A DIFFERENCE SCHEME FOR A STIFF SYSTEM OF CONSERVATION LAWS

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ABSTRACT. A stiff system of conservation laws is analyzed with a difference method. The existence of entropy-satisfying BV-solutions to the initial value problems is estabilished. Furthermore, we show that the solutions converge to the solutions of the corresponding equilibrium system as the relaxation time tends to zero.

1. INTRODUCTION

In this paper we use a difference scheme to study the following system of conservation laws with a stiff source term

$$\begin{aligned} &u_t + (f(u) - v)_x = 0, \\ &v_t + \frac{v - (1-\mu)f(u)}{\epsilon} = 0. \end{aligned} \tag{1.1}$$

Here f is a given function of u; μ is a parameter; the scalar variables u and v are unknown and ϵ is a small positive parameter, called the relaxation time.

The above system serves as a model for the equations of motion for a viscoelastic solid. Precisely, the relation of the system to the equations of motion for a nonlinear Maxwell solid is the same as the relation of the Burgers equation, $u_t + (u^2/2)_x = 0$, to the equations of motion for an ideal gas (see [2] and references cited therein). Furthermore, the system (1.1) is an example of general hyperbolic conservation laws with stiff source terms ([8]).

The main results of the present paper are the existence of entropy-satisfying BV-solutions (u^ϵ, v^ϵ) to the above system with proper initial data and the convergence of (u^ϵ, v^ϵ), as ϵ tends to zero, to the solution of the so-called equilibrium or reduced system of (1.1)

$$\begin{aligned} &u_t + \mu f(u)_x = 0, \\ &v - (1-\mu)f(u) = 0. \end{aligned} \tag{1.2}$$

Those results are based on the following condition on f and μ

$$f'(u) > 0 \quad \text{and} \quad 0 < \mu < 1. \tag{1.3}$$

This condition on system (1.1) is equivalent to the stability condition in [8] and the subcharacteristic condition in [4].

Here an entropy solution is defined in the following way.

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Definition 1.1. A pair of bounded measurable functions (u^ϵ, v^ϵ) is called an entropy solution of the system (1.1) if the following inequality

$$\eta(u^\epsilon, v^\epsilon)_t + q(u^\epsilon, v^\epsilon)_x \leq \frac{1}{\epsilon}\eta_v(u^\epsilon, v^\epsilon)((1-\mu)f(u^\epsilon) - v^\epsilon)$$

holds in the sense of distribution for any convex entropy pair (η, q), which is defined in section 3.

System (1.1) has been studied by several authors. In [2] Greenberg and Hsiao dealt with a class of Riemann problems of the above system. For the Cauchy problems with bounded measurable initial data, Chen and Liu ([1]) proved the existence of L^∞ entropy solutions by using the viscosity method and the theory of compensated compactness; furthermore they again used the theory of compensated compactness to discuss the limit as ϵ tends to zero.

Inspired by the papers [7] and [5] of Tveito and others, I prove the existence of entropy-satisfying BV-solutions for system (1.1) by demonstrating convergence of a difference scheme. Because of such an existence result, the study of the zero relaxation limit becomes a very simple matter. In fact, the existence result says that the family of solutions, indexed by the parameter ϵ, lies in a bounded subset of $L^\infty(R^+ \times R) \cap L^\infty(R^+, BV(R)) \cap Lip(R^+, L^1(R))$ with $R^+ = (0, \infty)$, which is compactly (see [6]) embeded into $L^1_{loc}(R^+ \times R)$. This compactness is naturally used to study the zero ralaxation limit.

We note that the Kružkov entropy condition was used by Tveito and others in [7] and [5] to study a simpler model. However, the entropy condition does not make sense for system (1.1). Thus, the approach in [7] does not seem to be valid for our problem. But the approach here works for the problem in the aforementioned papers and some more general systems than that given in (1.1).

The paper is organized as follows. Section 2 is devoted to the analysis of a difference scheme for system (1.1). In section 3, we analyze entropies for more general systems and establish an entropy inequality for the difference solution. The main results are shown in section 4.

