Qualitative Behavior of Conservation Laws with Reaction Term and Nonconvex Flux

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Abstract. The aim of the paper is to study qualitative behavior of solutions to equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(u),$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $u = u(x,t) \in \mathbb{R}$. The main new feature with respect to previous works is that the flux function f can have finitely many inflections, intervals in which it is affine, and corner points. The function g is supposed to be zero at 0 and 1, and positive in between.

We prove existence of heteroclinic travelling waves connecting the two constant states for opportune choice of speeds. Finally we analyze the large-time behavior of the Riemann problem with values 0 and 1, showing convergence to one of the travelling waves. The speed of the limiting profile is explicitly characterized.

Key words and phrases: balance laws, travelling waves, asymptotic behavior, Riemann problem.

1. Introduction

Recently many results have been proved on the qualitative behavior of the scalar balance law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(u), \tag{1.1}$$

where $(x,t) \in \mathbb{R} \times \mathbb{R}_+$, $u = u(x,t) \in \mathbb{R}$ ([1]-[3], [6], [7], [13]-[18], [21]-[23]). These results wanted to explain the new asymptotic patterns appearing in (1.1) when considering it as a conservation law with an additional source term. Since the aim was to observe the new features introduced by this term, it has been given a large freedom in the choice of g.

On the contrary the flux function f was usually supposed regular and strictly convex. At most it has been allowed to f to have one or two change of convexity ([4],[15], [22]). The convexity hypothesis is strong. Indeed it implies an ordering property on propagation speeds of characteristic curves, due to the fact that f' is increasing. In [22] it is explicitly showed that this hypothesis is too restrictive to understand the behavior in the general case. In fact it is proved that, for a case in which the flux is C^1 and has two changes of convexity, the solution of a Riemann problem for (1.1) is discontinuous until a finite time, and then remains continuous for any larger time; on the contrary in [1] it is proved that in the regular convex case, when a discontinuity comes into view, it cannot disappear.

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This leads to expect different behaviors in the qualitative analysis of (1.1) with weaker convexity assumptions.

The aim of the present paper is exactly to allow more freedom in the choice of the function f. In particular with our hypotheses the flux f can have the following properties: finetely many inflection points, intervals in which it is an affine function and corner points. Therefore the class of possible fluxes is widely enlarged with respect to previous results. Precise assumptions on f will be given later (see Section 2).

The main tool in the convex case is the technique of generalized characteristics ([1]-[2]). When f has one inflection point the technique can be still adapted, but it becomes quite complicated ([4]). In the case with two (or more) changes of convexity, there is no available extension of this tool to our knowledge.

On the contrary our approach applies to more general cases. The main difference is the following. Generalized characteristics gives an "almost explicit" representation of the solution at any time; then information on asymptotic behaviour can be extracted. Instead the method used in this work constists in constructing the expected large-time profile of the solution (looking for entropy travelling waves of the problem) and then, by a huge use of comparison principle for entropy solutions, in proving the convergence to the given profile.

We suppose that the function g has two zeros, and is positive in between.
The role of the term g can be easily understood by considering solutions of (1.1) independent of the space variable x. The evolution is governed by the ordinary differential equation

$$\frac{dU}{dt} = g(U). \tag{1.2}$$

Every $s \in \mathbb{R}$, such that $g(s) = 0$, is a constant solution of (1.1). The stability character of such solutions (with respect to equation (1.2)) depends on the sign of g near s. In our case the smaller zero is unstable, while the other is stable.

Qualitative analysis of equation (1.1) should give information on the evolution of solutions when the reactive phenomenon, above sketched, is coupled with the convective mechanism of transport of the quantity u, represented by the flux function f. We will refer to this kind of equations as reaction–convection equations (RC).

Similar studies had been made for a very long time for reaction–diffusion equations (RD), both for scalar case and for systems (see [24] and references therein). On the contrary, to our knowledge, the qualitative analysis of hyperbolic systems with a non-dissipative reaction term is still at the starting point.

In the scalar case RD and RC equations have some features in common. Both problems present the same property concerning heteroclinic travelling waves: once fixed the asymptotic states of the wave, there is a limiting speed, say c_0, such that there is existence of travelling fronts if and only if the speed belongs either to $(-\infty, c_0]$, or to $[c_0, +\infty)$, depending on the asymptotic states.

In [8] it was proved that the solution to a Riemann Problem for an RD equation converges (in a certain sense) to a travelling profile with limiting speed. In this work we prove an analogous result (Theorem 2.4) for RC equations.

Note that similar result on existence of travelling waves can be proved in the case of the simultaneous presence of reaction, diffusion and convection (RDC equations). It would be interesting, both from a mathematical and from an applicative point of view, to give a more complete analysis for this problem, for example concerning the large time behavior of solutions. Some new results on the subject are contained in [5].

Now we give a sketch of the new results contained in this work. The precise statements will be given in Section 2.

A typical question we want to answer is the following. Let us consider an initial datum for the Cauchy problem for (1.1) of Riemann type, with the two constant values given by the two constant solution of (1.1). How do these two different values of u interact? That is: which is the behavior of the solution as time increases?

Following the results of the convex case, we look for the possible asymptotic profiles as heteroclinic travelling waves solutions of (1.1). We prove that there exists an extremal speed such that this kind of fronts exists if and only if the speed is either greater than or smaller than this extremal value. Moreover we exhibit the existence of such waves in a constructive way. This permits us to have precise results on regularity and on the structure of jump sets of the profiles.

Then we turn our attention to the large–time behavior. We prove, by the construction of an opportune family of entropy super- and subsolution of (1.1), the convergence of the solution of Riemann problem to the travelling wave with the extremal speed.

The plan of the paper is the following.

In Section 2, we give the precise assumptions on the function f and g and the statements of the main results.

Section 3 concernes with entropy travelling waves for (1.1). In order to prove uniqueness of such profiles with a given speed, we need some results on entropy solution of ordinary differential equations. Then we prove the existence for opportune choices of speed.

The content of Section 4 is the asymptotic behavior of solutions. We construct some useful supersolution by joining together different entropy solution of (1.1). Results on large-time profiles are consequences of comparison results.

2. Mathematical Framework and Results

We consider the Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = g(u) & (x,t) \in \mathbb{R} \times \mathbb{R}_+, \\ u(x,0) = u_0(x) & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Here u is a real valued function of the variables (x,t).

The initial data u_0 will be chosen with range in $[0,1]$, therefore, by comparison principle, the solutions to (2.1) will have the same property. This allows us to give assumptions on f and g just in the interval $[0,1]$.

On the function f and g we make the following hypotheses:

$$\text{(F)} \quad \begin{cases} f \in C([0,1]), \quad f(0) = f'(0) = 0, \\ \text{there exist } 0 \equiv a_0 < a_1 < \cdots < a_{n+1} \equiv 1 \text{ such that } f \in C^1([a_i, a_{i+1}]) \text{ and} \\ f' \text{ is either strictly monotone or constant in any interval } (a_i, a_{i+1}); \end{cases}$$

$$\text{(G)} \qquad g \in C([0,1]), \quad g(0) = g(1) = 0, \quad g(s) > 0 \quad \forall s \in (0,1).$$

Note that no assumptions on the multiplicity of 0 and 1 as zeroes of g has been made. Moreover hypotheses $f(0) = f'(0)$ are not restrictive.
We denote with $f'_+(s)$ and $f'_-(s)$ the right, respectively, the left derivative of f at s.

It is well–known (see [9]) that under assumptions (F) and (G) the Cauchy problem has a unique entropy solution, i.e. for any positive $\psi \in C_0^\infty(\mathbb{R} \times \mathbb{R}_+)$ it holds

$$\text{(2.2)} \quad \iint\limits_{\mathbb{R}\times\mathbb{R}_+} \left[|u-k| \frac{\partial \psi}{\partial t} + \operatorname{sgn}(u-k)(f(u) - f(k)) \frac{\partial \psi}{\partial x} + \operatorname{sgn}(u-k) g(u) \psi \right] dx\, dt \geq 0 \quad \forall k \in \mathbb{R},$$

and

$$\operatorname*{ess\,lim}_{t \to 0^+} \int_a^b |u(x,t) - u_0(x)|\, dx \qquad -\infty < a < b < +\infty.$$

In the case of piecewise smooth solution, formula (2.2) can be substituted by an easier criterion. Let $N \in \mathbb{N}$ and $\Gamma_i := \{(\gamma_i(t), t) : t > 0\}$ $(i = 1, \dots, N)$ be smooth curves in $\mathbb{R} \times \mathbb{R}_+$, then $u \in C^1(\mathbb{R} \times \mathbb{R}_+ \setminus \bigcup_{i=1}^N \Gamma_i)$ is an entropy solution of (1.1) if and only if

(i) u is a classical solution of (1.1) in $\mathbb{R} \times \mathbb{R}_+ \setminus \bigcup_{i=1}^N \Gamma_i$;

(ii) for any $t > 0$, $i = 1, \dots, N$ the Rankine-Hugoniot condition holds

$$\frac{d\gamma_i}{dt}(t) = \frac{f(u_i^+(t)) - f(u_i^-(t))}{u_i^+(t) - u_i^-(t)},$$

where $u_i^\pm(t) := \lim_{\varepsilon \to 0^+} u(t, \gamma_i(t) \pm \varepsilon)$;

(iii) for any $t > 0$, $i = 1, \dots, N$ the Oleinik condition holds

$$(s - u_i^-(t)) \left(f(s) - \frac{d\gamma_i}{dt}(t)(s - u_i^-(t)) - f(u_i^-(t)) \right) \geq 0 \quad \forall s \in \mathcal{I}(u_i^-(t), u_i^+(t)),$$

where $\mathcal{I}(a,b) := (\min\{a,b\}, \max\{a,b\})$ $(a, b \in \mathbb{R})$.

