\right] |\sigma_\alpha|, \tag{2.33}$$

$$\begin{aligned} Q(u) \doteq &\sum_{i\le j} \iint_{x<y} \big|u_x^j(x)\big|\big|u_x^i(y)\big|\ dxdy + \sum_{k_\alpha \le k_\beta,\ x_\alpha > x_\beta} |\sigma_\alpha\sigma_\beta| \\ &+ \sum_\alpha \left[\sum_{i\le k_\alpha} |\sigma_\alpha| \int_{x_\alpha}^\infty \big|u_x^i(x)\big|dx + \sum_{i\ge k_\alpha} |\sigma_\alpha| \int_{-\infty}^{x_\alpha} \big|u_x^i(x)\big|dx\right], \end{aligned} \tag{2.34}$$

and κ_1, κ_2 are suitably large constants. Thinking of $\big|u_x^i(x)\big|dx$ as the strength of an infinitesimal i-wave in u located at x, the weight (2.32) can thus be interpreted as

$$\begin{aligned} W_i^u(x) = 1 + \kappa_1 &\big[\text{amount of waves in } u \text{ which approach an } i\text{-wave located at } x\big] \\ &+ \kappa_1\kappa_2 \big[\text{global interaction potential of waves in } u\big]. \end{aligned}$$

A general formula for the weighted length of a tangent vector, valid for functions u with arbitrary jumps, will be given in Section 8. Here and in the sequel there will be slight variations in the definitions of Q, due to the fact that couples of waves of the same family may or may not be regarded as approaching. The different choices made at various stages of the paper aim at simplifying the computations in the proofs. They are all essentially equivalent in that the basic interaction estimates (1.9) always hold.

In turn, the weighted length of a piecewise regular path can then be defined as in (1.47), by integrating the norm of a tangent vector. To make sense of the weighted length of an arbitrary path $\gamma : \theta \mapsto u^\theta$ in the case where the u^θ are general BV functions, we set

$$\|\gamma\|_\star \doteq \lim_{\varepsilon\to 0+} \inf \Big\{ \|\gamma'\|_\star\ ;\quad \gamma' \text{ is a piecewise regular path,}\quad \big\|\gamma'(\theta) - \gamma(\theta)\big\|_{\mathbf{L}^1} < \varepsilon \ \text{ for all } \theta \Big\}. \tag{2.35}$$

Proposition 8. *Let $\gamma_0 : \theta \mapsto \bar{u}^\theta$ be a regular path of initial data, say defined for $\theta \in \Theta \doteq [a, b]$. For some value $\bar{\theta}$, assume that the corresponding ε-solution $u^{\bar{\theta}}$ of (1.1) is structurally stable on $[0, T]$. Then there exists $\rho > 0$ such that the weighted length of the path $\theta \mapsto S_t^\varepsilon \bar{u}^\theta$, restricted to $\theta \in [\theta', \theta'']$, is a non-increasing function of time, for every subinterval $[\theta', \theta''] \subseteq [\bar{\theta} - \rho,\ \bar{\theta} + \rho]$.*

As a first step toward the proof of Proposition 8, we show that the conclusion holds in the special case where the solutions u^θ vary with θ only within a single isolating block.

Lemma 2. *Let Γ be as in (2.27) and let $\theta \mapsto u^\theta$ be a family of ε-solutions of (1.1), defined for $t \in [t', t''] \doteq [\tau - r',\ \tau + r'']$, $\theta \in \Theta \doteq [a, b]$. Assume that, at the initial time $t' = \tau - r'$, all functions $u^\theta(t', \cdot)$ satisfy one of the conclusions of Proposition 7: either (i), or (ii), or (iii). Moreover, assume that for all θ one has*

$$u^\theta(t', x) = u^{\bar{\theta}}(t', x) \qquad\qquad x \notin [\bar{x} - r^*,\ \bar{x} + r^*]. \tag{2.36}$$

Then the weighted length of the path $\gamma_{t''} : \theta \mapsto u^\theta(t'', \cdot)$ is smaller or equal to the length of the path $\gamma_{t'} : \theta \mapsto u^\theta(t', \cdot)$.

