Hyperbolic to Parabolic Relaxation Theory for Quasilinear First Order Systems

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Abstract

In this paper we study the limiting behavior of nonhomogeneous hyperbolic systems of balance laws when the relaxed equilibria are described by means of systems of parabolic type. In particular we obtain a complete theory for the 2×2 systems of genuinely nonlinear hyperbolic balance laws in *1-D* with a strong dissipative term. A different method, which combines the *div-curl lemma* with accretive operators, is then applied to study the limiting profiles in the case of nonhomogeneous isentropic gas dynamics. We also investigate relaxation results for some *2-D* cases, which include the Cattaneo model for nonlinear heat conduction and the compressible Euler flow. Moreover, convergence result is also obtained for general semilinear systems in *1-D*.

Contents

1 Introduction

This paper is concerned with the nonlinear system of partial differential equations

$$W_t + \nabla_x \cdot \mathcal{A}(W) = \mathcal{B}(W),$$

where $W = W(x,t)$ takes on values in $\mathbb{R}^N$ and denotes the density vector of some physical quantities over the space variable, $x \in \mathbb{R}^d$ and $\mathcal{A}(W)$ is a $N \times d$ matrix valued function. We wish to investigate the relaxation phenomena where the relaxed equibria are described by means of an equation (or a system) of parabolic type. A very simple example of this kind of phenomena is given (at a linear level) by the dissipative wave equation

$$\Box\, u + \alpha u_t = 0$$

where it can be shown that $\tau = 1/\alpha$ behaves as a relaxation time and $z(x', t') = \alpha u(x', \alpha t')$ converges (in many different norms, for instance H^s, C^k) to the solution of the heat equation

$$-\triangle_{x'} z + z_{t'} = 0$$

as α tends to ∞.

Therefore in this paper we are interested to study, in a systematic way, this kind of hyperbolic to parabolic relaxation limit. In particular we shall focus our attention on the investigation of the convergence problem. One way to understand the meaning of this phenomena is to consider it as the large time behavior of *dissipative* nonlinear hyperbolic systems and to look at the asymptotic profile as the relaxed equilibrium. This is the case of many relevant situations in mathematical physics and in applied mathematics where relaxation times are often built in the mathematical model as a remind of more accurate descriptions given at kinetic or mesoscopic level and then averaged in the partial differential equations at macroscopic level.

This is the case of the equations describing the porous media flow [20] or the nonlinear heat conduction [21] and it is also the same kind of limit investigated in the papers of Kurtz [14], Mckean [24], Lions and Toscani [19]. In that particular case the hydrodynamic version of the Carleman equations has been shown to converge towards a nonlinear diffusion equation which, at that level, plays the role of the natural hydrodynamic limit.

Section 2 of this paper is dedicated to show a general multidimensional framework of investigation for these kind of phenomena. At this stage we do not go into the convergence analysis but we will bound to a formal introduction.

Section 3 is devoted to show a fairly complete theory for the one dimensional 2×2 systems of genuinely nonlinear hyperbolic balance laws, with a *strong* dissipative nonlinear term. From the mathematical point of view we show two different type of scaling which produce the same asymptotic profiles and we analyze their limiting behavior by using the methods of compensated compactness (see [30, 31], [10]). We actually obtain the relaxation limit by analyzing the relaxation sequence associated L.C. Young measures and by controlling their supports by means of the Lax exponential entropies. Then this analysis is restricted to the genuinely nonlinear systems since, for instance in the case of isentropic gas dynamics, the Lax exponential entropies might blow up along the sonic line.

We dedicate Section 4 to study the case of nonhomogeneous isentropic gas dynamics, which cannot be investigated with the previous methods. The limiting profiles in this case are given by the solutions of convective nonlinear parabolic equations which include porous media type equations. The analysis of this case is done by using the interior structure of the system in a more specific way. We actually combine the *div-curl lemma* [26, 30] with the theory of monotone operators [16] and then following the ideas used in [20] for the case of the porous media equation we get the convergence results. The results in [19] can be obtained in a similar way.

Some results concerning the multi-D case will be given, for some specific examples in Section 5. In particular in subsection 5.1 we extend the results of [21] concerning the Cattaneo model for nonlinear heat conduction and in subsection 5.2 the results of [20] concerning the porous media equation to the *2-D* case. Finally in Section 6 we consider general semilinear systems and obtain a convergence result to the limit profile in the case of *1-D* space dimension.

2 Multidimensional Framework

2.1 Basic ideas

We will consider the following system of equations

$$W_s + \nabla_y \cdot \mathcal{A}(W) = \mathcal{B}(W) \tag{2.1}$$

where $s \geq 0$, $y \in \mathbb{R}^d$, $W \in \mathbb{R}^N$. Moreover we assume the following hypotheses hold:

(A.1) there exists an open set $\mathcal{O} \subset \mathbb{R}^N$ such that $\mathcal{A} \in C^1(\mathcal{O}, \mathcal{M}_{N\times d})$, $\mathcal{A}(0) = 0$ (here $\mathcal{M}_{N\times d}$ denotes the linear space of $N \times d$ matrices),

(A.2) $\mathcal{B} \in C^1(\mathcal{O}, \mathbb{R}^N)$,

(A.3) the system (2.1) is hyperbolic, namely for all nonzero vector $\xi \in \mathbb{R}^d$, the $N \times N$ matrix $D_W\mathcal{A}(W) \cdot \xi$ has real eigenvalues and is diagonalizable,

(A.4) there exists a matrix $P \in \mathcal{M}_{k\times N}$, $1 \leq k < N$, such that $P\mathcal{B}(W) = 0$, for all $W \in \mathcal{O}$ and $\operatorname{rank} P = k$.

Let us consider $[\sigma_1, \dots, \sigma_{N-k}]$ a vector basis for the subspace $\ker P$ and denote by Q the $(N-k)\times N$ matrix having σ_i as row vectors. Then we set

$$U = PW, \qquad V = QW, \qquad M = \begin{bmatrix} P \\ Q \end{bmatrix}^{-1},$$

hence we have defined a linear change of variable which allows us to rewrite the system (2.1) in the following way

$$\begin{cases} U_s + \nabla_y \cdot \mathcal{F}(U,V) = 0 \\ V_s + \nabla_y \cdot \mathcal{G}(U,V) = \mathcal{H}(U,V), \end{cases} \tag{2.2}$$

where we set

$$\mathcal{F}(U,V) = P\mathcal{A}\left(M(U,V)\right), \qquad \mathcal{G}(U,V) = Q\mathcal{A}\left(M(U,V)\right)$$
$$\mathcal{H}(U,V) = Q\mathcal{B}\left(M(U,V)\right), \qquad W = M(U,V),$$

since $P\nabla_y \cdot \mathcal{A} = \nabla_y \cdot P\mathcal{A}$.

We are going to consider two type of scaling which reflect phenomena of different nature but, from the mathematical point of view they can be treated in the same way. The first type of scaling is given by

$$\mathcal{U}(x,t) = U\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right), \qquad \mathcal{V}(x,t) = \frac{1}{\sqrt{\varepsilon}} V\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right), \tag{2.3}$$

for any $\varepsilon > 0$, $t \geq 0$, $x \in \mathbb{R}^d$. In this case the system (2.2) becomes as follows

$$\begin{cases} \partial_t \mathcal{U} + \varepsilon^{-1/2} \nabla_x \cdot \mathcal{F}\left(\mathcal{U}, \varepsilon^{1/2}\mathcal{V}\right) = 0 \\ \varepsilon \partial_t \mathcal{V} + \nabla_x \cdot \mathcal{G}\left(\mathcal{U}, \varepsilon^{1/2}\mathcal{V}\right) = \varepsilon^{-1/2} \mathcal{H}\left(\mathcal{U}, \varepsilon^{1/2}\mathcal{V}\right). \end{cases} \tag{2.4}$$

A different scaling can be used in the case (which is a common situation in several models in physics), where for some reasons we have a relaxation time appearing inside the problem in a natural way. Let us, for instance, consider the following situation

$$\mathcal{H}(U,V) = \mathcal{H}\left(U, \tau^{-1} V\right).$$

Then we can introduce a different type of scaling which is more strictly related with the asymptotic profiles of the system (2.1). Namely we define

$$\mathcal{U}(x,t) = U\left(x, t/\tau\right), \qquad \mathcal{V}(x,t) = \frac{1}{\tau} V\left(x, t/\tau\right)$$

which transform the system (2.1) into the new system

$$\begin{cases} \partial_t \mathcal{U} + \tau^{-1} \nabla_x \cdot \mathcal{F}\left(\mathcal{U}, \tau\mathcal{V}\right) = 0 \\ \tau^2 \partial_t \mathcal{V} + \nabla_x \cdot \mathcal{G}\left(\mathcal{U}, \tau\mathcal{V}\right) = \mathcal{H}\left(\mathcal{U}, \mathcal{V}\right). \end{cases}$$

2.2 Formal analysis of the singular limit

We shall restrict our analysis to the case of the system (2.4). Let us consider the following hypotheses:

(A.5) for all $W \in \ker Q$ one has $\mathcal{A}(W) \in \ker P$ and $\mathcal{B}(W) \in \ker Q$,

(A.6) let us define $\mathcal{H}_V^0(U) = \mathcal{H}_V(U,0) \in \mathcal{M}_{(N-k)\times(N-k)}$, then $\det \mathcal{H}_V^0(U) \neq 0$,

(A.7) let us denote by $\mathcal{F}^h$, $\mathcal{G}^l$, $h, l = 1, \ldots, d$ the columns of $\mathcal{F}$, $\mathcal{G}$ and define the $k \times k$ matrix

$$-\Pi_{hl}(U) = \mathcal{F}_V^h(U,0)\left(\mathcal{H}_V^0(U)\right)^{-1} \mathcal{G}_U^l(U,0).$$

If τ_j, $j = 1, \ldots, k$ are the eigenvalues of the principal symbol $-\sum_{h,l} \xi_h \xi_l \Pi_{hl}\left(\mathcal{U}\right)$, we require there exists $\lambda_0 > 0$ such that, for all $\xi \in \mathbb{R}^d$,

$$\operatorname{Re} \tau_j\left(\mathcal{U}, \xi\right) \leq -\lambda_0 |\xi|^2.$$

As $\varepsilon \downarrow 0$, we formally obtain system (2.4) relaxes to the reduced system

$$\begin{cases} \mathcal{U}_t + \nabla_x \cdot [\mathcal{F}_{\mathcal{V}}(\mathcal{U}, 0)\, \mathcal{V}] = 0 \\ \nabla_x \cdot [\mathcal{G}(\mathcal{U}, 0)] = \mathcal{H}^0_{\mathcal{V}}(\mathcal{U})\, \mathcal{V}. \end{cases}$$

So, by using condition (A.6) we have $\mathcal{U}$ satisfies the resulting system

$$\mathcal{U}_t + \nabla_x \cdot \left\{ \mathcal{F}_{\mathcal{V}}(\mathcal{U}, 0) \left(\mathcal{H}^0_{\mathcal{V}}(\mathcal{U})\right)^{-1} \nabla_x \cdot \mathcal{G}(\mathcal{U}, 0) \right\} = 0$$

which is Petrowski-parabolic (see Taylor [32] volume III, Eidel'man [12], Kreiss and Lorenz [13]) because of the condition (A.7).

Further investigations in order to develop the theory in the framework of H^s spaces will be carried out in a forthcoming paper. Here we shall get a fairly complete *1-D* theory and some relevant examples in the multi-D case (under some natural restrictions).

3 2×2 Systems in *1-D*

We are going to develop here a rigorous theory for the case of quasilinear 2×2 systems of balance laws. In this case we are able to provide the convergence to relaxed equation without any smallness assumption and including weak non smooth solutions for the hyperbolic system.

The basic ideas used in this section involves the methods of compensated compactness as they have been created by Tartar [30, 31] and DiPerna [8, 11, 10], and are somehow related with the spirit of the paper [21]. They are different from the methods used by Marcati and Milani [20], Marcati and Natalini [22], Rubino [28, 29], Lions and Toscani [19] which are specifically related to the possibility of using the *div-curl lemma* (see for instance [30]) and some monotonicity arguments to achieve the strong convergence of the relaxing sequence.

Instead of this approach we have to go back here to the original DiPerna's idea of using exponential entropies [15] and the Tartar identity for L.C. Young measures. The relaxation mechanism provides a simplified Tartar type identity which allows us to pass into the limit. Namely, this modified identity can be considered very weak form of asymptotic analysis where only the contributions of the leading terms of the relaxation mechanism is taken into account.

As we did in the general case, also in this situation we can implement two different types of scaling on nonhomogeneous hyperbolic systems depending whether or not there exists some internal relaxation time. The meaning of those two types of scaling is quite different although the final result is formally the same.