This criterion suggests to look for travelling waves with a finite number of possible jump points.

Definition 2.1. An *entropy travelling wave* for equation (1.1) is an entropy solution $u = u(x,t)$ of the form

$$u(x,t) = \phi(x - ct), \tag{2.3}$$

where ϕ is a monotone function belonging to $C^1(\mathbb{R} \setminus \{\xi_1, \dots, \xi_N\})$ for some $\xi_1, \dots, \xi_N$. The function ϕ is called *profile* of the wave, and c is called the *speed* of the wave.

It follows at once from the previous criterion that a function u of the form (2.3) is an entropy solution if and only if ϕ satisfies the differential equation

$$(f'(\phi) - c)\phi' = g(\phi),$$

where it is regular and the Rankine-Hugoniot and Oleinik jump conditions at any point ξ_i $(i = 1, \dots, N)$. Moreover we look for *heteroclinic* travelling waves, i.e. waves satisfying the additional property

$$\lim_{\xi \to \pm\infty} \phi(\xi) = \phi_\pm \in \{0, 1\} \quad \text{with } \phi_- \neq \phi_+. \tag{2.4}$$

This request is natural when considering the problem of interaction between two constant states of the equation. The values $\phi_\pm$ are referred to as the *asymptotic states* of the travelling wave ϕ.

Now we can state the result on existence of heteroclinic waves for (1.1).

Theorem 2.2. *Let assumptions (F) and (G) be satisfied and let c_* be defined by*

$$c_* := \inf_{s \in (0,1]} \frac{f(s)}{s}. \tag{2.5}$$

Then there exist heteroclinic travelling waves with $\phi_- = 0$ and $\phi_+ = 1$ if and only if $c \leq c_*$.

Moreover, fixed the speed c, the waves are unique (up to translations).

Note that the extremal speed c_* does not depend on the function g, except for the zeroes 0 and 1 and for the sign of g in between. Moreover, since $f'(0) = 0$, $c_* \leq 0$.

A similar result can be proved for heteroclinic waves with asymptotic states $\phi_- = 1$ and $\phi_+ = 0$. In this case there is existence if and only if the speed c is greater than or equal to c^*, where

$$c^* := \sup_{s \in (0,1]} \frac{f(s)}{s} \geq 0.$$

The proof of Theorem 2.2 is constructive. Thus, as a byproduct, we obtain a result concerning the structure and the regularity of the waves. As stated below, there is a critical speed separating continuous profiles from discontinuous ones.

Theorem 2.3. *Let (F) and (G) hold.*

Let $c_0 := \min\{f'_{\pm}(s) : s \in [0,1]\} \le c_*$ *and let* ϕ_c *be a travelling profile with* $\phi_- = 0$, $\phi_+ = 1$ *and speed* c.

(i) If $c < c_0$ *then* $\phi_c \in C^1(\mathbb{R})$;

(ii) if $c_0 \le c \le c_*$ *then* $\phi_c \in C(\mathbb{R} \setminus D) \cap C^1(\mathbb{R} \setminus C \cup D)$, *where* D *and* C *are finite disjoint sets.*

Moreover supp ϕ_c *is bounded from below if and only if* $c = c_*$ *and there holds*

$$\begin{cases} \textit{either } \dfrac{f(\sigma)}{\sigma} = c_* \textit{ for some } \sigma > 0, \\ \textit{or } c_* = 0 \textit{ and } \dfrac{f'(s)}{g(s)} \in L^1(0,\varepsilon) \textit{ for some } \varepsilon > 0. \end{cases}$$

Now we want to examine the qualitative behavior of entropy solutions of (2.1). In particular we want to analyze interaction between the two constant solutions of (1.1). To this aim, we consider the Riemann initial datum, given by

$$u_0^R(x) := \begin{cases} 0 & x < 0, \\ 1 & x \ge 0. \end{cases} \tag{2.6}$$

We make the following additional assumption:

$$\begin{cases} \text{either } \dfrac{f(\sigma)}{\sigma} = c_* \text{ for some } \sigma > 0, \\ \text{or } c_* = 0 \text{ and } \dfrac{f'(s)}{g(s)} \in L^1(0,\varepsilon) \text{ for some } \varepsilon > 0. \end{cases} \tag{H}$$

Note that if f is twice differentiable at 0, $g'(0)$ exists and is not zero, then assumption (H) is satisfied.

As stated in Theorem 2.3, (H) holds if and only if the wave with speed c_* has support bounded from below.

In the convex case, since $\dfrac{f(s)}{s}$ is increasing, $c_* = f'(0) = 0$. This is exactly the speed of the limiting profile of the solutions to the Riemann problem in that case. The following result shows that the same behavior holds also in the non-convex case.

Theorem 2.4. *Assume (F) , (G) and (H). Let* $u^R = u^R(x,t)$ *be the entropy solution of problem (2.1) with* u_0 *given by (2.6) and* $\phi_* = \phi_*(\xi)$ *be the travelling profile for (1.1) with asymptotic states* $\phi_- = 0, \phi_+ = 1$, *speed* c_* *and supp* $\phi_* = [0, +\infty)$.

Then, for any $\varepsilon > 0$, *there exists a set* N_ε *such that*

(i) meas $N_\varepsilon < \varepsilon$,

(ii)

$$\lim_{t\to+\infty} \|u^R(\cdot + c_*t, t) - \phi_*(\cdot)\|_{L^\infty(\mathbb{R}\setminus N_\varepsilon)} = 0. \tag{2.7}$$

If, in addition, $1-\phi_* \in L^1(a,+\infty)$ *for some* $a \in \mathbb{R}$*, then*

$$\lim_{t\to+\infty} \|u^R(\cdot,t)-\phi_*(\cdot - c_* t)\|_{L^p(\mathbb{R})} = 0, \qquad \text{for any } p \geq 1.$$

Finally, if $f'(s) > 0$ *for any* $s \in (0,1]$*, the convergence in (2.7) is uniform over all* $\mathbb{R}$.

Hypothesis (H) is necessary in order to have (2.7). In fact we can also prove that if (H) does not hold, then the solution of problem (2.1), (2.6) converges to zero uniformly in subsets of $\mathbb{R}$ bounded from above. The precise statement is the following.

Theorem 2.5. *Assume (F) and (G). Let* $u = u(x,t)$ *be the entropy solution of problem (2.1), with* $u_0 \leq u_0^R$*. Moreover suppose* $f(s) > 0$ *for any* $s > 0$ *and* $\dfrac{f'(s)}{g(s)} \notin L^1(0,\varepsilon)$ *for any* $\varepsilon > 0$.

Then

$$\lim_{t\to+\infty} \|u(\cdot,t)\|_{L^\infty(-\infty,M)} = 0, \qquad \forall M \in \mathbb{R}. \tag{2.8}$$

Note that assumptions of Theorem 2.5 implies $c_* = f'(0) = 0$, thus the solution cannot converge to an entropy travelling wave in the sense of Theorem 2.4.

This phenomenon can be read in the following way. Suppose $f(s) \sim s^{\alpha+1}$ and $g(s) \sim s^\beta$ for $s \in (0,\varepsilon)$ for some $\alpha,\beta,\varepsilon > 0$. Then assumptions of Theorem 2.5 are satisfied if and only if $\alpha < \beta$. Hence if 0, as a zero of g, is "strong" (i.e. its multiplicity β is large) and the speed of propagation of small values of u, given by f', is "large", (i.e. the multiplicity α of 0 as zero of f' is small) then we cannot expect convergence to a travelling wave, but to zero, since the convective term rapidly moves small values of u to the right.

