Triple Shock Entropy Theorem and Its Consequences

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For a convex equation of state, a general theorem on shock waves is proved: A sequence of two shocks has a lower entropy than a single shock to the same final pressure. We call this the *Triple Shock Entropy Theorem*. This theorem has important consequences for shock interactions. In 1-D the interaction of two shock waves of the opposite family always results in two outgoing shock waves. In 2-D the intersection of 3 shocks, such as a Mach configuration, must have a contact. Moreover, the state behind the Mach stem has a higher entropy than the state behind the reflected shock. For the transition between a regular and Mach reflection, this suggests that the von Neumann (mechanical equilibrium) criterion would be prefered based on thermodynamic stability, *i.e.*, maximum entropy subject to the system constraint that the total specific enthalpy is fixed. However, to explain the observed hysteresis of the transition we propose an analogy with phase transitions in which locally stable wave patterns (regular or Mach reflection) play the role of meta-stable thermodynamic states. The hysteresis effect would occur only when the transition threshold exceeds the background fluctuations. The transition threshold is affected by flow gradients in the neighborhood of the shock intersection point and the background fluctuations are due to acoustic noise. Consequently, the occurrence of hysteresis is sensitive to the experimental design, and only under special circumstances is hysteresis observed.

1. Introduction

It is well known that the entropy change across a weak shock is third order in shock strength, as measured, for example, by the shock pressure normalized to the initial pressure. As a consequence, a sequence of two weak shocks has a lower entropy than a single shock to the same final pressure. Moreover, if a fixed pressure jump is obtained with multiple shocks then the entropy change vanishes in the limit of a large number of weak shocks. For an ideal gas equation of state, there are simple analytic expressions for the Hugoniot locus. These expression can be used to show that a sequence of arbitrary strength shocks has a lower entropy than a single shock to the same final pressure. This property of shock wave entropy is generally assumed for an arbitrary equation of state.

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Here we prove that the shock entropy does indeed obey this property for a large class of materials.

We consider materials with a convex equation of state and satisfying the weak condition (defined in the next section after the appropriate notation is introduced). These restrictions guarantee that a shock wave is entropy increasing if and only if the shock is compressive and satisfies the necessary condition for 1-D stability that relative to the shock front the flow ahead is supersonic and the flow behind is subsonic. Except in the vicinity of phase transitions, real materials satisfy these conditions, more precisely, nearly all materials in a single phase satisfy the convexity condition Bethe (1942), while all materials known to us satisfy the weak condition. In fact, real materials satisfy the medium condition and the 2-D stability condition discussed below.

For the above class of materials we prove in section 2 that the entropy behind a sequence of N shocks is lower than the entropy behind a single shock to the same final pressure. The key result is for two shocks. We call this the *Triple Shock Entropy Theorem*. It has important consequences for shock interactions. In our analysis of shock interactions we further assume that the (modified) medium condition is satisfied. This is a sufficient condition for all 1-D Riemann problems (and hence 1-D shock interactions) to have a unique solution. Since real material satisfy the medium condition, this assumption does not limit the applicability of our analysis.

Using the triple shock entropy theorem we derive in section 3 a bound on the particle velocity along the Hugoniot locus for a sequence of two shocks compared to a single shock to the same final pressure. We then use this bound to show that the collision of two shocks of the opposite wave family always results in two outgoing shock waves. By contrast, it is well known that the collision of two shocks of the same family can result in either a shock or rarefaction of the opposite family in addition to a stronger shock of the original family, see *e.g.*, Courant and Friedrichs (1948), chpt. III.D, § 79.

A sequence of oblique shocks naturally occurs in 2-D whenever shock waves interact. Thus, 2-D wave patterns are analogous to shock interactions in 1-D. Several wave patterns are discussed in section 4. One important example is a Mach configuration or the wave pattern formed when 3 shock waves intersect at a triple point. Streamlines either go through the incident and reflected shock or through the Mach stem. The pressure is continuous on the downstream side of the triple point, and the triple shock entropy theorem implies that the entropy is higher behind the Mach stem than behind the reflected shock. As a consequence, a contact must occur whenever 3 shocks intersect. Moreover, we show that the difference in entropy along steady state streamlines implies that shock interactions generate vorticity.

