Uniqueness for discontinuous O.D.E. and conservation laws

Alberto Bressan(*) and Wen Shen(**)

(*) S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy.

(**) Department of Informatics, University of Oslo,
PO Box 1080 Blindern, N-0314 Oslo, Norway.

Abstract

Consider a scalar O.D.E. of the form $\dot{x} = f(t,x)$, where f is possibly discontinuous w.r.t. both variables t, x. Under suitable assumptions, we prove that the corresponding Cauchy problem admits a unique solution, which depends Hölder continuously on the initial data.

Our result applies in particular to the case where f can be written in the form $f(t,x) \doteq g(u(t,x))$, for some function g and some solution u of a scalar conservation law, say $u_t + F(u)_x = 0$. In turn, this yields the uniqueness and continuous dependence of solutions to a class of 2×2 strictly hyperbolic systems, with initial data in $\mathbf{L}^\infty$.

Keywords: well posedness, discontinuous ODE, conservation laws.

1 Introduction

In this paper we prove a new existence and uniqueness result for solutions to a scalar differential equation with discontinuous right hand side

$$\dot{x} = f(t, x) \qquad t \in [0, T]. \tag{1}$$

Solutions are understood in the sense of Carathéodory, i.e. as absolutely continuous functions which satisfy (1) almost everywhere. Equivalently, a function $x = x(t)$ is a solution of the Cauchy problem (1) with initial data

$$x(0) = \bar{x}, \tag{2}$$

iff it satisfies the integral equation

$$x(t) = \bar{x} + \int_0^t f(s, x(s))\, ds \qquad t \in [0, T].$$

For (possibly vector valued) O.D.E's with discontinuous right hand side, we recall two known conditions which imply the well posedness of the Cauchy problem:

- **Case 1.** (Carathéodory conditions [9]) The map $x \mapsto f(t, x)$ is Lipschitz continuous for a.e. t, while the map $t \mapsto f(t, x)$ is measurable for every x.

- **Case 2.** (Bounded Directional Variation [3]) For some constants L, M one has the uniform bound $|f(t, x)| \leq L < M$. Moreover the quantities

$$\sum_{i=1}^{N} \Big| f(t_i, x_i) - f(t_{i-1}, x_{i-1}) \Big| \tag{3}$$

  are uniformly bounded. In (3), $N \geq 1$ is any integer and the summation is taken over any $(N+1)$-tuple of points (t_i, x_i) which is "directionally increasing", i.e.

$$t_0 < t_1 < \cdots < t_N, \qquad |x_i - x_{i-1}| \leq M(t_i - t_{i-1}) \quad i = 1, \cdots, N.$$

In both cases, one can show that the solution of the Cauchy problem depends Lipschitz continuously on the initial data. On the contrary, for the equations considered in the present paper, only a Hölder continuous dependence on initial data will hold. Our study was originally motivated by the following system of conservation law

$$\begin{aligned} u_t + \left(\frac{u}{1+u+v}\right)_x &= 0, \\ v_t + \left(\frac{v}{1+u+v}\right)_x &= 0. \end{aligned} \tag{4}$$

Introducing the new variables $w = u + v$, $\quad \theta = v/u$, the system (4) takes the form

$$w_t + \left(\frac{w}{1+w}\right)_x = 0, \tag{5}$$

$$\theta_t + \frac{1}{1+w}\,\theta_x = 0. \tag{6}$$

For a given initial data $w(0) = \bar{w}$, $\theta(0) = \bar{\theta}$, one can first construct the unique entropy solution $w = w(t,x)$ to the scalar conservation law (5) according to Kruzkov [10]. The function θ is then obtained by setting $\theta(t,x) = \bar{\theta}(y(0;t,x))$, where $s \mapsto y(s;t,x)$ denotes the solution to the Cauchy problem

$$\frac{dy}{ds} = \frac{1}{1+w(s,y)}, \qquad y(t) = x. \tag{7}$$

Observe that (7) contains O.D.E. whose right hand side is discontinuous w.r.t. both variables t, x. More generally, we shall look at equations of the form

$$\frac{dx}{dt} = f(t,x) \doteq g(w(t,x)) \tag{8}$$

where g is any bounded function and w is a solution to some scalar conservation law. We remark that these equations do not fall in either of the cases [9, 3].

The key properties of the function f in (8) are the following.

(A1) For every point $(\bar{t}, \bar{x})$, there exists a slope $\lambda_{\bar{t},\bar{x}}$ such that the function f is constant along the segment $I_{\bar{t},\bar{x}}$ defined as

$$I_{\bar{t},\bar{x}} \doteq \Big\{(t,x);\quad t \in]\,0, \bar{t}\,],\ x - \bar{x} = (t - \bar{t}\,)\lambda_{\bar{t},\bar{x}}\Big\}.$$

Moreover, $(t,x) \in I_{\bar{t},\bar{x}}$ implies $\lambda_{t,x} = \lambda_{\bar{t},\bar{x}}$ and hence $I_{t,x} \subseteq I_{\bar{t},\bar{x}}$.

(A2) There exists disjoint intervals $[a,b]$ and $[c,d]$ such that $f(t,x) \in [a,b]$ and $\lambda_{t,x} \in [c,d]$ for all $(t,x) \in [0,T] \times I\!\!R$.