We conclude this Section giving some remarks on the multidimensional case.
As noted in [25], the analysis of multi-dimensional scalar conservation laws immediately leads to consider the nonconvex case for one-dimensional equations. Indeed let us consider the scalar equation

$$\frac{\partial u}{\partial t}(\mathbf{x},t) + \nabla_{\mathbf{x}} \cdot F(u(\mathbf{x},t)) = g(u(\mathbf{x},t)) \qquad \mathbf{x} \in \mathbb{R}^N, \quad t > 0, \tag{2.9}$$

where $N \in \mathbb{N}$ and $F : \mathbb{R} \to \mathbb{R}^N$ is given by the vector $(F_1,\ldots,F_N)$.

Given a unit vector $\mathbf{k} = (k_1,\ldots,k_N) \in S^{N-1} \subset \mathbb{R}^N$, suppose to consider the Riemann initial data for (2.9)

$$u(\mathbf{x},0) = u_0(\mathbf{x}) := \begin{cases} 0 & \mathbf{k}\cdot\mathbf{x} < 0, \\ 1 & \mathbf{k}\cdot\mathbf{x} \geq 0, \end{cases} \tag{2.10}$$

where the dot $\cdot$ represents the usual scalar product in $\mathbb{R}^N$.

In order to study asymptotic behavior, it is natural to look for planar travelling waves of (2.9), i.e. solutions of the form

$$u(\mathbf{x},t) = \Phi(\mathbf{k}\cdot\mathbf{x} - ct) \qquad (c \in \mathbb{R}),$$

with asymptotic states given by

$$\lim_{\xi\to-\infty}\Phi(\xi)=0,\qquad \lim_{\xi\to+\infty}\Phi(\xi)=1. \tag{2.11}$$

Substituting (2.11) in (2.9), we get the ordinary differential equation for the profile Φ

$$\Big(\sum_{i=1}^{N}k_iF_i'(\Phi)-c\Big)\Phi'=g(\Phi).$$

Thus the existence problem of travelling profiles for (2.9) is the same of the one-dimensional equation

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=g(u)$$

with

$$f(s):=\sum_{i=1}^{N}k_iF_i(s)\equiv F(s)\cdot\mathbf{k}. \tag{2.12}$$

Hence no convexity assumptions on the functions F_i can lead to convexity properties of the function f given by (2.12).

Our results on existence of entropy travelling waves can be easily applied to obtain existence of planar travelling waves for the multi-dimensional case for any direction $\mathbf{k}$ such that the function f, defined in (2.12), satisfies (F).

The critical speed obviously depends on $\mathbf{k}$, and precisely is given by

$$c_{\mathbf{k}}:=\inf_{s\in(0,1]}\frac{F(s)-F(0)}{s}\cdot\mathbf{k}.$$

Now we turn our attention to the large-time behavior of the solution to the Riemann problem (2.9), (2.10). The simmetry of the initial data suggests to construct the solution as follows. Let u^R be the solution of the one-dimensional Cauchy problem

$$\begin{cases}\dfrac{\partial v}{\partial t}+\dfrac{\partial f(v)}{\partial x}=g(v) & (x,t)\in\mathbb{R}\times\mathbb{R}_+,\\ v(x,0)=\begin{cases}0 & x<0,\\ 1 & x>0,\end{cases}\end{cases}$$

with f as in (2.12). It is easy to see that the unique entropy solution of (2.9), (2.10), say $u_{\mathbf{k}}=u_{\mathbf{k}}(\mathbf{x},t)$ can be written as

$$u_{\mathbf{k}}(\mathbf{x},t)=u^R(\mathbf{k}\cdot\mathbf{x},t).$$

Applying the results on convergence to travelling profiles of u^R, we get the convergence of the solution $u_{\mathbf{k}}$ to the travelling planar wave with profile Φ with speed $c_{\mathbf{k}}$ and supp $\Phi=[0,+\infty)$.

3. Entropy Travelling Waves

In this section we want to estabilish some results on the existence and the local structure of entropy travelling waves for (1.1). This leads to consider possibly discontinuous solutions to ordinary differential equations satisfying Rankine–Hugoniot and Oleinik conditions at the jump points. Precisely, the profile ϕ of a travelling wave is such that
(a) if $\xi \notin \{\xi_1, \dots, \xi_N\}$ the following ordinary differential equation is satisfied:

$$(f'(\phi(\xi)) - c)\,\phi'(\xi) = g(\phi(\xi));$$

(b) if $\xi \in \{\xi_1, \dots, \xi_N\}$ then

$$c(\phi^r - \phi^l) = f(\phi^r) - f(\phi^l)$$
$$(s - \phi^l)\big[f(s) - c(s - \phi^l) - f(\phi^l)\big] \geq 0 \quad \forall s \in \mathcal{I}(\phi^l, \phi^r),$$

where $\phi^l := \lim\limits_{\varepsilon \to 0^+} \phi(\xi - \varepsilon)$ and $\phi^r := \lim\limits_{\varepsilon \to 0^+} \phi(\xi + \varepsilon)$.

These properties suggest the following definition.

Definition 3.1. A function $\psi : I \equiv I_\delta(\xi_0) := \{\xi \in \mathbb{R} : |\xi - \xi_0| < \delta\} \to \mathbb{R}$ piecewise C^1 is an *entropy solution* of the ordinary differential equation

$$(f'(\psi) - c)\psi' = g(\psi), \tag{3.1}$$

if
(i) for any $\xi \in I$, such that there exists $J = I_\eta(\xi) \subset I$ with $\psi \in C^1(J)$, the equation (3.1) is satisfied;
(ii) for any $\xi \in I$, such that $\psi \notin C^1(I_\eta(\xi))$ for any $\eta > 0$,

$$c = \sigma(\psi^r, \psi^l) \tag{RH}$$

where

$$\sigma(\psi^r, \psi^l) := \begin{cases} \dfrac{f(\psi^r) - f(\psi^l)}{\psi^r - \psi^l} & \text{if } \psi^r \neq \psi^l, \\ f'(\psi^r) & \text{if } \psi^r = \psi^l, \end{cases}$$

and

$$(\psi^r - \psi^l)\{f(s) - [c(s - \psi^r) + f(\psi^r)]\} \geq 0 \quad \forall s \in \mathcal{I}(\psi^l, \psi^r) \tag{O}$$

An entropy solution of (3.1) defines locally an entropy solution of (1.1) by

$$u(x,t) := \psi(x - ct) \qquad \text{for } x - ct \in I_\delta(\xi_0). \tag{3.2}$$

Conversely, if u is a piecewise smooth entropy solution of (1.1) of the form (3.2), then ψ is an entropy solution of (3.1).

In the following Lemma, we assume $g > 0$. As showed by the following results, this hypothesis guarantees uniqueness of solution in some cases. We will apply these properties to obtain uniqueness of profiles of travelling waves.

It follows at once from the classical theory for ordinary differential equations that there exists locally a unique classical solution of (3.1) such that $\psi(\xi_0) = \psi_0$ if $f'(\psi_0) \neq c$ ($\xi_0 \in \mathbb{R}$). A detailed analysis of the integral curves of (3.1) shows that, if $f'(\psi_0) = c$, there is no local solution to the same problem. Thus the possibility to have global solution is due to the fact that it can be discontinuous.

Lemma 3.2. *Let ψ be an entropy solution of (3.1) in $I := (\xi_1, \xi_2)$ such that $\psi \in C^1(I \setminus \{\xi_0\})$ for some $\xi_0 \in I$.*

Then, if $f'_-(s) > c$, for s in a left neighborhood of $\psi(xi_2-)$), the solution ψ is uniquely determined by $\psi(\xi_2-)$ and it is monotone.

Proof. If $\psi \in C^1(I)$ the statement is a consequence of classical theory for ordinary differential equations.

Let $\psi \notin C^1(I)$. Set

$$\psi^{\pm} := \psi(\xi_0 \pm 0).$$

Then ψ^+ is uniquely determined by $\psi(\xi_2-)$. Since $f'_-(s) > c$ at the left of $\psi(\xi_2-)$, ψ is increasing in (ξ_0, ξ_2). Then $f'(s) > c$ for any $s \in (\psi^+, \psi(\xi_2-))$. The value ψ^- belongs to $\mathcal{S}$, where

$$\mathcal{S} := \{\sigma \in [0, \psi^+] : f(\sigma) = f(\psi^+) + c(\sigma - \psi^+), f(s) \geq f(\psi^+) + c(s - \psi^+) \quad \forall s \in (\sigma, \psi^+)\}.$$

Note that $\mathcal{S} \neq \emptyset$ since $\psi^+ \in \mathcal{S}$.

Let $\sigma_0 := \min \mathcal{S}$. Then $\psi^- = \sigma_0$. Indeed let us suppose, by contradiction, $\psi^- \in \mathcal{S} \setminus \{\sigma_0\}$. By definition of $\mathcal{S}$, in a neighborhood of ψ^-, either

$$(f'(s) - c)(s - \psi^-) \geq 0 \qquad \text{for } s \geq \psi^-,$$

or

$$f(s) = cs + \text{const.}.$$

In both cases such an f does not define ψ in any left neighborhood of ξ_0, and we obtain the contradiction. $\square$

Now we turn our attention to the construction of travelling waves.

