Convergence of Explicit Finite Volume Schemes for Friedrichs Systems

J.P. Vila [*], P. Villedieu [†]

Abstract

We study here the convergence of explicit Finite Volume schemes for general multidimensional Friedrichs systems. By generalizing a previous result of Kuznetsov for Finite Difference schemes applied to scalar conservation laws, we obtain under general assumptions error bounds in $h^{1/8}$ when the initial data lies in $H^s(R^d)$ for s large enough. Convergence follows for initial data in $L^2(R^d)$.

1 Introduction

We consider here a general linear Friedrichs system, with initial condition :

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} + \displaystyle\sum_{i=1,d} \frac{\partial (A^i(x)u)}{\partial x_i} + B(x)u = f(x,t), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}^+, \quad u \in \mathbb{R}^m \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^d, \quad u_0 \in \mathbb{R}^m \end{array} \right. \tag{1}$$

We suppose that A^i $(i=1,d)$ are $m \times m$ symmetric matrices, with coefficients in $\mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$ such that :

$$\begin{array}{ll} (i) & \exists \beta_i > 0\,, \forall x \in \mathbb{R}^d,\ \forall u \in \mathbb{R}^m\ |u^t.A^i.u| \leq \beta_i \|u\|^2 \\ (ii) & \exists K_i > 0\ , \forall x, y \in \mathbb{R}^d, \|A^i(x) - A^i(y)\| \leq K_i \|x-y\| \\ (iii) & \exists \beta_{m+2} > 0\,, \forall x \in \mathbb{R}^d,\ \forall u \in \mathbb{R}^m\ |u^t.\mathrm{div}A.u| \leq \beta_{m+2}\|u\|^2 \\ (iv) & \exists K_{m+1} > 0\ , \forall x, y \in \mathbb{R}^d, \|\mathrm{div}(A(x) - A(y))\| \leq K_{m+2}\|x-y\| \end{array} \tag{2}$$

where $\mathrm{div}A = \displaystyle\sum_{i=1}^{d} \frac{\partial A^i}{\partial x_i}$ and $\|x\|$ is the Euclidian norm of x in $\mathbb{R}^m$.
Taking $\beta = (\displaystyle\sum_{i=1,d} \beta_i^2)^{1/2}$ It follows that :

$$\forall n = (n_1, \ldots, n_d) \in S^{d-1},\ |{}^t u (\sum_{i=1,m} A^i n_i) u| \leq \beta \|u\|^2$$

where S^{d-1} is the unit sphere of $\mathbb{R}^d$.

Lastly, we suppose that B is a symmetric $m \times m$ matrice, with coefficients $\in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R})$, which satisfies similar requirements as in (2) :

$$\begin{array}{ll} (i) & \exists \beta_{m+1} > 0 \text{ such that } \forall x \in \mathbb{R}^d,\ \forall u \in \mathbb{R}^m\ |u^t.B.u| \leq \beta_{m+1}\|u\|^2 \\ (ii) & \exists K_{m+1} > 0\ \text{ such that } \forall x, y \in \mathbb{R}^d, \|B(x) - B(y)\| \leq K_{m+1}\|x-y\| \end{array}$$

[*]Institut National des Sciences Appliquées de Toulouse, Département Génie Mathématique, UMR CNRS 5640, Mathématiques pour l'Industrie et la Physique, 31077 TOULOUSE CEDEX

[†]ONERA-CERT, 2 Ave. Ed. Belin, 31055 Toulouse Cedex

We are interested in the numerical approximation of the weak solution of this problem. We recall (see for example [1]) that if $f \in \mathcal{C}^1([0,+\infty[, H^s(I\!R^d))$ $(s \in I\!R^+)$ and $u_0 \in H^s(I\!R^d)$, the system (1) admits an unique weak solution $u \in \mathcal{C}^0([0,+\infty[, H^s(I\!R^d)) \cap \mathcal{C}^1([0,+\infty[, H^{s-1}(I\!R^d))$ wich satisfies, for each $T > 0$ the following estimate :

$$||u||_{C^0(0,T,L^2)} \leq K(T)(||u^0||_{L^2} + ||f||_{L^2(0,T,L^2)}) \tag{3}$$

where $K(T)$ is some positive constant wich only depends on T. Moreover it is easy to verify that the weak solution u also satisfies the following weak energy estimates:

$$\begin{aligned}
&\forall z \in I\!R^m, \quad \forall (t_0,t_1) \in [0,+\infty[\times[0,+\infty[, \quad \forall \phi \in \mathcal{C}_c^\infty(I\!R^d \times [t_0,t_1]),\\
&\qquad \int_{I\!R^d} \frac{1}{2}||u(x,t_1)-z||^2 \ \phi(x,t_1)\ dx - \int_{I\!R^d} \frac{1}{2}||u(x,t_0)-z||^2\ \phi(x,t_0)\ dx\\
&\qquad -\int_{I\!R^d\times[t_0,t_1]} \frac{1}{2}||u(x,t)-z||^2\ \frac{\partial \phi}{\partial t}\ dxdt\\
&\qquad -\int_{I\!R^d\times[t_0,t_1]} \frac{1}{2}\sum_{i=1}^d (u(x,t)-z)^t.A^i(x).(u(x,t)-z)\ \frac{\partial \phi}{\partial x_i}\ dxdt\\
&\qquad -\int_{I\!R^d\times[t_0,t_1]} \frac{1}{2}(u(x,t)-z)^t.div(A)(x).(u(x,t)-z)\ \phi\ dxdt\\
&\qquad +\int_{I\!R^d\times[t_0,t_1]} (u(x,t)-z)^t.(B(x)+div(A)(x)).(u(x,t)\ \phi\ dxdt\\
&\qquad -\int_{I\!R^d\times[t_0,t_1]} (u(x,t)-z)^t.f(x,t)\ \phi\ dxdt \leq 0
\end{aligned} \tag{4}$$

This family of weak energy estimates, indexed by z, plays the same role for Friedrichs systems as the Kruskov entropy inequalities for scalar conservation laws [9]. This analogy will be a crucial argument in our convergence proof of Finite Volume methods. Note also that these estimates are sufficient to characterize the solution of (1).

We now consider Finite Volume explicit upwind schemes, defined as follows. Let T_h be a triangulation of $I\!R^d$ ($I\!R^d = \bigcup_{K\in T_h} K$).

We denote by m_K the measure in $I\!R^d$ of the cell K, by $e_i(K)(i = 1,2,\ldots,n_K)$ the faces of K, by $n_{e_i,K}$ the outward unit normal to the face $e_i(K)$, by m_e the measure in $I\!R^{d-1}$ of the face e. Let S_h denote the set of the faces of the triangulation.

The family of triangulation T_h is regular in the following sense: there exists two positive real a_- and a_+ such that :

$$\begin{array}{lllllll}
(i) & a_-\ h^d & \leq & m_K & \leq & a_+\ h^d & \forall\ K\ \in T_h\\
(ii) & a_-\ h^{d-1} & \leq & m_e & \leq & a_+\ h^{d-1} & \forall\ e\ \in S_h\\
(iii) & a_-\ h & \leq & diam_K & \leq & a_+\ h & \forall\ K\ \in T_h
\end{array}$$

These conditions are classical in Finite Element analysis. They are not stringent, they just avoid local degeneracy of the mesh. We also suppose that n_K is uniformly bounded by n_0.

Let $\{t^n, n \in N\}$ be an increasing sequence of time values, we consider a Finite Volume upwind method. It yields a piecewise constant approximation u_h :

$$\left\{\begin{array}{l}
u_h(x,t) = u_K^n \qquad \forall\ (x,t)\ \in\ K\times[t^n,t^{n+1}[\\
u_K^{n+1} = u_K^n - \dfrac{k^n}{m_K}\displaystyle\sum_{e\in\partial K} g(n_{e,K},u_K^n,u_{K_e}^n)m_e - k^n B_K.u_K^n + k^n f_K^n\\
u_h(x,0) = \dfrac{1}{m_K}\displaystyle\int_K u(x,0)dx \equiv u_K^0 \qquad \forall\ x\ \in\ K
\end{array}\right. \tag{5}$$

where $k^n = t^{n+1} - t^n$ is the time step.
The quantities B_K, A_e^i, f_K^n are defined as follows :

$$\begin{array}{llll} (i) & B_K = \dfrac{1}{m_K}\displaystyle\int_K B(x)dx & (ii) & A_e^i = \dfrac{1}{m_e}\displaystyle\int_e A^i(x(\sigma))d\sigma \\ (iii) & f_K^n = \dfrac{1}{m_K}\displaystyle\int_K f(x,t^n)dx & (iv) & A_{e,K} = \displaystyle\sum_{i=1,d} A_e^i n_{e,K}^i \end{array}$$

$g(n_{e,K}, u_K^n, u_{K_e}^n)$ (K_e is the neighbourgh of K along the face e) is the numerical flux of a 3 point upwind scheme for the 1D linear hyperbolic system of equations :

$$u_t(x,t) + \partial_x(A_{e,K}u(x,t)) = 0$$

such that :

$$\begin{array}{ll} (i) & g(n,u,v) = -g(-n,v,u) \\ (ii) & g(n,u,u) = A_{e,K}.u \end{array}$$

We choose the simplest upwind scheme. The incremental coefficients and the numerical viscosity of the scheme are defined as follow :

$$\begin{array}{llll} (i) & C_{e,K} = -(A_{e,K})^- & (ii) & D_{e,K} = (A_{e,K})^+ \\ (iii) & C_{e,K} = D_{e,K_e} & (iv) & Q_e = \dfrac{1}{2}(C_{e,K} + D_{e,K}) = \dfrac{1}{2}(C_{e,K_e} + D_{e,K_e}) \end{array}$$

(i) and (ii) make sense since A^i are symmetric matrices. By construction, we have

$$A_{e,K} = -C_{e,K} + D_{e,K}$$

Then $g(n_{e,K}, u, v)$ is defined classicaly as :

$$g(n_{e,K}, u, v) = -C_{e,K}.v + D_{e,K}.u \tag{6}$$

For technical reasons (see section 4), we shall also suppose that the matrixes A^i satisfy the following requirement: for any compact subset $K \cap I\!R^d$, there exists a constant $C(K)$ such that for any $e \in S_h \cap K$, any $x \in e$, any $v \in R^m$:

$$v^t.\left|\sum_{i=1}^{d}(A^i(x) - A_e^i)n_{e,K}^i\right|.v \le C(K)diam(e)v^t.Q_e.v \tag{7}$$

Remark 1.1 *This assumption is necessarily satisfied if A^i (i=1,...,d) are constant matrixes or if the smallest eigen-value of Q_e is greater than a constant $\lambda(K)$ which only depends on the compact set K.*

We study the convergence of u_h as $h \to 0$. We shall suppose that the sequence $\{k^n\}$ is such that there exists two positive functions of h, $k^+(h)$ and $k^-(h)$ satisfying :

$$0 < k^-(h) \le k^n \le k^+(h)$$

When the triangulation is a cartesian grid, the Finite Volume method reduces to the classical Finite-Difference upwind scheme. L^2 convergence can be easily analysed by using the equivalence theorem of Lax. For general triangulation and implicit time discretisation, variationnal methods can be used to study the convergence. This approach has been used in the Works of Lesaint [10] (discontinuous Galerkin methods), Johnson-Pitkaranka [7] (scalar case), and Croisille [5] (Friedrichs systems with constant coefficients). Here we take a different approach: we extend results of Vila [11], Cockburn, Coquel and Lefloch [4] for Finite Volume scheme applied to non linear scalar conservation laws in the spirit of the work of Kuznetsov [8] for Finite-Difference schemes and we prove the following result :

Theorem 1.1 *Let $u_h(x,t)$ be an approximate solution of (1), computed by the Finite Volume scheme (5). We suppose that the triangulation is regular and that the following CFL condition is satisfied (for some constant ϵ, $0 < \epsilon < 1$) :*

$$\sup_{K \in T_h, e \in \partial K} \frac{k^+(h)}{h_K} ||C_{e,K}|| \leq 1 - \epsilon$$

with $h_K = \dfrac{m_K}{m_{\partial K}}$. We also suppose that the following inverse CFL condition is satisfied (for some constant $\kappa > 0$) :

$$\inf_{K \in T_h} \frac{k^-(h)}{h_K} \geq \kappa$$

If the initial data lies in $H^s(\mathbb{R}^d)$ (with $s > 1 + d/2$) and has a compact support, if f lies in $\mathcal{C}^1([0, +\infty[, H^s(\mathbb{R}^d))$ and if there exists, for each $T \geq 0$, a positive constant $R(T)$ such that $f(x,t) = 0$ for $x > R(T)$ and $0 \leq t \leq T$, then there exists some positive constant $K(u^0, f, T)$ such that :

For any $t \leq T$

$$||u(.,t) - u_h(.,t)||_{L^2(\mathbb{R}^d)} \leq K(u^0, f, T)\, h^{1/8} \tag{8}$$

where $u(.,t)$ is the solution of (1). If the initial data only lies in $L^2(\mathbb{R}^d)$ and f in $\mathcal{C}^1([0, +\infty[, L^2(\mathbb{R}^d))$, then u_h converges uniformly (for bounded t) in $L^2(\mathbb{R}^d)$ towards u :

$$\lim_{h \to 0} \sup_{0 \leq t \leq T} ||u(.,t) - u_h(.,t)||_{L^2(\mathbb{R}^d)} = 0$$

Remark 1.2 *If we drop the assumptions of compact support on u_0 and f, the estimate (8) remain valid but $\mathbb{R}^d$ must be replaced by K where K is an arbitrary compact subset of $\mathbb{R}^d$.*

The paper is organized as follows. The second section introduces main features of the upwind scheme, in particular L^2 contraction properties. We also establish an estimate of the local energy production, involving the numerical viscosity, that extends the results in [2] to Friedrichs systems. This estimate is a crucial argument for the convergence proof since it provides a weak bound of the space derivatives of the approximate solution. In the third section, we use these local results to establish a family of global weak energy estimates which play the same role as (4) for the discrete solution. The last section is devoted to the proof of theorem 1.1.