3.1 Generalities

First, let us consider the following quasilinear nonhomogeneous 2×2 hyperbolic system

$$\begin{cases} w_s + f(w, z)_y = 0 \\ z_s + g(w, z)_y = h(w, z), \end{cases} \tag{3.1}$$

where $y \in \mathbb{R}$, $s \geq 0$. We assume that the following hypotheses are satisfied:

(B.1) $f, g, h :\ \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuously differentiable functions such that

$$f(w,0) = 0, \qquad\qquad h(w,0) = 0,$$

(B.2) denote by $\lambda_\pm(w,z)$ the characteristic speeds of (3.1) (namely the eigenvalues of $\frac{\partial(f,g)}{\partial(w,z)}$), we assume the system (3.1) is strictly hyperbolic, namely $\lambda_- < \lambda_+$, which is equivalent to assume

$$\Delta(w,z) := (f_w + g_z)^2 + 4g_w f_z > 0 \qquad\qquad \text{for all } (w,z),$$

(B.3) denote by

$$f^*(w) := \frac{\partial f}{\partial z}(w,0), \qquad h^*(w) := \frac{\partial h}{\partial z}(w,0), \qquad g^0(w) := g(w,0).$$

We assume, for all $w \in \mathbb{R}$,

$$h^*(w) < 0, \qquad\qquad (g^0(w))' f^*(w) > 0.$$

Here, in order to investigate the asymptotic behavior as $s \to +\infty$, we scale both space and time, keeping fixed the parabolic scale, namely x^2/t. Then we set

$$\begin{cases} u^\varepsilon(x,t) = w\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right) \\ v^\varepsilon(x,t) = \dfrac{1}{\sqrt{\varepsilon}} z\left(\frac{x}{\sqrt{\varepsilon}}, \frac{t}{\varepsilon}\right) \end{cases} \tag{3.2}$$

for all $\varepsilon > 0$, $x \in \mathbb{R}$, $t \geq 0$. The rescaled system is given by

$$\begin{cases} u_t + \dfrac{1}{\sqrt{\varepsilon}} f\left(u, \sqrt{\varepsilon} v\right)_x = 0 \\ \varepsilon v_t + g\left(u, \sqrt{\varepsilon} v\right)_x = \dfrac{1}{\sqrt{\varepsilon}} h\left(u, \sqrt{\varepsilon} v\right) \end{cases} \tag{3.3}$$

supplemented by the initial conditions

$$\begin{cases} u^\varepsilon(x,0) = u_0(x) \\ v^\varepsilon(x,0) = v_0(x). \end{cases}$$

We investigate the limiting behavior of $(u^\varepsilon, v^\varepsilon)$ as $\varepsilon \downarrow 0$ and we will be able to pass into the limit and to show that

$$\begin{aligned} u^\varepsilon &\longrightarrow u && \text{strongly in } L^p_{loc},\ p < +\infty, \\ v^\varepsilon &\rightharpoonup v && \text{weakly in } L^2, \\ \sqrt{\varepsilon} v^\varepsilon &\longrightarrow 0 && \text{strongly in } L^p_{loc},\ p < +\infty. \end{aligned}$$

Hence we will be able to pass into the limit into the nonlinear terms of the system (3.3) and to get the limit relaxed system

$$\begin{cases} u_t + \left(f^*(u)\, v\right)_x = 0 \\ g^0(u)_x = h^*(u)\, v. \end{cases} \tag{3.4}$$

The relaxation process still keeps the initial condition

$$u(x,0) = u_0(x)$$

while the other one is killed by an initial layer.

By the system (3.4) we can reconstruct the nonlinear parabolic equation

$$\begin{cases} u_t - (\beta(u))_{xx} = 0 \\ u(x,0) = u_0(x), \end{cases} \tag{3.5}$$

where

$$\beta'(u) = -(h^*(u))^{-1} f^*(u) \left(g^0(u)\right)' > 0$$

because of the hypothesis (B.3). The limiting procedure will be carried out by assuming the existence of a convex entropy which satisfies certain dissipativity conditions.

The second type of scaling that we are going to consider is related with systems which possess an internal relaxation time. This is, for instance, the case of the semiconductor devices [22, 23] or the nonlinear heat conduction [1] or the nonlinear diffusion limit considered by [19].

We investigate, in this case, the limiting behavior as $\tau \downarrow 0$, of the system

$$\begin{cases} w_s + f(w,z)_x = 0 \\ z_s + g(w,z)_x = h\left(w, \frac{1}{\tau} z\right) \end{cases} \tag{3.6}$$

under the initial conditions

$$\begin{cases} w(x,0) = w_0(x) \\ z(x,0) = z_0(x). \end{cases}$$

The scaling that we have to consider, in this case, is given by the following transformation

$$\begin{cases} u^\tau(x,t) = w\left(x, \frac{t}{\tau}\right) \\ v^\tau(x,t) = \frac{1}{\tau} z\left(x, \frac{t}{\tau}\right), \end{cases} \tag{3.7}$$

where $x \in \mathbb{R}$, $t \geq 0$, $\tau > 0$. It is worthy of mentioning that the scaling used in this case has a strong connection with the asymptotic behavior of the full system (3.6). Then the scaled system obtained in this way is given by

$$\begin{cases} u_t + \frac{1}{\tau} f(u, \tau v)_x = 0 \\ \tau^2 v_t + g(u, \tau v)_x = h(u,v) \end{cases} \tag{3.8}$$

under the initial conditions

$$\begin{cases} u^\tau(x,0) = w(x,0) = w_0(x) \\ v^\tau(x,0) = \frac{1}{\tau} z(x,0) = \frac{1}{\tau} z_0(x). \end{cases}$$

As before, we want to prove that, as $\tau \downarrow 0$, the weak solutions of the rescaled system satisfy

$$u^\tau \longrightarrow u \qquad \text{strongly in } L^p_{loc},\ p < +\infty,$$

$$v^\tau \rightharpoonup v \qquad\qquad \text{weakly in } L^2,$$
$$\tau v^\tau \longrightarrow 0 \qquad\qquad \text{strongly in } L^p_{loc},\ p < +\infty,$$

where (u, v) is a solution of the reduced system

$$\begin{cases} u_t + (f^*(u)\, v)_x = 0 \\ g^0(u)_x = h(u, v) \end{cases} \tag{3.9}$$

with initial condition $u(x, 0) = u_0(x)$.

In order to deal with this case we need to modify (B.3) in the following more restrictive condition:

(B.3') assume that f^*, g^0 are defined as in the condition (B.3) and for all $w, z \in \mathbb{R}$,

$$\frac{\partial h}{\partial z}(w, z) < 0, \qquad\qquad (g^0(w))' f^*(w) > 0.$$

On the other hand, condition (B.1) can be modified in the less restrictive:

(B.1') $f, g, h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuously differentiable functions such that $f(w, 0) = 0$.

Remark 3.1 As a consequence of the hypothesis (B.3') there exists a C^1 function $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}$, such that $\Phi(u, h(u, v)) = v$ and $h(u, \Phi(u, z)) = z$, for all $u, v, z \in \mathbb{R}$. Therefore the second identity of the reduced system (3.9) can be written in the following way

$$v = \Phi\left(u, g^0(u)_x\right),$$

so that u satisfies the nonlinear parabolic equation

$$u_t + \left(f^*(u)\, \Phi\left(u, g^0(u)_x\right)\right)_x = 0. \tag{3.10}$$

■

The two cases are actually very similar from the mathematical point of view, so we will bound ourselves to provide full details only in the first case.

3.2 A priori estimates

In this section we establish a priori estimates for the solutions of system (3.3) which are independent of ε.

First of all, by using the rescaling relation (3.2) we immediately obtain

Proposition 3.2 *Let us assume that the solution of the Cauchy problem associated to* (3.1) *is bounded, then there exists a constant, independent from* $\varepsilon > 0$*, such that*

$$\|u^\varepsilon\|_\infty + \sqrt{\varepsilon}\, \|v^\varepsilon\|_\infty \le const.$$

Remark 3.3 The boundedness of solutions for systems of this type can be achieved in several ways. In particular, the theory of invariant domains of Chueh, Conley and Smoller [3], the use of entropy method as in Dafermos [5] or the method of quasimonotonicity as in Natalini [27]. ■

Now we are going to show the existence of an energy bound which provides an *upper bound* for the initial layer on v. This bound will play a crucial role in our analysis; in order to get this estimate we need to assume the existence of at least one convex entropy.

Lemma 3.4 *Let us assume there exists a convex non negative entropy, entropy–flux pair (η^*, q^*), for the system (3.1), and a continuous function $\psi \geq 0$, such that*

$$\eta_z^*(w,z)h(w,z) \leq -\psi(z) \qquad \textit{for all } (w,z). \tag{3.11}$$

Then for any $0 < \varepsilon \leq 1$, there exists $c_0 > 0$, independent of ε, such that

$$\frac{1}{\varepsilon}\int_0^T \int_{-\infty}^{+\infty} \psi\left(\sqrt{\varepsilon}v^\varepsilon\right) dx\ dt \leq \int_{-\infty}^{+\infty} \eta^*\left(u_0(x), \sqrt{\varepsilon}v_0(x)\right) dx \leq c_0. \tag{3.12}$$

Proof. Let us consider the original, not scaled system (3.1). Then, by the entropy inequality related to η^*, we obtain

$$-\int_0^{+\infty}\int_{-\infty}^{+\infty} \eta_z^*(w,z)h(w,z)dy\ d\tau \leq \int_{-\infty}^{+\infty} \eta^*\left(w_0^\varepsilon(y), z_0^\varepsilon(y)\right) dy$$

where $w_0^\varepsilon(y) = u_0\left(\sqrt{\varepsilon}y\right)$, $z_0^\varepsilon(y) = \sqrt{\varepsilon}v_0\left(\sqrt{\varepsilon}y\right)$. Therefore by using the scaling transformation (3.2), we get

$$-\frac{1}{\varepsilon}\int_0^{+\infty}\int_{-\infty}^{+\infty} \eta_z^*(u,\sqrt{\varepsilon}v)h(u,\sqrt{\varepsilon}v)dx\ dt \leq \int_{-\infty}^{+\infty} \eta^*\left(u_0(x), \sqrt{\varepsilon}v_0(x)\right) dx \leq c_0.$$

Hence the assumption (3.11) yields

$$\frac{1}{\varepsilon}\int_0^{+\infty}\int_{-\infty}^{+\infty} \psi\left(\sqrt{\varepsilon}v^\varepsilon\right) dx\ dt \leq c_0.$$

■

By applying the previous lemma we obtain immediately the following asymptotic behavior

Theorem 3.5 *Assume (η^*, q^*) and ψ as in previous lemma. Moreover, let $\psi^{-1}(\{0\}) = \{0\}$. Then (extracting if necessary a subsequence) we have*

$$\sqrt{\varepsilon}v^\varepsilon \longrightarrow 0 \qquad \textit{a.e.} \tag{3.13}$$

$$\sqrt{\varepsilon}v^\varepsilon \longrightarrow 0 \qquad \textit{strongly in } L^p_{loc},\ p < +\infty. \tag{3.14}$$

In addition, if $\psi \in C^2$ and there exists $\gamma > 0$ such that $\psi'' > \gamma$, then

$$\int_0^{+\infty}\int_{-\infty}^{+\infty} |v^\varepsilon|^2\, dx\ dt \leq const. \tag{3.15}$$

Proof. By using the hypothesis that ψ is a continuous function, $\psi^{-1}(\{0\}) = \{0\}$ and the estimate (3.12), we immediately obtain (3.13) and consequently (3.14).

Since $\psi(0) = 0$, $\psi'(0) = 0$, then there exists a measurable function $\vartheta^\varepsilon = \vartheta^\varepsilon\left(v^\varepsilon(x,t)\right)$, $|\vartheta^\varepsilon| \leq \sqrt{\varepsilon}$, such that $\psi\left(\sqrt{\varepsilon}v^\varepsilon\right) = \varepsilon\,|v^\varepsilon|^2\,\psi''\left(\vartheta^\varepsilon v^\varepsilon\right)$. Since $\psi'' \geq \gamma$, from (3.12) we obtain immediately (3.15).
■

We need to characterize the entropy–entropy flux pairs associated to the rescaled system (3.3) in terms of the entropy–entropy flux pairs associated to the non scaled system (3.1).

For any entropy–entropy flux pair (η, q) of the system (3.1), let us define

$$
\begin{aligned}
&\eta^\varepsilon(u,v) = \sqrt{\varepsilon}\eta(u,\sqrt{\varepsilon}v)\\
&q^\varepsilon(u,v) = q(u,\sqrt{\varepsilon}v).
\end{aligned}
$$

It is easy to verify that the following result holds.

Lemma 3.6 *$(\eta^\varepsilon, q^\varepsilon)$ is an entropy–entropy flux pair to the rescaled system* (3.3) *if and only if (η, q) is an entropy–entropy flux pair for the system* (3.1).

We want now to prove that the entropy production of (3.3) lies in a compact subset of H^{-1}_{loc}, that is

Theorem 3.7 *Assume that the system* (3.1) *satisfies the hypotheses (B.1), (B.2), (B.3) and the hypotheses of Proposition 3.2 and of Theorem 3.5. For any entropy–entropy flux pair (η, q) of the system* (3.1) *such that $\eta_z(u,0) = 0$, along any solution to* (3.1) *which is limit of the vanishing viscosity method, one has*

$$
\{\eta^\varepsilon_t + q^\varepsilon_x\} \subset \mathcal{C},
$$

where $\mathcal{C}$ is a compact set of H^{-1}_{loc} independent of ε.

In order to prove the previous theorem, it is sufficient to prove the following result.