In contrast to 1-D, questions of nonuniqueness arise for 2-D shock interactions. In particular, a Mach configuration represents a splitting of an incoming shock into two outgoing shocks (Landau and Liftshitz (1959), chpt. X, § 102). Stability of shock waves in 2-D requires an additional constraint on the equation of state. The constraint for shock stability in multi-dimensions derived from a linear stability analysis Kontorovich (1959) and Fowles (1981) also implies the medium condition and is sufficient for 1-D shock interactions to be well behaved Menikoff and Plohr (1989). We show in subsection 4.1 that the 2-D stability condition can be given a physical interpretation related to a threshold for a Mach configuration to occur. Furthermore, we argue for the regular reflection to Mach reflection transition that stability favors the von Neumann (mechanical equilibrium) transition criterion. This is based on the observation that at fixed pressure the entropy of a regular reflection can be increased by creating a velocity shear layer which allows kinetic energy to be exchanged for enthalpy, *i.e.*, transition to Mach reflection.

Finally, in subsection 4.3 we discuss the hysteresis associated with the transition between regular and Mach reflection. A hysteresis effect was first observed by Henderson and Lozzi (1979) on curved surfaces and predicted by Hornung *et al.* (1979) for plane surfaces but not observed in the initial experiments of Hornung and Robinson (1982). Recent experiments and numerical simulations by Chpoun *et al.* (1995), Ivanov *et al.* (1995a), Ivanov *et al.* (1995b), Fomin *et al.* (1996), Skews (1996) shows that in some circumstances the transition does exhibit a hysteresis effect. In the parameter regime (shock strength and incident shock angle) in which either regular or Mach reflection can occur based on shock polars (local conservation about the intersection point of the shock waves), the hysteresis behavior implies that both types of reflection are locally stable. A related phenomena is the delayed transition due to boundary layer effects in pseudo-steady (shock tube) experiments Henderson *et al.* (1997). When the observation time is limited, a delayed transition can not be distinguished from a hysteresis effect. The delayed transition is not an issue for the steady state experiments.

To explain the experimental result that hysteresis is only observed under limited circumstances, we utilize an analogy with phase transitions. This leads to the hypothesis that the bifurcation or transition between locally stable wave patterns can be triggered only by a perturbation above a threshold amplitude. Since both the noise level and the threshold amplitude depend on the experimental design, it is not surprising that hysteresis behavior is observed in some experiments but not in others. In particular, transverse gradients from 3-D disturbances can have a significant effect on the transition Ivanov *et al.* (1995b), Fomin *et al.* (1996), Skews (1996).

2. Properties of Shock Hugoniot

In this section, properties of the Hugoniot locus are derived for a large class of materials. We assume a material is in local thermodynamic equilibrium and can be characterized by an equilibrium **equation of state** (EOS). We require the EOS to satisfy the following physical properties:

(1) Thermodynamic phase space is parameterized by (V, S) where $V = 1/\rho$ is the specific volume, ρ is the density and S is the entropy. A complete EOS is specified by the specific internal energy $e(V, S)$.

(2) Thermodynamic consistency requires that the fundamental thermodynamic identity is satisfied:

$$de = -P dV + T dS \;, \tag{2.1}$$

where P is the pressure and T is the temperature.

(3) The temperature is always positive. Since a temperature of absolute zero can not be physically obtained, we may assume that $T > 0$.

(4) Every isentrope satisfies the asymptotic conditions that (i) $P \to \infty$ as $V \to 0$, and (ii) $P \to 0$ as $V \to \infty$.

(5) For any V, the pressure becomes arbitrarily large as S increases; *i.e.*, $P(V, S) \to \infty$ as $S \to \infty$ for fixed V.

In addition, we assume that the equation of state is convex, *i.e.*, in the V–P plane every isentrope is convex.