We can now state our main existence and uniqueness result. In the following, we consider the family of functions

$$\mathcal{U} \doteq \Big\{u : [0,T] \mapsto I\!\!R;\quad u \text{ is Lipschitz continuous, } \dot{u}(t) \in [a,b] \text{ a.e.}\Big\}. \tag{9}$$

Theorem 1 *Let $f : [0,T] \times I\!\!R \mapsto I\!\!R$ be a measurable function satisfying the assumptions (A1) and (A2). Then the following holds*

(i) The map $t \mapsto f(t, u(t))$ is measurable for each $u \in \mathcal{U}$.

(ii) The Picard operator $(Pu)(t) = \int_0^t f(s,u(s))\,ds$ is continuous on $\mathcal{U}$.

(iii) For each initial data $\bar{x}$, the Cauchy problem (1)-(2) has a unique solution $x(\cdot)$. The map $\bar{x} \mapsto x(\cdot)$ is Hölder continuous from $I\!\!R$ into $\mathcal{C}^0([0,T])$.

The proof will be obtained by first deriving uniform estimates on the solutions, in the case where f is piecewise constant with jumps along a finite number of segments in the t-x plane. The result is then extended to general functions f, using an approximation argument in connection with a suitable topology.

From Theorem 1 one obtains the well posedness of the Cauchy problem for a class of strictly hyperbolic 2×2 systems of the form

$$w_t + F(w)_x = 0, \qquad w(0,x) = \bar{w}(x) \tag{10}$$

$$\theta_t + g(w)\theta_x = 0 \qquad \theta(0,x) = \bar{\theta}(x), \tag{11}$$

Theorem 2 *Let the initial data $\bar{w}$, $\bar{\theta}$ be bounded, say with $\bar{w}(x) \in [w^-, w^+]$ for all x, and let $\bar{\theta}$ be continuous. Let F be convex and assume that there exists disjoint intervals $[a, b]$ and $[c, d]$ such that*

$$g(w) \in [a, b], \quad F'(w) \in [c, d], \quad \text{for all } w \in [w^-, w^+].$$

Then the Cauchy problem (10)-(11) has a unique entropy solution, continuously depending on the initial data. More precisely, if $\bar{w}_n \to \bar{w}$ in $\mathbf{L}^1$ and $\bar{\theta}_n \to \bar{\theta}$ in $\mathcal{C}^0$, then for each $t > 0$, $\theta_n(t, x)$ converges to $\theta(t, x)$ uniformly on bounded sets.

For hyperbolic systems of the Temple class, the existence and the continuous dependence of solutions were established in [11] and [1], within the set of BV functions. A uniqueness result with L^∞ data, valid for some special 2×2 systems, was recently obtained in [2]. For 3×3 strictly hyperbolic systems, on the other hand, the Cauchy problem appears to be not well posed, in the class of $\mathbf{L}^\infty$ initial data. This is illustrated by a counterexample given at the end of this paper.

2 The piecewise constant case

In this section we study the case where, in addition to (A1)-(A2), the function f is piecewise constant, with jumps along a finite number of segments in the t-x plane. In this case, the conclusions (i)–(iii) of Theorem 1 are obvious, since all trajectories $t \mapsto u(t)$, $u \in \mathcal{U}$ cross transversally the discontinuity lines of f.

Our main goal is to derive a priori estimates on the dependence of solutions on the initial data. These bounds will depend only on the intervals $[a, b]$, $[c, d]$, and not on the number of jumps or on the total variation of f. To fix the ideas, in the following we shall assume $d < a$, the case $b < c$ being entirely similar.

The basic step is the analysis of tangent vectors. More precisely, consider any solution $t \mapsto x(t)$ of (1). For a fixed time t^*, denote by $t \mapsto x_\varepsilon(t)$ the solution of (1), with initial data

$$x_\varepsilon(t^*) = x(t^*) + \varepsilon.$$

Because of the simple geometrical structure of f, it is clear that the first order tangent vector

$$v(t) \doteq \lim_{\varepsilon \to 0+} \frac{x_\varepsilon(t) - x(t)}{\varepsilon} \tag{12}$$

is well defined for all $t \in [0, T]$. Moreover, the map $t \mapsto v(t)$ is piecewise constant, with jumps only at times τ where the reference solution $x(\cdot)$ crosses a discontinuity of f. Call v^-, v^+ the values of v before and after such a crossing time. Let $\lambda \in [c, d]$ be the slope of the segment where f is discontinuous, and call f^-, f^+ the values of f across the jump (see Figure 1). An elementary computation now yields the basic relation [8]

$$v^+ = \frac{f^+ - \lambda}{f^- - \lambda} v^- \, . \tag{13}$$

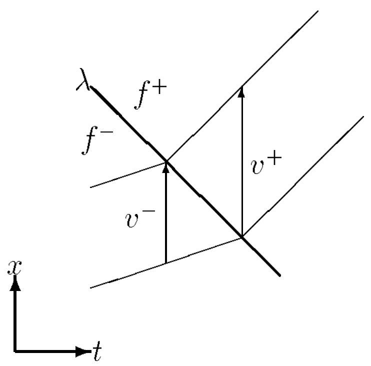

Figure 1: Change in tangent vector across a jump.