2 Upwind Finite Volume schemes

First, we write our Finite Volume scheme into a form wich is well suited for getting a local energy estimate. Let us define $\text{div}A_K$ as :

$$\text{div}A_K = \frac{1}{m_K} \int_K \text{div}A(x)dx$$

Thus, we get

$$\begin{aligned} \int_K \text{div}A(x)dx &= \int_{\partial K} A(x(\sigma)).n(x(\sigma))d\sigma \\ &= \sum_{e \in \partial K, i=1,d} m_e A_e^i.n_e^i = \sum_{e \in \partial K} m_e A_{e,K} \end{aligned}$$

It follows that :

$$\sum_{e\in\partial K} m_e A_{e,K} = m_K \mathrm{div} A_K$$

Consequently we have

$$\begin{aligned}\sum_{e\in\partial K} g(n_{e,K}, u_K^n, u_{K_e}^n) m_e &= \sum_{e\in\partial K} \left(g(n_{e,K}, u_K^n, u_{K_e}^n) - A_{e,K}.u_K^n \right) m_e \\ + m_K \mathrm{div} A_K . u_K^n &= \sum_{e\in\partial K} C_{e,K}.(u_K^n - u_{K_e}^n) m_e + m_K \mathrm{div} A_K . u_K^n\end{aligned}$$

and so the scheme (5) can be written in the non conservative form :

$$\begin{aligned}u_K^{n+1} = u_K^n - \frac{k^n}{m_K} &\sum_{e\in\partial K} C_{e,K}.(u_K^n - u_{K_e}^n) m_e \\ &- k^n \{(B_K + \mathrm{div} A_K) u_K^n - f_K^n\}\end{aligned}$$

2.1 From one-dimensional flux to Finite Volume schemes

For any edge of the cell K, we define a value $u_K^{n+1,e}$ in the following way :

$$\begin{aligned}u_K^{n+1,e} = u_K^n - \lambda_K^n &\left(g(n_{e,K}, u_K^n, u_{K_e}^n) - A_{e,K} u_K^n \right) \\ &- k^n (B_K + \mathrm{div} A_K) . u_K^n + k^n f_K^n\end{aligned}$$

where

$$\lambda_K^n = \frac{k^n m_{\partial K}}{m_K}$$

We can also define $u_K^{n+1,e}$ as :

$$u_K^{n+1,e} = u_K^n - \lambda_K^n C_{e,K} (u_K^n - u_{K_e}^n) - k^n (B_K + \mathrm{div} A_K) . u_K^n + k^n f_K^n$$

A straightforward calculation show that :

$$u_K^{n+1} = \sum_{e\in\partial K} \frac{m_e}{m_{\partial K}} u_K^{n+1,e} \tag{9}$$

We also introduce for any $u, v \in I\!R^m$ and for any $z \in I\!R^m$:

$$\begin{array}{llll} (i) & \|u\|^2_{C_{e,K}} = u^t . C_{e,K} . u & (iii) & h(n_{e,K}, u, v) = \frac{1}{2} u^t . D_{e,K} . u - \frac{1}{2} v^t . C_{e,K} . v \\ (ii) & (u, v)_{C_{e,K}} = \frac{1}{2} u^t . C_{e,K} . v & (iv) & h_z(n_{e,K}, u, v) = h(n_{e,K}, u - z, v - z) \end{array}$$

The Energy flux h as the same conservativity properties as the numerical flux g :

$$\begin{array}{ll} (i) & h(n, u, v) = -h(-n, v, u) \\ (ii) & h(n, u, u) = \frac{1}{2} u^t . A_{e,K} . u \end{array} \tag{10}$$

2.2 Local Energy dissipation

Here we evaluate the local energy dissipation of the scheme on each edge $e \in S_h$. We will need the following lemma:

Lemma 2.1 *Let us define $W(\lambda, u, v)$ as :*

$$W(\lambda, u, v) = u - \lambda C(u - v) = u - \lambda(g(u, v) - A.u) \tag{11}$$

Provided the following C.F.L restriction (for some ϵ, $0 < \epsilon < 1$) :

$$\lambda ||C|| \leq 1 - \epsilon \tag{12}$$

we have :

$$\frac{1}{2}||W||^2 - \frac{1}{2}||u||^2 + \lambda(h(u, v) - h(u, u)) \leq -\frac{1}{2}\epsilon\lambda||u - v||_C^2$$

where h(u,v) is the energy flux associated with the energy $\frac{1}{2}||u||^2$

$$h(u, v) = \frac{1}{2}(u^t.D.u - v^t.C.v)$$

Proof of lemma 2.1 On one hand, we have:

$$\begin{aligned}
(W(\lambda, u, v) - u, u) &= -u^t.\lambda C.(u - v) \\
&= \frac{-\lambda}{2}[u^t.C.u - v^t.C.v + (u - v)^t.C.(u - v)] \\
&= \lambda[\tfrac{1}{2}u^t.A.u - h(u, v) - \tfrac{1}{2}(u - v)^t.C.(u - v)]
\end{aligned}$$

On the other hand, we have:

$$\begin{aligned}
(W(\lambda, u, v) - u, u) &= \frac{1}{2}(||W||^2 - ||u||^2 - ||W - u||^2) \\
&= \frac{1}{2}||W||^2 - \frac{1}{2}||u||^2 - \frac{1}{2}\lambda^2(u - v)^t.C^2.(u - v)
\end{aligned}$$

so we get:

$$\frac{1}{2}||W||^2 - \frac{1}{2}||u||^2 + \lambda(h(u, v) - \frac{1}{2}u^t.A.u) = -\frac{1}{2}\lambda\left((u - v)^t.(C - \lambda C^2).(u - v)\right)$$

Since C and C^2 are symmetric positive matrices with same eigenvectors, it follows that : For any $x \in I\!R^d$, any $\epsilon > 0$ such that

$$\lambda||C|| \leq 1 - \epsilon$$

we have:

$$x^t.(C - \lambda C^2).x \geq \epsilon x^t.C.x$$

Consequently,

$$\frac{1}{2}||W||^2 - \frac{1}{2}||u||^2 + \lambda(h(u, v) - h(u, u)) \leq -\frac{1}{2}\epsilon\lambda||u - v||_C^2$$

This conclude the proof. •

We are now equiped to prove a local energy estimate on each edge of the mesh T_h. First let us introduce the following notations:

$$\Delta_{e,K}^n(u) = u_{K_e}^n - u_K^n$$
$$h_z(u, v) = h(u - z, v - z)$$

Proposition 2.1 *For any edge e of T_h and any $z \in I\!R^m$, provided the CFL condition:*

$$\frac{k^n}{h_K}||C_{e,K}|| \ \leq \ 1-\epsilon$$

we have:

$$\begin{array}{l} \frac{1}{2}||u_K^{n+1,e}-z||^2 - \frac{1}{2}||u_K^n - z||^2 + \lambda_K^n \left(h_z(n_{e,K}, u_K^n, u_{K_e}^n) \right. \\ \quad -\frac{1}{2}(u_K^n - z)^t.A_{e,K}.(u_K^n - z)) \\ \quad +k^n \left((B_K + divA_K).u_K^n - f_K^n, u_K^{n+1,e} - z \right) \leq -\frac{1}{2}\epsilon\lambda_K^n ||\Delta_{e,K}^n(u)||_{C_{e,K}}^2 \end{array} \tag{13}$$

Proof of Proposition 2.1

We start from the local energy production estimate of the lemma 2.1. Thanks to the linearity of the scheme (11) and under the CFL condition (12), we have for any $z \in I\!R^m$:

$$\frac{1}{2}||W-z||^2 - \frac{1}{2}||u-z||^2 + \lambda(h_z(u,v) - h_z(u,u)) \leq -\frac{1}{2}\epsilon\lambda||u-v||_C^2 \tag{14}$$

So applying (14) with $W = u_K^{n+1,e} + k^n\{(B_K + \mathrm{div}A_K).u_K^n - f_K^n$ we get :

$$\begin{array}{l} \frac{1}{2}||u_K^{n+1,e} + k^n\{(B_K + \mathrm{div}A_K).u_K^n - f_K^n\}||^2 - \frac{1}{2}||u_K^n - z||^2 + \\ \lambda_K^n \left(h_z(n_{e,K}, u_K^n, u_{K_e}^n) - h_z(n_{e,K}, u_K^n, u_K^n) \right) \leq -\frac{1}{2}\epsilon\lambda_K^n ||\Delta_{e,K}^n(u)||_{C_{e,K}}^2 \end{array}$$

Thus we have:

$$\begin{array}{l} \frac{1}{2}||u_K^{n+1,e}-z||^2 - \frac{1}{2}||u_K^n - z||^2 + \\ \quad \lambda_K^n \left(h_z(n_{e,K}, u_K^n, u_{K_e}^n) - \frac{1}{2}(u_K^n - z)^t.A_{e,K}.u_K^n - z) \right) \\ \quad +k^n \left((B_K + \mathrm{div}A_K).u_K^n - f_K^n, u_K^{n+1,e} - z \right) \leq -\frac{1}{2}\epsilon\lambda_K^n ||\Delta_{e,K}^n(u)||_{C_{e,K}}^2 \end{array}$$

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2.3 Stability estimates

In view of analysing the convergence of the scheme, we shall establish two results. The first one concerns L^2 and "weak" H^1 stability, the second one concerns $Lip(0,T;L^2)$ behaviour of approximate solutions :

Proposition 2.2 *Let $T > 0$, $u_0 \in L^2(I\!R^d)$, $f \in \mathcal{C}([0,T], L^2(I\!R^d))$ and $u_h(x,t)$ be the function defined by the scheme (5). We suppose that the following CFL condition is satisfied :*

$$\sup_{K\in T_h, e\in\partial K} \frac{k^+(h)}{h_K}||C_{e,K}|| \ \leq \ 1-\epsilon \tag{15}$$

then there exists some constants $C_1(T)$ and $C_1'(T)$ such that for all t, $0 \geq t \geq T$, $u_h(.,t) \in L^2(I\!R^d)$ and :

$$||u_h(.,t)||_{L^2(I\!R^d)} \leq C_1(T)||u_0||_{L^2} + C_1'(T)||f||_{C^0(0,T,L^2)} \tag{16}$$

and there exists some constant $C_2(T, u^0, f, \epsilon)$ such that :

$$\sum_{e\in S_h, \{n; t^n \leq T\}} m_e k^n \Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u) \leq C_2(T, u^0, f, \epsilon) \tag{17}$$

The $Lip(0,T;L^2)$ behaviour of approximate solutions is given by :

Proposition 2.3 *Let $T>0$, u_0 a Lipschitz function with compact support, f a function $\in \mathcal{C}^1([0,T],L^2(\mathbb{R}^d))$, and $u_h(x,t)$ be the function defined by the scheme (5). We suppose that the CFL condition (15) is satisfied, then there exists some positive constant $C_3(u_0,f,T)$ such that :*

For any t and $\tau \geq 0$ such that $\tau + t \leq T$

$$\|u_h(.,t+\tau) - u_h(.,t)\|_{L^2(\mathbb{R}^d)} \leq C_3(u_0,f,T)\max(\tau, k^+(h)) \tag{18}$$