Proposition 3.8 *Under the previous hypotheses, let us consider the parabolic approximation of* (3.1)*, namely, for all $\delta > 0$,*

$$
\begin{cases}
w_s + f(w,z)_y = \delta w_{yy}\\
z_s + g(w,z)_y = h(w,z) + \delta z_{yy}
\end{cases}
$$

with initial data

$$
\begin{cases}
w(x,0) = u_0\left(\sqrt{\varepsilon}x\right)\\
z(x,0) = \sqrt{\varepsilon}v_0\left(\sqrt{\varepsilon}x\right).
\end{cases}
$$

Then, for any entropy–entropy flux pair (η, q) of the system (3.1)*, we have*

$$
\left\{\eta^{\varepsilon,\delta}_t + q^{\varepsilon,\delta}_x\right\} \subset \mathcal{K}
$$

where $\mathcal{K}$ is a compact set of H^{-1}_{loc} independent of ε and δ and

$$
\eta^{\varepsilon,\delta}(u,v) = \sqrt{\varepsilon}\eta\left(u^{\varepsilon,\delta}, \sqrt{\varepsilon}v^{\varepsilon,\delta}\right), \qquad q^{\varepsilon,\delta}(u,v) = q\left(u^{\varepsilon,\delta}, \sqrt{\varepsilon}v^{\varepsilon,\delta}\right).
$$

Proof. By using the transformation (3.2) we come back to

$$\begin{cases} \sqrt{\varepsilon} u_t + f\left(u, \sqrt{\varepsilon} v\right)_x = \delta \sqrt{\varepsilon} u_{xx} \\ \varepsilon v_t + g\left(u, \sqrt{\varepsilon} v\right)_x = \frac{1}{\sqrt{\varepsilon}} h\left(u, \sqrt{\varepsilon} v\right) + \delta \varepsilon v_{xx}. \end{cases} \tag{3.16}$$

For a given entropy η, let us multiply system (3.16) by $\frac{1}{\sqrt{\varepsilon}} \nabla \eta^\varepsilon$ and obtain

$$\begin{aligned} &\partial_t \left\{ \sqrt{\varepsilon} \eta \left(u, \sqrt{\varepsilon} v\right)\right\} + \partial_x \left\{ q\left(u, \sqrt{\varepsilon} v\right)\right\} \\ &= \frac{1}{\varepsilon} h\left(u, \sqrt{\varepsilon} v\right) \eta_z \left(u, \sqrt{\varepsilon} v\right) \\ &+ \delta \sqrt{\varepsilon} \left\{ \eta_u \left(u, \sqrt{\varepsilon} v\right) u_{xx} + \eta_z \left(u, \sqrt{\varepsilon} v\right) v_{xx} \right\} \\ &= \frac{1}{\varepsilon} h\left(u, \sqrt{\varepsilon} v\right) \eta_z \left(u, \sqrt{\varepsilon} v\right) + \delta \sqrt{\varepsilon} \eta \left(u, \sqrt{\varepsilon} v\right)_{xx} \\ &- \delta \sqrt{\varepsilon} \left\langle \nabla^2 \eta \left(\left(u, \sqrt{\varepsilon} v\right)_x, \left(u, \sqrt{\varepsilon} v\right)_x \right) \right\rangle = I_1^{\varepsilon,\delta} + I_2^{\varepsilon,\delta} + I_3^{\varepsilon,\delta}. \end{aligned}$$

In a very standard way we know that the existence of a convex entropy η^* implies that

$$\delta^{1/2} \varepsilon^{1/4} \|u_x\|_{L^2_{(x,t)}} + \delta^{1/2} \varepsilon^{1/2} \|v_x\|_{L^2_{(x,t)}} \le const.$$

The estimates of $I_2^{\varepsilon,\delta}$, $I_3^{\varepsilon,\delta}$ are also quite standard (see [31] and [10]) and it follows easily that $\{I_2^{\varepsilon,\delta} + I_3^{\varepsilon,\delta}\}$ is relatively compact in H^{-1}_{loc} independent of ε, δ.

Then it is sufficient to estimate the term $I_1^{\varepsilon,\delta}$.

Since $\eta_z(u,0) = 0$, we have

$$\eta_z(u,z) = \left(\int_0^1 \eta_{zz}(u, \vartheta z) d\vartheta \right) z =: z\Gamma(u,z)$$

and $\eta_z = \frac{1}{\sqrt{\varepsilon}} \eta_v$,

$$\begin{aligned} \frac{1}{\varepsilon} \int_0^{+\infty} \int_{-\infty}^{+\infty} |\eta_z h| \; dx \; dt &\le \frac{1}{\varepsilon} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left| \Gamma\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right) \right| \left| \frac{h}{\sqrt{\varepsilon} v^\varepsilon} \right| (v^\varepsilon)^2 dx \; dt \\ &\le const \int_0^{+\infty} \int_{-\infty}^{+\infty} (v^\varepsilon)^2 dx \; dt, \end{aligned}$$

and we can conclude $\frac{1}{\sqrt{\varepsilon}} \eta_v^\varepsilon h$ lies in a bounded set of L^1. ■

3.3 Strong convergence

Our next step is to prove strong convergence for the sequence $\{u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\}$ in L^p_{loc}, for all $p \in [1, +\infty)$.

Let $K \subset \mathbb{R}^m$ and $\Omega \subset \mathbb{R}^n$ be bounded open sets, $\mathcal{U}^\varepsilon : \Omega \longrightarrow \mathbb{R}^m$ be measurable functions such that $\mathcal{U}^\varepsilon(y) \in K$ and w-lim$\,\mathcal{U}^\varepsilon = \mathcal{U}$ in the sense of weak topology in certain Banach space. Let us consider a continuous function $F : \mathbb{R}^m \longrightarrow \mathbb{R}$. Following [30] and [8, 9, 10, 11] there exists a family of probability measures $\{\nu_y\}_{y \in \Omega} \subset \text{Prob}\,(\mathbb{R}^m)$, such that supp$\,\nu_y \in \bar{K}$, and there exists a subsequence, still labeled $\mathcal{U}^\varepsilon$, satisfying

$$\text{w}^*\text{-}\lim_{\varepsilon \to 0^+} F\left(\mathcal{U}^\varepsilon\right) = \int F(\lambda) d\nu_y(\lambda) = \langle \nu_y(\lambda), F(\lambda) \rangle \tag{3.17}$$

where w*-lim denotes the weak limit in the sense of weak* topology in the L^∞ space.

The identity (3.17) is a representation formula for the weak limits in terms of nonlinear functions and Young measure. The uniformly bounded sequence $\mathcal{U}^\varepsilon$ in L^∞ converges to $\mathcal{U}$ a.e. if and only if the corresponding Young measure ν_y reduces to a Dirac mass concentrated at $\mathcal{U}(y)$, i.e., $\nu_y = \delta_{\mathcal{U}(y)}$.

In our case we intend to show that the Young measure $\nu_{(x,t)}$ associated to $\{u^\varepsilon, \sqrt{\varepsilon}v^\varepsilon\}$ reduces to a point mass.

The following lemma is simple consequence of Theorem 3.5 and reduces the level of the difficulty to the analysis of *1-D* Young measure.

Lemma 3.9 ([21] page 62) *Let us denote by $\{u_n\}$, $\{v_n\}$ two bounded sequences in L^∞; assume that $u_n \longrightarrow \bar{u}$ a.e.. Then the Young measure ν associated to $z_n = (u_n, v_n)$ can be decomposed in the following tensor product of measures*

$$\nu = \delta_{\bar{u}} \otimes \mu_2$$

namely, for any continuous $\Phi = \Phi(\lambda_1, \lambda_2)$ one has

$$\langle \nu, \Phi \rangle = \int \Phi(\bar{u}, \lambda_2)\, \mu_2(d\lambda_2),$$

where μ_2 is the Young measure associated to $\{v_n\}$.

Corollary 3.10 *The statement of Lemma 3.9 applied to the measure ν associated to $\{u^\varepsilon, \sqrt{\varepsilon}v^\varepsilon\}$, yields*

$$\nu = \mu \otimes \delta_0,$$

where μ is the Young measure associated to $\{u^\varepsilon\}$.

Let us finally recall a fundamental tool in the limit process, that is the celebrated lemma of Tartar [30] and Murat [26].

Lemma 3.11 (div-curl lemma) *Let us consider two sequences $\{\mathcal{U}^\varepsilon\}$ and $\{\mathcal{V}^\varepsilon\}$, uniformly bounded in $L^\infty(\mathbb{R}^N)$ and assume that $\{\operatorname{div} \mathcal{U}^\varepsilon\}$ and $\{\operatorname{curl} \mathcal{V}^\varepsilon\}$ belong to a compact set of H^{-1}_{loc}, independent from ε. Then $\mathcal{U}^\varepsilon \cdot \mathcal{V}^\varepsilon \longrightarrow \mathcal{U} \cdot \mathcal{V}$ in $L^\infty(\mathbb{R})$ weak*, where $\mathcal{U} = \text{w*-lim}\, \mathcal{U}^\varepsilon$ and $\mathcal{V} = \text{w*-lim}\, \mathcal{V}^\varepsilon$.*

For strictly hyperbolic systems of two equations like (3.1), it is possible to construct, on each compact subset of the state space $\mathbb{R}^2$, two infinite families of entropy–entropy flux pairs having the following asymptotic expansions (see Lax [15])

$$\eta^{\pm k} = e^{\pm kR} \left(\sum_{n=0}^{N} \frac{\eta_n}{k^n} + O\left(\frac{1}{k^{N+1}}\right) \right), \tag{3.18}$$

$$q^{\pm k} = e^{\pm kR} \left(\sum_{n=0}^{N} \frac{q_n}{k^n} + O\left(\frac{1}{k^{N+1}}\right) \right), \tag{3.19}$$

where the variable coefficients η_n, q_n are determined by a recursive procedure and the Riemann invariant $R = R(u, \sqrt{\varepsilon}v)$. For convenience we shall regard k as a positive parameter. The ratio of entropy flux to entropy is asymptotically equal to the corresponding eigenvalue λ:

$$\frac{q}{\eta} = \lambda + O\left(\frac{1}{k}\right).$$

The following technical lemma gives us the possibility to reduce to a special class of entropy–entropy flux pair of Lax type when we apply the compensated compactness theory.

Lemma 3.12 *Given an entropy–entropy flux pair* $\left(\eta^k, q^k\right)$ *of Lax type* (3.18)-(3.19), *we can construct another family of entropy–entropy flux pair* $\left(E^k, Q^k\right)$ *of Lax type which satisfies*

- $\frac{\partial E^k}{\partial v}(u,0) = 0,$
- $\eta^k(u,0) = E^k(u,0), \qquad q^k(u,0) = Q^k(u,0).$

Proof. The proof of this lemma is straightforward since it suffices to solve the linear second order equation for the entropy

$$\operatorname{curl}\ \left[\nabla \hat{\eta}^k D(f,g)\right] = 0$$

with the initial condition

$$\hat{\eta}^k(u,0) = 0, \qquad\qquad \partial_v \hat{\eta}^k(u,0) = \partial_v \eta^k(u,0)$$

and to define $E^k = \eta^k - \hat{\eta}^k$.

Moreover, since $\hat{\eta}^k(u,0) = 0$ and since $\frac{\hat{q}^k}{\hat{\eta}^k} = \lambda + O\left(\frac{1}{k}\right)$, it follows that $\hat{q}^k(u,0) = 0$, namely $q^k(u,0) = Q^k(u,0)$. ■

The next result provides the strong convergence of the sequence $\{u^\varepsilon\}$.

Theorem 3.13 *Under the hypotheses of Theorem 3.7, denote by* u *the weak limit of* u^ε. *Then the Young measure* ν *associated to* $\{u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\}$ *verifies*

$$\nu_{(x,t)} = \delta_{u(x,t)} \otimes \delta_0 = \delta_{(u(x,t),0)} \qquad\qquad \textit{for almost all } (x,t).$$

Proof. For any entropy–entropy flux pair (E,Q) satisfying the hypotheses of Proposition 3.8, one has

$$\begin{cases} u_t^\varepsilon + \frac{1}{\sqrt{\varepsilon}} f\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right)_x = 0 \\ \sqrt{\varepsilon} E\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right)_t + Q\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right)_x \in \text{compact set in } H^{-1}_{loc}. \end{cases}$$

By using the *div-curl lemma* to the previous relations, we have the commutation identity of Tartar, namely

$$\langle \nu, \lambda Q\left(\lambda,\sigma\right) - f\left(\lambda,\sigma\right) E\left(\lambda,\sigma\right)\rangle = \langle \nu, \lambda\rangle \langle \nu, Q(\lambda,\sigma)\rangle - \langle \nu, f(\lambda,\sigma)\rangle \langle \nu, E(\lambda,\sigma)\rangle.$$

On the other hand, by Theorem 3.5 we have, as $\varepsilon \downarrow 0$,

$$E\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right) f\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right) = \left\{ E\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right) \frac{f\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right)}{\sqrt{\varepsilon} v^\varepsilon} \right\} \sqrt{\varepsilon} v^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^2$$

and

$$\operatorname*{w-lim}_{\varepsilon \to 0^+} Q\left(u^\varepsilon, \sqrt{\varepsilon} v^\varepsilon\right) = \operatorname*{w-lim}_{\varepsilon \to 0^+} Q\left(u^\varepsilon, 0\right)$$

since the strong limit for the difference is 0. Then we can conclude

$$\langle\mu,\lambda\rangle\langle\mu,Q^0(\lambda)\rangle=\langle\mu,\lambda Q^0(\lambda)\rangle$$

for any flux Q, where $Q^0(\lambda)=Q(\lambda,0)$.