From the conservation laws for mass, momentum and energy, a shock wave satisfies the Hugoniot equation

$$e_1 - e_0 = \frac{1}{2}(P_0 + P_1)\,(V_0 - V_1) \;, \tag{2.2}$$

where the subscripts 0 and 1 denote the states ahead and behind the shock, respectively. The shock speed, σ, is determined by the relation

$$[\rho_0(\sigma - u_0)]^2 = [\rho_1(\sigma - u_1)]^2 = (P_1 - P_0)/(V_0 - V_1) \,, \tag{2.3}$$

and the change in the particle velocity, u, is given by

$$(u_1 - u_0)^2 = (P_1 - P_0)\,(V_0 - V_1) \,. \tag{2.4}$$

For the wave speed and particle velocity, the sign of the square root determines the wave family; *i.e.*, right facing (characteristic $u + c$) or left facing (characteristic $u - c$). We assume the initial state (V_0, e_0, u_0) is specified, and refer to the Hugoniot locus based on the state 0 as the set of all states that can be reached from state 0 by a shock wave. The shock state and wave speed for a given wave family are completely determined by (2.2), (2.3) and (2.4), and the EOS; *i.e.*, $P_0 = P(V_0, e_0)$ and $P_1 = P(V_1, e_1)$. Though the Hugoniot locus depends explicitly only on an incomplete EOS, $P = P(V, e)$, its properties depend on the existence of a complete EOS.

We will use three dimensionless parameters to characterize an equation of state: The **adiabatic exponent**

$$\gamma \equiv -\frac{V}{P}\,\frac{\partial P}{\partial V}\bigg|_S = \frac{c^2}{PV} \tag{2.5}$$

is a dimensionless form for the sound speed c. Thermodynamic stability requires $\gamma \geqslant 0$. The **Grüneisen coefficient**

$$\Gamma \equiv V\,\frac{\partial P}{\partial e}\bigg|_V = \frac{V}{T}\,\frac{\partial P}{\partial S}\bigg|_V \tag{2.6}$$

measures the spacing of the isentropes in the V–P plane.
Using these parameters, the fundamental thermodynamic relation can be expressed as

$$V dP = -\gamma P dV + \Gamma T dS \,. \tag{2.7}$$

The **fundamental derivative**

$$\mathcal{G} \equiv -\frac{1}{2} V \frac{\partial^2 P/\partial V^2|_S}{\partial P/\partial V|_S} \tag{2.8}$$

is a measure of the convexity of an isentrope in the V–P plane. For a convex EOS $\mathcal{G} > 0$ everywhere. Moreover, $\mathcal{G} > 0$ excludes $\gamma = 0$ and hence implies that $\gamma > 0$ everywhere.

In order for fluid flow (solution to the compressible Euler equations) to be well behaved, 1-D Riemann problems must have a unique solution. This requires that both the pressure and particle velocity are monotonic along the Hugoniot locus. Three constraints on an EOS have been proposed to ensure monotonicity of the state variables along the Hugoniot locus; they are called the strong, medium and weak condition, see Menikoff and Plohr (1989) and references contained therein. In addition, there is a slightly stronger variation or modified medium condition. The weak condition, $\Gamma \leqslant 2\gamma$, is sufficient for the pressure to be monotonic and the modified medium condition, $\Gamma \leqslant \gamma$, is sufficient for both pressure and particle velocity to be monotonic. A stronger stability condition, $\Gamma \leqslant \gamma - 1$ is needed in order for strong shock waves to be stable to small perturbations in multi-dimensions.

Except in the vicinity of a phase transition, most materials satisfy $\mathcal{G} > 0$. The triple shock entropy theorem also requires the weak condition. Furthermore, the analysis of the particle velocity along the Hugoniot locus needed for 1-D shock interactions requires the modified medium condition. Both the weak and modified medium condition follow from the 2-D stability condition. Thus, our assumptions are physically reasonable and are only mild constraints on an equation of state. We do not assume $\Gamma > 0$ everywhere. Consequently, the V–P plane does not uniquely parameterize phase space. Since $\gamma > 0$, a given isentrope can be parameterized by either V or P. Consequently, thermodynamic phase space can be uniquely parameterized by either the V–S plane or the P–S plane.

2.1. *Preliminary results*

The Hugoniot *locus* based on the state 0 corresponds to the zero-level set of the Hugoniot *function*. In the V–S plane, the Hugoniot function is given by

$$h(V,S) = e(V,S) - e_0 - \frac{1}{2}\left[P(V,S) + P_0\right](V_0 - V)\,. \tag{2.9}$$

The equivalent function in the P–S plane is

$$\begin{aligned}\tilde{h}(P,S) &= h(V(P,S),S)\\ &= H(P,S) - H_0 - \frac{1}{2}\left[V(P,S) + V_0\right](P - P_0)\,,\end{aligned} \tag{2.10}$$

where $H = e + PV$ is the specific enthalpy.