Proof of Proposition 2.2

Thanks to he local energy production estimate (13) and to the convexity of $u \longrightarrow \|u\|^2$ which leads to :

$$\|u_K^{n+1}\|^2 \leq \sum_{e\in\partial K} \frac{m_e}{m_{\partial K}}\|u_K^{n+1,e}\|^2$$

we get :

$$\begin{aligned}
&\frac{1}{2}\|u_K^{n+1}\|^2 - \frac{1}{2}\|u_K^n\|^2 + k^n\left((B_K + \mathrm{div}A_K).u_K^n - f_K^n, u_K^{n+1}\right) + \\
&\frac{k^n}{m_K}\sum_{e\in\partial K} m_e h(n_{e,K}, u_K^n, u_{K_e}^n) - \frac{k^n}{2m_K}(u_K^n)^t.\left(\sum_{e\in\partial K} m_e A_{e,K}\right).u_K^n \\
&\qquad \leq -\frac{\epsilon}{2}\sum_{e\in\partial K}\|\Delta_{e,K}^n(u)\|^2_{C_{e,K}}\lambda_K^n\frac{m_e}{m_{\partial K}},
\end{aligned}$$

and consequently :

$$\begin{aligned}
&\frac{1}{2}\|u_K^{n+1}\|^2 - \frac{1}{2}\|u_K^n\|^2 + k^n\left((B_K + \mathrm{div}A_K).u_K^n - f_K^n, u_K^{n+1}\right) + \\
&+\frac{k^n}{m_K}\sum_{e\in\partial K} m_e h(n_{e,K}, u_K^n, u_{K_e}^n) - \frac{k^n}{2}(u_K^n)^t.\mathrm{div}A_K.u_K^n \\
&\qquad \leq -\frac{\epsilon}{2}\frac{k^n}{m_K}\sum_{e\in\partial K} m_e\|\Delta_{e,K}^n(u)\|^2_{C_{e,K}}
\end{aligned}$$

Multiplying the previous inequality by m_K and summing over $K \in T_h$, we get :

$$\begin{aligned}
&\sum_{K\in T_h} m_K\|u_K^{n+1}\|^2 + \epsilon\sum_{K\in T_h} k^n\sum_{e\in\partial K}\|\Delta_{e,K}^n(u)\|^2_{C_{e,K}}m_e \leq \\
&\sum_{K\in T_h} m_K(1 + k^n\beta_{m+2})\|u_K^n\|^2 - 2k^n\sum_{K\in T_h} m_K\left((u_K^{n+1}), (B_K + \mathrm{div}A_K).u_K^n - f_K^n\right)
\end{aligned}$$

By using the Cauchy Schwartz inequality, we obtain :

$$\begin{aligned}
&\|u_h^{n+1}\|^2_{L^2} + \epsilon\sum_{K\in T_h} k^n\sum_{e\in\partial K}\|\Delta_{e,K}^n(u)\|^2_{C_{e,K}}m_e \\
&\leq (1 + k^n\beta_{m+2})\|u_h^n\|^2_{L^2} + 2k^n\|u_h^{n+1}\|_{L^2}\|(B_h + \mathrm{div}A_h).u_h^n - f_h^n\|_{L^2} \\
&\leq (1 + k^n\beta_{m+2})\|u_h^n\|^2_{L^2} + 2k^n\left((\beta_{m+1} + \beta_{m+2})\|u_h^n\|_{L^2} + \|f_h^n\|_{L^2}\right)\|u_h^{n+1}\|_{L^2} \\
&\leq (1 + k^n\beta_{m+2})\|u_h^n\|^2_{L^2} + k^n\left\{(\beta_{m+1} + \beta_{m+2})\|u_h^n\|^2_{L^2} + \|f_h^n\|^2_{L^2}\right. \\
&\qquad\qquad \left. +(1 + \beta_{m+1} + \beta_{m+2})\|u_h^{n+1}\|^2_{L^2}\right\}
\end{aligned}$$

which leads to :

$$\begin{aligned}\|u_h^{n+1}\|_{L^2}^2(1-k^n(\beta_{m+1}+\beta_{m+2}+1)) + \epsilon \sum_{K\in T_h} k^n \sum_{e\in\partial K} \|\Delta_{e,K}^n(u)\|_{C_{e,K}}^2 m_e \\ \leq \|u_h^n\|_{L^2}^2(1+k^n(2\beta_{m+2}+\beta_{m+1}) + k^n\|f_h^n\|_{L^2}^2\end{aligned} \tag{19}$$

We are now equiped to establish the desired results. For h small enough, we have thanks to the CFL condition:

$$k^n(1+\beta_{m+2}+\beta_{m+1}) \leq \frac{1}{2}$$

Thus:

$$\frac{1}{1-k^n(1+\beta_{m+2}+\beta_{m+1}} \leq 1+2k^n(1+\beta_{m+2}+\beta_{m+1}) \leq 2$$

And so (19) leads to:

$$\|u_h^{n+1}\|_{L^2}^2 \leq \|u_h^n\|_{L^2}^2(1+C_1k^n) + C_2k^n\|f_h^n\|_{L^2}^2 \tag{20}$$

with

$$C_1 = 2+4\beta_{m+1}+6\beta_{m+2} \qquad\qquad C_2 = 2$$

It follows that :

$$\|u_h^{n+1}\|_{L^2}^2 \quad \leq \quad \prod_{p=0,n}(1+C_1k^p)\left(\|u_h^0\|_{L^2}^2 + C_2\sum_{p=0,n} k^p\|f_h^p\|_{L^2}^2\right) \tag{21}$$

Finally, since $\displaystyle\prod_{p=0,n}(1+C_1k^p) \leq \exp(C_1T)$, (16) is valid with :

$$C_1(T) = \exp(\frac{1}{2}C_1T) \qquad\qquad C_1'(T) = \sqrt{C_2T}\exp(\frac{1}{2}C_1T)$$

To get the "weak H^1" estimate we start from (19). We sum these inequalities over n such that $t^n \leq T$ to get :

$$\begin{aligned}\|u_h^N\|_{L^2}^2 + \frac{\epsilon}{2}\sum_{K\in T_h, n\leq N} k^n \sum_{e\in\partial K}\|\Delta_{e,K}^n(u)\|_{C_{e,K}}^2 m_e \\ \leq \|u_h^0\|_{L^2}^2 + (2\beta_{m+2}+\beta_{m+1})\sum_{n=0,\dots,N-1} k^n\|u_h^n\|_{L^2}^2 \\ +(1+\beta_{m+2}+\beta_{m+1})\sum_{n=0,\dots,N-1} k^n\|u_h^{n+1}\|_{L^2}^2 + T\|f\|_{\mathcal{C}(0,T,L^2)}^2\end{aligned}$$

The second term of the left hand side of this inequality is equal to :

$$\frac{\epsilon}{2}\sum_{e\in S_h,\{n;t^n\leq T\}} m_e k^n \Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)$$

Taking account of the L^2 estimate (16) we get easily (17), with :

$$\begin{aligned}C_2(T,u^0,f,\epsilon) = \tfrac{2}{\epsilon}\left\{\|u^0\|_{L^2}^2 + T\|f\|_{\mathcal{C}(0,T,L^2)}\right. \\ \left. +\ (1+3\beta_{m+2}+2\beta_{m+1})T\left(C_1(T)\|u_0\|_{L^2} + C_1'(T)\|f\|_{C(0,T,L^2)}\right)^2\right\}\end{aligned}$$

•

Proof of Proposition 2.3

Let us define $\tilde{u}_K^{n+1}$, v_K^{n+1} and $\tilde{v}_K^{n+1}$ as

$$\tilde{u}_K^{n+1} = u_K^n - \frac{k^{n-1}}{m_K} \sum_{e \in \partial K} C_{e,K}.(u_K^n - u_{K_e}^n) m_e - k^{n-1} B_K u_K^n - k^{n-1} f_K^n$$

$$v_K^{n+1} = u_K^{n+1} - u_K^n \qquad\qquad \tilde{v}_K^{n+1} = \tilde{u}_K^{n+1} - u_K^n$$

We remark that :

$$\tilde{v}_K^{n+1} = \frac{k^{n-1}}{k^n} v_K^{n+1}$$

and that :

$$\begin{aligned}\tilde{v}_K^{n+1} = v_K^n - \frac{k^{n-1}}{m_K} &\sum_{e \in \partial K} C_{e,K}.(v_K^n - v_{K_e}^n) m_e \\ &-k^{n-1} B_K v_K^n - k^{n-1}(f_K^n - f_K^{n-1})\end{aligned}$$

Thus for $n \geq 1$, with similar calculations as in the proof of estimate (20), we establish :

$$\begin{aligned}\|\tilde{v}_h^{n+1}\|_{L^2}^2 \leq\ & (1 + k^{n-1}\beta_{m+2}) \|v_h^n\|_{L^2}^2 \\ & +2k^{n-1} \left((\beta_{m+1} + \beta_{m+2}) \|v_h^n\|_{L^2} + \|f_h^n - f_h^{n-1}\|_{L^2} \right) \|\tilde{v}_h^{n+1}\|_{L^2}\end{aligned}$$

It follows that :

$$\begin{aligned}\frac{1}{(k^n)^2} \|v_h^{n+1}\|_{L^2}^2 \leq\ & \frac{1 + k^{n-1}\beta_{m+2}}{(k^{n-1})^2} \|v_h^n\|_{L^2}^2 \\ & +2k^{n-1} \left(\frac{\beta_{m+1} + \beta_{m+2}}{k^{n-1}} \|v_h^n\|_{L^2} + \frac{\|f_h^n - f_h^{n-1}\|_{L^2}}{k^{n-1}} \right) \frac{\|v_h^{n+1}\|_{L^2}}{k^n}\end{aligned}$$

We set $w_h^n = \dfrac{v_h^{n+1}}{k^n}$. Using the same arguments as in the proof of estimate (16), we have for $0 \leq n \leq N-2$:

$$\|w_h^{n+1}\|_{L^2}^2 \quad \leq \quad \prod_{p=0,n} (1 + C_1 k^p) \left(\|w_h^0\|_{L^2}^2 + C_2 \sum_{p=0,n} k^p \frac{\|f_h^{n+1} - f_h^n\|_{L^2}^2}{(k^n)^2} \right)$$

Since we have

$$\frac{\|f_h^{n+1} - f_h^n\|_{L^2}}{k^n} \leq \|f\|_{\mathcal{C}^1(0,T,L^2)}$$

we obtain

$$\max_{n \leq N} (\|w_h^{n+1}\|_{L^2}^2) \leq \exp(C_1 T) \|w_h^0\|_{L^2}^2 + C_2 T \exp(C_1 T) \|f\|_{\mathcal{C}^1(0,T,L^2)}$$

It remains to control $\|w_h^0\|_{L^2}$. Taking account of :

$$w_K^0 = -\frac{1}{m_K} \sum_{e \in \partial K} C_{e,K}.(u_K^0 - u_{K_e}^0) m_e - (B_K + div\ A_K).u_K^0 + f_K^0$$

and that u^0 is Lipschitz with compact support, we get easily that :

$$\|w_h^0\|_{L^2} \leq C(u^0, f)$$

which leads to the desired estimate.•

We finally give a result which will be usefull to extend our convergence proof to general initial data in L^2.

Proposition 2.4 *Let $T > 0$, $u_0, v_0 \in L^2(\mathbb{R}^d)$, $f, g \in \mathcal{C}^1([0,T], L^2(\mathbb{R}^d))$ and $u_h(x,t)$ (resp. $v_h(x,t)$) be the function defined by the scheme (5) with the initial data u_0 and the R.H.S. f (resp. v_0,g) . We suppose that the following CFL condition is satisfied :*

$$\sup_{K \in T_h, e \in \partial K} \frac{k^+(h)}{h_K} \|C_{e,K}\| \leq 1$$

then for all $t \geq 0$, $u_h(.,t), v_h(.,t) \in L^2(\mathbb{R}^d)$ and for $t \leq T$:

$$\begin{aligned} \|u_h(.,t) - v_h(.,t)\|_{L^2(\mathbb{R}^d)} &\leq C_1(T)\|u_0 - v_0\|_{L^2(\mathbb{R}^d)} \\ &+ C_1'(T)\|f - g\|_{L^2([0,T],L^2)} + C"_1(T)\|f - g\|_{C^1([0,T],L^2)} k^+(h) \end{aligned} \tag{22}$$

Proof of Proposition 2.4 Let us introduce:

$$\tilde{f}_K^n = \frac{1}{k^n m_K} \int_{K \times [t^n, t^{n+1}]} f(x,t) dx dt$$

We start from (21). It leads to:

$$\|u_h^{n+1}\|_{L^2}^2 \leq \exp(C_1 T) \left(\|u_h^0\|_{L^2}^2 + 2C_2 \sum_{\substack{p=0,n \\ K \in T_h}} (\|\tilde{f}_K^p\|^2 + \|f_K^p - \tilde{f}_K^p\|^2) k^p m_K \right)$$

we have:

$$\|f_K^p - \tilde{f}_K^p\|^2 \leq \frac{1}{k^p m_K} \int_{K \times [t^n, t^{n+1}]} \|f(x,t) - f(x,t^p)\|^2 dx dt$$

so we get:

$$\begin{aligned} \sum_{\substack{p=0,n \\ K \in T_h}} \|f_K^p - \tilde{f}_K^p\|^2 k^p m_K &\leq \sum_{p=0,n} \int_{t^p}^{t^{p+1}} \|f(.,t) - f(.,t^p)\|_{L^2(\mathbb{R}^d)}^2 \\ &\leq \|f\|_{C^1(0,T,L^2)} T k^+(h) \end{aligned}$$

Moreover:

$$\sum_{\substack{p=0,n \\ K \in T_h}} \|\tilde{f}_K^p\|^2 k^p m_K \leq \|f\|_{L^2([0,T],L^2}^2$$

We then conclude easily by using the linearity of the scheme. •

3 Weak convergence

At this point, following a previous proof of convergence of finite volume scheme for linear scalar hyperbolic equations of Gallouet and al [3], we could easily get weak convergence in L^2 of u_h towards u the weak solution of our initial value problem (1). We omit the detail of the proof here.