2. The Difference Scheme

Let $u_0(x)$ and $v_0(x)$ be two bounded measurable functions of $x \in R$. Our difference approximation to (1.1) with initial data $(u_0(x), v_0(x))$ is the following semi-implicit scheme

$$\begin{aligned} &\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f(u_j^n) - v_j^n - f(u_{j-1}^n) + v_{j-1}^n}{\Delta x} = 0, \\ &\frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{v_j^{n+1} - (1-\mu)f(u_j^{n+1})}{\epsilon} = 0, \end{aligned} \tag{2.1}$$

together with

$$(u_j^0, v_j^0) = \frac{1}{\Delta x}\int_{x_j}^{x_{j+1}} (u_0(x), v_0(x))dx. \tag{2.2}$$

Here Δx, Δt are the increments respectively in x and t; u_j^n and v_j^n denote approximations of $u(x,t)$ and $v(x,t)$, respectively, over the grid-block $[x_j, x_{j+1}) \times [t_n, t_{n+1})$ with $x_j = j\Delta x$ and $t_n = n\Delta t$; $j = 0, \pm 1, \pm 2, \cdots$ and $n = 0, 1, 2, \cdots$.

Let C_1 and C_2 be such constants that the initial data take values in Σ, which means

$$C_1 \leq f(u_j^0) - v_j^0 \leq C_2,$$
$$\frac{1-\mu}{\mu} C_1 \leq v_j^0 \leq \frac{1-\mu}{\mu} C_2 \tag{2.3}$$

for all j. Throughout this paper, we assume that the grid sizes satisfy the CFL-condition

$$f'(u) \frac{\Delta t}{\Delta x} \leq 1 \tag{2.4}$$

for u, with some v, in the above domain Σ. It is interesting, from a computational point of view, to note that this condition is independent of the relaxation time ϵ.

It is obvious that for each (j, n), (u_j^n, v_j^n) is well-defined. Furthermore, we have

Lemma 2.1. *Assume the CFL-condition in (2.4). Then for each (j, n), (u_j^n, v_j^n) lies in Σ, namely,*

$$C_1 \leq f(u_j^n) - v_j^n \leq C_2,$$
$$\frac{1-\mu}{\mu} C_1 \leq v_j^n \leq \frac{1-\mu}{\mu} C_2.$$

Proof. Set $A_j^n = \int_0^1 f'(\theta u_j^{n+1} + (1-\theta) u_j^n) d\theta$ and $U_j^n = f(u_j^n) - v_j^n$. Then the difference scheme in (2.1) gives

$$\frac{U_j^{n+1} + v_j^{n+1} - U_j^n - v_j^n}{\Delta t} + A_j^n \frac{U_j^n - U_{j-1}^n}{\Delta x} = 0,$$
$$\frac{v_j^{n+1} - v_j^n}{\Delta t} - \frac{(1-\mu) U_j^{n+1} - \mu v_j^{n+1}}{\epsilon} = 0. \tag{2.5}$$

From (2.5) we obtain

$$\frac{\epsilon + \Delta t}{\epsilon + \mu \Delta t} U_j^{n+1} = (1 - \lambda A_j^n) U_j^n + \lambda A_j^n U_{j-1}^n + \frac{\mu \Delta t}{\epsilon + \mu \Delta t} v_j^n,$$
$$v_j^{n+1} = \frac{\epsilon}{\epsilon + \mu \Delta t} v_j^n + \frac{(1-\mu)\Delta t}{\epsilon + \mu \Delta t} U_j^{n+1}, \tag{2.6}$$

where $\lambda = \dfrac{\Delta t}{\Delta x}$. If we inductively assume that $C_1 \leq f(u_j^n) - v_j^n \leq C_2$ and $\dfrac{1-\mu}{\mu} C_1 \leq v_j^n \leq \dfrac{1-\mu}{\mu} C_2$ for all j and n, then it is easily to see from (2.6), by using the CFL-condition in (2.4) and the stability condition in (1.3), that

$$C_1 \leq f(u_j^{n+1}) - v_j^{n+1} \leq C_2,$$
$$\frac{1-\mu}{\mu} C_1 \leq v_j^{n+1} \leq \frac{1-\mu}{\mu} C_2.$$