Firstly, let us state the following Lemma.

Lemma 3.3. *Given $f \in Lip([0,1])$ with Lipschitz constant L and $c \leq L$, let $\mathcal{F}_c \subset C([0,1])$ be the the set of functions h such that*

(i) h is $L-$lipschitz;

(ii) $h \leq f$;

(iii) $h(s) - cs$ is monotone increasing.

Let

$$f_c(s) := \sup\{h(s) : h \in \mathcal{F}_c\}.$$

Then $f_c \in \mathcal{F}_c$.

Proof. The function f_c is well defined since the class $\mathcal{F}_c$ is bounded from above by f. We have to show that f_c enjoies properties (i), (ii) and (iii).

(i) For any s_1, s_2 and for any $h \in \mathcal{F}_c$, it holds

$$h(s_2) - L|s_1 - s_2| \leq h(s_1) \leq h(s_2) + L|s_1 - s_2|.$$

Taking the supremum in these inequalities, we get that f_c is $L-$lipschitz continuous.

(ii) From property (ii) we deduce that

$$h(s) \leq f(s) \qquad \forall h \in \mathcal{F}_c, \quad \forall s \in [0,1],$$

therefore the same property holds for the supremum.

(iii) Fix $0 \leq s_1 < s_2 \leq 1$. Then for any $h \in \mathcal{F}_c$

$$h(s_1) + c(s_2 - s_1) \leq h(s_2).$$

Taking the supremum in $\mathcal{F}_c$, we deduce

$$f_c(s_1) + c(s_2 - s_1) \leq f_c(s_2),$$

and so the monotonicity of $f_c(s) - cs$. $\square$

The main idea in the construction of the travelling wave is to substitute the flux function f with the function f_c. The monotonicity of $f_c(s) - cs$ permit us to construct the inverse of the profile of the wave. This function turns to be regular and monotone, thus, after an inversion, it gives the desired travelling wave.

To have a better understanding of the regularity of the wave and of the structure of the set of jumps, we need some more informations on f_c.

Lemma 3.4. *Let f and f_c as in Lemma 3.3. For any $s_0 \in (0,1)$, one of the following holds:*

(i) either $f_c(s_0) = f(s_0)$;

(ii) or there exists a left neighborhood of s_0, say I_0, such that

$$f_c(s) = f_c(s_0) + c(s - s_0) \qquad \forall s \in I_0.$$

Proof. If $c \leq -L$, then $f_c \equiv f$ and if $c = L$ then $f_c(s) = f(1) - c(1-s)$. In both cases (i) holds for any s.

Now let us suppose $-L < c < L$. Let s_0 be such that $f_c(s_0) < f(s_0)$, then, by continuity, there exists a neighborhood J of s_0 such that $f_c(s) < f(s)$ for any $s \in J$. Since $f_c \in \mathcal{F}_c$, by property (iii)

$$f_c(s) \leq f_c(s_0) + c(s - s_0) \qquad \forall s < s_0.$$

Let $0 < \eta \leq \delta$ to be fixed. Let us define

$$h(s) := \begin{cases} f_c(s_0 - \delta) + L(s - s_0 + \delta) & s_0 - \delta < s < s_0 - \eta, \\ f_c(s_0) + c(s - s_0) & s_0 - \eta < s < s_0, \\ f_c(s) & \text{elsewhere.} \end{cases}$$

Then, if δ is sufficently small,

$$f_c(s) \leq h(s) \leq f(s).$$

Now we fix η in such a way that h is continuous. Hence

$$f_c(s_0 - \delta) + L(\delta - \eta) = f_c(s_0) - c\eta,$$

and so η is given by

$$0 < \eta := \frac{f_c(s_0 - \delta) - f_c(s_0) + L\delta}{L - c} \leq \delta.$$

Since f_c is the supremum in the class $\mathcal{F}_c$, we derive that $f_c \equiv h$. $\square$

Lemma 3.5. *Let f and f_c as in Lemma 3.3. Suppose that f satisfies (F). Finally let $I = (s_1, s_2)$ such that*

$$f_c(s) = f_c(s_1) + c(s - s_1) \tag{3.3}$$

for any $s \in I$ and (3.6) false for any open set J containing I.

Then

(i) $f_c(s_2) = f(s_2)$ and, if $s_1 \neq 0$, $f_c(s_1) = f(s_1)$;
(ii) $f'_-(s_2) \leq c \leq f'_+(s_2)$;
(iii) $a_i \in [s_1, s_2]$ for some $i \in \{0, \dots, n-1\}$.

Proof. (i) Suppose by contradiction $f_c(s_1) \neq f(s_1)$. Then, by Lemma 3.4, there exists a left neighborhood I_1 of s_1 such that (3.3) holds in I_1. Hence (3.3) is true at any point of $I \cup I_1$, contradicting the maximality assumption on I.

Similar arguments can be used to prove $f_c(s_2) = f(s_2)$, if $s_2 < 1$. Let $s_2 = 1$ and assume $f(1) > f_c(1)$. Consider the function

$$h(s) := \begin{cases} f_c(s) & 0 \leq s \leq s_3, \\ f(1) - L(1-s) & s_3 \leq s \leq 1. \end{cases} \tag{3.4}$$

with s_3 given by

$$s_3 := 1 - \frac{f(1) - f_c(1)}{L - c}.$$

First of all we show that $s_3 \geq s_1$. Since f is $L-$Lipschitz,

$$\frac{f(1) - f(s_1)}{1 - s_1} \leq L. \tag{3.5}$$

Moreover, by assumptions on f_c,

$$\frac{f_c(1) - f(s_1)}{1 - s_1} = c. \tag{3.6}$$

Subtracting (3.5) to (3.6), we get

$$\frac{f(1) - f_c(1)}{1 - s_1} \leq L - c.$$

This implies

$$s_1 \leq 1 - \frac{f(1) - f_c(1)}{1 - s_1} \equiv s_3.$$

By definition of s_3, the function h, defined in (3.4), belongs to $\mathcal{F}_c$ and $h \geq f_c$. This contradicts the definition of f_c. Thus $f_c(1) = f(1)$.

(ii) From $f_c \leq f$ it follows $f'_-(s_2) \leq c$. If $f'_+(s_2) < c$, then there exists $s_3 \in (s_2, 1)$ such that

$$\frac{f(s_3) - f(s_1)}{s_3 - s_1} < c.$$

Hence

$$f_c(s_3) \le f(s_3) < f(s_1) + c(s_3 - s_1) = f_c(s_1) + c(s_3 - s_1),$$

implying

$$f_c(s_3) - cs_3 < f_c(s_1) - cs_1.$$

This contradicts the monotonicity of $f_c(s) - cs$.

(iii) Suppose by contradiction that $a_i \notin [s_1, s_2]$ for any i. Then $f \in C^1([s_1, s_2])$. If $f_c \equiv f$ over all I, the result is obvious. Now suppose that there is $s_3 \in (s_1, s_2)$ such that $f_c(s_3) < f(s_3)$. Then there are s_4 and s_5 such that $s_1 < s_4 < s_3 < s_5 < s_2$ and

$$f'(s_4) = \frac{f(s_3) - f(s_1)}{s_3 - s_1} > \frac{f_c(s_3) - f_c(s_1)}{s_3 - s_1} = c,$$
$$f'(s_5) = \frac{f(s_2) - f(s_3)}{s_2 - s_3} < \frac{f_c(s_2) - f_c(s_3)}{s_2 - s_3} = c.$$

Therefore we have proved that there are $s_4 < s_5 < s_2$ such that

$$f'(s_4) > f'(s_5) \quad \text{and} \quad f'(s_5) < f'(s_2).$$

Since f' is nonmonotonic in I, f changes convexity at some $s_0 \in I$. Thus $s_0 = a_i$ for some i. $\square$

Proof of Theorem 2.2. Let $c \le c_*$.