In the scalar case this will be sufficient to get strong convergence by using a uniqueness result of measure valued solutions. Such a result is actually limited to the non linear scalar case (see Diperna [6]). We shall study its extension to linear Friedrichs systems in a forecoming paper [12].

4 Weak energy estimates for the approximate solution

First we come back to the inequalities (4) which are satisfied by the exact solution. Letting z be a $\mathcal{C}(0,T,L^2(\mathbb{R}^d))$ function of y, s, and integrating over $\mathbb{R}^d \times [t_0, t_1]$ the estimates (4), we get:

$\forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R})$,

$$
\begin{array}{l}
\forall \phi \in \mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R}) \\
\displaystyle\int_{\mathbb{R}^{2d}\times[t_0,t_1]} \frac{1}{2}\|u(x,t_1) - z(y,s)\|^2 \; \phi(x-y, t_1 - s) \; dxdyds \\
\displaystyle - \int_{\mathbb{R}^{2d}\times[t_0,t_1]} \frac{1}{2}\|u(x,t_0) - z(y,s)\|^2 \; \phi(x-y, t_0 - s) \; dxdyds \\
\displaystyle - \int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \frac{1}{2}\|u(x,t) - z(y,s)\|^2 \\
\qquad\qquad \displaystyle\frac{\partial \phi(x-y,t-s)}{\partial t} \; dxdydtds \\
\displaystyle - \int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \frac{1}{2}\sum_{i=1}^{d}(u(x,t) - z(y,s))^t . A^i(x).(u(x,t) - z(y,s)) \\
\qquad\qquad \displaystyle\frac{\partial \phi(x-y,t-s)}{\partial x_i} \; dxdydtds \\
\displaystyle - \int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \frac{1}{2}(u(x,t) - z(y,s))^t . div(A)(x).(u(x,t) - z(y,s)) \\
\qquad\qquad \phi(x-y,t-s) \; dxdydtds \\
\displaystyle + \int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} (u(x,t) - z(y,s))^t . (B(x) + div(A)(x)).(u(x,t)) \\
\qquad\qquad \phi(x-y,t-s) \; dxdydtds \\
\displaystyle - \int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} (u(x,t) - z(y,s))^t . f(x,t) \; \phi(x-y,t-s) \; dxdydtds \leq 0
\end{array}
\tag{23}
$$

The aim of this section is to establish a family of global weak estimates wich play the same role as (23) for the approximate solution. The only difference is that the R.H.S. of these inequalities is not 0 but a function of h and ϕ, which has to be precisely evaluted. We shall prove the following:

Proposition 4.1 *Let $T \in R^+$, t_0, t_1 with $0 \leq t_0 \leq t_1 \leq T$. Let $v(y,s)$ any function in $L^\infty([0,T] \times \mathbb{R}^d)) \cap \mathcal{C}^0([0,T], L^2)$, $\omega(x,t)$ any positive, symmetric (i.e. $\omega(x,t) = \omega(-x,t) = \omega(x,-t)$) function in $\mathcal{C}_0^\infty(\mathbb{R}^d \times \mathbb{R})$. Under the assumptions that u^0 is a lipschitz function with a compact support and that f lies in $\mathcal{C}^1([0,T], L^2)$ and there exists a positive constant $R(T)$ such that $f(x,t) = 0$ for $x > R(T)$ and $0 \leq t \leq T$, then, provided the following restrictions on the time step (for $0 < \epsilon < 1$ and $\kappa > 0$):*

$$\sup_{K\in T_h, e\in\partial K} \frac{k^+(h)}{h_K}\|C_{e,K}\| \;\leq\; 1-\epsilon \qquad \inf_{K\in T_h} \frac{k^-(h)}{h_K} \;\geq\; \kappa$$

the approximate solution $u_h(x,t)$ computed with the scheme (5), satisfies:

$$
\begin{aligned}
&\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,t_1)-v(y,s)\|^2\ \omega(x-y,t_1-s)\ dxdyds\\
&-\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,t_0)-v(y,s)\|^2\ \omega(x-y,t_0-s)\ dxdyds\\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\|u_h(x,t)-v(y,s)\|^2\ \frac{\partial\omega(x-y,t-s)}{\partial t}\ dxdydtds\\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\sum_{i=1}^{d}(u_h(x,t)-v(y,s))^t.A^i(x).(u_h(x,t)-v(y,s))\\
&\qquad\qquad \frac{\partial\omega(x-y,t-s)}{\partial x_i}\ dxdydtds\\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}(u_h(x,t)-v(y,s))^t.divA(x).(u_h(x,t)-v(y,s))\\
&\qquad\qquad \omega(x-y,t-s)\ dxdydtds\\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h(x,t)-v(y,s))^t.(B(x)+div(A)(x)).(u_h(x,t)\\
&\qquad\qquad \omega(x-y,t-s)\ dxdydtds\\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h(x,t)-v(y,s))^t.f(x,t)\ \omega(x-y,t-s)\ dxdydtds\\
&\le K(u_0,f,v,t_0,t_1)\|\omega\|_{1,1}\sqrt{h}
\end{aligned}
\tag{24}
$$

Remark 4.1 *For the sake of clarity, we shall use the following notations:*

$$\xi = x-y \quad \chi = t-s$$

$$H(v,A^i,u) \equiv \frac{1}{2}(u-v)^t.A^i(x).(u-v)$$

$$H(v,div(A),u) \equiv \frac{1}{2}(u-v)^t.divA(x).(u-v)$$

$$H(v,A.\nabla_x\omega,u) = \frac{1}{2}\sum_{i=1,\dots,n}(u-v)^t.A^i(x).(u-v)\frac{\partial}{\partial x^i}(\omega(\xi,\chi))$$

We will also use the following convention (when there is no ambiguity) :

$$
\begin{aligned}
v &= v(y,s)\\
u_h &= u_h(x,t)\\
f &= f(x,t)\\
\omega &= \omega(\xi,\chi)
\end{aligned}
$$

Proof of proposition 4.1

For the sake of simplicity, we suppose that the time discretisation is such that $t^0 = t_0, t^N = t_1, k^n = t^{n+1} - t^n$, $n \in \{0, 1, \ldots, N\}$ For (y,s) $\in \mathbb{R}^d \times \mathbb{R}^+$, let us define $\mathcal{H}, \mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$ as :

$$
\begin{aligned}
\mathcal{H}_0(v, u_h, \omega)(y, s) &= -\int_{\mathbb{R}^d} \frac{1}{2}\|u_h(x, t_0) - v\|^2 \omega(\xi, t_0 - s) dx \\
\mathcal{H}_1(v, u_h, \omega)(y, s) &= \int_{\mathbb{R}^d} \frac{1}{2}\|u_h(x, t_1) - v\|^2 \omega(\xi, t_1 - s) dx \\
\mathcal{H}(v, u_h, \omega)(y, s) &= -\int_{\mathbb{R}^d \times [t_0, t_1]} \left\{ \frac{1}{2}\|u_h - v\|^2 \omega_t + H(v, A.\nabla_x \omega, u_h) \right\} dx dt \\
\mathcal{S}_1(v, u_h, \omega)(y, s) &= \int_{\mathbb{R}^d \times [t_0, t_1]} (u_h - v)^t.[B(x) + divA(x)].u_h \ \omega dx dt \\
\mathcal{S}_2(v, u_h, \omega)(y, s) &= -\int_{\mathbb{R}^d \times [t_0, t_1]} (u_h - v)^t.f(x, t) \ \omega \ dx dt \\
\mathcal{S}_3(v, u_h, \omega)(y, s) &= -\int_{\mathbb{R}^d \times [t_0, t_1]} H(v, div(A), u_h) \ \omega \ dx dt
\end{aligned}
$$

Considering that $u_h(x, t)$ is constant over $K \times [t^n, t^{n+1}[$, we get that

$$
\mathcal{H}(v, u_h, \omega)(y, s) = -\sum_{\substack{n=0,\ldots,N-1 \\ K \in T_h}} \int_{K \times [t^n, t^{n+1}]} \frac{1}{2}\|u_K^n - v\|^2 \omega_t dx dt + \sum_{\substack{n=0,\ldots,N-1 \\ K \in T_h}} \mathcal{H}_K^n(y, s)
$$

with $\mathcal{H}_K^n$ defined as follows :

$$
\mathcal{H}_K^n(y, s) = -\int_{K \times [t^n, t^{n+1}]} H(v, A.\nabla_x \omega, u_K^n) dx dt
$$

Considering now some averaged values of ω :

$$
\begin{aligned}
\omega_{K,n}(y, s) &= \frac{1}{m_K} \int_K \omega(\xi, t^n - s) dx, \\
\tilde{\omega}_{K,n}(y, s) &= \frac{1}{k^n m_K} \int_{K \times [t^n, t^{n+1}]} \omega(\xi, t^n - s) dx dt, \\
\omega_{e,n}(y, s) &= \frac{1}{k^n m_e} \int_{e \times [t^n, t^{n+1}]} \omega(x(\sigma) - y, \chi) d\sigma dt
\end{aligned}
$$

we get :

$$
-\int_{K \times [t^n, t^{n+1}]} \frac{1}{2}\|u_K^n - v\|^2 \omega_t dx dt = \frac{1}{2}\|u_K^n - v\|^2 m_K (\omega_{K,n} - \omega_{K,n+1})
$$

and

$$
\begin{aligned}
&\mathcal{H}_K^n(y, s) \\
&= R_K^{n,3}(y, s) - \int_{K \times [t^n, t^{n+1}]} H(v, \sum_{i=1,d} \frac{\partial(A^i(x)\, \omega(\xi, \chi))}{\partial x_i}, u_K^n) dx dt \\
&= R_K^{n,3}(y, s) - \int_{\partial K \times [t^n, t^{n+1}]} H(v, \sum_{i=1,d} (A^i.n^i)(x(\sigma)), u_K^n)\ \omega(x(\sigma) - y, \chi) d\sigma dt \\
&= R_K^{n,3}(y, s) - \sum_{e \in \partial K} \int_{e \times [t^n, t^{n+1}]} H(v, \sum_{i=1,d} A^i(x).n^i_{e,K}, u_K^n)\ \omega(x(\sigma) - y, \chi) d\sigma dt \\
&= R_K^{n,3}(y, s) + R_K^{n,0}(y, s) - \sum_{e \in \partial K} H(v, A_{e,K}, u_K^n) m_e k^n \omega_{e,n}
\end{aligned}
$$

where

$$R_K^{n,0}(y,s) = \sum_{e\in\partial K} \int_{e\times[t^n,t^{n+1}]} H(v, \sum_{i=1,d} (A^i(x) - A_e^i).n_{e,K}^i, u_K^n\ ) \\ \omega(x(\sigma) - y, \chi))d\sigma dt$$

and

$$R_K^{n,3}(y,s) = \int_{K\times[t^n,t^{n+1}]} H(v, div(A), u_K^n)\omega(\xi,\chi)dxdt$$

A discrete integration by parts leads to:

$$\begin{aligned}
\sum_{\substack{n=0,\ldots,N-1\\K\in T_h}} \frac{1}{2}\|u_K^n - v\|^2 m_K(\omega_{K,n} - \omega_{K,n+1}) = \\
\sum_{K\in T_h} m_K \frac{1}{2}\|u_K^0 - v\|^2\omega_{K,0} - \sum_{K\in T_h} m_K\frac{1}{2}\|u_K^N - v\|^2\omega_{K,N} \\
+ \sum_{\substack{n=1,\ldots,N\\K\in T_h}} m_K\omega_{K,n}(\frac{1}{2}\|u_K^n - v\|^2 - \frac{1}{2}\|u_K^{n-1} - v\|^2)
\end{aligned}$$

Noticing that :

$$\begin{aligned}
\sum_{K\in T_h} m_K\frac{1}{2}\|u_K^0 - v\|^2\omega_{K,0} &= \int_{\mathbb{R}^d} \frac{1}{2}\|u_h(x,t_0) - v\|^2\omega(\xi, t_0 - s)dx \\
&= -\mathcal{H}_0(v, u_h, \omega)(y,s), \\
\sum_{K\in T_h} m_K\|u_K^N - v\|^2\omega_{K,N} &= \int_{\mathbb{R}^d} \|u_h(x,t_1) - v\|^2\omega(\xi, t_1 - s)dx \\
&= \mathcal{H}_1(v, u_h, \omega)(y,s), \\
\sum_{\substack{n=0,\ldots,N-1\\K\in T_h}} R_K^{n,3}(y,s) &= \int_{\mathbb{R}^d\times[t_0,t_1]} \frac{1}{2}(u_h - v)^t.div(A)(x)(u_h - v)\omega(\xi,\chi)dxdt \\
&= -\mathcal{S}_3(v, u_h, \omega)(y,s),
\end{aligned}$$

and taking into account the conservativity property (10)(i) we get :

$$\begin{aligned}
&\mathcal{H}(v, u_h, \omega) + \mathcal{H}_0(v, u_h, \omega) + \mathcal{H}_1(v, u_h, \omega) + \mathcal{S}_3(v, u_h, \omega) \\
&= \sum_{\substack{n=0,\ldots,N-1\\K\in T_h}} \Big\{ m_K\omega_{K,n+1}(\frac{1}{2}\|u_K^{n+1} - v\|^2 - \frac{1}{2}\|u_K^n - v\|^2) + \\
&\sum_{e\in\partial K} m_e k^n \omega_{e,n}[h_v(n_{e,K}, u_K^n, u_{K_e}^n) - H(v, A_{e,K}, u_K^n)]\Big\} + \mathcal{R}_0(v, u_h, \omega)
\end{aligned}$$

with:

$$\mathcal{R}_0(v, u_h, \omega)(y,s) = \sum_{\substack{n=0,\ldots,N-1\\K\in T_h}} R_K^{n,0}(y,s)$$