This proves the lemma. □

Lemma 2.1 implies an L^∞-bound of the difference solution. We will prove the existence of a BV-solution by demonstrating convergence of the difference solution via the standard argument in [6]. Thus we need to show that the difference solution has bounded total variation in x and is L^1-continuous in time. To this end, we further assume that the initial data satisfy

$$
\begin{aligned}
&TV(u^0) + TV(v^0) \stackrel{\text{def}}{=} \sum_j (|u_j^0 - u_{j-1}^0| + |v_j^0 - v_{j-1}^0|) < \infty; \\
&\|v^0 - (1-\mu) f(u^0)\|_{L^1} \stackrel{\text{def}}{=} \sum_j |v_j^0 - (1-\mu) f(u_j^0)| \Delta x = O(\epsilon),
\end{aligned} \tag{2.7}
$$

where the second relation means that the initial data are close to equilibruim.

In what follows, for convenience we often use the Riemann invariant $U = f(u) - v$ instead of u, since there are positive contants M_1, M_2 such that

$$
M_1|u_1 - u_2| - |v_1 - v_2| \le |U_1 - U_2| \le M_2|u_1 - u_2| + |v_1 - v_2| \tag{2.8}
$$

for $(u_i, v_i) \in \Sigma (i = 1, 2)$.

Firstly, we establish

Lemma 2.2. *For $n = 1, 2, \cdots$,*

$$
TV(U^n) + TV(v^n) \le TV(U^0) + TV(v^0).
$$

Proof. Let $\Delta U_j^n = U_j^n - U_{j-1}^n$ and $\Delta v_j^n = v_j^n - v_{j-1}^n$. Then it follows from (2.6) that

$$
\begin{aligned}
\frac{\epsilon + \Delta t}{\epsilon + \mu \Delta t} \Delta U_j^{n+1} = &\quad (1 - \lambda A_j^n) \Delta U_j^n + \lambda A_{j-1}^n \Delta U_{j-1}^n + \frac{\mu \Delta t}{\epsilon + \mu \Delta t} \Delta v_j^n, \\
\Delta v_j^{n+1} = &\quad \frac{\epsilon}{\epsilon + \mu \Delta t} \Delta v_j^n + \frac{(1-\mu)\Delta t}{\epsilon + \mu \Delta t} \Delta U_j^{n+1}.
\end{aligned}
$$

Furthermore, using the CFL-condition in (2.4) and the stability condition in (1.3) yields

$$
\begin{aligned}
\frac{\epsilon + \Delta t}{\epsilon + \mu \Delta t} |\Delta U_j^{n+1}| \le &(1 - \lambda A_j^n) |\Delta U_j^n| + \lambda A_{j-1}^n |\Delta U_{j-1}^n| + \frac{\mu \Delta t}{\epsilon + \mu \Delta t} |\Delta v_j^n|, \\
|\Delta v_j^{n+1}| \le &\frac{\epsilon}{\epsilon + \mu \Delta t} |\Delta v_j^n| + \frac{(1-\mu)\Delta t}{\epsilon + \mu \Delta t} |\Delta U_j^{n+1}|.
\end{aligned} \tag{2.9}
$$

Notice that $\lim_{j \to \pm\infty} |\Delta U_j^n| = 0$, if we inductively assume $\sum_j |\Delta U_j^n| < +\infty$. Now adding the two inequalities in (2.9) gives

$$
\sum_j (|\Delta v_j^{n+1}| + |\Delta U_j^{n+1}|) \le \sum_j (|\Delta v_j^n| + |\Delta U_j^n|),
$$

or

$$
TV(U^{n+1}) + TV(v^{n+1}) \le TV(U^n) + TV(v^n).
$$

The proof is complete. $\square$

Next we show that the difference solution remains close to equilibrium for all time. From now on, C denotes a generic positive constant which may change from relation to relation and is independent of Δx, Δt and ϵ.