The proof of Lemma 2 is achieved by constructing a path γ' of piecewise Lipschitz approximate solutions, arbitrarily close to γ, whose length does not increase in time. Observe that, inside the trapezoid Γ, all solutions u^θ have the same wave-front structure as $u^{\bar{\theta}}$. Namely:

- in Case 1 no large shock is present,
- in Case 2 there is exactly one large shock,
- in Case 3 there are initially two large incoming shocks. At some time t these shocks interact, generating a number of outgoing waves determined by the corresponding Riemann problem.

Inside the region Γ, the construction of approximate solutions, continuously depending on the initial data, can thus be carried out by the same algorithm used in the proof of Proposition 5. Actually, in Cases 1 and 2, we are dealing with a set of functions all contained in one of the domains $\mathcal{D}_{(u^-, u^+)}$ considered at (2.8) (if suitably extended outside Γ). Case 3, on the other hand, forces us to consider a domain $\mathcal{D}$ of functions which possibly contain two large approaching shocks. In this more general case, the construction of piecewise Lipschitz approximate solutions will be given in Section 8.

Outside the stabilizing block Γ, our approximate solutions will be constructed by wave-front tracking. Observe that, in this outer region, nothing is known about the wave-front structure of the functions u^θ. However, by (2.36) and finite propagation speed, all these solutions coincide. Their continuous dependence on the parameter θ is thus trivial.

In order to apply Lemma 2, we shall need to replace an arbitrary path γ with a new path $\tilde{\gamma} : \theta \mapsto \tilde{u}^\theta$ such that the values $\tilde{u}^\theta(x)$ locally vary with θ only inside one single stabilizing block. A suitable localization procedure is described below.

Definition 6. A path $\gamma : \theta \mapsto u^\theta$ has *Localized Variation* if, for every $\varepsilon^* > 0$ and every θ^*, there exists $\delta > 0$ and a point x^* such that

$$u^\theta(x) = u^{\theta^*}(x) \qquad \text{whenever } |\theta - \theta^*| < \delta, \quad |x - x^*| \geq \varepsilon^*. \tag{2.37}$$

In other words, as θ varies in a neighborhood of θ^*, the values of u^θ are allowed to change only inside a small neighborhood of some point x^*. For example, if u, v are two distinct continuous functions, then the path $\theta \mapsto u \cdot \chi_{]-\infty,\theta]} + v \cdot \chi_{]\theta,\infty[}$ has localized variation, while the path $\theta \mapsto \theta u + (1-\theta)v$ does not. The approximation of piecewise regular paths with paths having localized variation will play a key role in the sequel.

Lemma 3. *Let $\gamma : \theta \mapsto u^\theta$ be a piecewise regular path. Then, for every $\varepsilon^* > 0$, there exists another piecewise regular path γ' with localized variation such that*

$$\|\gamma'\|_\star \leq \|\gamma\|_\star + \varepsilon^*, \qquad \big\|\gamma'(\theta) - \gamma(\theta)\big\|_{\mathbf{L}^1} < \varepsilon^* \qquad \forall \theta. \tag{2.38}$$

The proof of Proposition 8 now goes as follows. For t in a small interval $[0, t_0]$, all solutions u^θ remain piecewise Lipschitz, and the result is an immediate consequence of Proposition 4. Choose M large enough so that all supports of the functions $u^\theta(t,\cdot)$ are contained inside $[-M, M]$. We then apply Lemma 1 to the structurally stable solution $u^{\bar\theta}$ and obtain a covering of the rectangle $[t_0, T] \times [-M, M]$ in terms of stabilizing blocks Γ_{ij}. Let (τ_i, x_{ij}) and r_{ij} be as in (2.29). For some $\rho > 0$ small and all $\theta \in \Theta \doteq [\bar\theta - \rho, \ \bar\theta + \rho]$, we can thus assume that every suitably accurate approximation to a solution u^θ has the same wave-front structure as $u^{\bar\theta}$, on each trapezoid Γ_{ij}.

By Lemma 3, the path $\gamma_{t_0} : \theta \mapsto S^\varepsilon_{t_0} \bar{u}^\theta$ can be replaced by a new path $\gamma^+_{t_0} : \Theta \mapsto BV$, having localized variation and almost the same length as γ_{t_0}. By compactness, we can cover the interval Θ with finitely many subintervals $\Theta_k \doteq [\theta_{k-1}, \theta_k]$, $k = 1, \ldots, m_1$, so that, as θ varies inside Θ_k, the values of $u^\theta(t_0+, \cdot) \doteq \gamma^+_{t_0}(\theta)$ vary inside a single interval, say $I_{1j} \doteq [x_{1j} - r_{1j}, \ x_{1j} + r_{1j}]$. Here $j = j(k)$.