Since u is bounded in L^∞, the support of μ will be bounded from above and from below, hence we can assume that there exist $u_-,u_+\in\mathbb{R}$ such that

$$\operatorname{supp}\mu\subset\{u\mid u_-\le u\le u_+\}.$$

Our goal is to prove that $u_-=u_+$. Taking into account Lemma 3.12, for any Lax entropy–entropy flux pair (η^k,q^k), we have

$$\langle\mu,\lambda\rangle=\frac{\langle\mu,\lambda q^k(\lambda,0)\rangle}{\langle\mu,q^k(\lambda,0)\rangle}.$$

Let us define two projection measures $\gamma_\pm$ as follows

$$\langle\gamma_\pm,\varphi(\lambda)\rangle=\lim_{k\to\infty}\frac{\langle\mu,\varphi q^0_{\pm k}\rangle}{\langle\mu,q^0_{\pm k}\rangle}.$$

Therefore we obtain $\gamma_+=\delta_{u_+}$, $\gamma_-=\delta_{u_-}$ from which $\langle\delta_{u_+},\lambda\rangle=\langle\gamma_+,\lambda\rangle=\langle\mu,\lambda\rangle=\langle\gamma_-,\lambda\rangle=\langle\delta_{u_-},\lambda\rangle$ and we conclude $u_+=u_-$. ∎

Corollary 3.14 *Under the previous hypotheses, the sequence $\{u^\varepsilon\}$ converges (eventually extracting a subsequence) strongly in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R})$, for any $p<+\infty$.*

We can conclude this section by stating the main theorem which summarize the previous convergence results and it describes the effects of them on the singular behavior of our system (3.3), as $\varepsilon\downarrow 0$. We obtain that the hyperbolic system relaxation profiles are described by the parabolic equation (3.5).

Theorem 3.15 *Assume that the hypotheses (B.1), (B.2), (B.3) hold; moreover assume that*

(B.4) *there exists a convex entropy, entropy–flux, pair (η^*,q^*), for the system (3.1), and a continuous function $\psi\in C^2$, $\psi^{-1}(\{0\})=\{0\}$ and a constant $\gamma>0$ such that $\psi''>\gamma$ and we have*

$$\eta^*_z(w,z)h(w,z)\le-\psi(z)\qquad \text{for all } (w,z).$$

(B.5) *$(u^\varepsilon,v^\varepsilon)$ is a solution to (3.3), limit of the vanishing viscosity method.*

Then one has, as $\varepsilon\downarrow 0$

$$u^\varepsilon\longrightarrow u\qquad \text{strongly in } L^p_{loc},\ p<+\infty,$$
$$v^\varepsilon\rightharpoonup v\qquad \text{weakly in } L^2,$$

and it follows that (u,v) verifies the relations

$$u_t+(f^*(u)\,v)_x=0 \tag{3.20}$$
$$g^0(u)_x=h^*(u)\,v \tag{3.21}$$

where (3.20) is verified in the sense of distributions and (3.21) in L^2.

In a similar way we can investigate the relaxation problem for the system (3.6) where the relaxation profile can be regarded as a weak solution of the parabolic equation (3.10).

Theorem 3.16 *Assume that the hypotheses (B.1'), (B.2), (B.3'), (B.4) hold and*

(B.5') (u^τ, v^τ) *is a solution to* (3.8), *limit of the vanishing viscosity method.*

Then one has, as $\tau \downarrow 0$,

$$
\begin{aligned}
& u^\tau \longrightarrow u && \text{strongly in } L^p_{loc},\ p<+\infty, \\
& v^\tau \rightharpoonup v && \text{weakly in } L^2,
\end{aligned}
$$

and it follows that (u,v) *verifies the relations*

$$u_t + \left(f^*\left(u\right)v\right)_x = 0 \tag{3.22}$$

$$g^0\left(u\right)_x = h\left(u,v\right) \tag{3.23}$$

where (3.22) *is verified in the sense of distributions and* (3.23) *in* L^2.

In the same spirit of [21] we also relate the existence of a convex entropy with the existence of a Lyapounov functional for the limit parabolic equation.

Proposition 3.17 *Let* η^* *be an entropy for* (3.1) *which verifies the hypothesis (B.4). If we set*

$$\mathcal{F}\left(\phi\right) = \int_{-\infty}^{+\infty} \eta^*\left(0, \phi(x)\right)\, dx,$$

$\mathcal{F}$ *is a Lyapounov functional for the corresponding relaxed parabolic equation, that is*

$$\frac{d\mathcal{F}}{dt} \leq 0$$

along the solutions of the corresponding problem (3.5).

Proof. An entropy-entropy flux pair (η^*, q^*) associated to (3.1) with convex η^* satisfies the inequality

$$\eta^*_t + q^*_y \leq h\eta_z + \delta\left(\eta_w w_y + \eta_z z_y\right)_y$$

so that we have, after integration on $\mathbb{R}_y \times [T, T+h]$,

$$
\begin{aligned}
& \int_{-\infty}^{+\infty} \eta\left(w\left(x,T+h\right), z\left(x,T+h\right)\right)\, dx - \int_{-\infty}^{+\infty} \eta\left(w\left(x,T\right), z\left(x,T\right)\right)\, dx \\
& \leq \int_T^{T+h} \left(\int_{-\infty}^{+\infty} h\left(w,z\right)\eta_z\left(w(x,t), z(x,t)\right) dx\right) dt \\
& \leq \int_T^{T+h} \left(\int_{-\infty}^{+\infty} -\psi\left(z(x,t)\right) dx\right) dt \\
& \leq -\frac{\gamma}{2}\int_T^{T+h} \left(\int_{-\infty}^{+\infty} z^2(x,t) dx\right) dt =: \int_T^{T+h} \theta^{\varepsilon,\delta} dt.
\end{aligned}
$$

On passing to the limits as $\delta \downarrow 0$, from the hypothesis (B.4) we obtain

$$\theta^\varepsilon := \underset{\delta\to 0+}{\text{w-lim}}\, \theta^{\varepsilon,\delta} \leq 0$$

so that, for all $T \geq 0$, $h > 0$, if we set

$$E^\varepsilon(t) := \int_{-\infty}^{+\infty} \eta^\varepsilon\left(u^\varepsilon(x,t), v^\varepsilon(x,t)\right)\, dx,$$

we obtain

$$E^\varepsilon(T+h) \leq E^\varepsilon(T).$$

Therefore, by using Lemma 3.6 we have

$$\sqrt{\varepsilon}\int_{-\infty}^{+\infty} \eta\left(u^\varepsilon(x,T+h), \sqrt{\varepsilon}v^\varepsilon(x,T+h)\right)\, dx \leq \sqrt{\varepsilon}\int_{-\infty}^{+\infty} \eta\left(u^\varepsilon(x,T), \sqrt{\varepsilon}v^\varepsilon(x,T)\right)\, dx$$

and, passing to the limits as $\varepsilon \downarrow 0$ we have, by Theorem 3.15,

$$\int_{-\infty}^{+\infty} \eta\left(u(x,T+h),0\right)\, dx \leq \int_{-\infty}^{+\infty} \eta\left(u(x,T),0\right)\, dx.$$

The proof is thus complete. ∎

4 Isentropic Gas Dynamics

Let us consider the following inhomogeneous system:

$$\begin{cases} \rho_s + \mu_x = F(\rho,\mu) \\ \mu_s + \left(\frac{\mu^2}{\rho} + p(\rho)\right)_x = G\left(\rho, \frac{\mu}{\tau}\right), \quad p(\rho) = \frac{1}{\gamma}\rho^\gamma \end{cases} \tag{4.1}$$

where $x \in \mathbb{R}$, $s \geq 0$, $\tau > 0$, $\gamma > 1$, and $F, G \in C^1(\mathbb{R}^+ \times \mathbb{R})$.

We are going to scale this system in such a way to deduct a relaxed parabolic equation following the same ideas of the previous sections. Due to the presence of an internal relaxation time $\tau > 0$, in the equation describing the balance of the linear momentum, we will scale in order to relate the relaxation process with the asymptotic behavior of the hydrodynamic equations. The previous system is a prototype of many interesting physics phenomena, particularly it is a simple but interesting model for gas reacting flows and has strong connection with hydrodynamic modeling for semiconductors (see [22, 23]).

Also in this case we will prove

1. *a priori* bounds,

2. strong convergence of the density ρ and initial layer on the momentum μ,

3. parabolic relaxation limit.

In order to investigate the asymptotic behavior for (4.1) as $s \uparrow +\infty$, we scale the time as in (3.7), namely

$$\begin{cases} r^\tau(x,t) = \rho\left(x, \frac{t}{\tau}\right) \\ m^\tau(x,t) = \frac{1}{\tau}\mu\left(x, \frac{t}{\tau}\right). \end{cases} \tag{4.2}$$

Then the Cauchy problem given for the system

$$\begin{cases} r_t + m_x = \frac{1}{\tau}F(r, \tau m) \\ \tau^2 m_t + \left(\tau^2 \frac{m^2}{r} + p(r)\right)_x = G(r,m), \end{cases} \tag{4.3}$$

supplemented by the initial data

$$\begin{cases} r(x,0) = r_0(x) \\ m(x,0) = m_0(x), \end{cases} \tag{4.4}$$

is equivalent to the initial system (4.1) with initial data

$$\begin{cases} \rho(x,0) = \rho_0(x) = r_0(x) \\ \mu(x,0) = \mu_0^\tau(x) = \tau m_0(x). \end{cases} \tag{4.5}$$

An essential feature of the system (4.1) is a nonstrictly hyperbolicity, that is a pair of wave speeds coalesce on the vacuum $\rho = 0$. Define $R_\mp = \frac{\mu}{\rho} \mp \frac{1}{\vartheta}\rho^\vartheta$ the Riemann invariants associated to (4.1), $\vartheta := \frac{1}{2}(\gamma - 1)$. We recall that for the Cauchy problem of the corresponding homogeneous system, P.L. Lions, Perthame and Tadmor [18] (see also DiPerna [9], sect. 4) proved the following result

Proposition 4.1 *Let us consider the Cauchy problem*

$$\begin{cases} \rho_s + \mu_x = \delta\rho_{xx} \\ \mu_s + \left(\frac{\mu^2}{\rho} + p(\rho)\right)_x = \delta\mu_{xx}, \\ (\rho,\mu)_{|x=0} = \left(\rho_0, \mu_0^1\right). \end{cases} \tag{4.6}$$

Assume the initial data $\left(\rho_0, \mu_0^1\right)$ *are such that* $\rho_0 \geq 0$ *and* $R_\mp^0 = R_\mp\left(\rho_0, \mu_0^1\right)$ *are well defined for any* $x \in \mathbb{R}$ *with compact support.*

Then we

$$\Sigma_1 = \left\{(\rho,\mu) \mid \rho \geq 0, \text{ and } \min_x R_-^0 \leq R_-(\rho,\mu) \leq R_+(\rho,\mu) \leq \max_x R_+^0\right\}$$

is an invariant region for (4.6) *uniformly in* $\delta > 0$.

Starting by these a priori estimates DiPerna established in [9] the existence of solutions for the homogeneous Cauchy problem corresponding to (4.1) in the case $\gamma = 1 + \frac{2}{2m+1}$, $m \geq 2$ integer, using the methods of vanishing viscosity and compensated compactness. Subsequently the existence theorem has been extended by Ding, Chen and Luo [6] and Chen [2] to the case $1 < \gamma \leq 5/3$ and large data containing the vacuum. Recently P.L. Lions, Perthame and Tadmor in [18] and P.L. Lions, Perthame and Souganidis in [17] proved the existence theorem in the general case $\gamma > 1$

by using the kinetic formulation. The existence of global solutions to (4.1)-(4.5) with large data has been established by Ding, Chen and Luo in [7] for $1 < \gamma \leq 5/3$ and inhomogeneous terms (F, G) satisfy technical hypotheses (see [7] page 72, conditions $C1^0$–$C3^0$) by using the fractional step Lax–Friedrichs scheme and Godunov scheme. Of course, in the light of the technics of [17], the results of [7] can be extended to the general case $\gamma > 1$.

An important role in these results has had the existence of the mechanical energy for (4.1),

$$\eta^* (\rho, \mu) = \frac{\mu^2}{2\rho} + \frac{1}{\gamma(\gamma-1)}\rho^\gamma. \tag{4.7}$$

Here, in order to obtain the singular behavior, as $\tau \downarrow 0$, of the solution of the system (4.3), we need to have a priori estimates in L^∞ and in L^2 on the solution. In particular we prove the following result concerning the existence of the solutions for fixed τ and the L^∞ estimates independent from τ.