Lemma 2.3. *There is a constant C such that for all n,*

$$
\|v^n - (1-\mu) f(u^n)\|_{L^1} \le C\epsilon.
$$

Proof. Set $Q_j^n = (1-\mu)U_j^n - \mu v_j^n$. Then from (2.5),

$$\begin{aligned} Q_j^{n+1} - Q_j^n = & \ (1-\mu)(U_j^{n+1} - U_j^n) - \mu(v_j^{n+1} - v_j^n) \\ = & \ -(1-\mu)\lambda A_j^n (U_j^n - U_{j-1}^n) - \frac{\Delta t}{\epsilon} Q_j^{n+1}. \end{aligned}$$

Thus,

$$(1 + \frac{\Delta t}{\epsilon})|Q_j^{n+1}| \le |Q_j^n| + \lambda(1-\mu)A_j^n |U_j^n - U_{j-1}^n|,$$

and thereby,

$$(1 + \frac{\Delta t}{\epsilon})\|Q^{n+1}\|_{L^1} \le \|Q^n\|_{L^1} + \Delta x \sum_j |\Delta U_j^n|_{L^1} \le \|Q^n\|_{L^1} + C\Delta t. \tag{2.10}$$

Here we have used Lemma 2.2 and $\Delta x = O(\Delta t)$. From (2.10) it is easy to verify that

$$\|Q^n\|_{L^1} \le (1 + \frac{\Delta t}{\epsilon})^{-n} \|Q^0\|_{L^1} + C\epsilon.$$

Hence the lemma follows from the assumption $\|Q^0\|_{L^1} \le C\epsilon$ in (2.7). □

Lemma 2.3 implies the following two lemmas.

Lemma 2.4. *For* $n = 0, 1, 2, \cdots$,

$$\|U^{n+1} - U^n\|_{L^1} + \|v^{n+1} - v^n\|_{L^1} \le C\Delta t.$$

Proof. It follows from the scheme in (2.5) that

$$\begin{aligned} |U_j^{n+1} - U_j^n| \le & \ \lambda A_j^n |U_j^n - U_{j-1}^n| + \frac{\Delta t}{\epsilon}|Q_j^{n+1}|, \\ |v_j^{n+1} - v_j^n| \le & \ \frac{\Delta t}{\epsilon}|Q_j^{n+1}|. \end{aligned}$$

By using the last two lemmas, we obtain

$$\|U^{n+1} - U^n\|_{L^1} + \|v^{n+1} - v^n\|_{L^1} \le C\Delta t.$$

The proof is complete. □

Lemma 2.5.

$$\|u^{n,\epsilon} - u^{n,0}\|_{L^1} + \|v^{n,\epsilon} - v^{n,0}\|_{L^1} \le n\Delta t C \frac{\epsilon}{\Delta t}.$$

Proof. The first equation of the scheme in (2.1) can be rewritten as

$$u_j^{n+1,\epsilon} = u_j^{n,\epsilon} + \lambda(\mu f(u_{j-1}^{n,\epsilon}) - \mu f(u_j^{n,\epsilon})) + \lambda(Q_{j-1}^{n,\epsilon} - Q_j^{n,\epsilon}). \tag{2.11}$$

Here we have used the superscript ϵ to denote the scheme depending on the parameter ϵ. On the other hand, fix Δt and Δx and let ϵ go to zero. It is easy to see that the difference solution $(u_j^{n,\epsilon}, v_j^{n,\epsilon})$ converges to $(u_j^{n,0}, v_j^{n,0})$, which satisfies

$$\begin{aligned} u_j^{n+1,0} &= u_j^{n,0} + \lambda(\mu f(u_{j-1}^{n,0}) - \mu f(u_j^{n,0})), \\ v_j^{n,0} &= (1-\mu) f(u_j^{n,0}). \end{aligned} \tag{2.12}$$

This is just the first-order upwind scheme for the reduced system in (1.2).