Our analysis utilizes the differential of the Hugoniot function. Taking the differential of (2.9) and (2.10) and substituting the thermodynamic relations (2.1) and (2.7) we obtain

$$dh = \left(1 + \frac{1}{2}\Gamma\frac{\Delta V}{V}\right)TdS - \frac{1}{2}\left(\gamma\frac{\Delta V}{V} + \frac{\Delta P}{P}\right)PdV, \tag{2.11}$$

$$d\tilde{h} = \left(1 - \frac{\Gamma}{2\gamma}\frac{\Delta P}{P}\right)TdS + \frac{1}{2\gamma}\left(\gamma\frac{\Delta V}{V} + \frac{\Delta P}{P}\right)VdP, \tag{2.12}$$

where $\Delta f = f - f_0$ is the difference of the variable f with respect to its value at the initial state. From (2.12) we obtain

$$\left.\frac{\partial\tilde{h}}{\partial S}\right|_P = \left(1 - \frac{\Gamma}{2\gamma}\frac{\Delta P}{P}\right)T\,. \tag{2.13}$$

We note that $0 \leqslant \frac{\Delta P}{P} < 1$ if $P \geqslant P_0$ and $\frac{\Gamma}{2\gamma} \leqslant 1$ if the weak condition holds. The importance of the weak condition stems from the fact that

$$\left.\frac{\partial\tilde{h}}{\partial S}\right|_P > 0\,, \qquad \text{if } P \geqslant P_0 \text{ and } \Gamma \leqslant 2\gamma. \tag{2.14}$$

The Hugoniot function reduces the problem of determining the Hugoniot locus to finding the solution of a single equation in two variables; either V and S, or P and S. In order for σ and u_1 to be real, and hence to have a solution to the three Rankine-Hugoniot jump conditions, it is necessary that $\Delta P \cdot \Delta V \leqslant 0$.

Lemma 1. *The Hugoniot function associated with the state 0 has no zeros when $\Delta P \cdot \Delta V \geqslant 0$ except for the point $(V,P) = (V_0,P_0)$.*

Proof. Let (V,P) be any point in the quadrant $\Delta P \geqslant 0$, $\Delta V \geqslant 0$ with $(V,P) \neq (V_0,P_0)$. If $S < S_0$ then the Hugoniot function can be expressed as

$$h(V,S) = h(V_0,S_0) + \int_{S_0}^{S} \partial_S h\, dS\Big|_{V_0} + \int_{V_0}^{V} \partial_V h\, dV\Big|_S\,.$$

We note that (i) $h(V_0,S_0) = 0$, (ii) $\partial_S h|_{V_0} = T > 0$, and (iii) $\partial_V h|_S = -\frac{1}{2}(\gamma\Delta V/V + \Delta P/P)P < 0$. Consequently, $h(V,S) < 0$.

Similarly, if $S > S_0$ then

$$\tilde{h}(P,S) = \tilde{h}(P_0,S_0) + \int_{S_0}^{S} \partial_S \tilde{h}\, dS\Big|_{P_0} + \int_{P_0}^{P} \partial_P \tilde{h}\, dP\Big|_S .$$

Now we note that (i) $\tilde{h}(P_0,S_0) = 0$, (ii) $\partial_S \tilde{h}|_{P_0} = T > 0$, and (iii) $\partial_P \tilde{h}|_S = (2\gamma)^{-1}\,(\gamma \Delta V/V + \Delta P/P) V > 0$. Consequently, $\tilde{h}(P,S) > 0$. Hence, the Hugoniot function is not zero in the quadrant $\Delta P \geqslant 0$, $\Delta V \geqslant 0$ provided that the point (V_0,P_0) is excluded.