For each t define $\lambda(t) \doteq \lambda_{t,x(t)}$ and let $y(t) = x(t) - t\lambda(t)$ be the intersection of the x-axis with the segment $I_{t,x(t)}$ where f is constantly equal to $f(t, x(t))$. By assumption, the function $t \mapsto y(t)$ is nondecreasing. It is not restrictive to assume that it is also piecewise constant. To keep track of $v(t)$ during the whole interval $[0, T]$, introduce the function

$$\varphi(t) \doteq \frac{b - \lambda(t)}{f(t, x(t)) - \lambda(t)} \, . \tag{14}$$

Observe that, for $f \in [a,b]$ and $\lambda \in [c,d]$, we have

$$\frac{b-\lambda}{f-\lambda} \in \left[1,\ \frac{b-d}{a-d}\right], \qquad \frac{d}{d\lambda}\left(\frac{b-\lambda}{f-\lambda}\right) = \frac{b-f}{(f-\lambda)^2} \geq 0.$$

We now investigate how the product $z(t) \doteq \varphi(t) \cdot v(t)$ varies in time. First, we notice that $z(t)$ is unaffected by the change in f across a discontinuity. Indeed, from (13)-(14) it follows

$$\varphi^+ \cdot v^+ = \frac{b-\lambda}{f^+ - \lambda} \cdot v^+ = \frac{b-\lambda}{f^- - \lambda} \cdot v^- = \varphi^- \cdot v^-.$$

Moreover, at times where y jumps upward, hence $y^+ > y^-$ and $\lambda^+ < \lambda^-$, we have

$$\varphi^+ \cdot v = \frac{b-\lambda^+}{f-\lambda^+} \cdot v \leq \frac{b-\lambda^-}{f-\lambda^-} \cdot v = \varphi^- \cdot v.$$

Finally, consider the evolution $z(t)$ during a time interval where $y(t)$ remains constant. Since $\lambda(t) = [x(t) - y(t)]/t$, we have

$$\dot\lambda(t) = \frac{1}{t}\dot x(t) - (x-y)\frac{1}{t^2} = \frac{f-\lambda}{t},$$

hence

$$\dot\varphi(t) = \frac{b-f}{(f-\lambda)t}\ .$$

As long as the trajectory $t \mapsto x(t)$ does not cross lines where f is discontinuous, the tangent vector $v(t)$ is a constant. Hence

$$\dot z(t) = \dot\varphi(t) \cdot v(t) = \frac{b-f(t)}{b-\lambda(t)} \cdot \frac{z(t)}{t} \leq \gamma \cdot \frac{z(t)}{t}, \tag{15}$$

where $\gamma \doteq (b-a)/(b-d) < 1$. From the differential inequality (15) it follows

$$\frac{z(t)}{z(s)} \leq \left(\frac{t}{s}\right)^\gamma \qquad 0 < s < t \leq T. \tag{16}$$

Since $\varphi(t) \in [1, M]$ with $M \doteq (b-d)/(a-d)$, by (16) we have

$$\frac{v(t)}{v(s)} \leq M\left(\frac{t}{s}\right)^\gamma \qquad 0 < s < t \leq T. \tag{17}$$

Now consider any two solutions $x_1, x_2 : [0,T] \mapsto I\!R$ of (1) with initial data $\bar{x}_1, \bar{x}_2$ respectively. From the estimate (17) on tangent vectors, for every $0 < \tau \leq t \leq T$ we deduce

$$\begin{aligned} |x_1(t) - x_2(t)| &\leq M\left(\frac{t}{\tau}\right)^{\gamma} \cdot |x_1(\tau) - x_2(\tau)| \\ &\leq M\left(\frac{t}{\tau}\right)^{\gamma} \cdot \Big\{|\bar{x}_1 - \bar{x}_2| + \tau(b-a)\Big\}. \end{aligned} \tag{18}$$

If $|\bar{x}_1 - \bar{x}_2| \leq t$, in (18) we can choose $\tau \doteq |\bar{x}_1 - \bar{x}_2|$ and obtain

$$|x_1(t) - x_2(t)| \leq (1 + b - a) M t^{\gamma} |\bar{x}_1 - \bar{x}_2|^{1-\gamma}. \tag{19}$$

If instead $|\bar{x}_1 - \bar{x}_2| > t$, we have the trivial estimate

$$|x_1(t) - x_2(t)| < (1 + b - a)|\bar{x}_1 - \bar{x}_2|. \tag{20}$$

Toward the proof of Theorem 1, we shall need an additional estimate concerning solutions to the perturbed equation

$$\dot{w}(t) = \pi_{[a,b]}\big(f(t, w(t)) + e(t)\big), \tag{21}$$

where

$$\pi_{[a,b]}(x) \doteq \begin{cases} b & \text{if} \quad x > b, \\ x & \text{if} \quad x \in [a,b], \\ a & \text{if} \quad x < a. \end{cases}$$

and e is a measurable function, with

$$|e(t)| \leq b - a \qquad t \in [0,T]. \tag{22}$$

Lemma 1 *Let $f : [0,T] \times I\!R \mapsto I\!R$ be a piecewise constant function satisfying the assumptions (A1)-(A2) and let the perturbation $e : [0,T] \mapsto I\!R$ satisfy (22). Let $x(\cdot)$, $w(\cdot)$ be solutions of (1), (21) respectively, with the same initial data $x(0) = w(0) = \bar{x}$. Then, for every $t \in [0,T]$, their distance can be estimated by*

$$|w(t) - x(t)| \leq \int_0^t M\left(\frac{t}{s}\right)^{\gamma} |e(s)|\, ds \leq \frac{M t^{\gamma}(b-a)^{\gamma}}{1-\gamma} \cdot \left(\int_0^t |e(s)|\, ds\right)^{1-\gamma} \tag{23}$$