With similar computations we prove easily that :

$$
\begin{aligned}
\mathcal{S}_1(v,u_h,\omega)(y,s) &= \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} m_K k^n \omega_{K,n+1}(u_K^{n+1}-v)^t.(B_K+divA_K).u_K^n \\
&\quad +\mathcal{R}_1(v,u_h,\omega)(y,s)\\
\mathcal{S}_2(v,u_h,\omega)(y,s) &= -\sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} m_K k^n \omega_{K,n+1}(u_K^{n+1}-v)^t.f_K^n \\
&\quad +\mathcal{R}_2(v,u_h,\omega)(y,s)
\end{aligned}
$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined as follows :

$$
\mathcal{R}_1(v,u_h,\omega) = \sum_{i=1,3} \mathcal{R}_{1,i}(v,u_h,\omega)
$$

$$
\begin{aligned}
\mathcal{R}_{1,1}(v,u_h,\omega) = \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} \int_{K\times[t^n,t^{n+1}]} (u_K^n-v)^t. \\
((B-B_K)(x)-(\mathrm{div}A(x)-\mathrm{div}A_K.u_K^n)\ \omega\ dxdt
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{1,2}(v,u_h,\omega) &= -\sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} m_K k^n \tilde{\omega}_{K,n}(u_K^{n+1}-u_K^n)^t.(B_K+\mathrm{div}A_K).u_K^n\\
\mathcal{R}_{1,3}(v,u_h,\omega) &= -\sum_{\substack{n=0,\ldots,N-1\\ K\in T_h;K\subset\Omega}} m_K k^n(\tilde{\omega}_{K,n}-\omega_{K,n+1}).\\
&\qquad (u_K^{n+1}-v)^t.(B_K+\mathrm{div}A_K).u_K^n\\
\mathcal{R}_2(v,u_h,\omega) &= \sum_{i=1,3} \mathcal{R}_{2,i}(v,u_h,\omega)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{R}_{2,1}(v,u_h,\omega) &= \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} \int_{K\times[t^n,t^{n+1}]} \omega\ (u_K^n-v)^t.(f_K^n-f(x,t))dxdt\\
\mathcal{R}_{2,2}(v,u_h,\omega) &= \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} m_K k^n \tilde{\omega}_{K,n}(u_K^{n+1}-u_K^n)^t.f_K^n\\
\mathcal{R}_{2,3}(v,u_h,\omega) &= \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h;K\subset\Omega}} m_K k^n(\omega_{K,n+1}-\tilde{\omega}_{K,n})(u_K^{n+1}-v)^t.f_K^n
\end{aligned}
$$

Let us now define $\mathcal{G}$

$$
\begin{aligned}
\mathcal{G}(v,u_h,\omega) = \ & \mathcal{H}(v,u_h,\omega)+\mathcal{H}_0(v,u_h,\omega)+\mathcal{H}_1(v,u_h,\omega)\\
&+\mathcal{S}_1(v,u_h,\omega)+\mathcal{S}_2(v,u_h,\omega)+\mathcal{S}_3(v,u_h,\omega)
\end{aligned}
$$

As a consequency of (9) and of the convexity of $u \longrightarrow \|u\|^2$ we have :

$$
\begin{aligned}
\mathcal{G}(v,u_h,\omega) \ \leq & \sum_{\substack{n=0,\ldots,N-1\\ K\in T_h}} m_K \sum_{e\in\partial K} \frac{m_e}{m_{\partial K}} \omega_{K,n+1} \Big\{ \frac{1}{2}(\|u_K^{n+1,e}-v\|^2-\|u_K^n-v\|^2)\\
& \lambda_K^n[h_v(n_{e,K},u_K^n,u_{K_e}^n)-H(v,A_{e,K},u_K^n)]\\
& +\ k^n(u_K^{n+1,e}-v)^t.[(B_K+divA_K).u_K^n-f_K^n]\Big\}\\
& +\mathcal{R}_0(v,u_h,\omega)+\mathcal{R}_0'(v,u_h,\omega)+\mathcal{R}_1(v,u_h,\omega)+\mathcal{R}_2(v,u_h,\omega)
\end{aligned}
$$

where:

$$\mathcal{R}'_0(v,u_h,\omega) = \sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; e\in\partial K}} m_e k^n(\omega_{e,n}-\omega_{K,n+1})[h_v(n_{e,K},u_K^n,u_{K_e}^n) - H(v,A_{e,K},u_K^n)]$$

Taking in account the local energy production estimate (13) of proposition 2.1 we get :

$$\mathcal{G}(v,u_h,\omega) \le \mathcal{R}_0(v,u_h,\omega) + \mathcal{R}'_0(v,u_h,\omega) + \mathcal{R}_1(v,u_h,\omega) + \mathcal{R}_2(v,u_h,\omega)$$

Let us now defines $\mathcal{L}(v,u_h,\omega)$ as :

$$\mathcal{L}(v,u_h,\omega) = \int_{I\!R^d\times[t_0,t_1]} \mathcal{G}(v,u_h,\omega)(y,s)dyds$$

By construction $\mathcal{L}(v,u_h,\omega)$ is the L.H.S. of the estimate (24). Integrating the last inequality over $I\!R^d\times[t_0,t_1]$ we finally obtain :

$$\begin{aligned}\mathcal{L}(v,u_h,\omega) \le \int_{I\!R^d\times[t_0,t_1]}\{\mathcal{R}_0(v,u_h,\omega) \quad & +\mathcal{R}'_0(v,u_h,\omega)+ \\ & +\mathcal{R}_1(v,u_h,\omega)+\mathcal{R}_2(v,u_h,\omega)\}dyds\end{aligned}$$

Now we give some technical lemmata which allow us to control the error terms $\mathcal{R}_i$ in the previous estimates.

Lemma 4.1 *The approximations of the symmetric test function ω satisfy the following estimates :*

$$\int_{I\!R^d\times[t_0,t_1]} \tilde{\omega}_{K,n}(y,s)dyds = \|\omega\|_1 \tag{25}$$

$$\int_{I\!R^d\times[t_0,t_1]} \mid \omega_{K,n+1}(y,s) - \tilde{\omega}_{K,n}(y,s) \mid dyds \le K\ h\ \|\omega\|_{1,1} \tag{26}$$

$$\int_{I\!R^d\times[t_0,t_1]} \mid \omega_{K,n+1}(y,s) - \omega_{e,n}(y,s) \mid dyds \le K\ h\ \|\omega\|_{1,1} \tag{27}$$

where K is some positive constant which only depends on a^+, a^-, d.

proof of Lemma 4.1 Standard calculations give the desired estimates, we omit the detailled proof.•

Lemma 4.2 *The error terms $\mathcal{R}_i$ satisfy the following bounds :*

$$\begin{array}{ll}
(i) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{1,1}(v,u_h,\omega)(y,s)\mid dyds \le hC_{1,1}(u_0,v,f,t_0,t_1)\|\omega\|_1 \\
(ii) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{1,2}(v,u_h,\omega)(y,s)\mid dyds \le hC_{1,2}(u_0,v,f,t_0,t_1)\|\omega\|_1 \\
(iii) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{1,3}(v,u_h,\omega)(y,s)\mid dyds \le hC_{1,3}(u_0,v,f,t_0,t_1)\|\omega\|_{1,1} \\
(iv) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{2,1}(v,u_h,\omega)(y,s)\mid dyds \le hC_{2,1}(u_0,v,f,t_0,t_1)\|\omega\|_1 \\
(v) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{2,2}(v,u_h,\omega)(y,s)\mid dyds \le hC_{2,2}(u_0,v,f,t_0,t_1)\|\omega\|_1 \\
(vi) & \int_{I\!R^d\times[t_0,t_1]} \mid \mathcal{R}_{2,3}(v,u_h,\omega)(y,s)\mid dyds \le hC_{2,3}(u_0,v,f,t_0,t_1)\|\omega\|_{1,1}
\end{array} \tag{28}$$

Proof of lemma 4.2 u^0 and f are supposed to have a compact support, thus thanks to the inverse CFL condition, we get easily that $u_h(x,t)=0$ for $t\in[t_0,t_1]$ and $x\notin\Omega$, where Ω is a bounded subset of R^d which only depends on u^0 and f.

$\mathcal{R}_{1,1}$ term We start by proving the estimate (28). For any $x\in K$ we have:

$$||B(x)-B_K||\leq K_{m+1}a_+h$$

and

$$||divA(x)-divA_K||\leq K_{m+2}a_+h$$

Thus, we obtain :

$$\begin{aligned}|\mathcal{R}_{1,1}(v,u_h,\omega)(y,s)| &\leq a^+h(K_{m+1}+K_{m+2})\times \\ &\sum_{\substack{n=0,\dots,N-1\\ K\in T_h}}\int_{K\times[t^n,t^{n+1}]}||u_K^n-v||||u_K^n||\omega\; dxdt \\ &\leq a^+h(K_{m+1}+K_{m+2})\left(\sum_{\substack{n=0,\dots,N-1\\ K\in T_h\cap\Omega}}\int_{K\times[t^n,t^{n+1}]}2(||u_K^n||^2+||v||^2)\omega\; dxdt\right)^{1/2} \\ &\times\left(\sum_{\substack{n=0,\dots,N-1\\ K\in T_h\cap\Omega}}\int_{K\times[t^n,t^{n+1}]}||u_K^n||^2\omega\; dxdt\right)^{1/2}\end{aligned}$$

It follows that

$$\begin{aligned}&\int_{\mathbb{R}^d\times[t_0,t_1]}|\mathcal{R}_{1,1}(v,u_h,\omega)(y,s)|dyds \\ &\leq a^+h(K_{m+1}+K_{m+2})\left(2||\omega||_1\sum_{\substack{n=0,\dots,N-1\\ K\in T_h\cap\Omega}}k^n m_K(||u_K^n||^2+m||v||_\infty^2)\right)^{1/2} \\ &\times\left(||\omega||_1\sum_{\substack{n=0,\dots,N-1\\ K\in T_h\cap\Omega}}k^n m_K||u_K^n||^2\right)^{1/2}\end{aligned}$$

Now, taking in account the L^2 stability of the scheme (estimate (16) of Proposition 2.2 we get :

$$\begin{aligned}\int_{\mathbb{R}^d\times[t_0,t_1]}|\;\mathcal{R}_{1,1}(v,u_h,\omega)(y,s)\;|\;dyds &\leq\sqrt{2}a^+h(K_{m+1}+K_{m+2})C(u^0,f,t_0,t_1)\times \\ &[C(u^0,f,t_0,t_1)^2+m||v||_\infty^2(t_1-t_0)]^{1/2}m(\Omega)||\omega||_1\end{aligned}$$

$\mathcal{R}_{1,2}$ term By using (25) of Lemma 4.1, we obtain :

$$\begin{aligned}&\int_{\mathbb{R}^d\times[t_0,t_1]}|\;\mathcal{R}_{1,2}(v,u_h,\omega)(y,s)\;|\;dyds \\ &\leq||\omega||_1\sum_{\substack{n=0,\dots,N-1\\ K\in T_h;K\subset\Omega}}m_Kk^n|(u_K^{n+1}-u_K^n)^t.(B_K+divA_K).u_K^n|\end{aligned}$$

By using the Cauchy Schwartz inequality, the r.h.s. of the previous inequality is bounded by :

$$\|\omega\|_1(\beta_{m+1}+\beta_{m+2})\left(\sum_{n=0,\dots,N-1} k^n\|u_h^{n+1}-u_h^n\|_{L^2}^2\right)^{1/2}$$
$$\times\beta_{m+1}\left(\sum_{n=0,\dots,N-1} k^n\|u_h^n\|_{L^2}^2\right)^{1/2}$$

Now, taking in account the $Lip(0,T,L^2)$ behaviour of the approximate solution (estimate (18) of proposition 2.3) and the L^2 bounds (16) of Proposition 2.2, we get easily the desired inequality.

$\mathcal{R}_{1,3}$ term By using (26) of Lemma 4.1, we obtain as in the previous paragraphs that :

$$\int_{\mathbb{R}^d\times[t_0,t_1]} \mid \mathcal{R}_{1,3}(v,u_h,\omega)(y,s)\mid dyds \le hC_{1,3}(u^0,v,f,t_0,t_1)\|\omega\|_{1,1}$$

$\mathcal{R}_{2,i}$ terms (i=1,2,3): We omit the proof of these estimates since they use the same arguments as for the $\mathcal{R}_{1,i}$ (i=1,2,3).