The Hugoniot equation is invariant when the initial and final states are interchanged. Interchanging the initial and final shock states interchanges the two quadrants $\Delta P \geqslant 0$, $\Delta V \geqslant 0$ and $\Delta P \leqslant 0$, $\Delta V \leqslant 0$. Hence the Hugoniot function is not zero when $\Delta P \cdot \Delta V \geqslant 0$ except for the point (V_0,P_0). □

This lemma implies that there are no extraneous solutions to the Hugoniot equation, *i.e.*, there is a one-to-one correspondence between solutions to the Hugoniot equation and points on Hugoniot locus for each wave family.

The Bethe-Weyl theorem is important for a convex EOS. It implies that the Hugoniot locus can be parameterized by the entropy and consists of a single curve connected to the initial state. We correct an error in the proof given in Menikoff and Plohr (1989).† It is useful to consider points on the Hugoniot locus based on the state 0 with a given entropy S. These points correspond to the zero-level set of the restricted Hugoniot function

$$h_S(V) = e(V,S) - e_0 - \frac{1}{2}\left[P(V,S) + P_0\right](V_0 - V)\,, \tag{2.15}$$

or equivalently

$$\tilde{h}_S(P) = H(P,S) - H_0 - \frac{1}{2}\left[V(P,S) + V_0\right](P - P_0)\,.$$

Taking the first two derivatives of (2.15) we obtain

$$h'_S(V) = -\frac{1}{2}\Big(P(V,S) - P_0 - \frac{\partial P}{\partial V}\Big|_S (V - V_0)\Big)\,, \tag{2.16}$$

$$h''_S(V) = \frac{1}{2}\,(V - V_0)\,\frac{\partial^2 P}{\partial V^2}\Big|_S \,. \tag{2.17}$$

The initial isentrope is an important reference curve. We note that

$$h_{S_0}(V_0) = h'_{S_0}(V_0) = h''_{S_0}(V_0) = 0\,.$$

Moreover, for a convex EOS $\frac{\partial^2 P}{\partial V^2}|_S > 0$, and by (2.17)

$$h''_{S_0}(V) \begin{cases} < 0\,, & \text{for } V < V_0; \\ > 0\,, & \text{for } V > V_0. \end{cases}$$

Therefore, $h_{S_0}(V)$ has an inflection point at $V = V_0$ and is monotonically increasing over the entire range $0 < V < \infty$. In particular,

$$h_{S_0}(V) < 0\,, \qquad \text{for } V < V_0. \tag{2.18}$$

† In (4.13) of Menikoff and Plohr (1989), P_0 which is equal to $P(V_0,S_0)$ has been inadvertently substituted for $P(V_0,S)$. As a consequence, an additional condition is needed to rule out a disconnected branch of the Hugoniot locus with entropy increasing expansive shocks or entropy decreasing compressive shocks. Either Bethe's condition $\Gamma \geqslant -2$ or the weak condition $\Gamma \leqslant 2\gamma$ are sufficient to recover theorem 4.1 of Menikoff and Plohr (1989).

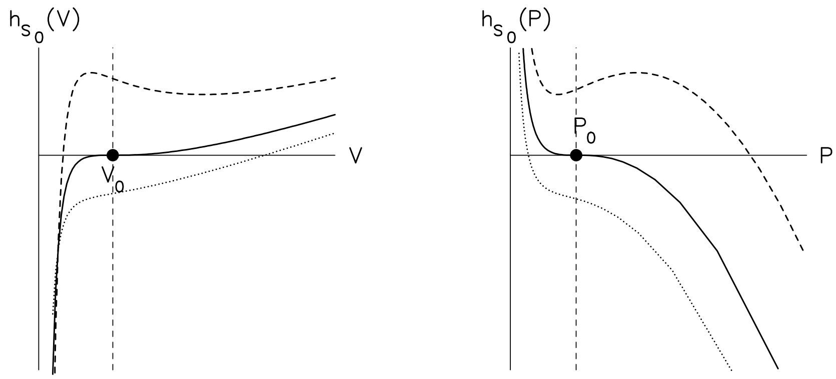

FIGURE 1. The essential geometry of the restricted Hugoniot function. The dotted, solid and dashed lines correspond to $S < S_0$, $S = S_0$ and $S > S_0$, respectively.