Proposition 4.2 *Let us consider the Cauchy problem*

$$\begin{cases} \rho_s + \mu_x = F(\rho, \mu) + \delta\rho_{xx} \\ \mu_s + \left(\frac{\mu^2}{\rho} + p(\rho)\right)_x = G\left(\rho, \frac{\mu}{\tau}\right) + \delta\mu_{xx}, \\ (\rho, \mu)_{|x=0} = (\rho_0, \mu_0^\tau). \end{cases} \tag{4.8}$$

Let us define $c_\tau > 0$ a constant such that

$$\Sigma_\tau = \{(\rho, \mu) \mid \rho \geq 0 \text{ and } -c_\tau \leq R_-(\rho, \mu) \leq R_+(\rho, \mu) \leq c_\tau\}$$

is an invariant region for (4.8) with $F = G \equiv 0$ uniformly in $\delta > 0$.

Assume the following hypotheses:

(C.1) *for $0 < \tau < 1$ the inequality*

$$F(\rho, \mu)\left(\frac{\vartheta+1}{\vartheta}\rho^\vartheta - c_\tau\right) \pm G\left(\rho, \frac{\mu}{\tau}\right) \leq 0 \qquad \text{for } \pm\mu \geq 0$$

holds,

(C.2) *there exists a constant $C > 0$ such that*

$$\eta^*_\rho F(\rho, \mu) + \eta^*_\mu G\left(\rho, \frac{\mu}{\tau}\right) \leq -C\frac{\mu^2}{\tau\rho^2}.$$

Then we have:

(a) *for any $0 < \tau < 1$, Σ_τ is an invariant region for (4.8) uniformly in $\delta > 0$,*

(b) *there exists a constant $\widetilde{C} > 0$ such that for any $T > 0$, the following energy type estimate*

$$\int_{-\infty}^{+\infty} \left\{\frac{\mu^2}{\rho}(x, T) + \frac{1}{\gamma(\gamma-1)}\rho^\gamma(x, T)\right\} dx + \int_0^T \int_{-\infty}^{+\infty} \frac{\mu^2}{\tau\rho^2}(x, s) dx\, ds \leq \widetilde{C}$$

holds.

In particular, for the corresponding solution (r, m) *of the system* (4.3), *with data* (4.4),

(c) *there exists a constant* $c > 0$ *such that* $||r^\tau||_\infty + \tau\,||m^\tau||_\infty \leq c$ *uniformly with respect to* $0 < \tau < 1$,

(d) *as* $\tau \downarrow 0$ *we have* $\tau m^\tau(x,t) \longrightarrow 0$ *a.e. in* $\mathbb{R} \times \mathbb{R}^+$.

Proof. From Proposition 4.1 we know that Σ_τ is an invariant region for (4.6) with data (ρ_0, μ_0^τ). Away from the origin the outward normal to the boundary $\partial\Sigma_\tau$ in the half plane $\{\pm\mu > 0\}$ is well defined by

$$n_\pm(\rho,\mu) = \left(\frac{\vartheta+1}{\vartheta}\rho^\vartheta - c_\tau, \pm 1\right)^T.$$

Applying the theory of invariant domains [3] it is sufficient to verify that

$$\left\langle \left(F(\rho,\mu), G(\rho,\mu/\tau)\right)^T, n_\pm(\rho,\mu)\right\rangle \leq 0 \qquad \text{for } \pm\mu \geq 0,$$

that is (C.1), so that the point (a) is proved. By using the rescaling transformation (4.2) we obtain immediately point (c).

To conclude point (b) it is sufficient a procedure similar to that used in Lemma 3.4 by using the mechanical energy (4.7) and condition (C.2). Finally, as a consequence of the previous results, we get point (d). ■

The last part of this section concerns the singular behavior, as $\tau \downarrow 0$, of the solution of the system (4.3). As in the previous subsection 3.3, we will apply the *div-curl lemma* 3.11 using the estimates obtained in Proposition 4.2 and the theory of compensated compactness.

Theorem 4.3 *Assume the hypotheses (C.1), (C.2) of Proposition 4.2 and in addition*

(C.3) $F, G \in C^1(\mathbb{R}^+ \times \mathbb{R})$, $F(\alpha, 0) = 0$, $\dfrac{\partial G}{\partial \beta}(\alpha,\beta) < 0$ *for all* $\alpha \geq 0$, $\beta \in \mathbb{R}$.

Then there exist $r \in L^\infty$ *and* $m \in L^2$, *such that, as* $\tau \downarrow 0$, *we have*

$$\begin{aligned} r^\tau &\longrightarrow r && \textit{strongly in } L^p_{loc},\ p < +\infty, \\ m^\tau &\rightharpoonup m && \textit{weakly in } L^2, \\ \tau m^\tau &\longrightarrow 0 && \textit{strongly in } L^p_{loc},\ p < +\infty, \end{aligned}$$

and moreover (r, m) *verifies the relaxed system*

$$r_t + m_x = \frac{\partial F}{\partial m}(r,0)m, \tag{4.9}$$

$$p(r)_x = G(r,m), \tag{4.10}$$

where (4.9) *is satisfied in the sense of distributions and* (4.10) *in* L^2. *Denote by* $m = H(r,z)$ *the unique solution to the equation* $G(r,m) = z$. *Therefore it follows* r *verifies the parabolic equation*

$$r_t + H(r, p(r)_x)_x = \frac{\partial F}{\partial m}(r,0)H(r,p(r)_x) \tag{4.11}$$

in the sense of distributions.

Proof. We can apply the L^2 part of the *div-curl lemma* 3.11, to the uniformly bounded $L^2(\Omega)$ sequences $(\mathcal{U}^\tau, \mathcal{V}^\tau)$, where

$$\mathcal{U}^\tau = \{r^\tau, m^\tau\} \qquad\qquad \mathcal{V}^\tau = \{-\tau^2 \frac{(m^\tau)^2}{r^\tau} - p(r^\tau), \tau^2 m^\tau\},$$

for any given region $\Omega \subset\subset \mathbb{R} \times \mathbb{R}^+$. We have

$$\operatorname{div}\ \mathcal{U}^\tau = \frac{1}{\tau} F(r^\tau, \tau m^\tau) \qquad\qquad \operatorname{curl}\ \mathcal{V}^\tau = G(r^\tau, m^\tau),$$

and

$$\left\| \frac{1}{\tau} F(r^\tau, \tau m^\tau) \right\|_\infty \leq c,$$

$$\begin{aligned}\iint_\Omega |(r^\tau, m^\tau)| &\leq \iint_{\{|m^\tau|<1\}} |G(r^\tau, m^\tau)| + \iint_{\{|m^\tau|\geq 1\}} m^\tau G(r^\tau, m^\tau) \\ &\leq \sup_{\{|r^\tau|\leq c, |m^\tau|\leq 1\}} |G(r^\tau, m^\tau)| + \iint_\Omega m^\tau G(r^\tau, m^\tau),\end{aligned}$$

so that they are bounded in $L^1(\Omega)$, for all Ω as above. We deduct that the product $\mathcal{U}^\tau \cdot \mathcal{V}^\tau = -r^\tau p(r^\tau)$ converges in $\mathcal{D}'$ to $\mathcal{U} \cdot \mathcal{V} = -r\widetilde{p}$, since

$$\begin{aligned}&\mathcal{U} = \operatorname*{w-lim}_{\tau\to 0^+} \mathcal{U}^\tau = (r, m), \\ &\mathcal{V} = \operatorname*{w-lim}_{\tau\to 0^+} \mathcal{V}^\tau = (-\widetilde{p}, 0), \qquad\qquad \text{where } \widetilde{p} := \operatorname*{w-lim}_{\tau\to 0^+} p(r^\tau).\end{aligned}$$

Since the function p is monotone, by using Minty argument (see for instance [16]) as in [20], we obtain $\widetilde{p} = p(r)$.

Our next step is to prove the strong convergence for r^τ. To this end, let $\{\nu_{(x,t)}\}$ be the family of Young's probability measures associated to the sequence $\{r^\tau\}$: since $r^\tau \longrightarrow r$ in L^∞ weak*, we can find a closed interval $[a, b]$, $0 \leq a \leq b$, such that $\operatorname{supp} \nu_{(x,t)} \subseteq [a, b]$. Since $p(\alpha) = \frac{\alpha^\gamma}{\gamma}$, $\gamma > 1$, we have three possibilities:

1. $p \in C^2(\mathbb{R} \setminus \{0\})$ and $p''(\alpha) \uparrow +\infty$ as $\alpha \downarrow 0$, if $1 < \gamma < 2$;
2. $p \in C^2(\mathbb{R})$ and $p''(0) = 1$, if $\gamma = 2$;
3. $p \in C^2(\mathbb{R})$ and $p''(0) = 0$, if $\gamma > 2$.

Let us suppose $1 < \gamma \leq 2$. Then we can write for any λ, λ_0

$$p(\lambda) - p(\lambda_0) = p'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2} p''(\lambda^*)(\lambda - \lambda_0)^2,$$

where λ^* belongs to the segment between λ and λ_0. If we choose

$$\lambda_0 = \int_a^b \lambda \nu_{(x,t)}(d\lambda) = r(x,t),$$

we have

$$p(\lambda_0) = \int_a^b p(\lambda)\, \nu_{(x,t)}(d\lambda),$$

so that

$$\int_a^b \{p(\lambda) - p(\lambda_0)\} \nu_{(x,t)}(d\lambda) = 0.$$

On the other hand, we also have

$$\int_a^b p'(\lambda_0)\,(\lambda - \lambda_0)\, \nu_{(x,t)}(d\lambda) = 0$$

so that

$$\int_a^b p''(\lambda^*)\,(\lambda - \lambda_0)^2\, \nu_{(x,t)}(d\lambda) = 0$$

and therefore, since $\min\limits_{\lambda\in[a,b]} p''(\lambda) > 0$, we obtain

$$\int_a^b (\lambda - \lambda_0)^2\, \nu_{(x,t)}(d\lambda) = 0,$$

that is $a = b$ and $\nu_{(x,t)}$ is a point mass.

To conclude, we remark that in the case $\gamma > 2$ this result can be obtained all the same by using the inverse function $-p^{-1}$. ■

Remark 4.4 We want to remark that, on the contrary, we can use the previous result to associate to any parabolic equation in the form

$$r_t + H(r, p(r)_x)_x = f(r)H(r, p(r)_x), \tag{4.12}$$

$f \in C(\mathbb{R}^+)$, $H \in C^1(\mathbb{R}^+ \times \mathbb{R})$ such that $\frac{\partial H}{\partial m}(r,m) < 0$, a corresponding nonlinear hyperbolic 2×2 system in the form (4.1), or in particular

$$\begin{cases} \rho_s + \mu_x = f(\rho)\mu \\ \mu_s + \left(\frac{\mu^2}{\rho} + p(\rho)\right)_x = G\left(\rho, \frac{\mu}{\tau}\right), \end{cases} \tag{4.13}$$

where $z = G(r,m)$ is the unique solution of $m = H(r,z)$. In fact via the rescaling (4.2), system (4.13) transforms into

$$\begin{cases} r_t + m_x = f(r)m \\ \tau^2 m_t + \left(\tau^2 \frac{m^2}{r} + p(r)\right)_x = G(r,m), \end{cases}$$

which, as proved in Theorem 4.3, relaxes into (4.12). ■

5 Multidimensional Examples

In this section we will investigate the two dimensional version of the nonlinear heat conduction and of the compressible Euler flow. These one will be only model problems in order to show how our hyperbolic–parabolic relaxation theory can be extended in the multidimensional cases when the original system verifies suitable hypotheses which include boundedness in L^∞ and energy estimates for the solutions. Some of these results will be extended in a forthcoming paper.

In these examples we apply ideas from compensated compactness due to Tartar [30, 31] and Murat [26] (see also Dacorogna [4]) and in particular the following theorem

Theorem 5.1 *Let us consider*

- *a bounded open set* $\Omega \subset \mathbb{R}^n$*;*
- *a sequence* $\{l^\nu\}_{\nu=1}^\infty$, $l^\nu : \ \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$*;*
- *a symmetric matrix* $\mathcal{M} : \ \mathbb{R}^m \longrightarrow \mathbb{R}^m$*;*
- *constants* $a^i_{jk} \in \mathbb{R}$, $i = 1, \dots, q$, $j = 1, \dots, m$, $k = 1, \dots, n$.

Let us define

- $f(a) = \langle \mathcal{M}a, a\rangle$, *for all* $a \in \mathbb{R}^m$*;*
- $\Lambda = \left\{ \underline{\lambda} \in \mathbb{R}^m : \ \exists \underline{\xi} \in \mathbb{R}^n \setminus \{\underline{0}\}, \ \sum\limits_{j,k} a^i_{jk}\lambda_j\xi_k = 0, i = 1, \dots, q \right\}.$

Assume that

(a) *there exists* $\widetilde{l} \in L^2_m(\Omega)$ *such that* $l^\nu \rightharpoonup \widetilde{l}$ *in* $L^2_m(\Omega)$ *as* $\nu \uparrow \infty$*;*

(b) $\mathcal{A}^i l^\nu = \sum\limits_{j,k} a^i_{jk} \frac{\partial l^\nu_j}{\partial x_k}$, $i = 1, \dots q$ *are relatively compact in* $H^{-1}_{loc}(\Omega)$*;*

(c) $f_{|\Lambda} \equiv 0$*;*

(d) *there exists* $\widetilde{f} \in \mathbb{R}$ *such that* $f(l) \rightharpoonup \widetilde{f}$ *in the sense of measures* $\mathcal{M}(\Omega)$.