Now it remains to control the term $\mathcal{R}_0$ and $\mathcal{R}_0'$. This is possible, thanks to the "weak H_1" estimate of the section 2.3. We have :

Lemma 4.3 *The errors terms $\mathcal{R}_0$ and $\mathcal{R}_0'$ satisfy the following bounds :*

$$\begin{array}{ll} (i) & \displaystyle\int_{\mathbb{R}^d\times[t_0,t_1]} \mathcal{R}_0(v,u_h,\omega)(y,s)dyds \le \|\omega\|_1 C_0(u^0,v,f,t_0,t_1)\sqrt{h} \\ (ii) & \displaystyle\int_{\mathbb{R}^d\times[t_0,t_1]} \mathcal{R}_0'(v,u_h,\omega)(y,s)dyds \le \|\omega\|_{1,1} C_0'(u^0,v,f,t_0,t_1)\sqrt{h} \end{array} \tag{29}$$

Proof of lemma 4.3

$\mathcal{R}_0'$ term

$$\begin{aligned}\mathcal{R}_0' &= \sum_{\substack{n=0,\dots,N-1\\ K\in T_h;e\in\partial K}} m_e k^n(\omega_{e,n}-\omega_{K,n+1})[h_v(n_{e,K},u_K^n,u_{K_e}^n)-H(v,A_{e,K},u_K^n)] \\ &= \sum_{\substack{n=0,\dots,N-1\\ K\in T_h;e\in\partial K\, K\subset\Omega}} \frac{m_e k^n}{2}(\Delta\omega_{e,K}^n)\left\{\Delta_{e,K}^n(u)^t.C_{e,K}.(u_K^n+u_{K_e}^n)-2v^t.C_{e,K}.\Delta_{e,K}^n(u)\right\}\end{aligned}$$

where

$$\Delta\omega_{e,K}^n=\omega_{e,n}-\omega_{K,n+1}$$

By using a Cauchy Schwartz discrete inequality (for the measure $m_e k^n \mid \Delta\omega^n_{e,K} \mid$ we therefore get :

$$2 \mid \mathcal{R}'_0 \mid \leq \left(\sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} m_e k^n \mid \Delta\omega^n_{e,K} \mid \Delta^n_{e,K}(u)^t.C_{e,K}.\Delta^n_{e,K}(u) \right)^{1/2} \times$$
$$\left(\sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} m_e k^n \mid \Delta\omega^n_{e,K} \mid (u^n_K + u^n_{K_e})^t.C_{e,K}.(u^n_K + u^n_{K_e}) \right)^{1/2}$$
$$+2 \left(\sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} m_e k^n \mid \Delta\omega^n_{e,K}) \mid v^t.C_{e,K}.v \right)^{1/2} \times$$
$$\left(\sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} m_e k^n \mid \Delta\omega^n_{e,K} \mid \Delta^n_{e,K}(u)^t.C_{e,K}.\Delta^n_{e,K}(u) \right)$$

Taking in account the estimates (17) and (27) (Lemma 4.1) and using a Cauchy Schwartz inequality (for the measure $dyds$) we get :

$$\begin{aligned}
&\int_{\mathbb{R}^d\times[t_0,t_1]} \mathcal{R}'_0(y,s)dyds \\
&\leq C(u^0,v,f,\epsilon,t_0,t_1)\|\omega\|_{1,1}h^{1/2} \left(\sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} 2\beta m_e k^n h(\|u^n_K\|^2 + \|u^n_{K_e})\|^2) \right)^{1/2} \\
&\qquad +C(u^0,v,f,\epsilon,t_0,t_1)\|\omega\|_{1,1}h^{1/2} \left(2m\beta\|v\|_\infty \sum_{\substack{n=0,\dots,N-1 \\ K\in T_h; K\subset\Omega; e\in\partial K}} m_e k^n h \right)^{1/2}
\end{aligned} \tag{30}$$

Thanks to the assumptions on the family of triangulations T_h, we have:

$$\forall K \in T_h,\ \forall e \in \partial K,\ \ m_e h \leq \frac{a^+}{a^-}, m_K \qquad m_e h \leq \frac{a^+}{a^-} m_{K_e}$$

So taking account of the L^2 estimate (16), the inequality (30 leads easily to the estimate (29)(ii).

$\mathcal{R}_0$ term We have :

$$\int_{\mathbb{R}^d\times[t_0,t_1]} \mathcal{R}_0(y,s)dyds \leq \mathcal{R}^u_0 + \mathcal{R}^v_0$$

where

$$\mathcal{R}^v_0 = \int_{\mathbb{R}^d\times[t_0,t_1]} \left(\sum_{\substack{n=0,\dots,N-1 \\ e\in S_h\cap\Omega}} \int_{e\times[t^n,t^{n+1}[} \Big| v^t. \right.$$
$$\left. (\sum_{i=1,d} (A^i(x) - A^i_e).n^i_e).\Delta^n_{e,K}(u) \Big| \omega d\sigma dt \right) dyds$$

and

$$\mathcal{R}_0^u = \frac{1}{2}\int_{I\!R^d\times[t_0,t_1]}\left(\sum_{\substack{n=0,\ldots,N-1\\ e\in S_h\cap\Omega}}\int_{e\times[t^n,t^{n+1}[}|\Delta_{e,K}^n(u)^t.\right.$$

$$\left.(\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i).(u_K^n+u_{K_e}^n)\right|\omega d\sigma dt\Bigg)\,dyds$$

We first deal with the term $\mathcal{R}_0^v$. Taking account of the following inequalities :

$$|v^t.(\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i).\Delta_{e,K}^n(u)|$$

$$\leq\left(v^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.v\right)^{1/2}\times$$

$$\left(\Delta_{e,K}^n(u)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.\Delta_{e,K}^n(u)\right)^{1/2}$$

$$\leq\left(2m\|v\|_\infty a^+h\sup_{i=1,\ldots,d}K_i\right)^{1/2}\times$$

$$\left(|\Delta_{e,K}^n(u)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.\Delta_{e,K}^n(u)\right)^{1/2}$$

we get :

$$\mathcal{R}_0^v\leq C(v)\sqrt{h}\|\omega\|_1\times$$

$$\sum_{\substack{n=0,\ldots,N-1\\ e\in S_h\cap\Omega}}\left(\int_{e\times[t^n,t^{n+1}[}\Delta_{e,K}^n(u)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.\Delta_{e,K}^n(u)d\sigma dt\right)^{1/2}$$

Now taking account of the hypothesis (7): for $x\in\Omega$

$$\Delta_{e,K}^n(u)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.\Delta_{e,K}^n(u)\leq C(\Omega)h\Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)$$

and of the "weak H_1" estimate of the section 2.3 which gives :

$$\sqrt{C(\Omega)}\sum_{\substack{n=0,\ldots,N-1\\ K\in T_h,e\in S_h\cap\Omega}}m_ek^nh^{1/2}\left(\Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)\right)^{1/2}$$

$$\leq\left(\sum_{\substack{n=0,\ldots,N-1\\ e\in S_h\cap\Omega}}m_ek^n\Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)\right)^{1/2}\left(\sum_{\substack{n=0,\ldots,N-1\\ K\in T_h,e\in S_h\cap\Omega}}m_ek^nh\right)^{1/2}$$

$$\leq\sqrt{C(\Omega)m(\omega)}C(u^0,f,\epsilon,t_0,t_1)C(a^-,a^+)$$

we finally obtain :

$$\mathcal{R}_0^v\leq C(v)C(u^0,f,\epsilon,t_0,t_1)C(a^-,a^+)\sqrt{C(\Omega)m(\Omega)}\|\omega\|_1h^{1/2}$$

wich is the desired estimate.

We deal similarly with the term $\mathcal{R}_0^u$. We have :

$$\mathcal{R}_0^u \le \frac{1}{2}\|\omega\|_1 \sum_{\substack{n=0,\dots,N-1\\ e\in S_h\cap\Omega}} \int_{e\times[t^n,t^{n+1}[} \left|\Delta_{e,K}^n(u)^t.(\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i).(u_K^n+u_{K_e}^n)\right| d\sigma dt$$

Taking account of:

$$\begin{aligned}
&|(u_K^n+u_{K_e}^n)^t.(\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i).\Delta_{e,K}^n(u)| \\
&\le \left((u_K^n+u_{K_e}^n)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.(u_K^n+u_{K_e}^n)\right)^{1/2} \\
&\qquad\qquad \left(\Delta_{e,K}^n(u)^t.\left|\sum_{i=1,d}(A^i(x)-A_e^i).n_e^i\right|.\Delta_{e,K}^n(u)\right)^{1/2} \\
&\le \left(C(\Omega)h(u_K^n+u_{K_e}^n)^t.Q_e.(u_K^n+u_{K_e}^n)\right)^{1/2}\left(C(\Omega)h\Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)\right)^{1/2}
\end{aligned}$$

we get :

$$\begin{aligned}
\mathcal{R}_0^u \le C(\Omega)\|\omega\|_1 h^{1/2} &\left(\sum_{\substack{n=0,\dots,N-1\\ e\in S_h\cap\Omega}} (u_K^n+u_{K_e}^n)^t.Q_e.(u_K^n+u_{K_e}^n)m_e h k^n\right) \\
&\qquad \left(\sum_{\substack{n=0,\dots,N-1\\ e\in S_h\cap\Omega}} \Delta_{e,K}^n(u)^t.Q_e.\Delta_{e,K}^n(u)m_e k^n\right)
\end{aligned}$$

So taking in account the L^2 and "weak H_1" bounds of the section 2.3, we easily conclude. •

This completes the proof of proposition 4.1 •

5 Convergence proof for approximate solutions

Before giving the detailed proof of theorem 1.1, we first briefly indicate the main idea of the proof. The detailed proof will be given in the second part of this section.

5.1 Sketch of the proof

Let $u(y,s)$ the exact weak solution of (1). When there is no ambiguity we take a similar convention as in the previous section :

$$u = u(y,s) \quad u_h = u_h(x,t) \quad f = f(x,t) \quad \omega = \omega(\xi,\chi)$$

First step:
Choosing $z(x,t) = u_h(x,t)$ and exchanging the role of (x,t) and (y,s), the weak estimates (23) lead to:

$$\begin{aligned}
&\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u(y,t_1)-u_h\|^2\ \omega(\xi,t_1-t)\ dydxdt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u(y,t_0)-u_h)\|^2\ \omega(x-y,t_0-t)\ dydxdt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\|u-u_h)\|^2\ \frac{\partial\omega(\xi,\chi)}{\partial s}\ dydxdsdt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\sum_{i=1}^{d}(u-u_h)^t.A^i(y).(u-u_h)\ \frac{\partial\omega(\xi,\chi)}{\partial y_i}\ dydxdsdt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}(u-u_h)^t.div(A)(y).(u-u_h)\ \omega(\xi,\chi)\ dydxdsdt \\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} (u-u_h)^t.(B(y)+div(A)(y)).u\ \omega(\xi,\chi)\ dydxdsdt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} (u-u_h)^t.f(y,s)\ \omega(\xi,\chi)\ dydxdsdt \le 0
\end{aligned} \tag{31}$$

On the other hand, choosing $v(y,s) = u(y,s)$ the estmates (24) lead to:

$$\begin{aligned}
&\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,t_1)-u\|^2\ \omega(\xi,t_1-s)\ dxdyds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,t_0)-u\|^2\ \omega(\xi,t_0-s)\ dxdyds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\|u_h-u\|^2\ \frac{\partial\omega(\xi,\chi)}{\partial t}\ dxdydtds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}\sum_{i=1}^{d}(u_h-u)^t.A^i(x).(u_h-u)\ \frac{\partial\omega(\xi,\chi)}{\partial x_i}\ dxdydtds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}(u_h-u)^t.divA(x).(u_h-u)\ \omega(\xi,\chi)\ dxdydtds \\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.(B(x)+div(A)(x)).u_h\ \omega(\xi,\chi)\ dxdydtds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.f(x,t)\ \omega(\xi,\chi)\ dxdydtds \\
&\le K_1(u^0,f,t_0,t_1)\|\omega\|_{1,1}\sqrt{h}
\end{aligned} \tag{32}$$

Summing up together (31) and (32) we finally obtain

$$
\begin{aligned}
&\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}||u_h(x,t_1)-u||^2\omega(\xi,t_1-s)dxdyds \\
&+\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}||u(y,t_1)-u_h||^2\omega(\xi,t_1-t)dxdydt \\
&-\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}||u_h(x,t_0)-u||^2\omega(\xi,t_0-s)dxdyds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}||u(y,t_0)-u_h||^2\omega(\xi,t_0-t)dxdydt \\
&\leq \int_{I\!R^{2d}\times[t_0,t_1]^2} \frac{1}{2}||u_h-u||^2(\omega_t+\omega_s) \\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} \{H(u_h,[A(x)-A(y)].\nabla_x\omega,u) \\
&\qquad +H(u_h,A(y).(\nabla_x\omega+\nabla_y\omega),u)\}\ dxdydtds \\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h,[divA(x)+divA(y)],u)\ \omega(\xi,\chi)\ dxdydtds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.(B(x)-B(y)+divA(x)-divA(y)). \\
&\qquad u_h\ \omega(\xi,\chi)\ dxdydtds \\
&-\int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h,2B(y)+2divA(y),u)\ dxdydtds \\
&+\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.(f(x,t)-f(y,s))\ \omega(\xi,\chi)\ dxdydtds \\
&\leq K(u^0,f,t_0,t_1)||\omega||_{1,1}\sqrt{h}
\end{aligned}
\tag{33}
$$

Second step

Inequality (33) is the key ingredient in the proof of our main theorem. It is quite similar to the inequality used by Kruzkov in its uniqueness proof, and to the one used by Kutznetsov in its error estimates for finite difference schemes.