Since $\tilde{h}_{S_0}(P) = h_{S_0}(V(P, S_0))$ and $\frac{\partial P}{\partial V}|_S < 0$, it follows that

$$\tilde{h}_{S_0}(P) < 0 \,, \qquad \text{for } P > P_0. \tag{2.19}$$

As an illustrative example, for an ideal gas EOS the restricted Hugoniot function along the initial isentrope is shown in figure 1 by the solid line.

We now exclude anomalous shocks.

LEMMA 2. *If $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ then there are no entropy increasing expansive shocks and no entropy decreasing compressive shocks.*

Proof. The Hugoniot function can be expressed as

$$\tilde{h}(P, S) = \tilde{h}_{S_0}(P) + \int_{S_0}^{S} \frac{\partial \tilde{h}}{\partial S}\bigg|_P dS \,.$$

From (2.19), the first term on the right hand side is negative for $P > P_0$. From (2.14) the integrand is positive. Hence the integral is negative for $S < S_0$. Therefore, $\tilde{h}(P, S) < 0$ for entropy decreasing compressive shocks. Since interchanging the initial and final states converts an entropy decreasing compressive shock into an entropy increasing expansive shock, both of these anomalous shocks are excluded. □

We note that for a non-convex EOS entropy increasing expansive shocks are required for the existence of solutions to Riemann problems since a rarefaction curve ends when $\mathcal{G} = 0$, see *e.g.,* Zel'dovich and Raizer (1967), chpt. XI, § 20. Thus, anomalous shocks are allowed by thermodynamics but are excluded by our assumption that the equation of state is convex.

Next we show that the weak condition implies that compressive shocks can be parameterized by P and that the Hugoniot locus extends up to $P = \infty$.

LEMMA 3. *If $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ then the Hugoniot locus based on state 0 contains a unique shock for any $P \geqslant P_0$. Moreover, pressure increasing shocks are entropy increasing.*

Proof. By (2.14), for any fixed $P \geqslant P_0$, $\tilde{h}(P, S)$ is a monotonic function of S, and therefore has at most one zero. Moreover, by (2.19), $\tilde{h}(P, S_0) < 0$ for $P > P_0$. If we can find an S' such that $\tilde{h}(P, S') > 0$ then by continuity $\tilde{h}(P, S)$ would have a unique zero for any $P > P_0$. From EOS property (5) and continuity, there exists a $S' > S_0$ such that

$P = P(V_0, S')$. Since $h(V_0, S_0) = 0$ and from (2.11), $\frac{\partial h}{\partial S}|_V(V_0, S) = T > 0$, it follows that $\tilde{h}(P, S') = h(V_0, S') > 0$. Hence, we can conclude that $\tilde{h}(P, S)$ has a unique zero for any $P > P_0$. Therefore, there is a unique shock for any $P > P_0$, and it is entropy increasing. $\square$

For a convex EOS satisfying the weak condition we can obtain the first key result.

THEOREM 4 (BETHE-WEYL). *If $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ then the Hugoniot locus based on any state 0 intersects every isentrope exactly once. Moreover, for entropy increasing shocks $V < V_0$ and $|u - \sigma| < c$, while for entropy decreasing shocks $V > V_0$ and $|u - \sigma| > c$.*

Proof. From the asymptotic EOS conditions, it follows that $h_S(V) \to -\infty$ as $V \to 0$ and $h_S(V) \to \infty$ as $V \to \infty$; see Menikoff and Plohr (1989). By continuity, there is at least one shock state for every S. By Lemmas 1 and 2 shocks are either compressive and entropy increasing or expansive and entropy decreasing. First, we consider the entropy increasing case; $S > S_0$ and $V < V_0$. From (2.15) and (2.1)

$$h_S(V_0) = e(V_0, S) - e_0 = \int_{S_0}^{S} T dS \bigg|_{V_0} > 0 \, .$$

Moreover, $h_S(V) \to -\infty$ as $V \to 0$ and from (2.17) $h''_S(V) < 0$. Hence, $h_S(V)$ is concave in the domain $0 < V < V_0$. Therefore, $h_S(V)$ has a unique zero. Furthermore, at the zero crossing $h'_S > 0$. The dashed line in figure 1 illustrates the restricted Hugoniot function for this case.