Then we have $\widetilde{f} = f(\widetilde{l})$.

Remark 5.2 The *div-curl lemma* 3.11 is a consequence of Theorem 5.1: the hypothesis $\mathcal{A}^i l^\nu$ in a compact set of $H^{-1}_{loc}(\Omega)$ is weaker than the hypothesis $\mathcal{A}^i l^\nu$ in a bounded set of $L^2(\Omega)$. ■

5.1 Two dimensional nonlinear heat conduction

Let us consider the two–dimensional model of nonlinear heat conduction (see Cattaneo [1]) which has been investigated in [21] in the *1-D* case

$$\Phi_{ss} - \mathrm{div}_{(x,y)} \ \left(\vartheta \left(\left| D_{(x,y)}\Phi \right| \right) D_{(x,y)}\Phi \right) + \frac{\alpha}{\tau}\Phi_s = 0,$$

where $\Phi = \Phi(x,y,s)$, $(x,y,s) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}^+$, $\alpha > 0$ is a constant and $\tau > 0$ is the relaxation parameter. By using the following setting

$$v = \Phi_s, \qquad\qquad u = \Phi_x, \qquad\qquad w = \Phi_y,$$

we obtain the equivalent first order system

$$\begin{cases} u_s - v_x = 0 \\ w_s - v_y = 0 \\ v_s - (\vartheta(r)u)_x - (\vartheta(r)w)_y = -\frac{\alpha}{\tau}v, \end{cases} \tag{5.1}$$

where $r = \sqrt{u^2+w^2}$. We remark there is an other equation as a consequence of the previous change of unknowns, that is

$$w_x - u_y = 0. \tag{5.2}$$

If we apply the following rescaling transformation on the time variable,

$$\begin{cases} \mathcal{U}^\tau(x,y,t) = u(x,y,t/\tau) \\ \mathcal{W}^\tau(x,y,t) = w(x,y,t/\tau) \\ \mathcal{V}^\tau(x,y,t) = \frac{1}{\tau}v(x,y,t/\tau), \end{cases} \tag{5.3}$$

if $\rho^\tau = \sqrt{(\mathcal{U}^\tau)^2 + (\mathcal{W}^\tau)^2}$, we obtain that (5.1) is equivalent to

$$\begin{cases} \mathcal{U}^\tau_t - \mathcal{V}^\tau_x = 0 \\ \mathcal{W}^\tau_t - \mathcal{V}^\tau_y = 0 \\ \tau^2\mathcal{V}^\tau_t - (\vartheta(\rho^\tau)\mathcal{U}^\tau)_x - (\vartheta(\rho^\tau)\mathcal{W}^\tau)_y = -\alpha\mathcal{V}^\tau \end{cases} \tag{5.4}$$

which, passing into the limit when $\tau \downarrow 0$, reduces to

$$\begin{cases} \mathcal{U}^0_t - \mathcal{V}^0_x = 0 \\ \mathcal{W}^0_t - \mathcal{V}^0_y = 0 \\ \left(\vartheta(\rho^0)\mathcal{U}^0\right)_x + \left(\vartheta(\rho^0)\mathcal{W}^0\right)_y = \alpha\mathcal{V}^0, \end{cases} \tag{5.5}$$

where $\rho^0 = \sqrt{(\mathcal{U}^0)^2 + (\mathcal{W}^0)^2}$, while the equation (5.2) after the transformation (5.3) relaxes into

$$\mathcal{W}^0_x - \mathcal{U}^0_y = 0. \tag{5.6}$$

The mechanical energy functional is given by

$$\mathcal{E}(\mathcal{U},\mathcal{W},\mathcal{V}) = \frac{1}{2}|\mathcal{V}|^2 + \left(\mathcal{U}^2+\mathcal{W}^2\right)\int_0^1 \vartheta\left(\lambda\left(\mathcal{U}^2+\mathcal{W}^2\right)^{1/2}\right)\lambda\, d\lambda,$$

therefore the total energy at time t is given by

$$E(t) = \int_{-\infty}^{+\infty} \mathcal{E}\left(\mathcal{U}(x,t),\mathcal{W}(x,t),\mathcal{V}(x,t)\right)dx.$$

In the next result we are going to investigate the behaviour as $\tau \downarrow 0$ of the solution to the system (5.4).

Theorem 5.3 *Let us consider the sequence* $\{\mathcal{U}^\tau, \mathcal{W}^\tau, \mathcal{V}^\tau\}$ *of the solutions to the system* (5.4) *with initial data* $\{\mathcal{U}_0^\tau, \mathcal{W}_0^\tau, \mathcal{V}_0^\tau\} \in [L^\infty(\mathbb{R})]^3$. *Assume that the initial data satisfy the following finite energy condition*

$$\int_{-\infty}^{+\infty} \mathcal{E}\left(\mathcal{U}_0^\tau(x), \mathcal{W}_0^\tau(x), \mathcal{V}_0^\tau(x)\right) dx < +\infty$$

and moreover the solutions to (5.4) *exist on* $[0,T)$, $T \leq +\infty$ *and are uniformly bounded in* $L^\infty[(0,T)\times\mathbb{R}]$ *(uniformly with respect to* τ*). Assume the function* ϑ *satisfies the following condition*

(D) *Define* $\sigma(\alpha) = \alpha\vartheta(\alpha)$*; then* $\sigma \in C^2(\mathbb{R}^+)$ *and there exists* $\sigma^* > 0$ *such that* $\sigma'(\alpha) \geq \sigma^*$ *for all* α, $\sigma''(\alpha) > 0$ *for all* $\alpha \neq 0$.

Then there exists $\mathcal{U}^0, \mathcal{V}^0 \in L^\infty[(0,T)\times\mathbb{R}]$, *such that (extracting if necessary a subsequence)*

$$(\mathcal{U}^\tau, \mathcal{W}^\tau) \longrightarrow \left(\mathcal{U}^0, \mathcal{W}^0\right) \quad \textit{strongly in } L^p_{loc} \textit{ for all } p < +\infty.$$

Moreover there exists $\mathcal{V}^0$ *such that*

$$\mathcal{V}^\tau \rightharpoonup \mathcal{V}^0 \quad \textit{weakly in } L^2.$$

The limit functions $\left\{\mathcal{U}^0, \mathcal{W}^0, \mathcal{V}^0\right\}$ *satisfy the system* (5.5), *with initial data*

$$\mathcal{W}^0(x,0) = \mathcal{W}_0^0(x), \quad \mathcal{U}^0(x,0) = \mathcal{U}_0^0(x),$$

where

$$\mathcal{W}_0^0 = \text{w}^*\text{-lim}\, \mathcal{W}_0^\tau, \quad \mathcal{U}_0^0 = \text{w}^*\text{-lim}\, \mathcal{U}_0^\tau, \quad \textit{in } L^\infty(\mathbb{R}).$$

Remark 5.4 The first two equations of the limit system (5.5) plus the additional identity (5.6), imply that $\operatorname{curl}\left(\mathcal{V}^0, \mathcal{U}^0, \mathcal{W}^0\right) = 0$ (with respect to the variables (t,x,y)). Therefore there exists $\varphi \in W^{1,\infty}$, such that $\varphi(x,t)$ is a weak solution of the parabolic equation

$$\varphi_t = \operatorname{div}_{(x,y)} \, \left(\vartheta(|D\varphi|)D\varphi\right).$$

■

Proof. Let us consider the following vector fields

$$A^\tau(x,y,t) = \left(\mathcal{V}^\tau(x,y,t), \mathcal{U}^\tau(x,y,t), \mathcal{W}^\tau(x,y,t)\right)$$

$$B^\tau(x,y,t) = \left(0, \vartheta(\rho^\tau(x,y,t))\mathcal{U}^\tau(x,y,t), \vartheta(\rho^\tau(x,y,t))\mathcal{W}^\tau(x,y,t)\right).$$

Therefore the system (5.4) can be rewritten in the following way

$$\begin{aligned} \operatorname{curl}_{(x,y,t)} A^\tau &= 0 \\ \operatorname{div}_{(x,y,t)} B^\tau &= \alpha\mathcal{V}^\tau + \tau^2\mathcal{V}_t^\tau. \end{aligned}$$

Since $\{V^\tau\}$ is uniformly bounded in L^2 and $\left\{\tau^2 \mathcal{V}_t^\tau\right\}$ is relatively compact in H^{-1}_{loc}, we can assert that

$$\text{div}\ \ B^\tau \quad \text{is relatively compact in } H^{-1}.$$

By using the classical *div-curl lemma*, recalled in the Lemma 3.11, we can conclude that

$$A^\tau \cdot B^\tau \rightharpoonup A \cdot B \quad \text{in the sense of measures } \mathcal{M}(\Omega), \tag{5.7}$$

where

$$\begin{aligned} A &= \left(\mathcal{V}^0, \mathcal{U}^0, \mathcal{W}^0\right) = \text{w-lim } A^\tau \\ B &= (0, B_2, B_3) = \text{w-lim } B^\tau. \end{aligned}$$

In other terms, we have that the map

$$T\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{W} \end{smallmatrix}\right) = \vartheta(\sqrt{\mathcal{U}^2 + \mathcal{W}^2})\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{W} \end{smallmatrix}\right)$$

is a maximal monotone map in the sense of Minty and Browder (see [16]). Indeed

$$\left\langle \vartheta(\rho)\left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{W} \end{smallmatrix}\right) - \vartheta(\overline{\rho})\left(\begin{smallmatrix} \overline{\mathcal{U}} \\ \overline{\mathcal{W}} \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{U} \\ \mathcal{W} \end{smallmatrix}\right) - \left(\begin{smallmatrix} \overline{\mathcal{U}} \\ \overline{\mathcal{W}} \end{smallmatrix}\right)\right\rangle \geq \sigma^* \left\|\left(\mathcal{U} - \overline{\mathcal{U}}, \mathcal{W} - \overline{\mathcal{W}}\right)^T\right\|^2$$

where $\rho = \sqrt{\mathcal{U}^2 + \mathcal{W}^2}$, implies that T is monotone and coercive. The maximality follows from the closed graph property of T.
Therefore, as a consequence of (5.7), it follows

$$\left\langle T\left(\begin{smallmatrix} \mathcal{U}^\tau \\ \mathcal{W}^\tau \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{U}^\tau \\ \mathcal{W}^\tau \end{smallmatrix}\right)\right\rangle \longrightarrow \left\langle \left(\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right), \left(\begin{smallmatrix} \mathcal{U}^0 \\ \mathcal{W}^0 \end{smallmatrix}\right)\right\rangle.$$

By using the classical Minty's Theorem, we get

$$T\left(\begin{smallmatrix} \mathcal{U}^0 \\ \mathcal{W}^0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} B_2 \\ B_3 \end{smallmatrix}\right),$$

where, as before, $\mathcal{U}^0 = \text{w-lim}\, \mathcal{U}^\tau$, $\mathcal{W}^0 = \text{w-lim}\, \mathcal{W}^\tau$.

Our next step is to prove the strong convergence for $(\mathcal{U}^\tau, \mathcal{W}^\tau)$ in L^p_{loc}. Since we have the weak* L^∞ convergence, to conclude it is sufficient to prove $\rho^\tau \longrightarrow \rho^0$ in L^p_{loc}. Let us define $\varphi(\alpha) = \alpha^2 \vartheta(\alpha)$, then from (D) we have

$$\varphi''(\alpha) = 2\vartheta(\alpha) + 4\alpha\vartheta'(\alpha) + \alpha^2\vartheta''(\alpha) > 0 \qquad\qquad \text{for all } \alpha \geq 0.$$

On the other hand we know that

$$\rho^\tau \overset{*}{\rightharpoonup} \rho^0 \qquad\qquad \text{and } \varphi\left(\rho^\tau\right) \overset{*}{\rightharpoonup} \varphi\left(\rho^0\right) \qquad\qquad \text{in } L^\infty \text{ weak}^*,$$

then we get the conclusion by applying the same arguments used in the Theorem 4.3 (see also [20], page 141). ■

5.2 Two dimensional compressible Euler flow

In this example we consider a two dimensional compressible Euler flow with a polytropic equation of state, that is the natural *2-D* generalization of the model considered by [20]:

$$\begin{cases} \rho_s + \mu_x + \nu_y = 0 \\ \mu_s + \left(\frac{\mu^2}{\rho} + p(\rho)\right)_x + \left(\frac{\mu\nu}{\rho}\right)_y = -\frac{1}{\tau}\frac{\mu}{\rho} \\ \nu_s + \left(\frac{\mu\nu}{\rho}\right)_x + \left(\frac{\nu^2}{\rho} + p(\rho)\right)_y = -\frac{1}{\tau}\frac{\nu}{\rho}, \end{cases} \tag{5.8}$$

where $(x, y, s) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, ρ is the density, $\mu = \rho u_1$, $\nu = \rho u_2$ and (u_1, u_2) the Eulerian velocity, $p = \frac{1}{\gamma}\rho^\gamma$ the pressure, $\gamma > 1$, and $\tau > 0$ the relaxation parameter. We shall study the behavior of solution at infinity of the corresponding Cauchy problem as $\tau \downarrow 0$. To do that, we will scale the time in this way:

$$\begin{cases} r^\tau(x, y, t) = \rho(x, y, t/\tau) \\ m^\tau(x, y, t) = \frac{1}{\tau}\mu(x, y, t/\tau) \\ n^\tau(x, y, t) = \frac{1}{\tau}\nu(x, y, t/\tau). \end{cases}$$

Consequently system (5.8) transforms into

$$\begin{cases} r_t + m_x + n_y = 0 \\ \tau^2 m_t + \left(\tau^2 \frac{m^2}{r} + p(r)\right)_x + \left(\tau^2 \frac{mn}{r}\right)_y = -\frac{m}{r} \\ \tau^2 n_t + \left(\tau^2 \frac{mn}{r}\right)_x + \left(\tau^2 \frac{n^2}{r} + p(r)\right)_y = -\frac{n}{r}. \end{cases} \tag{5.9}$$

When $\tau \downarrow 0$, formally we obtain the relaxed system

$$\begin{cases} r^0_t = \left(r^0 p(r^0)_x\right)_x + \left(r^0 p(r^0)_y\right)_y \\ m^0 = -r^0 p(r^0)_x \\ n^0 = -r^0 p(r^0)_y. \end{cases} \tag{5.10}$$

More precisely, we are able to show the following result.