By applying Lemma 2 to each subinterval Θ_k, we conclude that the weighted length of the path

$$\gamma^-_{t_1} : \theta \mapsto S^\varepsilon_{t_1 - t_0}\big(\gamma^+_{t_0}(\theta)\big)$$

is less or equal than the weighted length of $\gamma_{t_0^-}$.

We now apply Lemma 3, replacing $\gamma_{t_1}^-$ with a new path $\gamma_{t_1}^+$, having localized variation and almost the same length as $\gamma_{t_1^-}$. Then we use Lemma 2 to estimate the length of the path $\gamma_{t_2}^- \doteq S_{t_2-t_1}^\varepsilon(\gamma_{t_1}^+)$, etc... Since at every step the increase in the weighted length $\|\gamma_{t_i}^+\|_\star - \|\gamma_{t_i}^-\|_\star$ can be kept arbitrarily small, after N steps we obtain a path $\gamma_{t_N}^-$, arbitrarily close to $\gamma_{_T} : \theta \mapsto S_{_T}^\varepsilon \bar{u}^\theta$, and whose weighted length is bounded by the length of γ_0 up to an arbitrary small correction. This achieves the proof.

From Proposition 8 and the continuity of the semigroup S^ε, one obtains

Corollary 1. *If u^θ is structurally stable on $]0,T]$ for all but finitely many values of θ, then the weighted length of the path $\gamma_t : \theta \mapsto u^\theta$ is non-increasing in time.*

The proof of Theorem 3 is completed by showing that the above assumption of structural stability can be removed, relying on a perturbation argument.

Proposition 9. *Let $\gamma_0 : \theta \mapsto \bar{u}^\theta$ be a regular path of initial conditions. Then the conclusion of Proposition 8 remains valid, even without assuming the structural stability of the ε-solution $u^{\bar\theta}$.*

The proof is based on the following argument. Call Θ^* the set of all values of θ for which u^θ is structurally unstable. If $\bar\theta \notin \Theta^*$, or if $\bar\theta$ is an isolated point of Θ^*, the result is clear. Consider the remaining case, where $\bar\theta$ is a limit point of Θ^*. Observe that there exists only finitely many points (t_ℓ, x_ℓ) in the t-x plane where $u^{\bar\theta}$ is unstable. Indeed, by Definition 5, at every such point an amount of interaction $> \varepsilon^{18}$ must take place.

To fix the ideas, assume, for example, that for infinitely many $\theta \in \Theta^*$ in a neighborhood of $\bar\theta$, the corresponding ε-solution u^θ contains three large shocks interacting at the single point (t^θ, x^θ), with $(t^\theta, x^\theta) \to (t^{\bar\theta}, x^{\bar\theta})$ as $\theta \to \bar\theta$. At a suitable time $\tau < t^{\bar\theta}$ we can then perform an arbitrarily small perturbation of the path $\gamma_\tau : \theta \mapsto u^\theta(\tau)$, in such a way that all but finitely many of the perturbed solutions are structurally stable. More precisely, given any $\varepsilon' > 0$, we consider a smooth scalar map $\varphi = \varphi(\theta, x)$ with

$$\|\varphi\|_{\mathcal{C}^3} < \varepsilon' \tag{2.39}$$

and construct a new path $\tilde\gamma_\tau : \theta \mapsto \tilde{u}^\theta(\tau)$, where

$$\tilde{u}^\theta(\tau, x) = u^\theta\big(\tau-,\ x + \varphi(\theta, x)\big). \tag{2.40}$$

Because of (2.39), the new path will be close to the old one and have almost the same weighted length. In addition, by (2.40) the locations of the three large shocks in each u^θ will be slightly shifted. Therefore, an application of the coarea formula [E-G] will show that, for a "generic" function φ, these three shocks will no longer interact at a single point, but only two at a time, for all except finitely many values of the parameter θ. The previous estimates can thus be applied to the perturbed path $\tilde{\gamma}$, for $t > \tau$.

This will complete the proof of Theorem 3. Letting $\varepsilon \to 0$, since the constants $L, \eta_0 > 0$ are independent of ε, we establish Theorem 1.