Indeed, for every $\tau \in [0,T]$, call x^τ the solution to the Cauchy problem

$$\dot{x}(s) = f(s, x(s)), \qquad x^\tau(\tau) = w(\tau).$$

Since $x^t(t) = w(t)$, $x^0(t) = x(t)$, from the bounds (17) on the growth of tangent vectors we derive the estimate

$$|w(t) - x(t)| \leq \int_0^t \left| \frac{d}{d\tau} x^\tau(t) \right| \, d\tau \leq \int_0^t M \left(\frac{t}{\tau} \right)^\gamma |e(\tau)| \, d\tau.$$

The second inequality is obtained integrating by parts and observing that

$$\int_0^t |e(s)| \, ds \leq E^*(t) \doteq \max \left\{ (b-a)t, \int_0^T |e(s)| \, ds \right\}.$$

3 Proof of Theorem 1

The main purpose of this section is to show that the a priori estimates valid for O.D.E's with piecewise constant right hand side can be extended to general functions f satisfying the assumptions (A1)-(A2).

Recalling the definition of $\mathcal{U}$ at (9), we define a family of measurable functions $\mathcal{F}$ by setting

$$\begin{aligned} \mathcal{F} \quad \doteq \quad & \Big\{ f : [0,T] \times I\!\!R \mapsto [a,b]; \ f \text{ is jointly measurable and the map} \\ & \quad t \mapsto f(t, u(t)) \ \text{ is measurable for each } u \in \mathcal{U} \Big\}. \end{aligned}$$

The distance between two functions $f, g \in \mathcal{F}$ is defined as

$$\mathrm{dist}(f,g) := \sup_{u \in \mathcal{U}} \int_0^T \Big| f(t,u(t)) - g(t,u(t)) \Big| \, dt. \tag{24}$$

The proof of Theorem 1 is based on the following approximation lemma.

Lemma 2 *Let $f : [0,T] \times I\!\!R \mapsto [a,b]$ be any measurable function which satisfies the assumptions (A1)-(A2). Then, for every $\varepsilon > 0$, there exists a piecewise constant function $\tilde{f} \in \mathcal{F}$, still satisfying (A1)-(A2), with dist$(f, \tilde{f}) < \varepsilon$.*

Proof. Let ε be given. The measurability assumption on f implies that, for some $t_0 \in [\varepsilon/2, \varepsilon]$, the map $x \mapsto f(t_0, x)$ is measurable. Therefore, we can find a piecewise constant function $\psi : \mathbb{R} \mapsto \mathbb{R}$, with finitely many jumps on any compact set, such that

$$\int |\psi(x) - f(t_0, x)| \, dx < \varepsilon^2. \tag{25}$$

To fix the ideas, let

$$\psi(x) = \psi_j, \quad x \in \,]x_{j-1}, x_j], \quad j \in \mathbb{Z}.$$

For each integer j, define the region

$$\Gamma_j \doteq \Big\{(t,x); \;\; t \in [t_0, T], \; x - (t - t_0)\lambda_{t,x} \in \,]x_{j-1}, x_j]\Big\}.$$

The assumption $\lambda_{t,x} \in [c,d]$ for all (t,x) implies that the boundaries of the sets Γ_j are Lipschitz continuous. More precisely, there exists functions $y_j : [t_0, T] \mapsto \mathbb{R}$ such that

$$\Gamma_j \subseteq \Big\{(t,x); \;\; t \in [t_0, T]; \; y_{j-1}(t) < x < y_j(t)\Big\},$$

and

$$c \leq \frac{y_j(t) - y_j(s)}{t - s} \leq d, \quad \text{for every } t_0 \leq s < t.$$

By suitably approximating the curves y_j, we can find polygonal lines $z_j : [0,T] \mapsto \mathbb{R}$ with

$$\sum_{j \in \mathbb{Z}} \left(\sup_{t \in [t_0, T]} |z_j(t) - y_j(t)| \right) < \varepsilon, \tag{26}$$

and such that the piecewise constant function

$$\tilde{f}(t,x) \doteq \psi_j \quad \text{if } x \in \;]z_{j-1}(t), z_j(t)]$$

satisfies the assumptions (A1)-(A2).

Using (25)-(26), we now estimate the distance between f and $\tilde{f}$. Define the set of times

$$I = \Big\{t \in [t_0, T]; \;\; u(t) \in \;]z_{j-1}(t), z_j(t)] \cap \;]y_{j-1}(t), y_j(t)] \;\text{ for some } j\Big\},$$

and call $I^c = [0,T]\backslash I$ its complement. Since $\dot y_j, \dot z_j \in [c,d]$ while $\dot u \in [a,b]$, from (26) we deduce

$$\mathrm{meas}(I^c) \le t_0 + \frac{\varepsilon}{a-d} \le \varepsilon\left[1 + \frac{1}{a-d}\right]. \tag{27}$$

For $t \in I$, consider the points

$$\begin{aligned} x(t) &\doteq u(t) - (t-t_0)\lambda_{t,u(t)}, \\ x_0(t) &\doteq u(t) - t\lambda_{t,u(t)}. \end{aligned}$$