By the definition of the class $\mathcal{F}_c$ it follows that

$$f_c'(s) \ge c \qquad \text{a.e..}$$

Therefore, given $s_0 \in (0,1)$, the following function is well-defined

$$\xi(s) := \int_{s_0}^{s} \frac{f_c'(\sigma) - c}{g(\sigma)}\, d\sigma, \qquad \forall s \in (0,1). \tag{3.7}$$

and the derivative of ξ is

$$\frac{d\xi}{ds} = \frac{f'(s) - c}{g(s)} \ge 0 \quad \text{a.e..}$$

Set $\xi_0 := \xi(0)$ and $\xi_1 := \xi(1)$. Then

$$-\infty \le \xi_0 \le 0 \le \xi_1 \le +\infty,$$

and

$$\begin{cases} \xi_0 > -\infty \text{ if and only if } \dfrac{f_c'(s) - c}{g(s)} \in L^1(0, \varepsilon) \text{ for some } \varepsilon > 0, \\[2ex] \xi_1 < +\infty \text{ if and only if } \dfrac{f_c'(s) - c}{g(s)} \in L^1(1 - \varepsilon, 1) \text{ for some } \varepsilon > 0. \end{cases}$$

From hypothesis (F) and Lemma 3.5(iii), there exist at most finitely many intervals, say $[s_1, t_1], \dots, [s_N, t_N]$, such that

$$\frac{d\xi}{ds}(s) \equiv \frac{f_c'(s) - c}{g(s)} = 0 \qquad \forall s \in \bigcup_{i=1}^{N} [s_i, t_i].$$

We define the following multivalued function

$$\hat{\phi}(\xi) := \begin{cases} 0 & \xi \le \xi_0, \\ \xi^{-1}(\xi) & \xi_0 < \xi < \xi_1, \\ 1 & \xi \ge \xi_1. \end{cases}$$

If the function $\hat{\phi}$ is multivalued at $\bar{\xi} \in (\xi_0, \xi_1)$, then

$$\xi^{-1}(\xi) \in \bigcup_{i=1}^{N} [s_i, t_i].$$

Set

$$\xi_i := \xi^{-1}(s_i) \qquad i = 1, \dots, N.$$

Then we can define the profile ϕ of the wave:

$$\phi(\xi) := \begin{cases} \hat{\phi}(\xi) & \xi \in \mathbb{R} \setminus \{\xi_1, \dots, \xi_N\}, \\ s_i & \xi = \xi_i \quad i = 1, \dots, N. \end{cases}$$

If $s_i \neq t_i$, then ϕ is discontinuous at ξ_i and

$$\phi(\xi_i -) = s_i, \qquad \phi(\xi_i +) = t_i.$$

If $s_i = t_i$, then ϕ is continuous at ξ_i, but not differentiable.

If $s \notin \bigcup (s_i, t_i)$, then, by Lemma 3.4, $f_c(s) = f(s)$. Thus

$$\frac{d\xi}{ds} = \frac{f'(\phi) - c}{g(\phi)}$$

and (3.1) is satisfied.

If $s \in (s_i, t_i)$ for some i and $s_i \neq t_i$, the wave ϕ is discontinuous. We have to check that condition (RH) and (O) hold.

Let $s_i \neq 0$. Then (RH) is an immediate consequence of Lemma 3.5(i), indeed

$$\frac{f(s_i) - f(t_i)}{s_i - t_i} = \frac{f_c(s_i) - f_c(t_i)}{s_i - t_i} = c.$$

Moreover

$$f(s) - f_c(s) \ge 0.$$

Since for any $s \in (s_i, t_i)$

$$f_c(s) = f(t_i) + c(s - t_i),$$

we get

$$f(s) - f(t_i) - c(s - t_i) \geq 0 \qquad \forall s \in (s_i, t_i),$$

that is exactly (O).

Now suppose $s_i = 0$. Then $c = c_*$. In fact, let $c < c_*$. Then

$$f(s) - cs > 0 \qquad \forall s \in [0, 1]. \tag{3.8}$$

But, at t_i, we have $f(t_i) = ct_i$, that contradicts (3.8).

Thus it remains to show that if $c = c_*$ and $s_i = 0$ the jump interval (s_i, t_i) is admissible. Condition (O) can be proved as in the case $s_i > 0$. For the Rankine-Hugoniot condition we have to check that

$$c_* \equiv \frac{f_c(t_i) - f_c(0)}{t_i} = \frac{f(t_i)}{t_i}.$$

By Lemma 3.5(i), $f_c(t_i) = f(t_i)$, thus we only have to show that $f_c(0) = 0$. By definition of c_*, we easily obtain that $c_* s \in \mathcal{F}_c$. Since $f_c(s) \geq h(s)$ for any $h \in \mathcal{F}_c$, we have $f_c(s) \geq c_* s$. Then $f_c(0) = 0$ and the conclusion follows.

Now we prove that for $c > c_*$ there is no travelling waves with asymptotic states 0 and 1. On the contrary let ϕ be an entropy traveling wave with speed c. By definition of c_* there exists $s \in [0, 1]$ such that

$$\frac{f(s)}{s} < c. \tag{3.9}$$

Let

$$s_0 := \inf\{s \in [0, 1] : f(s) - cs = \min_{t \in [0,1]} (f(t) - ct)\}.$$

If $s_0 = 0$, then $f(t) - ct \geq 0$ for any $t > 0$, that contradicts (3.9). Hence $s_0 > 0$. There exists $\delta > 0$ such that

$$(f'(s) - c)(s - s_0) \geq 0 \qquad \forall s \in I_\delta(s_0).$$

There is no local classical solution of (3.1) satisfying $\psi(\xi_0) = s_0$. Thus there is $\xi_0 \in \mathbb{R}$ and $s_1 \leq s_0 \leq s_2$ such that the wave ϕ is discontinuous in ξ_0 and it has has left and right limits at ξ_0 the values s_1 and s_2 respectively.

Since $f(s) - cs$ has a minimum at s_0, condition (O) implies

$$f(s_1) - cs_1 = f(s_2) - cs_2 = f(s_0) - cs_0 = \min_{s \in [0,1]} (f(s) - cs).$$

From the definition of s_0 it follows that $s_1 = s_0$. Then we get the contradiction from the fact that the integral curves of (3.1) are decreasing at the left of ξ_0, and thus $\phi(\xi) \geq s_0 > 0$ for any $\xi \in \mathbb{R}$.

To prove uniqueness, first of all, let us suppose $f'(s) \leq c$ in a left neighborhood of 1. Then (3.1) does not define an increasing function converging to 1 at $+\infty$. Therefore ther exists $\xi_0 \in \mathbb{R}$ such that $\phi(\xi_0-) < 1 = \phi(\xi_0+)$. Moreover $f'(s) < c$ in a left neighborhood of $\phi(\xi_0-)$. By Lemma 3.2, coming back from ξ_0 to $-\infty$, we get that the solution is uniquely detrmined by ξ_0 and the conclusion follows.

Suppose $f'(s) > c$ in a left neighborhood of 1. If $\phi = \phi(\xi)$ is regular for any ξ, uniqueness is a consequence of results for ordinary differential equations. Otherwise there is ξ_0 such that ϕ is discontinuous at ξ_0. Then, applying Lemma 3.2, we obtain the uniqueness of the profile. $\square$

Proof of Theorem 2.3. Let $c < c_0$. Then, since

$$f'_{\pm}(s) > c_0 \qquad \forall s \in [0,1], \tag{3.10}$$

it follows that $f_c \equiv f$. By (3.10) we deduce that for the function ξ, given in (3.7), there holds

$$\frac{d\xi}{ds}(s) > 0, \qquad \forall s.$$

Therefore ξ globally invertible and the travelling wave ϕ belongs to $C^1(\mathbb{R})$.

If $c_0 \leq c \leq c_*$, then there are a finite number of points, say $\xi_1, \ldots, \xi_N$, such that

$$\phi(\xi_i-) = s_i, \quad \phi(\xi_i+) = t_i,$$

with s_i, t_i given as in the proof of Theorem 2.2.

Set

$$C := \{\xi_i : s_i = t_i\},$$
$$D := \{\xi_i : s_i \neq t_i\}.$$

Then $\phi \in C(\mathbb{R} \setminus D) \cap C^1(\mathbb{R} \setminus C \cup D)$.

From the definition of ϕ, we deduce that the support of the wave is bounded from below if and only if

$$\xi_0 \equiv \xi(0) \geq -\infty.$$

This condition is equivalent to the following

$$\frac{f'_c(s) - c}{g(s)} \in L^1(0,\varepsilon) \quad \text{for some } \varepsilon > 0. \tag{3.11}$$

Since $g(0) = 0$, this implies $f'_c(0) = c$. Hence $s_1 = 0$ and $c = c_*$.

Now there are two possibilities: either $f_c(s) = c_* s$ for any $s \in [0,\sigma)$ for some $\sigma > 0$ or $f_c(s) = f(s)$ for any $s \in [0,\sigma)$ for some $\sigma > 0$. In the former case (3.11) holds. In the latter

$$c = c_* = f'_c(0) = f'(0) = 0,$$

and (3.11) is equivalent to

$$c_* = 0, \qquad \frac{f'(s)}{g(s)} \in L^1(0,\varepsilon).$$

$\square$

4. Asymptotic Behavior of Solutions

This section is devoted to the proof of Theorem 2.4. We always assume (F), (G), (H). In the following we denote with u^R the entropy solution of the Cauchy problem (1.1), (2.6) and with f_* the function f_{c_*}, defined in Section 3.