We remark that :

$$(\omega_t+\omega_s)(x,t,y,s)=0$$

and that :

$$(grad_x(\omega)+grad_y(\omega))(x,t,y,s)=0$$

Formally, if we choose for $\omega=\theta_d\theta$, θ_d as the dirac function in $I\!R^d$ and θ as the dirac function in $I\!R$, the l.h.s. of inequality (33) becomes :

$$||(u-u_h)(.,t_1)||^2_{L^2(I\!R^2)}-||(u-u_h)(.,t_0)||^2_{L^2(I\!R^d)}$$

If we suppose that the r.h.s of (33) is small, we get a formal error estimate. Although this formal calculation is not correct, it gives the wright way to get exact error estimates. To this ends, the test function ω has to be chosen very carefully. Practically we take regularization (via mollifier) of the previous dirac functions .

5.2 Proof of the main theorem

Let ζ be a cut-off (or mollifier) function in $I\!R$. We assume that $\zeta \in \mathcal{C}_0^\infty(I\!R)$ satifies :

$$(i)\ \ \zeta(x) = \zeta(-x), \quad (ii)\ \ support\ \zeta \subset [-1,1], \quad (iii)\ \ \int_{I\!R} \zeta(x)dx = 1$$

we fix $\epsilon > 0$, and we take :

$$\begin{array}{llll} (i) & \theta_d^\epsilon(z) & = & \dfrac{1}{\epsilon^d}\prod_{i=1}^{d}\zeta(\dfrac{z^i}{\epsilon}) \quad \text{for } z=(z^1,\ldots,z^d)\in I\!R^d \\ (ii) & \theta^\epsilon(t) & = & \dfrac{1}{\epsilon}\zeta(\dfrac{t}{\epsilon}) \qquad\quad\ \text{for } t \in I\!R \end{array} \tag{34}$$

We remark that :

$$\|\omega\|_{1,1} \leq \frac{C}{\epsilon} \quad \|\omega\|_1 = 1$$

We suppose that u_0 has a compact support and belongs to $H^s(I\!R^d)$ with $s > 1 + d/2$ and that f belongs to $C^1([0,\infty[, L^2(I\!R^d))$. We also suppose that for each $T \in R^+$, there exits some positive constant $R(T)$ such that for $|x| \geq R$ and $0 \leq t \leq T$, $f(x,t) = 0$. Under these assumptions, we know that the weak solution $u(t,.)$ of (1) belongs to $C^0([0,\infty[, H^s(I\!R^d)) \cap C^1([0,\infty[, H^{s-1}(I\!R^d))$. Since the embedding of $H^s(I\!R^d)$ $(s > 1 + d/2)$ in $B^1(I\!R^d))\left\{u \in C^1(I\!R^d));\ \lim_{\|x\|\to+\infty}\|D^\alpha u\| = 0,\ \alpha \leq 1\right\}$ is continuous, $u_0 \in Lip(I\!R^d)$ and $u(.,t) \in L^\infty([0,T]\times I\!R^d)$, for each $T > 0$.

Now for $t_0, t_1 \in I\!R^+$ with $0 \leq t_0 \leq t_1 \leq T$, let us define $J(u, u_h, \tau, \omega)$ as:

$$\begin{aligned} J(u,u_h,\tau,\omega) &= \int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,t) - u(y,\tau)\|^2\omega(\xi,\tau-t)dxdydt \\ &+ \int_{I\!R^{2d}\times[t_0,t_1]} \frac{1}{2}\|u_h(x,\tau) - u(y,s)\|^2\omega(\xi,\tau-s)dxdyds \end{aligned}$$

$L_1(u,\omega)$ as :

$$L_1(u,\omega) = \int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h, [A(x)-A(y)].\nabla_x\omega, u)dxdydtds$$

$L_2(u,\omega)$ as :

$$L_2(u,\omega) = -\int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.[B(x)-B(y)+divA(x)-divA(y)].u_h\omega dxdydtds$$

$L_3(u,\omega)$ as :

$$L_3(u,\omega) = \int_{I\!R^{2d}\times[t_0,t_1]^2} (u_h-u)^t.(f(x,t)-f(y,s))\omega dxdydtds$$

$L_4(u,\omega)$ as :

$$L_4(u,\omega) = \int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h, [divA(x)-divA(y)], u)\ \omega(\xi,\chi)\ dxdydtds$$

$L_5(u,\omega)$ as :

$$L_5(u,\omega) = -2\int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h, B(y), u)\ \omega(\xi,\chi)\ dxdydtds$$

With these notations (33) can be written:

$$J(u,u_h,t_1,\omega) \le J(u,u_h,t_0,\omega) \ + \sum_{i=1,5} L_i(u,\omega) \le K(u_0,f,t_0,t_1)\|\omega\|_{1,1}\sqrt{h}$$

Now we are equipped to estimate $J(u,u_h,t_i,\omega)$ (i=0,1). With the particular choice (34) for ω we get :

$$\begin{aligned} J(u,u_h,t_i,\omega) &= \frac{1}{2}\int_{t_0}^{t_1}\theta_\epsilon(t-t_i)\{\int_{\mathbb{R}^{2d}}\|u_h(x,t)-u(y,t_i)\|^2\zeta_\epsilon(\xi)dxdy\}dt \\ &+ \frac{1}{2}\int_{t_0}^{t_1}\theta_\epsilon(t_i-s)\{\int_{\mathbb{R}^{2d}}\|u_h(x,t_i)-u(y,s)\|^2\zeta_\epsilon(\xi)dxdy\}ds \\ &= \frac{1}{2}\int_{t_0}^{t_1}\theta_\epsilon(t-t_i)\{\int_{\mathbb{R}^{2d}}\{\|u_h(x,t)-u(y,t_i)\|^2+ \\ &\qquad \|u_h(x,t_i)-u(y,t)\|^2\}\zeta_\epsilon(\xi)dxdy\}dt \end{aligned}$$

Now, by repeated use of the following inequalities valid for any $a,b,c\in\mathbb{R}^d$ and any $\eta\in\mathbb{R}$:

$$\begin{aligned} \|a+b+c\|^2 &\ge \|a\|^2(1-2\eta^2)-\frac{1}{\eta^2}\|b\|^2-\frac{1}{\eta^2}\|c\|^2 \\ \|a+b+c\|^2 &\le \|a\|^2(1+2\eta^2)+(2+\frac{1}{\eta^2})\|b\|^2+(2+\frac{1}{\eta^2})\|c\|^2 \end{aligned}$$

we get :

$$\begin{aligned} &\|u_h(x,t)-u(y,t_0)\|^2+\|u_h(x,t_0)-u(y,t)\|^2 \le \\ &\quad (2+4\eta^2)\|u(x,t_0)-u_h(x,t_0)\|^2+(4+\frac{2}{\eta^2})\|u(x,t_0)-u(y,t_0)\|^2 \\ &\quad +(2+\frac{1}{\eta^2})\|u_h(x,t)-u_h(x,t_0)\|^2+(2+\frac{1}{\eta^2})\|u(y,t_0)-u(y,t)\|^2 \end{aligned}$$

and

$$\begin{aligned} &\|u_h(x,t)-u(y,t_1)\|^2+\|u_h(x,t_1)-u(y,t)\|^2 \ge \\ &\quad (2-4\eta^2)\|u(x,t_1)-u_h(x,t_1)\|^2-\frac{2}{\eta^2}\|u(x,t_1)-u(y,t_1)\|^2 \\ &\quad -\frac{1}{\eta^2}\|u_h(x,t)-u_h(x,t_1)\|^2-\frac{1}{\eta^2}\|u(y,t_1)-u(y,t)\|^2 \end{aligned}$$

Thus, we obtain the following estimates for $J(u,u_h,t_i,\omega)$ (i=0,1) :

$$\begin{aligned} (i)\quad J(u,u_h,t_0,\omega) \le\ & (2+4\eta^2)J_1(u,u_h,t_0,\omega)+(4+\tfrac{2}{\eta^2})J_2(u,t_0,\omega) \\ & +(2+\frac{1}{\eta^2})J_3(u_h,t_0,\omega)+(2+\frac{1}{\eta^2})J_4(u,t_0,\omega) \\ (ii)\quad J(u,u_h,t_1,\omega) \ge\ & (2-4\eta^2)J_1(u,u_h,t_1,\omega)-\frac{2}{\eta^2}J_2(u,t_1,\omega) \\ & -\frac{1}{\eta^2}J_3(u_h,t_1,\omega)-\frac{1}{\eta^2}J_4(u,t_1,\omega) \end{aligned}$$

with :

$$J_1(u,u_h,t_i,\omega)=\frac{1}{2}\int_{t_0}^{t_1}\theta_\epsilon(t-t_i)\{\int_{\mathbb{R}^{2d}}\|u(x,t_i)-u_h(x,t_i)\|^2\zeta_\epsilon(\xi)dxdy\}dt$$

$$J_2(u,t_i,\omega) = \frac{1}{2}\int_{t_0}^{t_1} \theta_\epsilon(t-t_i)\{\int_{I\!R^{2d}} \|u(x,t_i)-u(y,t_i)\|^2 \zeta_\epsilon(\xi)dxdy\}dt$$

$$J_3(u_h,t_i,\omega) = \frac{1}{2}\int_{t_0}^{t_1} \theta_\epsilon(t-t_i)\{\int_{I\!R^{2d}} \|u_h(x,t)-u_h(x,t_i)\|^2 \zeta_\epsilon(\xi)dxdy\}dt$$

$$J_4(u,t_i,\omega) = \frac{1}{2}\int_{t_0}^{t_1} \theta_\epsilon(t-t_i)\{\int_{I\!R^{2d}} \|u(y,t_i)-u(y,t)\|^2 \zeta_\epsilon(\xi)dxdy\}dt$$

It remains to evaluate the corresponding integrals.

The first one :

$$\begin{aligned} &J_1(u,u_h,t_i,\omega) = \\ &= \tfrac{1}{2}\left(\int_{t_0}^{t_1}\theta_\epsilon(t-t_i)dt\right)\left[\int_{I\!R^d}\|u(x,t_i)-u_h(x,t_i)\|^2\left(\int_{I\!R^d}\zeta_\epsilon(\xi)d\xi\right)dx\right] \\ &= \frac{1}{4}\int_{I\!R^d}\|u(x,t_i)-u_h(x,t_i)\|^2dx = \frac{1}{4}\|u(.,t_i)-u_h(.,t_i)\|_{L^2(I\!R^d)} \end{aligned}$$

Thus, we have established :

$$J_1(u,u_h,t_i,\omega) = \frac{1}{4}\|u(.,t_i)-u_h(.,t_i)\|_{L^2(I\!R^d)}$$

The second one :
We will need the following lemma:

Lemma 5.1 *Let $u\in H^1(I\!R^d)$, the following estimate holds for each $z\in I\!R^d$:*

$$\int_{I\!R^d}|u(x+z)-u(x)|^2dx \le \|u\|^2_{H^1(I\!R^d)}\|z\|^2$$

Proof of lemma 5.1 Let $\tilde{u}(\xi)$ be the Fourrier transform of $u(x)$. We have :

$$\int_{I\!R^d}|u(x+z)-u(x)|^2dx = \int_{I\!R^d}|1-\exp(i\xi.z)|^2|\tilde{u}(\xi)|^2d\xi$$

Since :

$$|1-\exp(i\xi.z)|^2 = (1-cos(\xi.z))^2 + sin(\xi.z)^2 = 4sin(\frac{\xi.z}{2})^2 \le \|\xi\|^2\|z\|^2$$

finally we have:

$$\int_{I\!R^d}\|u(x+z)-u(x)\|^2dx \le \|z\|^2\int_{I\!R^d}\|\xi\|^2|\tilde{u}(\xi)|^2d\xi \le \|u\|^2_{H^1(I\!R^d)}\|z\|^2$$

•

$$\begin{aligned} J_2(u,t_i,\omega) &= (\int_{t_0}^{t_1}\theta_\epsilon(t-t_i)dt)\int_{I\!R^d\times[-\epsilon,\epsilon]^d}\|u(y+\xi,t_i)-u(y,t_i)\|^2\zeta_\epsilon(\xi)dyd\xi \\ &= \frac{1}{2}\int_{I\!R^d\times[-\epsilon,\epsilon]^d}\|u(y+\xi,t_i)-u(y,t_i)\|^2\zeta_\epsilon(\xi)dyd\xi \end{aligned}$$