Since the map $t \mapsto x_0(t)$ is nondecreasing, for any $s < t$ we have

$$x(s) = x_0(s) + \frac{t_0}{s}\left[u(s) - x_0(s)\right]$$

and

$$\begin{aligned} x(t) &= \frac{t_0}{t}u(t) + \frac{t-t_0}{t}x_0(t) \\ &\ge \frac{t_0}{t}[u(s) + a(t-s)] + \frac{t-t_0}{t}x_0(s) \\ &= x(s) + \left(\frac{t_0}{t} - \frac{t_0}{s}\right)[u(s) - x_0(s)] + \frac{t_0}{t}a(t-s) \\ &\ge x(s) - \left(\frac{t_0}{s} - \frac{t_0}{t}\right)ds + \frac{t_0}{t}a(t-s). \end{aligned}$$

Therefore

$$x(t) - x(s) \ge t_0\left(\frac{t-s}{t}\right)(a-d) \ge \frac{(a-d)t_0}{T}(t-s).$$

From the measurability of the map $x \mapsto f(t_0, x)$ and the relations

$$f(t,u(t)) = f(t_0, x(t)), \qquad \frac{dx(t)}{dt} \ge (a-d)\frac{t_0}{T} \quad \text{a.e.}, \tag{28}$$

it follows that the restriction of $t \mapsto f(t,u(t))$ to $[t_0, T]$ is measurable.

Using (28) to change the variable of integration and then (25), we now obtain the estimate

$$\begin{aligned}
&\int_I \left| f(t,u(t)) - \tilde f(t,u(t)) \right| dt \\
&\quad = \int_I \left| f(t_0,x(t)) - \psi(x(t)) \right| dt \\
&\quad \leq \frac{T}{(a-d)t_0} \int_{\mathbb{R}} \left| f(t_0,x) - \psi(x) \right| dx \\
&\quad \leq \frac{T\varepsilon^2}{(a-d)\varepsilon/2}. \qquad (29)
\end{aligned}$$

Moreover, (27) implies

$$\int_{I^c} \left| f(t,u(t)) - \tilde f(t,u(t)) \right| dt \leq (b-a)\mathrm{meas}(I^c) \leq (b-a)\varepsilon\left[1 + \frac{1}{a-d}\right]. \qquad (30)$$

Together, (29) and (30) imply

$$\int_0^T \left| f(t,u(t)) - \tilde f(t,u(t)) \right| dt \leq \varepsilon\left[\frac{2T}{a-d} + (b-a)\left(1 + \frac{1}{a-d}\right)\right],$$

which can be made arbitrarily small by a suitable choice of ε.

Thanks to the previous approximation lemma we can now give a proof of Theorem 1. Let f satisfy the assumptions (A1)-(A2) and consider a sequence of piecewise constant functions $f_n \in \mathcal{F}$ also satisfying (A1)-(A2), with $\mathrm{dist}(f, f_n) \to 0$. By Lemma 2, the corresponding Picard operator

$$(Pu)(t) = \int_0^t f(s,u(s))\,ds, \quad u \in \mathcal{U},$$

is the uniform limit of the operators

$$(P_n u)(t) = \int_0^t f_n(s,u(s))ds, \quad u \in \mathcal{U}.$$

Therefore, P itself is well defined and continuous. For any given initial data $\bar x$, the continuous operator

$$(\bar P u)(t) = \bar x + \int_0^t f(s,u(s))\,ds,$$

maps the closed convex set $\mathcal{U}$ compactly into itself. By Schauder's theorem, $\bar{P}$ has a fixed point, which is a solution to the Cauchy problem (1)-(2).

To prove the uniqueness of this solution and its continuous dependence on the initial data $\bar{x}$, let $\bar{x}_1, \bar{x}_2 \in I\!R$ and $\varepsilon > 0$ be given. Let $x_1(\cdot)$, $x_2(\cdot)$ be solutions of (1) with initial data $x_1(0) = \bar{x}_1$, $x_2(0) = \bar{x}_2$ repectively. Using Lemma 2, construct a piecewise constant $\tilde{f}$ satisfying (A1)-(A2), with $\mathrm{dist}(f, \tilde{f}) < \varepsilon$. For $i = 1, 2$, introduce the quantities

$$e_i(t) = f(t, x_i(t)) - \tilde{f}(t, x_i(t)) \quad t \in [0, T],$$

and observe that $x_i(\cdot)$ is a solution to

$$\dot{x}(t) = \pi_{[a,b]}\left(\tilde{f}(t, x(t)) + e_i(t)\right) \quad x(0) = \bar{x}_i.$$

Moreover, the assumption $\mathrm{dist}(f, \tilde{f}) < \varepsilon$ imples

$$\int_0^T |e_i(t)|\, dt < \varepsilon. \tag{31}$$

For $i = 1, 2$, call $y_i(\cdot)$ the solution to the Cauchy problem

$$\dot{y}(t) = \tilde{f}(t, y(t)), \quad y(0) = \bar{x}_i.$$

Using the estimates (23), (31) and (19)-(20), we now obtain

$$\begin{aligned} &|x_1(t) - x_2(t)| \\ &\leq \ |x_1(t) - y_1(t)| + |y_1(t) - y_2(t)| + |y_2(t) - x_2(t)| \qquad (32) \\ &\leq \ \frac{Mt^\gamma (b-a)^\gamma}{1-\gamma} \cdot 2\varepsilon^{1-\gamma} + (1 + b - a) \cdot \max\left\{Mt^\gamma |\bar{x}_1 - \bar{x}_2|^{1-\gamma}, |\bar{x}_1 - \bar{x}_2|\right\}. \end{aligned}$$

Since $\varepsilon > 0$ in (32) is arbitrary, this proves the uniqueness of solution of (1)-(2) and their Hölder continuity on the initial data, with exponent $1 - \gamma = (a-d)/(b-d) > 0$.