Set $e_j^n = u_j^{n,\epsilon} - u_j^{n,0}$. Then from the equation in (2.11) and the first equation in (2.12) it follows that

$$e_j^{n+1} = e_j^n + \lambda\mu f'(\tilde{u}_{j-1}^n)e_{j-1}^n - \lambda\mu f'(\tilde{u}_j^n)e_j^n + \lambda(Q_{j-1}^{n,\epsilon} - Q_j^{n,\epsilon}),$$

where $\tilde{u}_j^n$ satisfies

$$f(u_j^{n,\epsilon}) - f(u_j^{n,0}) = (u_j^{n,\epsilon} - u_j^{n,0})f'(\tilde{u}_j^n).$$

Therefore,

$$|e_j^{n+1}| \le |e_j^n| + \lambda\mu f'(\tilde{u}_{j-1}^n)|e_{j-1}^n| - \lambda\mu f'(\tilde{u}_j^n)|e_j^n| + \lambda(|Q_{j-1}^{n,\epsilon}| + |Q_j^{n,\epsilon}|).$$

Note that $\|u_j^{n,\epsilon} - u_j^{n,0}\|_{L^1} < +\infty$ implies $\lim_{j\to\pm\infty} |u_j^{n,\epsilon} - u_j^{n,0}| = 0$. Using Lemma 2.3 and the last inequality gives

$$\|e^{n+1}\|_{L^1} \le \|e^n\|_{L^1} + 2\lambda\|Q^n\|_{L^1} \le \|e^n\|_{L^1} + C\epsilon.$$

Since $\|u_j^{0,\epsilon} - u_j^{0,0}\|_{L^1} = 0$, we obtain

$$\|e^{n+1}\|_{L^1} \le (n+1)C\epsilon \le (n+1)\Delta t C\frac{\epsilon}{\Delta t}.$$

This completes the proof. □

By combining Lemma 2.5 and the well-known error estimate in [3], we immediately obtain

Lemma 2.6. *There is a constant C_T, depending only on T, such that for all n with $n\Delta t \le T$,*

$$\|u^{n,\epsilon} - u^0(\cdot, n\Delta t)\|_{L^1} + \|v^{n,\epsilon} - v^0(\cdot, n\Delta t)\|_{L^1} \le C_T(\frac{\epsilon}{\Delta t} + \sqrt{\Delta t}).$$

Here $(u^0, v^0)(x,t)$ is the unique solution of the reduced system in (1.2) with initial data $(u_0, v_0)(x)$.

3. Entropies

In this section, we discuss entropies for general systems of the form

$$\begin{aligned} u_t + F(u,v)_x &= 0, \\ v_t + Q(u,v) &= 0. \end{aligned} \tag{3.1}$$

We assume that there is a function $h(u)$ such that $Q(u,v) = 0$ implies $v = h(u)$. In addition, we call (3.1) a full system and

$$u_t + F(u, h(u))_x = 0 \tag{3.2}$$

its equilibrium or reduced system. Now, the stability condition like that in (1.3) for the system (1.1) becomes

$$Q_v(u,v) > 0 \quad \text{and} \quad F_v Q_u(F_u Q_v - F_v Q_u) > 0 \tag{3.3}$$

for all (u,v) under consideration. See [8] for this condition. Note that

$$h'(u) = -Q_u(u,h(u))/Q_v(u,h(u)). \tag{3.4}$$

Let (η, q) be a smooth entropy pair for the system in (3.1). Then by the definition we have

$$\eta_u(u,v)F_u(u,v) = q_u(u,v) \quad , \quad \eta_u(u,v)F_v(u,v) = q_v(u,v). \tag{3.5}$$

Set $\phi = \eta_u(u,v)$. From the last two relations it immediately follows that

$$\phi_v F_u - \phi_u F_v = 0.$$

Thus, ϕ is a function of $F(u,v)$, that is, $\phi = \phi(F(u,v))$. Consequently, the general representation of the entropy pairs is

$$\begin{aligned} \eta &= \int^u \phi(F(\xi,v))d\xi + \psi(v), \\ q &= \int^{F(u,v)} \phi(\xi)d\xi, \end{aligned} \tag{3.6}$$

where ϕ and ψ are two arbitrary smooth functions of a single variable.