References

[B-B] P. Baiti and A. Bressan, Lower semicontinuity of weighted path length in BV, to appear in *Geometrical Optics and Related Topics*, F. Colombini and N. Lerner Eds., Birkhäuser.

[B-J] P. Baiti and H. K. Jenssen, On the front tracking algorithm, Preprint S.I.S.S.A., Trieste 1996.

[B1] A. Bressan, Contractive metrics for nonlinear hyperbolic systems, *Indiana Univ. Math. J.* **37** (1988), 409-421.

[B2] A. Bressan, Global solutions to systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* **170** (1992), 414-432.

[B3] A. Bressan, A contractive metric for systems of conservation laws with coinciding shock and rarefaction curves. *J. Differential Equations* **106** (1993), 332-366.

[B4] A. Bressan, A locally contractive metric for systems of conservation laws, *Ann. Scuola Norm. Sup. Pisa* **IV-22** (1995), 109-135.

[B5] A. Bressan, The unique limit of the Glimm scheme, *Arch. Rational Mech. Anal.* **130** (1995), 205-230.

[B6] A. Bressan, *Lecture Notes on Systems of Conservation Laws,* S.I.S.S.A., Trieste 1996.

[B-C1] A. Bressan and R. M. Colombo, The semigroup generated by 2×2 conservation laws, *Arch. Rational Mech. Anal.* **133** (1995), 1-75.

[B-C2] A. Bressan and R. M. Colombo, Unique solutions of 2×2 conservation laws with large data, *Indiana Univ. Math. J.* **44** (1995), 677-725.

[B-C3] A. Bressan and R. M. Colombo, Decay of positive waves in nonlinear systems of conservation laws, Preprint S.I.S.S.A., Trieste 1996.

[B-LF1] A. Bressan and P. LeFloch, Uniqueness of weak solutions to systems of conservation laws, *Arch. Rational Mech. Anal.*, to appear.

[B-LF2] A. Bressan and P. LeFloch, Structural stability of solutions to systems of conservation laws, to appear.

[B-M1] A. Bressan and A. Marson, A variational calculus for shock solutions of systems of conservation laws, *Comm. Part. Diff. Equat.* **20** (1995), 1491-1552.

[B-M2] A. Bressan and A. Marson, Error bounds for a deterministic version of the Glimm scheme, *Arch. Rational Mech. Anal.*, to appear.

[Cl] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[C] M. Crandall, the semigroup approach to first-order quasilinear equations in several space variables, *Israel J. Math.* **12** (1972), 108-132.

[D] C. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation laws, *J. Math. Anal. Appl.* **38** (1972), 33-41.

[DP1] R. DiPerna, Singularities of solutions of nonlinear hyperbolic systems of conservation laws, *Arch. Rational Mech. Anal.* **60** (1975), 75-100.

[DP2] R. DiPerna, Global existence of solutions to nonlinear hyperbolic systems of equations, *J. Differential Equations* **20** (1976), 187-212.

[E-G] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine properties of Functions,* CRC Press, Boca Raton, 1992.

[G] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure Appl. Math.* **18** (1965), 697-715.

[K] S. Kruzkov, First order quasilinear equations with several space variables, *Math. USSR Sbornik* **10** (1970), 217-243.

[Lx] P. Lax, Hyperbolic systems of conservation laws II, *Comm. Pure Appl. Math.* **10** (1957), 537-566.

[L] T. P. Liu, The deterministic version of the Glimm scheme, *Comm. Math. Phys.* **57** (1977), 135-148.

[O] O. Oleinik, Discontinuous solutions of nonlinear differential equations, *Usp. Mat. Nauk.* **12** (1957), 3-73; English transl. in *Amer. Math. Soc. Transl. Ser. 2,* **26**, 95-172.

[R] N. H. Risebro, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.* **117** (1993), 1125-1139.

[Sch] M. Schatzman, Continuous Glimm functionals and uniqueness of solutions of the Riemann problem, *Indiana Univ. Math. J.* **34** (1985), 533-589.

[Sm] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Springer-Verlag, New York, 1983.

[T1] B. Temple, Systems of conservation laws with invariant submanifolds, *Trans. Amer. Math. Soc.* **280** (1983), 781-795.

[T2] B. Temple, No L^1-contractive metrics for systems of conservation laws, *Trans. Amer. Math. Soc.* **288** (1985), 471-480.