Moreover let ϕ_* be the heteroclinic travelling wave with $\phi_- = 0$, $\phi_+ = 1$, speed c_* and supported in $[0, +\infty)$. Let $C \cup D = \{\xi_1 < \cdots < \xi_N\}$ with C and D as in Theorem 2.3. Finally set $\xi_{N+1} = +\infty$.

Our main tools in proving large-time behavior for the Riemann problem are comparison results for equation (1.1). For completeness, we recall these properties of entropy solutions of balance laws (see [18]).

Let us consider the following Cauchy problem

$$
\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{\partial f(u)}{\partial x} = g(u) & \text{in } \mathbb{R} \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}, \end{cases} \tag{4.1}
$$

where $T > 0$, $u_0 \in L^\infty_{\text{loc}}(\mathbb{R})$.

Moreover let us set

$$
[r]_\pm := \frac{1}{2}(|r| \pm r), \qquad H_\pm(r) := \frac{1}{2}(\text{sgn} r \pm 1).
$$

Definition 4.1. A measurable function u is called an *entropy supersolution* of problem (4.1) if

(i) $u \in L^\infty_{\text{loc}}(\mathbb{R} \times (0, T))$;

(ii) for any nonnegative function $\eta \in C_0^\infty(\mathbb{R} \times (0, T))$ and $k \in \mathbb{R}$ there holds

$$
\iint_{\mathbb{R} \times (0,T)} \{[u - k]_- \frac{\partial \eta}{\partial t} + H_-(u - k)(f(u) - f(k)) \frac{\partial \eta}{\partial x} + H_-(u - k) g(u) \eta\} \, dx \, dt \geq 0;
$$

(iii) for any interval $(a, b) \subseteq \mathbb{R}$

$$
\operatorname{ess} \lim_{t \to 0+} \int_a^b [u(x, t) - u_0(x)]_- \, dx = 0.
$$

An *entropy subsolution* of problem (4.1) is defined replacing $H_-(\cdot)$ by $H_+(\cdot)$ and $[\cdot]_-$ by $[\cdot]_+$ in (ii) and (iii).

Then the following comparison result can be proved.

Theorem 4.2. *(Comparison principle) Let assumptions (F) and (G) be satisfied. If $\underline{u}, \overline{u}$ are an entropy subsolution, respectively supersolution of problem (4.1) and if*

$$
\underline{u}(x, 0) \leq \overline{u}(x, 0) \qquad \text{a.e. } in\ \mathbb{R},
$$

then

$$
\underline{u}(x, t) \leq \overline{u}(x, t) \qquad \text{a.e. } in\ \mathbb{R} \times \mathbb{R}_+.
$$

Since we want to construct some explicit supersolution u of (1.1), we proceed on by considering some additional assumptions on the function u.

First of all, let $u = u(x,t)$ be a classical C^1 solution of (1.1) in $\mathbb{R} \times \mathbb{R}_+ \setminus \Gamma$ where $\Gamma := \{(\gamma(t), t) : t > 0\}$ and γ is regular. Then by Definition 4.1, u is a supersolution if and only if

$$\gamma'(t)([u^+ - s]_- - [u_- - s]_-) \geq H_-(u_+ - s)[f(u_+) - f(s)] - H_-(u_- - s)[f(u_-) - f(s)],$$

for any $s \in \mathcal{I}(u_+, u_-)$ and for any $t > 0$, where $u_\pm := \lim_{\varepsilon \to 0^+} u(\gamma(t) \pm \varepsilon, t)$.

If we suppose further $u_- < u_+$ for any $t > 0$, we get that u is a supersolution if and only if

$$f(s) \geq \gamma'(t)(s - u_-) + f(u_-), \qquad \forall s \in [u_-, u_+], \quad t > 0. \tag{4.2}$$

Let ϕ_* be the unique entropy travelling wave with asymptotic states $\phi_- = 0$, $\phi_+ = 1$, speed c_* and supp $\phi_* = [0, +\infty)$. We define, locally in time, a family of supersolutions with the above structure, where γ is an affine function. We set

$$V(x,t;y,k) := \begin{cases} \phi_*(x - c_* t) & x < kt + y, \\ 1 & x > kt + y, \end{cases} \tag{4.3}$$

with $x, k \in \mathbb{R}$ and $y, t > 0$.

Let K be defined by

$$K(s) := \inf_{\sigma \in (s,1]} \frac{f(\sigma) - f(s)}{\sigma - s} \qquad s \in [0,1). \tag{4.4}$$

Proposition 4.3. *Let $y > 0$ such that $c_* < K(\phi_*(y))$. Given $k \in (c_*, K(\phi_*(y)))$ then the function $u(x,t) := V(x,t;y,k)$ is an entropy supersolution of (1.1) in $(0,T) \times \mathbb{R}$ with*

$$T \equiv T(y,k) := \sup\{t \geq 0 : \forall \tau \leq t \quad K(\phi_*^\tau) \geq k\} > 0, \tag{4.5}$$

where $\phi_*^t := \phi_*(y + (k - c_*)t)$.

Proof. For the family of functions defined by (4.3), condition (4.2) takes the form

$$f(s) \geq k(s - \phi_*^t) + f(\phi_*^t) \qquad \forall s \in [\phi_*^t, 1].$$

Therefore $V = V(x,t;y,k)$ is a supersolution if and only if, for any $t \in (0,T)$,

$$k \leq K(\phi_*^t).$$

This immediately leads to (4.5).

To complete the proof we have to show that, under the assumptions of Proposition 4.3, $T = T(y,k) > 0$. We do this by proving that if $c_* < k < K(\phi_*(y))$, then there exists $t_0 > 0$ such that $K(\phi_*^t) > k$ for any $t \in (0, t_0)$.

On the contrary there is a decreasing sequence $\{t_n\}$ such that

$$\lim_{n\to+\infty} t_n = 0 \quad \text{and} \quad K(\phi_*^{t_n}) \le k \quad \forall n.$$

From the definition of K (see (4.4)), we get that there is a subsequence of $\{t_n\}$ (for simplicity we again denote it by $\{t_n\}$) such that either

$$f'_+(\phi_*^{t_n}) = K(\phi_*^{t_n}) \qquad \forall n, \tag{4.6}$$

or

$$\frac{f(\phi_*^{t_n}) - f(\sigma_n)}{\phi_*^{t_n} - \sigma_n} = K(\phi_*^{t_n}) \qquad \forall n, \tag{4.7}$$

for some $\sigma_n > \phi_*^{t_n}$.

If (4.6) holds, as $n \to +\infty$, we obtain the following contradiction

$$K(\phi_*(y)) \le f'_+(\phi_*(y)) \le k < K(\phi_*(y)),$$

in fact $K(s) \ge f'_+(s)$ and f'_+ is right continuous.

If (4.7) is true, then we can take a subsequence such that, for some σ_0, $\lim_{n\to+\infty} \sigma_n = \sigma_0$. Then

$$\lim_{n\to+\infty} \frac{f(\phi_*^{t_n}) - f(\sigma_n)}{\phi_*^{t_n} - \sigma_n} = \begin{cases} \dfrac{f(\phi_*^{t_n}) - f(\sigma_0)}{\phi_*^{t_n} - \sigma_0} & \text{if } \sigma_0 \ne \phi_*(y), \\ f'_+(\phi_*(y)) & \text{if } \sigma_0 = \phi_*(y). \end{cases}$$

Hence

$$K(\phi_*(y)) \le \lim_{n\to+\infty} \frac{f(\phi_*^{t_n}) - f(\sigma_n)}{\phi_*^{t_n} - \sigma_n} \le k < K(\phi_*(y)).$$

The proof is complete. $\square$

Lemma 4.4. *Assume hypotheses of Proposition 4.3. Let $y \in (\xi_i, \xi_{i+1})$ with $i \in \{1, \dots, n\}$. Then*

$$\lim_{k\to c_*^+} y + (k - c_*)T(y,k) = \xi_{i+1}.$$

Proof. Given $\xi \in \mathbb{R}$, let $\sigma > \phi_*(\xi)$. Then $f_*(\phi_*(\xi)) = f(\phi_*(\xi))$ and

$$\frac{f(\sigma) - f(\phi_*(\xi))}{\sigma - \phi_*(\xi)} \ge \frac{f_*(\sigma) - f_*(\phi_*(\xi))}{\sigma - \phi_*(\xi)} \ge c_* \quad \forall \sigma > \phi_*(\xi),$$

This implies

$$K(\phi_*(\xi)) \ge c_* \qquad \forall \xi \in \mathbb{R}.$$

If $c_* < K(\phi_*(\xi))$, then $f'(\phi_*(\xi)) > c_*$. Suppose, by contradiction, that $\phi_*(\xi) = s_i$ for some i. Then there would exist some $t_i > s_i$ such that

$$c_* = \frac{f(t_i) - f(s_i)}{t_i - s_i} \ge K(\phi_*(\xi)) > c_*.$$

If $c_* = K(\phi_*(\xi))$ then either $f'(\phi_*(\xi)) = c_*$ or there is $\sigma > \phi_*(\xi)$ such that

$$f(\sigma) = f(\phi_*(\xi)) + c_*(\sigma - \phi_*(\xi)).$$

In the first case $\phi_* = s_i$ for some i. In the latter

$$f_*(\sigma) - c_*\sigma \leq f_*(\phi_*(\xi)) - c_*\phi_*(\xi).$$

By monotonicity of $f_*(s) - c_*s$ we obtain that

$$f_*(s) = f_*(\phi_*(\xi)) + c_*(s - \phi_*(\xi)) \qquad \forall s \in (\phi_*(\xi), \sigma),$$

and $\phi_*(\xi) = s_i$ for some i.