Note that both ϕ and q are functions of a single variable. Having these explicit expressions, one can easily see the following basic properties about the entropy pairs

(a). $\eta_u(u,v) = \phi(F(u,v))$;
(b). $q'(\xi) = \phi(\xi)$;
(c). If η is convex, then $\phi(F(u,v))$ is increasing with respect to u.

Obviously, there exist convex entropies for the systems in (3.1). Furthermore, we have

Lemma 3.1. *Let (η, q) be an entropy pair for the full system and η_v vanish along $v = h(u)$. Then $(\eta, q)(u, h(u))$ is an entropy pair for the reduced system. Conversely, given an entropy pair (η_*, q_*) for the reduced system, there is an entropy pair (η, q) for the full system, such that*

$$(\eta, q)(u, h(u)) = (\eta_*, q_*)(u) \quad \textit{and} \quad \eta_v(u, h(u)) = 0.$$

Proof. For the first part, we only need to show that

$$q(u, h(u))' = \eta(u, h(u))' F(u, h(u))'.$$

In fact, the relations in (3.5), satisfied by η and q, give us

$$\begin{aligned} q(u,h(u))' &= q_u(u,h(u)) + q_v(u,h(u))h'(u) \\ &= \eta_u(u,h(u))F_u(u,h(u)) + \eta_u(u,h(u))F_v(u,h(u))h'(u) \\ &= \eta_u(u,h(u))F(u,h(u))' \\ &= \eta(u,h(u))'F(u,h(u))'. \end{aligned}$$

For the other part, by the general representation in (3.6) of the entropy pairs we need to choose ϕ and ψ such that

$$\int^u \phi(F(\xi, h(u)))d\xi + \psi(h(u)) = \eta_*(u) \tag{3.7}$$

and

$$\int^u \phi'(F(\xi, h(u)))F_v(\xi, h(u))d\xi + \psi'(h(u)) = 0. \tag{3.8}$$

Differentiating the equation in (3.7) with respect to u and using the equation in (3.8), we get

$$\eta_*'(u) = \phi(F(u, h(u))).$$

The stability condition in (3.3) with the relation in (3.4) implies $F(u,h(u))' \neq 0$. Therefore, as a function of u, $F(u,h(u))$ has an inverse, say g. Namely, $F(g(v),h(g(v))) = v$. Thus, we take

$$\phi(\xi) = \eta'_*(g(\xi)).$$

Having ϕ, ψ can be chosen from the equation in (3.7) as

$$\psi(v) = \eta_*(h^{-1}(v)) - \int^{h^{-1}(v)} \phi(F(\xi,v))d\xi,$$

where we have used the fact that h has an inverse h^{-1}, which is also implied by the stability condition in (3.3) with the relation in (3.4). This proves the lemma. □

The stability condition in (3.3) also implies $F_u(u,v) \neq 0$. Without loss of generality, we assume that $F_u(u,v) > 0$. Thus, the scheme for the system in (1.1) is an example of the following scheme for the system in (3.1)

$$\begin{aligned} \frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{F(u_j^n, v_j^n) - F(u_{j-1}^n, v_{j-1}^n)}{\Delta x} &= 0, \\ \frac{v_j^{n+1} - v_j^n}{\Delta t} + Q(u_j^{n+1}, v_j^{n+1}) &= 0. \end{aligned} \tag{3.9}$$

And the CFL-condition now becomes

$$\lambda F_u(u,v) \leq 1 \tag{3.10}$$

for (u,v) under consideration.

Now we turn to prove the main result of this section.