Theorem 5.5 *Let $\{r^\tau, m^\tau, n^\tau\}$ be a sequence of weak entropy solutions to the system* (5.9) *such that $\{r^\tau, \tau m^\tau, \tau n^\tau\}$ is uniformly bounded in L^∞. Assume that the pressure $p(\rho)$ is a non negative C^2 function such that $p'(\rho) > 0$, $p''(\rho) > 0$, for all $\rho > 0$. Then it follows*

(i) *if the initial data $u_{1,0}(x, y) = \frac{\mu(x,y,0)}{\rho(x,y,0)}$, $u_{2,0}(x, y) = \frac{\nu(x,y,0)}{\rho(x,y,0)}$, $\rho_0(x, y) = \rho(x, y, 0)$ satisfy the initial finite energy condition*

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathcal{E}(u_{1,0}, u_{2,0}, \rho_0)dx\ dy \le C,$$

where $\mathcal{E}(u_1, u_2, \rho) = \frac{1}{2}\rho\left(u_1^2 + u_2^2\right) + \int_0^\rho \int_0^\sigma \frac{p'(\eta)}{\eta} d\eta\ d\sigma$, therefore one has

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathcal{E}(u_1(\cdot, t), u_2(\cdot, t), \rho(\cdot, t))dx\ dy \le C, \tag{5.11}$$

and

$$\frac{1}{\tau^2}\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{\mu^2(x,y,s)}{\rho(x,y,s)}+\frac{\nu^2(x,y,s)}{\rho(x,y,s)}\right\}dx\ dy\ ds \le C; \tag{5.12}$$

(ii) *there exists $r^0 \in L^\infty$, such that (extracting if necessary a subsequence)*

$$r^\tau \longrightarrow r^0 \qquad \text{in } L^p_{loc},\ p<+\infty.$$

Moreover if we denote by $\left(m^0,n^0\right)=\underset{\tau\to 0^+}{\text{w-lim}}(m^\tau,n^\tau)$ in L^2, therefore the triplet $\left(m^0,n^0,r^0\right)$ satisfies the system (5.10) *in the sense of distributions.*

Proof. The standard entropy inequality can be applied to the physical entropy $\mathcal{E}$ and it yields to the energy estimate given in (5.11), (5.12).

To prove the second part we want to use the Theorem 5.1. Let us point out that from the relations

$$\begin{aligned}
&r^\tau_t+m^\tau_x+n^\tau_y=0\\
&\quad p(r^\tau)_x=-\tau^2\left(m^\tau\right)_t-\tau^2\left(\frac{(m^\tau)^2}{r^\tau}\right)_x-\frac{m^\tau}{r^\tau}-\tau^2\left(\frac{m^\tau n^\tau}{r^\tau}\right)_y\\
&\quad p(r^\tau)_y=-\tau^2\left(n^\tau\right)_t-\tau^2\left(\frac{(n^\tau)^2}{r^\tau}\right)_y-\frac{n^\tau}{r^\tau}-\tau^2\left(\frac{m^\tau n^\tau}{r^\tau}\right)_x
\end{aligned}$$

and the L^∞ estimates and the energy estimates (5.11), (5.12) it follows that

$$\begin{pmatrix} r^\tau_t+m^\tau_x+n^\tau_y\\ p(r^\tau)_x\\ p(r^\tau)_y\end{pmatrix} \qquad \text{is relatively compact in } \left(H^{-1}_{loc}\right)^3. \tag{5.13}$$

In order to fit into the framework of the Theorem 5.1 we use the previous information in the following way. Define $\Omega=\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\subset\mathbb{R}^3$, $x_1=x$, $x_2=y$, $x_3=t$, the sequence $l^\tau=(m^\tau,n^\tau,r^\tau,p(r^\tau))$, hence $m=4$. By the previous argument in (5.13) we know that the number q of differential constraints is $q=3$. Then we can define the matrices $\mathcal{A}^1,\mathcal{A}^2,\mathcal{A}^3\in\mathcal{M}_{3\times 4}$ (where $A^i=\left\{a^i_{jk}\right\}$, $i=1,2,3$, $j=1,\dots,4$, $k=1,2,3$) as follows

$$\mathcal{A}^1=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{pmatrix},\quad \mathcal{A}^2=\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix},\quad \mathcal{A}^3=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}.$$

The characteristic manifold Λ is then given by

$$\Lambda=\left\{\lambda\in\mathbb{R}^4\mid \exists\xi\in\mathbb{R}^3\setminus\{0\},B(\xi,\lambda)=0\right\}$$

where

$$B(\xi,\lambda)=\begin{pmatrix}\lambda_1\xi_1+\lambda_2\xi_2+\lambda_3\xi_3\\ \lambda_4\xi_1\\ \lambda_4\xi_2\end{pmatrix}.$$

Therefore

$$\Lambda = \left\{ \lambda \in \mathbb{R}^4 \mid \det \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & 0 & 0 \\ 0 & \lambda_4 & 0 \end{pmatrix} = 0 \right\} = \left\{ \lambda \in \mathbb{R}^4 \mid \lambda_3 \lambda_4 = 0 \right\}.$$

We can define $M = \frac{1}{2} \left(\begin{smallmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{smallmatrix} \right) \in \mathcal{M}_{4\times 4}$, thus $f(\lambda) = \lambda^T M \lambda = \lambda_3 \lambda_4$ and, of course, $f_{|\Lambda} \equiv 0$.

By using the Theorem 5.1 we conclude with w-lim $f(l^\tau) = f$ (w-lim l^τ), namely in our case

$$r^\tau p(r^\tau) \rightharpoonup r^0 p^0,$$

where $p^0 = $ w-lim $p(r^\tau)$. At this point we can follow exactly the methods of [20] used in the Section 4, and we get the strong convergence of $\{r^\tau\}$. The remaining part of the proof follows exactly the same arguments used in the previous sections so is omitted. ■

6 Semilinear Systems

6.1 Multidimensional framework

Consider the semilinear problem

$$W_s + \sum_{j=1}^{d} E_j \partial_j W = \mathcal{B}(W) \tag{6.1}$$

where $E_j \in \mathcal{M}_{N\times N}$, $j = 1, \ldots, d$ and $B: \ \mathcal{O} \longrightarrow \mathbb{R}^N$ as in the general settings of Section 2. Of course we can reformulate the previous problem as a special case of (2.1) by setting, for all $\xi \in \mathbb{R}^d$, $W \in \mathcal{O} \subset \mathbb{R}^N$,

$$\mathcal{A}(W)\,\xi = \sum_{j=1}^{d} \xi_j E_j W.$$

Let $P: \ \mathbb{R}^N \longrightarrow \mathbb{R}^k$ be a linear map such that $P\mathcal{B}(W) = 0$. We define $Q: R^N \longrightarrow \mathbb{R}^{N-k}$ and $M: \ \mathbb{R}^k \times \mathbb{R}^{N-k} \longrightarrow \mathbb{R}^N$ as before, namely $U = PW$, $V = QW$, $W = M(U,V)$.

Then there exists matrices $H_j^1 \in \mathcal{M}_{k\times k}$, $K_j^1 \in \mathcal{M}_{k\times (N-k)}$ and $H_j^2 \in \mathcal{M}_{(N-k)\times k}$, $K_j^2 \in \mathcal{M}_{(N-k)\times(N-k)}$ such that

$$\begin{aligned} H_j^1 U + K_j^1 V &= PE_j M(U,V) \\ H_j^2 U + K_j^2 V &= QE_j M(U,V). \end{aligned}$$

Hence the system (6.1) becomes

$$\begin{cases} \partial_s U + \sum\limits_{j=1}^{d} H_j^1 \partial_j U + \sum\limits_{j=1}^{d} K_j^1 \partial_j V = 0 \\ \partial_s V + \sum\limits_{j=1}^{d} H_j^2 \partial_j U + \sum\limits_{j=1}^{d} K_j^2 \partial_j V = \mathcal{R}(U,V), \end{cases} \tag{6.2}$$

where $\mathcal{R}(U,V) = Q\mathcal{B}(M(U,V))$. The relevant assumption which is needed to apply the theory is

(F.1) $H_j^1 \equiv 0$ for all $j = 1, \ldots, d$.

Without this assumption we can expect a nontrivial relaxation limit under the hyperbolic scale which prevents the system to relax directly on a parabolic system.

It is sufficient to apply a rescaling of the first type (2.3) to transform system (6.2) in the form

$$\begin{cases} \partial_t \mathcal{U}^\varepsilon + \sum\limits_{j=1}^d K_j^1 \partial_j \mathcal{V}^\varepsilon = 0 \\ \varepsilon \partial_s \mathcal{V}^\varepsilon + \sum\limits_{j=1}^d H_j^2 \partial_j \mathcal{U}^\varepsilon + \varepsilon^{1/2} \sum\limits_{j=1}^d K_j^2 \partial_j \mathcal{V}^\varepsilon = \varepsilon^{-1/2} \mathcal{R}\left(\mathcal{U}^\varepsilon, \varepsilon^{1/2} \mathcal{V}^\varepsilon\right). \end{cases} \tag{6.3}$$

As established in the general case (see subsection 2.2), if we define $\mathcal{R}_\mathcal{V}^0(\cdot) = \partial_\mathcal{V} \mathcal{R}(\cdot, 0)$, if we denote by $\left(\mathcal{U}^0, \mathcal{V}^0\right)$ the limit profile, the formal limit in this particular case is

$$\begin{cases} \partial_t \mathcal{U}^0 + \sum\limits_{j=1}^d K_j^1 \partial_j \mathcal{V}^0 = 0 \\ \sum\limits_{j=1}^d H_j^2 \partial_j \mathcal{U}^0 = \mathcal{R}_\mathcal{V}^0\left(\mathcal{U}^0\right) \mathcal{V}^0. \end{cases} \tag{6.4}$$

Assume the following additional hypothesis on $R_\mathcal{V}^0$:

(F.2) $\mathcal{R}_\mathcal{V}^0 \in \mathcal{M}_{(N-k)\times(N-k)}$ and $\det \mathcal{R}_\mathcal{V}^0(\mathcal{U}) \neq 0$ for all $\mathcal{U}$.

Under this condition, system (6.4) transforms into the following

$$\begin{cases} \mathcal{V}^0 = \sum\limits_{j=1}^d \left(\mathcal{R}_\mathcal{V}^0\left(\mathcal{U}^0\right)\right)^{-1} H_j^2 \partial_j \mathcal{U}^0 \\ \partial_t \mathcal{U}^0 + \sum\limits_{j=1}^d K_j^1 \partial_j \left\{ \sum\limits_{h=1}^d \left(\mathcal{R}_\mathcal{V}^0\left(\mathcal{U}^0\right)\right)^{-1} H_h^2 \partial_h \mathcal{U}^0 \right\} = 0. \end{cases}$$

To write the system for $\mathcal{U}^0$ in a more compact form, let us define the matrix L with component

$$L_{hj}\left(\mathcal{U}^0\right) = -K_j^1 \left(\mathcal{R}_\mathcal{V}\left(\mathcal{U}^0\right)\right)^{-1} D_{\mathcal{U}^0} \mathcal{R}_\mathcal{V}^0\left(\mathcal{U}^0\right) \left(\mathcal{R}_\mathcal{V}\left(\mathcal{U}^0\right)\right)^{-1} H_h^2.$$

Under this notation, the final equation for $\mathcal{U}^0$ can be given by

$$\partial_t \mathcal{U}^0 + \sum_{h,j=1}^d L_{hj}\left(\mathcal{U}^0\right) \partial_{hj}^2 \mathcal{U}^0 = 0. \tag{6.5}$$

To assure the strong parabolicity of the reduced system (6.5), we require that L verifies the following strongly parabolic condition (see Eidel'man [12] or Taylor [32]):

(F.3) if we define λ_k the eigenvalues of the matrix $-\sum_{h,k=1}^d \xi_h \xi_k L_{hk}\left(\mathcal{U}^0\right)$, we assume there exists $\alpha_0 > 0$ such that

$$\operatorname{Re} \lambda_k\left(\mathcal{U}^0, \xi\right) \leq -\alpha_0 \left|\xi\right|^2.$$

We will show the strong convergence of the relaxing sequence in the next section for the *1-D* case.