Therefore

$$\phi_*(\xi) \neq s_i \quad \forall i \quad \text{if and only if} \quad K(\phi_*(\xi)) > c_*. \tag{4.8}$$

Now let $\{k_j\}$ be a sequence converging to c_* from the right as $j \to +\infty$ such that

$$\lim_{j \to +\infty} y + (k_j - c_*)T(y, k_j) = z,$$

for some z. Since for any $t < T$

$$\xi_i < y \leq y + (k_j - c_*)T(y, k_j) \leq \xi_{i+1},$$

we have

$$\xi_i < y \leq z \leq \xi_{i+1}. \tag{4.9}$$

If $K(\phi_*(z)) > c_*$, arguing as in the proof of Proposition 4.3, we can choose $k \in (c_*, K(\phi_*(y)))$ and $t_0 > 0$ such that

$$T(y, k) + t_0 \in \{t > 0 : \forall \tau \leq t \quad K(\phi_*^\tau) \geq k\},$$

contradicting the definition of $T(y, k)$. Thus $K(\phi_*(z)) = c_*$.

From (4.8) we get that $z = \xi_j$ for some j. Since (4.9) holds, $z = \xi_{i+1}$. $\square$

Lemma 4.5. *Let assumptions of Proposition 4.3 be satisfied. Let y be such that $\phi_* \in C^1([y, +\infty))$. Then there exists $k_* \geq c_*$ such that*

$$\lim_{k \to k_*^+} T(y, k) = +\infty.$$

Moreover if $f'(1) \neq c_*$, then $k_* > c_*$.

Proof. First of all we prove that the function $T = T(y, k)$ is decreasing with respect to the variable k. Hereafter to enphasize the dependence on k, we denote with $\phi_*^{t,k}$ the following

$$\phi_*^{k,t} := \phi_*(y + (k - c_*)t).$$

Moreover set

$$\mathcal{T}_k := \{t \geq 0 : \forall \tau \leq t \quad K(\phi_*^{k,\tau}) \geq k\}.$$

Let $c_* < k_1 < k_2 < K(\phi_*(y))$.

Let $t \in \mathcal{T}_{k_2}$. We will show that $t \in \mathcal{T}_{k_2}$. Given $\tau \leq t$, since ϕ_* is an increasing function,

$$\phi_*^{0,k_1} = \phi_*^{0,k_2} \leq \phi_*^{\tau,k_1} \leq \phi_*^{\tau,k_2}.$$

Then there exists $\theta \leq \tau$ such that $\phi_*^{\theta,k_2} = \phi_*^{\theta,k_1}$.

Since $t \in \mathcal{T}_{k_2}$ and $\theta \leq \tau$, we have

$$K(\phi_*^{\theta,k_2}) \geq k_2 \geq k_1.$$

Hence

$$K(\phi_*^{\theta,k_1}) \geq k_1.$$

Then we obtain $\mathcal{T}_{k_2} \subseteq \mathcal{T}_{k_1}$ and

$$T(y,k_2) \leq T(y,k_1) \qquad \text{if } k_2 < k_1. \tag{4.10}$$

Now suppose y be such that $\phi_* \in C^1([y,+\infty))$.

If there is $k > c_*$ such that $T(y,k) = +\infty$, the conclusion follows from (4.10) by defining $k_* := \sup\{k > c_* : T(y,k) = +\infty\}$.

Suppose that $T(y,k) < +\infty$ for any $k > c_*$. We show that in this case $f'(1) = c_*$.

For any k there exist $t_k < T(y,k)$ and $\tau_k < t_k$ such that

$$K(\phi_*^{\tau_k,k}) < k.$$

Therefore

$$k > \frac{f(s) - f(\phi_*^{\tau_k,k})}{s - \phi_*^{\tau_k,k}} \qquad \forall s > \phi_*^{\tau_k,k}. \tag{4.11}$$

Now we pass to the limit as $k \to c_*^+$ on a sequence such that $\lim \phi_*^{\tau_k,k} = \phi_*^\infty$ for some ϕ_*^∞. If $\phi_*^\infty < 1$, then

$$c_* \geq \frac{f(s) - f(\phi_*^\infty)}{s - \phi_*^\infty} \qquad \forall s > \phi_*^\infty.$$

Moreover, since ϕ_* is regular in $[y,+\infty)$, $f(s) = f_*(s)$ for any $s \geq \phi_*^\infty$. Hence

$$c_* \geq \frac{f_*(s) - f_*(\phi_*^\infty)}{s - \phi_*^\infty} \geq c_*.$$

Then

$$f_*(s) = f_*(\phi_*^\infty) + c_*(s - \phi_*^\infty) \qquad \forall s \geq \phi_*^\infty,$$

and ϕ_* would be discontinuous. This contradicts the hypothesis on ϕ_*. Thus $\phi_*^\infty = 1$.

Moreover from (4.13) with $s = 1$, and passing to the limit, we get $c_* \geq f'(1)$. Since ϕ_* is C^1 at $+\infty$, $f'(1) \geq c_*$. Therefore $f'(1) = c_*$.

To complete the proof, we have to show that, if $f'(1) = c_*$ and $T(y,k) < +\infty$ for any $k > c_*$, then

$$\lim_{k \to c_*^+} T(y,k) = +\infty.$$

Suppose by contradiction that

$$\lim_{k \to c_*^+} T(y,k) = T_* < +\infty.$$

Then for any $k > c_*$, there is $\tau_k \in [T(y,k), 2T_*)$ such that

$$k > \frac{f(s) - f(\phi_*^{\tau_k,k})}{s - \phi_*^{\tau_k,k}} \qquad \forall s > \phi_*^{\tau_k,k}. \tag{4.12}$$

Let $k \to c_*$. Then there are two cases: either there exists a sequence $\{k_j\}$ such that

$$\lim_{j \to +\infty} \tau_{k_j} > 0;$$

or

$$\lim_{k \to c_*^+} \tau_k = 0.$$

In the first case

$$\lim_{j \to +\infty} \phi_*^{\tau_{k_j},k_j} = \phi_*(y),$$

hence, by (4.12), $c_* \geq K(\phi_*(y))$ that contradicts the choice of y.

In the second case, we get

$$\lim_{k \to c_*^+} T(y,k) = 0,$$

from which we deduce $T(y,k) = 0$ for any k. This contradicts the positivity of $T(y,k)$, stated in Proposition 4.3. Hence $T_* = +\infty$. $\square$

Proof of Theorem 2.4. We construct an opportune set of supersolutions of the type (4.5).

Let η and k be such that

$$0 < \eta < \xi_{i+1} - \xi_i \text{ and } c_* < k < K(\phi(\xi_i + \eta)) \qquad i = 1, \ldots, N.$$

Next we define

$$V_1(x,t) = V(x + \delta_1, t; \xi_1 + \eta, k), \tag{4.13}$$

$$V_i(x,t) = V(x - c_* \sum_{j=1}^{i-1} T_j + \delta_i, t; \xi_i + \eta, k), \quad t \in [0, T_i], \quad i = 2, \ldots, N, \tag{4.14}$$

where

$$\begin{gathered}T_i := T(\xi_i + \eta, k), \qquad i = 1, \dots, N,\\ \delta_0 := \eta,\\ \delta_{i+1} := \delta_i + \xi_{i+1} - \xi_i - (k - c_*)T_i, \qquad i = 1, \dots, N-1.\end{gathered}$$

The definition of δ_i leads to

$$0 < \delta_{i+1} = \eta + \sum_{j=1}^{i} \{\xi_{i+1} - [\xi_j + (k - c_*)T_j]\} \leq \delta_N = N\eta + \sum_{j=1}^{N-1} \{\xi_{j+1} - [\xi_j + \eta + (k - c_*)T_j]\}.$$

Then, applying Lemma 4.4,

$$\lim_{\eta \to 0^+} \lim_{k \to c_*^+} \delta_i = 0 \qquad i = 1, \dots, N.$$

From (4.13) and (4.14) it follows

$$\begin{aligned}&u_0^R(x) \leq V_1(x, 0) \qquad \forall x \in \mathbb{R},\\ &V_i(x, T_i) \leq V_{i+1}(x, 0) \qquad \forall x \in \mathbb{R}, \forall i = 1, \dots, N-1.\end{aligned}$$