Thanks to lemma 5.1, we have for $\|\xi\| \leq \sqrt{d}\epsilon$:

$$\int_{I\!R^d} \|u(y+\xi,t_i) - u(y,t_i)\|^2 dy \leq d\|u\|^2_{C^0(0,T,H^1(I\!R^d))}\epsilon^2$$

Thus we get:

$$J_2(u,t_i,\omega) \leq \frac{1}{2} d\epsilon^2 \|u\|^2_{C^0(0,T,H^1(I\!R^d))}$$

The third one :

$$\begin{aligned} J_3(u_h,t_i,\omega) &= \int_{t_0}^{t_1} \theta_\epsilon(t-t_i)\{\int_{I\!R^d} \|u_h(x,t) - u_h(x,t_i)\|^2 (\int_{I\!R^d} \zeta_\epsilon(\xi)d\xi)dx\}dt \\ &= \int_{t_0}^{t_1} \theta_\epsilon(t-t_i)\{\int_{I\!R^d} \|u_h(x,t) - u_h(x,t_i)\|^2 dx\}dt \end{aligned}$$

We recall that support $\theta_\epsilon \subset [-\epsilon,\epsilon]$ and that

$$\int_{I\!R} \theta_\epsilon(t)dt = 1$$

Thus,

$$\begin{aligned} J_3(u_h,t_i,\omega) &\leq (\int_{t_0}^{t_1} \theta_\epsilon(t-t_i)dt) \sup_{0\leq\tau\leq\epsilon}\{\int_{I\!R^d} \|u_h(x,t_i-\tau) - u_h(x,t_i)\|^2 dx\} \\ &\leq \frac{1}{2} \sup_{0\leq\tau\leq\epsilon}\{\int_{I\!R^d} \|u_h(x,t_i-\tau) - u_h(x,t_i)\|^2 dx\} \end{aligned}$$

Taking in account the $Lip(0,T,L^2(I\!R^d))$ behaviour of the approximate solution (see Proposition 2.3) we finally get :

$$J_3(u_h,t_i,\omega) \leq \frac{1}{2} C_3(u^0,f,T)[\max(\epsilon,k^+(h))]^2$$

The fourth one :

The exact solution belongs to $C^1(0,T,L^2(I\!R^d))$, thus we get similarly :

$$J_4(u,t_i,\omega) \leq \frac{1}{2}\|u\|_{C^1(0,T,L^2(I\!R^d))}\epsilon^2$$

We also need to evaluate the terms $L_i(u,\omega)$.

The term L_1: Taking in account that $support(\zeta_\epsilon) \subset [-\epsilon,\epsilon]^d$ we have:

$$\begin{aligned} |L_1(u,\omega)| &= \left|\int_{I\!R^{2d}\times[t_0,t_1]^2} H(u_h,[A(x)-A(y)].\nabla_x\omega,u)dxdydtds\right| \\ &\leq \int_{I\!R^{2d}\times[t_0,t_1]^2} (\sum_{i=1}^d K_i\epsilon\|u_h(x,t) - u(y,s)\|^2 |\omega_{x^i}(\xi,\chi)|)dxdydtds \\ &\leq \int_{I\!R^{2d}\times[t_0,t_1]^2} (\sum_{i=1}^d 3K_i\epsilon\|u_h(x,t) - u(x,t)\|^2 |\omega_{x^i}|)dxdydtds \\ &\quad + \int_{I\!R^{2d}\times[t_0,t_1]^2} (\sum_{i=1}^d 3K_i\epsilon\|u(x,t) - u(y,t)\|^2 |\omega_{x^i}|)dxdydtds \\ &\quad + \int_{I\!R^{2d}\times[t_0,t_1]^2} (\sum_{i=1}^d 3K_i\epsilon\|u(y,t) - u(y,s)\|^2 |\omega_{x^i}|)dxdydtds \end{aligned}$$

The first term of the R.H.S. of the last inequality is bounded by:

$$3 \sup_{i=1,d} K_i \ \epsilon \|\omega\|_{1,1} \int_{t_0}^{t_1} \|u(.,t) - u_h(.,t)\|^2_{L^2(\mathbb{R}^d)} dt$$

the second one by:

$$3C \sup_{i=1,d} K_i \ \epsilon^3 \|\omega\|_{1,1} \|u\|_{C^0(0,T,H^1(\mathbb{R}^d))}$$

and the third one by:

$$3 \sup_{i=1,d} K_i \ \epsilon^3 \|\omega\|_{1,1} \|u\|_{C^1(0,T,L^2(\mathbb{R}^d))}$$

Finally we have proved that:

$$|L_1(u,\omega)| \leq 3 \sup_{i=1,d} K_i \epsilon \|\omega\|_{1,1} \int_{t_0}^{t_1} \|u(.,t) - u_h(.,t)\|^2_{L^2} dt + C_1(u^0, f, T) \|\omega\|_{1,1} \epsilon^3$$

The term L_2: We have:

$$\begin{aligned} &|L_2(u,\omega)| = \\ &\left| \int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} (u_h - u)^t . [B(x) - B(y) + divA(x) - divA(y)] . u_h \omega dx dy dt ds \right| \\ &\qquad \leq \int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \frac{3}{2} (K_{m+1} + K_{m+2}) \epsilon \|u_h(x,t)\|^2 \omega dx dy dt ds \\ &\qquad + \int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \frac{1}{2} (K_{m+1} + K_{m+2}) \epsilon \|u(y,s)\|^2 \omega dx dy dt ds \end{aligned}$$

The first term of the R.H.S. of the last inequality is bounded by:

$$\frac{3}{2} T (K_{m+1} + K_{m+2}) \epsilon \|\omega\|_1 [C_1(T) \|u_0\|_{L^2} + C_1'(T) \|f\|_{C(0,T,L^2)}]$$

the second one by:

$$\frac{1}{2} T (K_{m+1} + K_{m+2}) \epsilon \|\omega\|_1 \|u\|^2_{C^0(0,T,L^2(\mathbb{R}^d))}$$

Finally we have proved that:

$$|L_2(u,\omega)| \leq C_2(u^0, f, T) \epsilon$$

The term L_3: We have:

$$\begin{aligned} |L_3(u,\omega)| &\leq \int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \|u_h - u\| \|f(x,t) - f(y,s)\| \omega dx dy dt ds \\ &\leq \left(\int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \|u_h - u\|^2 \omega dx dy dt ds \right)^{1/2} \times \\ &\qquad \left(\int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \|f(x,t) - f(y,s)\|^2 \omega dx dy dt ds \right)^{1/2} \end{aligned}$$

Using Lemma 5.1 we have:

$$\int_{\mathbb{R}^{2d} \times [t_0,t_1]^2} \|f(x,t) - f(y,s)\|^2 \omega dx dy dt ds \quad \leq \quad \epsilon^2 \|\omega\|_1 \|f\|^2_{C^0(0,T,H^1)}$$

We also have:

$$
\begin{aligned}
&\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \|u_h-u\|^2\omega dxdydtds \\
&\le 3\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \|u_h(x,t)-u(x,t)\|^2\omega dxdydtds \\
&\qquad +3\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \|u(x,t)-u(y,t)\|^2\omega dxdydtds \\
&\qquad +3\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2} \|u(y,t)-u(y,s)\|^2\omega dxdydtds \\
&\le 3\|\omega\|_1\left(\int_{t_0}^{t_1}\|u_h-u\|_{L^2}^2\,dt+\epsilon^2\|u\|^2_{C^0(0,T,H^1)}+\epsilon^2\|u\|^2_{C^1(0,T,L^2)}\right)
\end{aligned}
$$

Finally, we get:

$$
|L_3(u,\omega)| \le C_3(u^0,f,T)\epsilon^2+\frac{3}{2}\int_{t_0}^{t_1}\|u_h-u\|_{L^2}^2\,dt
$$

The term L_4: We have:

$$
|L_4(u,\omega)| \le K_{m+2}\epsilon\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2}\|u_h(x,t)-u(y,s)\|^2\,\omega(\xi,\chi)\,dxdydtds
$$

So using the same argument as above we get:

$$
|L_4(u,\omega)| \le C_4(u^0,f,T)\epsilon^3+3K_{m+2}\epsilon\int_{[t_0,t_1]}\|u(.,t)-u_h(.,t)\|_{L^2}^2dt
$$

The term L_5: We have:

$$
|L_5(u,\omega)| \le \beta_{m+1}\int_{\mathbb{R}^{2d}\times[t_0,t_1]^2}\|u_h(x,t)-u(y,s)\|^2\,\omega(\xi,\chi)\,dxdydtds
$$

So using the same argument as above we get:

$$
|L_5(u,\omega)| \le 3\beta_{m+1}\int_{t_0}^{t_1}\|u(.,t)-u_h(.,t)\|_{L^2}^2dt+C_5(u^0,f,T)\epsilon^2
$$

Now setting $\alpha=\eta^2$ and taking in account all the previous inequalities, we get for each $^t\le T$:

$$
\begin{aligned}
1/2\|u(.,t)-u_h(.,t)\|^2_{L^2(\mathbb{R}^d)} \le\ & 1/2\|u(x,0)-u_h(x,0)\|^2_{L^2(\mathbb{R}^d)} \\
& +D_1\int_0^t\|u(.,\tau)-u_h(.,\tau)\|^2_{L^2(\mathbb{R}^d)}\,d\tau \\
& +D_2\alpha+\frac{D_3}{\alpha}\max(\epsilon^2,k^{+2}(h))+\frac{D_4\epsilon^2}{\alpha} \\
& +D_5\epsilon+D_6\epsilon^{-1}h^{1/2}
\end{aligned}
\tag{35}
$$

where D_i, $i=1,\dots,6$ are positive constants wich only depend on u^0,f,T. Taking $\epsilon=h^{1/4}$ and $\alpha=h^{1/4}$, using Gronwald lemma and taking into account that, $\|u(x,0)-u_h(x,0)\|^2_{L^2(\mathbb{R}^d)}\le Ch$, (35) leads to:

$$
\|u(.,t)-u_h(.,t)\|^2_{L^2(\mathbb{R}^d)} \quad\le\quad K(u^0,f,T)h^{1/8} \tag{36}
$$

where $K(u^0,f,T)$ is a positive constant wich only depends on u^0,f,T.

As a consequency of the error bound (36), we also get a convergence proof for approximate solutions of (1) with initial data u^0 in $L^2(I\!R^d)$ and $f \in C^1([0,+\infty[, L^2)$.

For $u_0 \in L^2(I\!R^d)$ and $f \in C^1([0,+\infty[, L^2)$, we choose $u_{0,m} \in H^s(I\!R^d)$ (s sufficiently large) with compact support and $f_m \in C^1([0,+\infty[, H^s)$ (with $f(x,t) = 0$ for $||x|| \leq R_m(T)$, $0 \leq t \leq T$) such that:

$$||u_0 - u_{0,m}||_{L^2(I\!R^d)} \rightarrow 0 \text{ as } m \rightarrow +\infty$$

and for each $T > 0$:

$$||f - f_m||_{L^2([0,T],L^2)} \rightarrow 0 \text{ as } m \rightarrow +\infty$$

and

$$||f_m||_{C^1([0,T],L^2)} \leq C(f)$$

where $C(f)$ is some positive constant. This can be done by standard methods. Denote by $u_{h,m}$ the Finite Volume solution corresponding to the data $u_{0,m}, f_m$. Considering all functions as functions of t with values in $L^2(I\!R^d)$, we know from (36) that for a given value of m :

$$lim_{h\rightarrow 0}||u_{h,m}(.,t) - u_m(.,t)||_{C^0([0,T],L^2)} = 0 \tag{37}$$

The L^2 contraction property (3) of the exact solutions gives:

$$||u(.,t) - u_m(.,t)||_{L^2} \leq K(T)(||u_0 - u_{0,m}||_{L^2} + ||f - f_m||_{L^2(0,T,L^2)}) \tag{38}$$

and he L^2 contraction property (22) of the approximate solutions gives:

$$\begin{aligned} ||u_h(.,t) - u_{h,m}(.,t)||_{L^2} \leq C_1(T)||u_0 - u_{0,m}||_{L^2} + C_1'(T)||f - f_m||_{L^2([0,T],L^2} \\ +C"_1(T)[||f||_{C^1([0,T],L^2)} + C(f)]k^+(h) \end{aligned} \tag{39}$$

We are now equipped to get our convergence result. We have for each $t \leq T$:

$$\begin{aligned} ||u(.,t) - u_h(.,t)||_{L^2} \leq ||u(.,t) - u_m(.,t)||_{L^2} + \\ ||u_{h,m}(.,t) - u_m(.,t)||_{L^2} + ||u_h(.,t) - u_{h,m}(.,t)||_{L^2} \end{aligned}$$

Thanks to (37) and the estimates (38),(39), it follows easily that u_h converges for $0 \leq t \leq T$ uniformly in $L^2(I\!R^d)$ towards u. This completes the proof.•

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