Remark: An entirely similar estimate on the Hölder continuous dependence holds for solutions to the backward Cauchy problem:

$$x(\tilde{t}) = \tilde{x}, \qquad \dot{x} = f(t, x) \quad t \in [0, \tilde{t}].$$

To see this, we first consider the case where f is a piecewise constant function satisfying (A1)-(A2). Let $t \mapsto x(t)$ be a solution of (1) and let $t \mapsto v(t)$ be a corresponding tangent vector, defined as in (12). Call $z(t) \doteq \psi(t) \cdot v(t)$, with

$$\psi(t) \doteq \frac{a - \lambda(t)}{f(t, x(t)) - \lambda(t)} \;\in \left[\frac{a-d}{b-d},\ 1\right] = \left[\frac{1}{M},\ 1\right].$$

By the equation (13) and the fact that $\lambda(t) \in [c, d]$, it follows that z can have only upward jumps. Moreover,

$$\dot{z}(t) \geq -\beta \cdot \frac{z(t)}{t} \qquad \text{a.e.,}$$

with $\beta \doteq (b-a)/(a-d)$. Therefore

$$z(s) \leq z(t) \left(\frac{t}{s}\right)^{\beta}, \qquad v(s) \leq M \left(\frac{t}{s}\right)^{\beta} v(t) \qquad \text{for all } 0 < s < t < \tilde{t} \tag{33}$$

If x_1, x_2 are two solutions of (1), with $x_1(\tilde{t}) = \tilde{x}_1$, $x_2(\tilde{t}) = \tilde{x}_2$, from (33) we obtain

$$\begin{aligned} |x_1(0) - x_2(0)| &\leq (b-a)s + |x_1(s) - x_2(s)| \\ &\leq (b-a)s + M \left(\frac{\tilde{t}}{s}\right)^{\beta} |\tilde{x}_1 - \tilde{x}_2|, \end{aligned}$$

valid for every $s \in \,]0, \tilde{t}]$. By choosing

$$s = \min\left\{\tilde{t},\quad |\tilde{x}_1 - \tilde{x}_2|^{1/(1+\beta)}\right\},$$

one obtains

$$|x_1(0) - x_2(0)| \leq \left[(b-a) + M\tilde{t}^{\beta}\right] |\tilde{x}_1 - \tilde{x}_2|^{1/(1+\beta)}, \tag{34}$$

and

$$|x_1(0) - x_2(0)| \leq (b-a)|\tilde{x}_1 - \tilde{x}_2|^{1/(1+\beta)} + M|\tilde{x}_1 - \tilde{x}_2|, \tag{35}$$

respectively in the cases where $\tilde{t}$ is greater or smaller than $|\tilde{x}_1 - \tilde{x}_2|^{1/(1+\beta)}$.

Now let $x(\cdot)$ and $w(\cdot)$ be solutions to (1) and (21) respectively, with the same terminal data $x(\tilde{t}) = w(\tilde{t}) = \tilde{x}$. From the second inequality in (33), for every $s \in\,]0, \tilde{t}]$ we deduce

$$\begin{aligned} |x(0) - w(0)| &\leq (b-a)s + \int_s^{\tilde{t}} M \left(\frac{t}{s}\right)^\beta |e(t)|\, dt. \\ &\leq (b-a)s + M \left(\frac{\tilde{t}}{s}\right)^\beta \int_0^{\tilde{t}} |e(t)|\, dt. \end{aligned}$$

Choosing $s \doteq \left(\int |e(t)|\, dt\right)^{1/(1+\beta)}$ we obtain

$$|x(0) - w(0)| \leq \left[(b-a) + M\tilde{t}^\beta\right] \cdot \left[\int_0^{\tilde{t}} |e(\tau)|\, d\tau\right]^{1/(1+\beta)}.$$

By the approximation technique described in Lemma 2, the same estimates hold for general functions f (not necessarily piecewise constant), satisfying (A1)-(A2). In particular, if x_1, x_2 are solutions to

$$\dot{x}_i(t) = f_i(t, x_i(t)), \qquad x_i(\tilde{t}) = \tilde{x} \qquad i = 1, 2,$$

for functions f_i satisfying (A1)-(A2), then

$$|x_1(0) - x_2(0)| \leq \left[(b-a) + M\tilde{t}^\beta\right] \cdot \mathrm{dist}(f_1, f_2)^{1/(1+\beta)}. \tag{36}$$

4 Application to conservation laws

In this section we prove Theorem 2 on the continuous dependence of solutions to the conservation laws (10)-(11). Let the initial data $\bar{w}, \bar{\theta}$ be given, according to the assumptions in Theorem 2. Then the component $w = w(t, x)$ is uniquely determined, being the unique entropy solution of (10) in the sense of Kruzkov [10]. Moreover, the component $\theta = \theta(t, x)$ provides a solution to the linear system (11). It is thus determined by the explicit formula

$$\theta(t, x) = \bar{\theta}(y(0; t, x))$$

where $s \mapsto y(s; t, x)$ is the solution to the backward Cauchy problem

$$\dot{y}(s) = g(w(s, y(s))) \qquad y(t) = x. \tag{37}$$

The uniqueness and Hölder continuous dependence of the solution of (37), stated at (34)-(35), imply the uniqueness of θ and its continuity w.r.t. both variables t, x.