Lemma 3.2. *Suppose (η, q) is a convex entropy pair for the system in (3.1), the CFL-condition in (3.10) holds and (u_j^n, v_j^n) is the solution of the difference scheme in (3.9). Then the following discrete entropy inequality*

$$\frac{\eta(u_j^{n+1}, v_j^{n+1}) - \eta(u_j^n, v_j^n)}{\Delta t} + \frac{q(u_j^n, v_j^n) - q(u_{j-1}^n, v_{j-1}^n)}{\Delta x} \leq -\eta_v(u_j^{n+1}, v_j^{n+1}) Q(u_j^{n+1}, v_j^{n+1})$$

holds for each n and j.

Proof. The convexity of η implies that $\eta_{vv} \geq 0$ and thereby

$$\begin{aligned} \eta(u_j^{n+1}, v_j^{n+1}) - \eta(u_j^{n+1}, v_j^n) &\leq (v_j^{n+1} - v_j^n)\eta_v(u_j^{n+1}, v_j^{n+1}) \\ &\leq -\Delta t \eta_v(u_j^{n+1}, v_j^{n+1}) Q(u_j^{n+1}, v_j^{n+1}), \end{aligned} \tag{3.11}$$

where we have used the scheme.

On the other hand, set $\delta = u_j^{n+1} - u_j^n$ and consider

$$\eta(u_j^{n+1}, v_j^n) - \eta(u_j^n, v_j^n).$$

Since $\delta = \lambda(F(u_{j-1}^n, v_{j-1}^n) - F(u_j^n, v_j^n))$ due to the scheme, by using the properties (a) and (b) of the entropies in the previous paragraph we have

$$
\begin{aligned}
&\eta(u_j^{n+1}, v_j^n) - \eta(u_j^n, v_j^n) \\
&= (u_j^{n+1} - u_j^n) \int_0^1 \eta_u(\theta u_j^{n+1} + (1-\theta) u_j^n, v_j^n) d\theta \\
&= \delta \int_0^1 \phi(F(u_j^n + \theta\delta, v_j^n)) d\theta \\
&= \lambda(F(u_{j-1}^n, v_{j-1}^n) - F(u_j^n, v_j^n)) \int_0^1 q'(\theta F(u_{j-1}^n, v_{j-1}^n) + (1-\theta) F(u_j^n, v_j^n)) d\theta \\
&\quad + \delta \int_0^1 \{\phi(F(u_j^n + \theta\delta, v_j^n)) - \phi(\theta F(u_{j-1}^n, v_{j-1}^n) + (1-\theta) F(u_j^n, v_j^n))\} d\theta \\
&\equiv \lambda(q(F(u_{j-1}^n, v_{j-1}^n)) - q(F(u_j^n, v_j^n))) + \delta \int_0^1 \gamma(\theta\delta) d\theta,
\end{aligned}
\tag{3.12}
$$

where $\gamma(\theta\delta) \stackrel{\text{def}}{=} \phi(F(u_j^n + \theta\delta, v_j^n)) - \phi(F(u_j^n, v_j^n) + \theta\delta/\lambda)$.

Next we show that

$$
\delta \int_0^1 \gamma(\theta\delta) d\theta \le 0. \tag{3.13}
$$

Because $F_u(u, v) > 0$ and η is convex, the property (c) of entropies implies that ϕ is increasing. Thus, it suffices to show that

$$
\delta(F(u_j^n + \theta\delta, v_j^n) - F(u_j^n, v_j^n) - \theta\delta/\lambda) \le 0,
$$

which is implied by the CFL condition in (3.10), since

$$
\delta(F(u_j^n + \theta\delta, v_j^n) - F(u_j^n, v_j^n) - \theta\delta/\lambda) = \theta\delta^2 \frac{1}{\lambda}(\lambda F_u - 1) \le 0.
$$

Combining (3.12) and (3.13) yields

$$
\eta(u_j^{n+1}, v_j^n) - \eta(u_j^n, v_j^n) \le \lambda(q(F(u_{j-1}^n, v_{j-1}^n)) - q(F(u_j^n, v_j^n))).
$$

Adding this to the inequality in (3.11) leads to the discrete entropy inequality in the lemma. □