6.2 $1-\mathrm{D}$ semilinear systems

We want now find sufficient conditions in order to justify the formal limit from (6.3) to (6.4). Let us reduce to the particular situation when $d = 1$. In this case we can rewrite the rescaled system (6.3) in the form

$$\begin{cases} \partial_t \mathcal{U}^\varepsilon + K^1 \partial_x \mathcal{V}^\varepsilon = 0 \\ \varepsilon \partial_t \mathcal{V}^\varepsilon + H^2 \partial_x \mathcal{U}^\varepsilon + \varepsilon^{1/2} K^2 \partial_x \mathcal{V}^\varepsilon = \varepsilon^{-1/2} \mathcal{R}\left(\mathcal{U}^\varepsilon, \varepsilon^{1/2} \mathcal{V}^\varepsilon\right). \end{cases} \tag{6.6}$$

We will obtain a convergence result for the solutions of the system (6.6). To this end, we make on (6.6) the following hypotheses:

(G.1) $K^2 \in \mathcal{M}_{(N-k)\times(N-k)}$ is symmetric,

(G.2) there exists a symmetric positive definite matrix $B_0 \in \mathcal{M}_{k\times k}$ such that $\left(K^1\right)^T B_0 = H^2$,

(G.3) there exists $\lambda_1 > 0$ such that for any $\mathcal{U} \in \mathbb{R}^k$, $\mathcal{R}\left(\mathcal{U}, 0\right) \equiv 0$ and $\left\|\mathcal{R}_\mathcal{V}\left(\mathcal{U}, 0\right)\right\|_\infty \leq \lambda_1$,

(G.4) there exist $\lambda_2, \lambda_3 > 0$ such that for any $(\mathcal{U}, \mathcal{Z}) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$, $\mathcal{R}_\mathcal{V}\left(\mathcal{U}, \mathcal{Z}\right) \leq -\lambda_2 \mathcal{Z}$ and $\left\|\mathcal{R}_{\mathcal{V}\mathcal{V}}\left(\mathcal{U}, \mathcal{Z}\right)\right\|_\infty \leq \lambda_3$

(G.5) $\det\left(\left(H^2\right)^T H^2\right) \neq 0$.

Since $H^2 \in \mathcal{M}_{(N-k)\times k}$, from elementary linear algebra we deduce condition (G.5) is violated whenever $k > N/2$.

The following convergence result will be shown by using only the information provided by the energy estimates.

Theorem 6.1 *Let us consider the Cauchy problem for the system* (6.6). *Assume that the hypotheses (G.1), (G.2), (G.3), (G.4) and (G.5) hold. Then there exists* $\left(\mathcal{U}^0, \mathcal{V}^0\right) \in L^2$ *such that, as* $\varepsilon \downarrow 0$, *one has*

$$\mathcal{U}^\varepsilon \longrightarrow \mathcal{U}^0 \qquad \text{strongly in } L^2_{loc}, \tag{6.7}$$

$$\mathcal{V}^\varepsilon \rightharpoonup \mathcal{V}^0 \qquad \text{weakly in } L^2, \tag{6.8}$$

$$\sqrt{\varepsilon}\mathcal{V}^\varepsilon \longrightarrow 0 \qquad \text{strongly in } L^2_{loc} \tag{6.9}$$

and the limit profile $\left(\mathcal{U}^0, \mathcal{V}^0\right)$ *verifies in* $\mathcal{D}'$ *the system*

$$\begin{cases} \partial_t \mathcal{U}^0 + K^1 \partial_x \mathcal{V}^0 = 0 \\ H^2 \partial_x \mathcal{U}^0 = \mathcal{R}^0_\mathcal{V}\left(\mathcal{U}^0\right) \mathcal{V}^0. \end{cases}$$

Proof. We will use the energy estimates in L^2. To do this, we multiply the first equation by $B_0 \mathcal{U}^\varepsilon$ and the second equation by $\mathcal{V}^\varepsilon$ and we sum the two relation. Let us denote by $||\cdot||$ the norm of $L^2\left(\mathbb{R}_x\right)$ and by $\langle\cdot,\cdot\rangle$ the $L^2\left(\mathbb{R}_x\right)$ product. We obtain, taking into account (G.1), (G.2) and (G.3),

$$\partial_t \left\{ \frac{\varepsilon}{2} \left\|\mathcal{V}^\varepsilon\right\|^2 + \frac{1}{2} \left\| B_0^{1/2} \mathcal{U}^\varepsilon \right\|^2 \right\} + \partial_x \left\{ \left\langle H^2 \mathcal{U}^\varepsilon, \mathcal{V}^\varepsilon \right\rangle + \frac{\sqrt{\varepsilon}}{2} \left\langle K^2 \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon \right\rangle \right\}$$

$$= \left\langle \left[\int_0^1 \mathcal{R}_{\mathcal{V}} \left(\mathcal{U}^\varepsilon, \Theta\sqrt{\varepsilon}\mathcal{V}^\varepsilon \right) d\Theta \right] \mathcal{V}^\varepsilon, \mathcal{V}^\varepsilon \right\rangle .$$

Integrating on $[0,T] \times \mathbb{R}$ and by using the first part of (G.4), we have

$$\int_{-\infty}^{+\infty} \varepsilon \left|\left|\mathcal{V}^\varepsilon(T)\right|\right|^2 + \left|\left| B_0^{1/2} \mathcal{U}^\varepsilon(T) \right|\right|^2 dx + 2\lambda_2 \int_0^T \int_{-\infty}^{+\infty} \left|\left|\mathcal{V}^\varepsilon(x,t)\right|\right|^2 dx \ dt$$
$$\leq \int_{-\infty}^{+\infty} \varepsilon \left|\left|\mathcal{V}_0^\varepsilon\right|\right|^2 + \left|\left| B_0^{1/2} \mathcal{U}_0^\varepsilon \right|\right|^2 dx.$$

So in particular we obtain (6.8) and (6.9). Let us now start again from the system (6.6). Multiplying the second equation by $\left(H^2\right)^T$ first and after by $\left(\left(H^2\right)^T H^2\right)^{-1}$ we have (6.6) is equivalent to

$$\begin{cases} \partial_t \mathcal{U}^\varepsilon + K^1 \partial_x \mathcal{V}^\varepsilon = 0 \\ \partial_t \left\{ \varepsilon \left(\left(H^2\right)^T H^2 \right)^{-1} \left(H^2\right)^T \mathcal{V}^\varepsilon \right\} + \partial_x \mathcal{U}^\varepsilon + \varepsilon^{1/2} \left(\left(H^2\right)^T H^2 \right)^{-1} \left(H^2\right)^T K^2 \partial_x \mathcal{V}^\varepsilon \\ \qquad = \left(\left(H^2\right)^T H^2 \right)^{-1} \left(H^2\right)^T \varepsilon^{-1/2} \mathcal{R}\left(\mathcal{U}^\varepsilon, \varepsilon^{1/2}\mathcal{V}^\varepsilon\right). \end{cases}$$

By using the second part of (G.4), we can now apply the *div-curl lemma* and obtain for any $i = 1, \dots, k$, that

$$\left(\mathcal{U}_i^\varepsilon\right)^2 \rightharpoonup \left(\mathcal{U}_i^0\right)^2 \qquad \text{weakly in } L^2.$$

Since we also have $\mathcal{U}^\varepsilon \rightharpoonup \mathcal{U}^0$ weakly in L^2, we can conclude (6.7). So we can pass to the limit into the nonlinear term $\mathcal{R}$ and conclude with the relaxation result. ∎

References

[1] C. Cattaneo. Sulla conduzione del calore. *Atti Sem. Mat. Fisico Univ. Modena*, 3:83–101, 1948.

[2] G.Q. Chen. Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics III. *Acta Math. Sci.*, 6:75–120, 1986.

[3] K.N. Chueh, C.C. Conley, and J.A. Smoller. Positively invariant regions for systems of nonlinear diffusion equations. *Indiana Univ. Math. J.*, 26:372–411, 1977.

[4] B. Dacorogna. *Weak Continuity and Weak Lower Semicontinuity of Nonlinear Functional.* Springer–Verlag, 1982.

[5] C.M. Dafermos. Estimates for conservation laws with little viscosity. *SIAM J. Math. Anal.*, 18:409–421, 1987.

[6] X. Ding, G.Q. Chen, and P. Luo. Convergence of the Lax–Friedrichs scheme for isentropic gas dynamics I, II. *Acta Math. Sci.*, 5:415–432 and 433–472, 1985.

[7] X. Ding, G.Q. Chen, and P. Luo. Convergence of the fractional step Lax–Friedrichs scheme and Godunov scheme for the isentropic system of gas dynamics. *Comm. Math. Phys.*, 121:63–84, 1989.

[8] R.J. DiPerna. Convergence of approximate solutions to conservation laws. *Arch. Rational Mech. Anal.*, 82:27–70, 1983.

[9] R.J. DiPerna. Convergence of the viscosity method for isentropic gas dynamics. *Comm. Math. Phys.*, 91:1–30, 1983.

[10] R.J. DiPerna. Compensated compactness and general systems of conservation laws. *Trans. Amer. Math. Soc.*, 292:383–420, 1985.

[11] R.J. DiPerna. Measure–valued solutions to conservation laws. *Arch. Rational Mech. Anal.*, 88:223–270, 1985.

[12] S.D. Eidel'man. *Parabolic Systems.* North-Holland Publishing Company, 1969.

[13] H.-0. Kreiss and J. Lorenz. *Initial–boundary Value Problems and the Navier-Stokes Equations.* Academic Press, 1989.

[14] T.G. Kurtz. Convergence of sequences of semigroups of nonlinear operators with an application to gas kinetics. *Trans. Amer. Math. Soc.*, 186:259–272, 1973.

[15] P.D. Lax. Shock waves and entropy. In E.A. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, pages 603–634. Academic Press, 1971.

[16] J.L. Lions. *Perturbations singulieres dans les problèmes aux limites et en controle optimale*, volume 323. Springer–Verlag, Berlin, 1973.

[17] P.L. Lions, B. Perthame, and P.E. Souganidis. Existence and stability of entropy solutions for the hyperbolic systems of isentropic gas dynamics in Eurelian and Lagrangian coordinates. *Comm. Pure and Appl. Math.*, 49:599–638, 1996.

[18] P.L. Lions, B. Perthame, and E. Tadmor. Kinetic formulation of the isentropic gas dynamics and p-systems. *Comm. Math. Physics*, 163:415–431, 1994.

[19] P.L. Lions and G. Toscani. Diffusive limit for finite velocity Boltzmann kinetic models. *Rev. Mat. Iberoamericana*, 1996. To appear.

[20] P. Marcati and A. Milani. The one-dimensional Darcy's law as the limit of a compressible Euler flow. *J. Differential Equations*, 13:129–147, 1990.

[21] P. Marcati, A. Milani, and P. Secchi. Singular convergence of weak solutions for a quasilinear nonhomogeneous hyperbolic system. *Manuscripta Math.*, 60:49–69, 1988.

[22] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors and relaxation to the drift-diffusion equation. *Arch. Rational Mech. Anal.*, 129:129–145, 1995.

[23] P. Marcati and R. Natalini. Weak solutions to a hydrodynamic model for semiconductors: the Cauchy problem. *Proc. Roy. Soc. Edinburgh*, 125A:115–131, 1995.

[24] H.P. McKean. The central limit theorem for Carleman's equation. *Israel J. Math.*, 21:54–92, 1975.

[25] A.J. Milani. Long time existence and singular perturbations results for quasilinear hyperbolic equations with small parameter and dissipation term I. *Nonlinear Anal.*, 10:1237–1248, 1986.

[26] F. Murat. Compacité par compensation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 5:489–507, 1978.

[27] R. Natalini. Convergence to equilibrium for the relaxation approximations of conservation laws. *Comm. Pure and Appl. Math.*, 49:795–823, 1996.

[28] B. Rubino. Porous media flow as the limit of a nonstrictly hyperbolic system of conservation laws. *Comm. Partial Differential Equations*, 21:1–21, 1996.

[29] B. Rubino. Weak solutions to quasilinear wave equations of Klein-Gordon or sine-Gordon type and relaxation to reaction-diffusion equations. *Nonlinear Differential Equations and Appl.*, 4, 1997. To appear.

[30] L. Tartar. *Compensated compactness and applications to partial differential equations*, volume 39, pages 136–210. Research Notes in Math., 1979.

[31] L. Tartar. The compensated compactness method applied to partial differential equations. In J.M. Ball, editor, *Systems of Nonlinear Partial Differential Equations*, Reidel, Dordrecht, 1983. NATO ASI.

[32] M.E. Taylor. *Partial Differential Equations I, II, III*, volume 115-117 of *Appl. Math. Sciences*. Springer-Verlag, 1996.