By comparison principle (Theorem 4.2) applied to $u^R(x,t)$ and $V_1(x,t)$ we get the following

$$u^R(x, t) \leq V_1(x, t) \qquad \text{a.e. in } \mathbb{R}, \forall t \in [0, T_1].$$

Thus

$$u^R(x, T_1) \leq V_1(x, T_1) \leq V_2(x, 0).$$

Now we use Theorem once again to obtain

$$u^R(x, T_1 + t) \leq V_2(x, t) \qquad \text{a.e. in } \mathbb{R}, \forall t \in [0, T_2].$$

By applying n times the comparison principle, we derive

$$u^R(x, \sum_{j=1}^{n-1} T_j + t) \leq V_n(x, t) \qquad \text{a.e. in } \mathbb{R}, \forall t \in [0, T_n]. \tag{4.15}$$

Moreover, since $u_0^R(x) \geq \phi_*(x)$,

$$\phi_*(x - c_* t) \leq u^R(x, t) \qquad \text{a.e. in } \mathbb{R}, \forall t \in [0, +\infty). \tag{4.16}$$

Given $\varepsilon > 0$, let us set

$$N_\varepsilon := \bigcup_{i=1}^{N} (\xi_i - \frac{\varepsilon}{4N}, \xi_i + \frac{\varepsilon}{4N}),$$

then meas $N_\varepsilon = \varepsilon/2$.

From formulas (4.15) and (4.16), we obtain

$$|u^R(y + c_*t, t) - \phi_*(y)| \le V_n(y + c_*t, t - \sum_{j=1}^{N-1} T_j) - \phi_*(y), \tag{4.17}$$

for almost any $y \in \mathbb{R}$ and for any $t \in [\sum_{j=1}^{N-1} T_j, \sum_{j=1}^{N} T_j]$.

From (4.14) we get

$$V_N(y + c_*t, t - \sum_{j=1}^{N-1} T_j) - \phi_*(y) = \begin{cases} \phi_*(y + \delta_N) - \phi_*(y) & y < (k - c_*)t, \\ 1 - \phi_*(y) & y > (k - c_*)t. \end{cases}$$

Given $\theta > 0$, we can choose $\eta = \eta(\theta)$ and $k = k(\theta)$ such that

$$\sup_{\mathbb{R}\setminus N_\varepsilon} |\phi_*(y + \delta_N) - \phi_*(y)| < \theta.$$

Moreover since the function $1 - \phi_*(y)$ decreases monotonically to zero as $y \to +\infty$, there is $t_0 = t_0(\eta, k)$ such that

$$1 - \phi_*(y) \le 1 - \phi_*((k - c_*)t) < \theta \qquad \forall y > (k - c_*)t, \tag{4.18}$$

for any $t > t_0$.

Coming back to (4.17), we deduce

$$\operatorname*{ess\,sup}_{\mathbb{R}\setminus N_\varepsilon} |u^R(y + c_*t, t) - \phi_*(y)| < \theta, \tag{4.19}$$

for any $t > t_0$.

If $T_n = +\infty$, passing to the limit as $t \to +\infty$, we get the conclusion. In the case $T_n < +\infty$ for any $k < c_*$, the same estimate follows from Lemma 4.4. In fact it implies

$$\lim_{k \to c_*^+} y_{T_N} = +\infty.$$

Thus we can choose k sufficiently close to c_* in such a way that (4.18) holds. From (4.19) we get (2.7).

To prove L^1-convergence note that, by (4.17),

$$\|u^R(\cdot, t) - \phi_*(\cdot - c_*t)\|_{L^1(\mathbb{R})} \le \int_{-\infty}^{(k-c_*)t} |\phi_*(y + \delta_n) - \phi_*(y)|\, dy + \int_{(k-c_*)t}^{+\infty} (1 - \phi_*(y))\, dy.$$

Since

$$\int_{\mathbb{R}} |\phi_*(y + \delta) - \phi_*(y)|\, dy = \delta(\phi_*(+\infty) - \phi_*(-\infty)) = \delta,$$

then

$$\|u^R(\cdot,t)-\phi_*(\cdot-c_*t)\|_{L^1(\mathbb{R})}\leq \delta_n+\int\limits_{(k-c_*)t}^{+\infty}(1-\phi_*(y))\,dy.$$

For any $\varepsilon>0$, choose k such that $\delta_n<\varepsilon/2$. Then there exists $t_0=t_0(\varepsilon,k)$ such that

$$\int\limits_{(k-c_*)t}^{+\infty}(1-\phi_*(y))\,dy<\frac{\varepsilon}{2}.$$

Therefore

$$\|u^R(\cdot)-\phi_*(\cdot-c_*t)\|_{L^1(\mathbb{R})}<\varepsilon\qquad\forall t>t_0.$$

The L^p-convergence follows at once from L^1-convergence and L^∞ global boundedness, in fact

$$\begin{aligned}\|u(\cdot,t)-\phi_*(\cdot-c_*t)\|^p_{L^p(\mathbb{R})}&=\int\limits_{\mathbb{R}}|u(x,t)-\phi_*(x-c_*t)|^{p-1}\cdot|u(x,t)-\phi_*(x-c_*t)|\,dx\\&\leq\|u(\cdot,t)-\phi_*(x-c_*t)\|_{L^1(\mathbb{R})}.\end{aligned}$$

The proof is complete. $\square$

To conclude we prove Theorem 2.5.

Proof of Theorem 2.5. By assumptions, we get $c_*=0$ and f' strictly increasing in $[0,a_1]$. Then there exists $\varepsilon_0\in(0,a_1)$ such that for any $\varepsilon\in(0,\varepsilon_0)$, there is $h_\varepsilon\in(0,f'(\varepsilon))$ such that

$$f(s)>f(\varepsilon)+h_\varepsilon(s-\varepsilon)\qquad\forall s>\varepsilon. \tag{4.20}$$

In fact suppose by contradiction that for any $\delta>0$ there exists $\varepsilon\in(0,\delta)$ such that for any $h\in(0,f'(\varepsilon))$ there is $s_{h,\varepsilon}$ such that

$$f(s_{h,\varepsilon})=f(\varepsilon)+h(s_{h,\varepsilon}-\varepsilon). \tag{4.21}$$

Let $H(s):=f(s)-f(\varepsilon)-h(s-\varepsilon)$. Then $H(\varepsilon)=0$, $H'(\varepsilon)>0$, $H''(\varepsilon)>0$. Hence $H(s)>0$ for any $s\in(\varepsilon,a_1]$ and $s_{h,\varepsilon}>a_1$.

Let h_n be a sequence converging to zero as $n\to+\infty$, and let $s_n:=s_{h_n,\varepsilon}$. Then (taking a subsequence if needed) s_n converges to some $s_\varepsilon\geq a_1$ as $n\to+\infty$. For such an s_ε we deduce from (4.21)

$$f(s_\varepsilon)=f(\varepsilon),$$

with $\varepsilon\in(0,\delta)$, $s_\varepsilon\geq a_1$.

Letting $\delta\to0$, we deduce that there exists $s_0\geq a_1$ such that $f(s_0)=f(0)$, that contradicts the hypothesis. Therefore (4.20) holds.

Now, given $\varepsilon \in (0, \varepsilon_0)$, we define

$$\int_{\varepsilon}^{\phi_\varepsilon(x)} \frac{f'(s)}{g(s)}\, ds = x,$$

and

$$V_\varepsilon(x,t) = \begin{cases} \phi_\varepsilon(x) & x < h_\varepsilon t, \\ 1 & x > h_\varepsilon t. \end{cases}$$

Since (4.20) holds, the function V_ε proves to be an entropy supersolution of problem (2.1) with $u_0 \le u_0^R$ for any $\varepsilon \in (0, \varepsilon_0)$.

Then, for any $M \in \mathbb{R}$,

$$\|u(\cdot,t)\|_{L^\infty(-\infty,M)} \le \|u^R(\cdot,t)\|_{L^\infty(-\infty,M)} \le \|V_\varepsilon(\cdot,t)\|_{L^\infty(-\infty,M)} = \phi_\varepsilon(M) \quad \forall t > \frac{M}{h_\varepsilon}.$$

Since f'/g is not integrable at 0, we deduce

$$\lim_{\varepsilon \to 0} \phi_\varepsilon(M) = 0.$$

Thus, for any $\varepsilon > 0$ there is $T_\varepsilon = M/h_\varepsilon$ such that

$$\|u(\cdot,t)\|_{L^\infty(-\infty,M)} < \varepsilon \qquad \forall t > T_\varepsilon.$$

□

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