We now study the dependence of the solution w, θ on the initial data $\bar w, \bar\theta$. Assume that

$$\bar w_n \to \bar w \quad \text{in } \mathbf{L}^1, \quad \bar\theta_n \to \bar\theta \quad \text{in } \mathcal{C}^0.$$

By the contractivity of the flow generated by scalar conservation law [7, 10], for every $t \geq 0$ the corresponding solutions of (10) satisfy

$$\|w_n(t,\cdot) - w(t,\cdot)\|_{\mathbf{L}^1} \leq \|\bar w_n - \bar w\|_{\mathbf{L}^1}.$$

To establish the convergence of the corresponding solutions θ_n of (11), call

$$f_n(t,x) \doteq g(w_n(t,x)), \qquad f(t,x) \doteq g(w(t,x)).$$

We claim that the distance between f_n and f, defined at (24), can be precisely estimated, namely,

$$\begin{aligned} \mathrm{dist}(f_n, f) &\doteq \sup_{u\in\mathcal{U}} \int_0^T \Big| g(w_n(t,u(t))) - g(w(t,u(t)))\Big|\, dt \\ &\leq \frac{1}{a-d}\, \|\bar w_n - \bar w\|_{\mathbf{L}^1} \cdot \max_{w\in[w^-,w^+]} |g'(w)|. \end{aligned} \tag{38}$$

To prove (38), consider the vector

$$\Psi \doteq \Big(|w_n - w|,\ [F(w_n) - F(w)]\mathrm{sgn}\,(w_n - w)\Big).$$

Fix any $u \in \mathcal{U}$. Let $v(t) = u(T) + c(t-T)$ denote the straight line passing through the point $(T, u(T))$ with slope c, and consider the triangular domain

$$\Omega \doteq \Big\{(t,x);\ \ t\in[0,T],\ u(t) \leq x \leq v(t)\Big\}.$$

By entropy inequality $\mathrm{div}\,\Psi \leq 0$ one has

$$\int_{\partial\Omega} \Psi\cdot\vec n\, d\sigma \leq 0, \tag{39}$$

where $\partial\Omega$ is the boundary of Ω and $\vec{n}$ denotes the outer normal. Writing out (39) for each of the three lines composing the boundary of Ω, we get

$$-I_1 + I_2 + I_3 \leq 0,$$

where

$$\begin{aligned} I_1 &= \int_{u(0)}^{u(T)-cT} |w_n(0,x) - w(0,x)|\, dx, \\ I_2 &= \int_0^T \Big\{ \dot{u}(t)|w_n - w| - (F(w_n) - F(w))\mathrm{sgn}\ (w_n - w) \Big\}_{x=u(t)} dt, \\ I_3 &= \int_0^T \Big\{ -c|w_n - w| + (F(w_n) - F(w))\mathrm{sgn}\ (w_n - w) \Big\}_{x=v(t)} dt. \end{aligned}$$

Observing that

$$(F(w_n) - F(w))\, \mathrm{sgn}\ (w_n - w) = F'(w^*)|w_n - w|$$

for some $w^* \in [w_n, w]$, we conclude that $I_3 \geq 0$ because $F' \in [c, d]$. Moreover,

$$I_2 \geq (a-d) \int_0^T \Big| w_n(t, u(t)) - w(t, u(t)) \Big|\, dt.$$

Therefore,

$$\int_0^T \Big| w_n(t, u(t)) - w(t, u(t)) \Big|\, dt \leq \frac{1}{a-d} \int_{u(0)}^{u(T)-cT} |\bar{w}_n(x) - \bar{w}(x)|\, dx.$$

Since $u \in \mathcal{U}$ was arbitrary, the estimate (38) follows.

We can now apply (36) with $f_1 \doteq f$ and $f_2 \doteq f_n$. From the convergence $\mathrm{dist}(f, f_n) \to 0$ it thus follows

$$\lim_{n\to\infty} |y_n(0; t, x) - y(0; t, x)| = 0,$$

uniformly for $(t, x) \in [0, T] \times I\!R$. Therefore, the assumption $\bar{\theta}_n \to \bar{\theta}$ in $\mathcal{C}^0$ implies the convergence $\theta_n(t, x) \to \theta(t, x)$, uniformly for (t, x) in bounded sets.

5 Counterexamples

For general 3×3 quasilinear systems, the Cauchy problem with $\mathbf{L}^\infty$ initial data is not well posed, as shown by the following counterexample.

Example 1. Consider the hyperbolic system

$$\begin{cases} u_t + u_x = 0 \\ v_t - v_x = 0 \\ w_t + \dfrac{1-uv}{6} w_x = 0 \end{cases} \tag{40}$$

with initial data

$$u(0,x) = v(0,x) = \varphi(x), \qquad\qquad w(0,x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

where

$$\varphi(x) \doteq \begin{cases} 1, \quad \text{if } \ 2^{-(2n+1)} < |x| < 2^{-2n}, \\ \\ -1, \quad \text{if } \ 2^{-(2n)} < |x| < 2^{-2n-1}, \end{cases} \tag{41}$$