4. Main Results

The main results of this paper are two theorems to be shown here. Having the estimates in lemmas 2.1–2.4 and Lemma 3.2 and recalling the inequalities in (2.8), we follow the standard argument in [6] to obtain

Theorem 4.1. *Suppose the initial data (u_0, v_0) are bounded measurable, have bounded total variations and are close to equilibrium as indicated in (2.7), and the grid sizes $\Delta t, \Delta x$ always satisfy the CFL-condition in (2.4). Then when the grid sizes $\Delta t, \Delta x$ tend to zero, there is a subsequence of the family $\{(u^{\Delta,\epsilon}, v^{\Delta,\epsilon})\}$, generated by the scheme in (2.1)–(2.2), converging in $(L^1_{loc}(R \times R^+))^2$ to an entropy solution (u^ϵ, v^ϵ) of the system in*

(1.1) with initial data (u_0, v_0). Furthermore, u^ϵ and v^ϵ satisfy

$$\begin{aligned} &C_1 \leq f(u^\epsilon(x,t)) - v^\epsilon(x,t) \leq C_2, \\ &\frac{1-\mu}{\mu} C_1 \leq v^\epsilon(x,t) \leq \frac{1-\mu}{\mu} C_2; \end{aligned} \tag{4.1}$$

$$TV(U^\epsilon(\cdot,t)) + TV(v^\epsilon(\cdot,t)) \leq TV(U_0) + TV(v_0); \tag{4.2}$$

$$\|u^\epsilon(\cdot,t) - u^\epsilon(\cdot,\tau)\|_{L^1} + \|v^\epsilon(\cdot,t) - v^\epsilon(\cdot,\tau)\|_{L^1} \leq C|t-\tau|; \tag{4.3}$$

$$\|v^\epsilon(\cdot,t) - (1-\mu)f(u^\epsilon(\cdot,t))\|_{L^1} \leq C\epsilon;$$

$$\eta(u^\epsilon, v^\epsilon)_t + q(u^\epsilon, v^\epsilon)_x + \frac{1}{\epsilon}\eta_v(u^\epsilon, v^\epsilon)(v^\epsilon - (1-\mu)f(u^\epsilon)) \leq 0$$

in the sense of distribution.

Here is an explanation of the notations in the above theorem. The definition of $(u^{\Delta,\epsilon}, v^{\Delta,\epsilon})$ from $(u_j^{n,\epsilon}, v_j^{n,\epsilon})$ is that in [6]. C_1, C_2 are found as in (2.3). U is the Riemann invariant $f(u) - v$. $t \geq 0, \tau \geq 0$ and the constant C is independent of t (and τ) as well.

In this framework of BV-solutions, the zero relaxation limit can be very easily discussed. In fact, the standard argument in [6] proves actually that the embedding of $L^\infty(R^+ \times R) \cap L^\infty(R^+, BV(R)) \cap Lip(R^+, L^1(R))$ into $L^1_{loc}(R^+ \times R)$ is compact. On the other hand, the relations in (4.1)–(4.3) show that $\{(u^\epsilon, v^\epsilon)\}_{\epsilon>0}$ lies in a bounded subset of $L^\infty(R^+ \times R) \cap L^\infty(R^+, BV(R)) \cap Lip(R^+, L^1(R))$. Thus, we have

Theorem 4.2. *Assume the conditions of Theorem 4.1. Then when the relaxation time ϵ tends to zero, there is a subsequence of the family $\{(u^\epsilon, v^\epsilon)\}_{\epsilon>0}$ converging in $(L^1_{loc}(R \times R^+))^2$ to a weak solution (u^0, v^0) of the reduced system in (1.2). Furthermore, u^0 and v^0 are bounded measurable, have bounded total variations and satisfy*

$$\|u^0(\cdot,t) - u^0(\cdot,\tau)\|_{L^1} + \|v^0(\cdot,t) - v^0(\cdot,\tau)\|_{L^1} \leq C|t-\tau|.$$

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