For this problem, the components u and v of the solution can be written out explicitly:

$$u(t,x) = u(0, x-t), \quad v(t,x) = v(0, x+t).$$

On the other hand, the function w is given by

$$w(t,x) = \begin{cases} 0 & \text{if } x < y(t), \\ 1 & \text{if } x > y(t), \end{cases} \tag{42}$$

where $t \mapsto y(t)$ is a solution of

$$\dot y = \frac{1-uv}{6} \qquad y(0) = 0. \tag{43}$$

Note that u and v only take the values 1 and -1, therefore, $\dot y = 0$ if sgn $(u) =$ sgn (v), and $\dot y = 1/3$ if sgn $(u) \neq$ sgn (v). The integral curves of this ODE are illustrated in Figure 2. It is clear that the problem (43) has the two distinct solutions $y(t) = 0$ and $y(t) = t/3$ (together with infinitely many other solutions). Inserting these functions in (42), we obtain two distinct solutions of the system (40) with the same initial data. Observe that, since all three characteristic fields are linearly degenerate, both of these solutions can be obtained as limits of smooth solutions, in $\mathbf{L}^1_{loc}$.

We remark that in this example the non-uniqueness of solutions is essentially due to the unbounded variation of the components u, v. Indeed, if both initial data $\bar{u}, \bar{v} : I\!\!R \mapsto [0,1]$ have bounded variation, then the function

$$f(t,x) \doteq \frac{1 - \bar{u}(x-t)\bar{v}(x+t)}{6}$$

has bounded directional variation and the uniqueness theorem [3] can be applied.

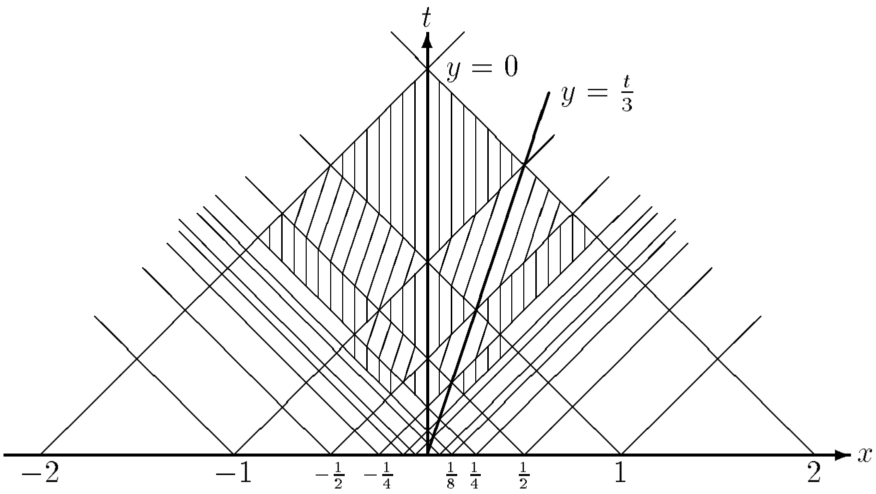

Figure 2:

The system (40) is not in conservative form. For strictly hyperbolic systems of conservation laws, when the initial data has unbounded variation even the existence of solutions with values in $\mathbf{L}^1_{loc}(I\!\!R)$ may fail.

Example 2. Consider the system

$$\begin{cases} u_t + u_x = 0 \\ v_t - v_x = 0 \\ w_t + \left(\dfrac{1-uv}{6} \cdot w\right)_x = 0 \end{cases} \tag{44}$$

with initial data

$$u(0,x) = \varphi(x+1), \qquad v(0,x) = \varphi(x-1), \qquad w(0,x) = 1.$$

where φ is the function in (41). Clearly we have

$$u(t,x) = \varphi(x+1-t), \qquad v(t,x) = \varphi(x-1+t).$$

Call $t \mapsto y(t,x)$ the characteristic line through $(0,x)$, i.e. the solution to the Cauchy problem

$$\dot y(t) = f(t,x) \doteq \frac{1-u(t,x)v(t,x)}{6}, \qquad y(0) = x.$$

For $t \in [0,1]$ this Cauchy problem has a unique solution. In particular,

$$y(t,-1/3) = \frac{t-1}{3}, \qquad y(t,0) = 0.$$

For any $\varepsilon > 0$, applying the divergence theorem on the domain

$$\Omega_\varepsilon \doteq \Big\{(t,x);\ \ t \in [0,\ 1-\varepsilon],\ \ (t-1)/3 \le x \le 0\Big\}$$

we obtain

$$\int_{-\varepsilon/3}^{0} w(1-\varepsilon, x)\ dx = \int_{-1/3}^{0} w(0,x)\ dx = \frac{1}{3}.$$

Therefore, as $t \to 1$, the solution $w(t,\cdot)$ has no limit in $\mathbf{L}^1_{loc}$. Its weak limit is a measure with a mass of 1/3 concentrated at the origin. This measure-valued solution can be prolonged for times $t > 1$ by letting the atomic mass travel along any characteristic line through the point $(1,0)$. However, the analysis in Example 1 shows that the Cauchy problem

$$\dot y(t) = f(t,x), \qquad y(1) = 0$$

has the two solutions $y(t) = 0$ and $y(t) = (t-1)/3$. Summarizing, this is an example where

- For $t \in [0,1[$ there exists a unique solution. This solution cannot be prolonged as a continuous function $[0,T] \mapsto \mathbf{L}^1_{loc}(I\!R)$ for $T > 1$.
- For $t > 1$ infinitely many measure-valued solutions exist.

Acknowledgements: This research was partially supported by TMR project HCL # ERBFMRXCT960033.

The second author wishes to thank the Department of Informatics, University of Oslo for supporting her visit at SISSA.

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