At the zero crossing, using (2.3) and (2.5), we can rewrite (2.16) as

$$\begin{aligned} 0 < 2\frac{h'_s(V)}{V_0 - V} &= \frac{\Delta P}{\Delta V} - \frac{\partial P}{\partial V}\bigg|_S \\ &= \rho^2 \left[-(\sigma - u)^2 + c^2 \right] \, . \end{aligned}$$

Therefore, $|\sigma - u| < c$. A similar argument can be applied to obtain the analogous result in the entropy decreasing case. $\square$

A continuity argument allows us to characterize the Hugoniot locus as follows.

COROLLARY 5. *If $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ then the Hugoniot locus can be parameterized by S and consists of a single curve connected to the base state.*

Proof. From the Bethe-Weyl theorem, the base state 0 is the only sonic state on the Hugoniot locus. By (2.12), along the Hugoniot locus

$$\frac{\partial \tilde{h}}{\partial P}\bigg|_S = \frac{1}{2\gamma} \left(\gamma \frac{\Delta V}{V} + \frac{\Delta P}{P} \right) V \neq 0$$

for $S \neq S_0$. Since the Hugoniot locus corresponds to $\tilde{h}(P, S) = 0$, the implicit value theorem (see appendix A) implies that the Hugoniot locus can locally be parameterized by S except at S_0. From the Bethe-Weyl theorem, there is a unique shock for each S. Therefore, the local patches can be pieced together to give a connected entropy increasing branch and a connected entropy decreasing branch. In the vicinity of S_0, an additional argument is needed because of the well known fact that for weak shocks the entropy change is third order. We note that $\frac{\partial \tilde{h}}{\partial S}|_P(P_0, S_0) = T_0 > 0$. The implicit value theorem implies that weak shocks can locally be parameterized by P. Because there is unique shock for each S, the Hugoniot locus can be parameterized by S in a local neighborhood of S_0. This connects the the entropy increasing and entropy decreasing branch. Therefore, the Hugoniot locus consists of a single branch. It can be parameterized by S and is connected to the initial state. $\square$

The following is an important consequence of the Bethe-Weyl theorem that we will need for the triple shock entropy theorem.

COROLLARY 6. *If $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ then for an entropy increasing shock the ahead state is supersonic and the behind state is subsonic.*

Proof. From the Bethe-Weyl theorem an entropy increasing shock is compressive. The behind state is subsonic since $c > |\sigma - u|$. Interchanging the initial and final state results in an entropy decreasing expansive shock. The equations for the wave speed and particle velocity, (2.3) and (2.4), are symmetric in the shock states. Hence, it follow from the Bethe-Weyl theorem that $c_0 < |\sigma - u_0|$ and the ahead state is supersonic. $\square$

2.2. *Main theorem*

We can now proof our main result.

THEOREM 7 (TRIPLE SHOCK ENTROPY THEOREM).
Suppose $\mathcal{G} > 0$ and $\Gamma \leqslant 2\gamma$ are satisfied everywhere. Consider only the physical entropy increasing shocks. Then a sequence of two shocks has a smaller entropy than a single shock to the same final pressure.

Proof. Suppose the material is initially in state 0. Let state 1 be a point on the shock Hugoniot based on state 0, and let state 2 be a point on the shock Hugoniot based on state 1. By Lemma 3 there is a unique state $2'$ on the Hugoniot locus based on state 0 such that $P_{2'} = P_2$. With this notation, the theorem can be stated as $S_2 < S_{2'}$

We specify a shock by its end states; *e.g.*, shock 0-1 denotes the shock connecting state 0 to state 1. From the Hugoniot equations for shocks 1-2 and 0-1

$$e_2 - e_1 = \frac{1}{2}\left(P_1 + P_2\right)\left(V_1 - V_2\right),$$
$$e_1 - e_0 = \frac{1}{2}\left(P_0 + P_1\right)\left(V_0 - V_1\right).$$

Adding these equation we obtain

$$e_2 - e_0 = \frac{1}{2}\left(P_1 + P_2\right)\left(V_1 - V_2\right) + \frac{1}{2}\left(P_0 + P_1\right)\left(V_0 - V_1\right). \tag{2.20}$$

The restricted Hugoniot function based on state 0 with entropy S_2 is

$$h_{S_2}(V) = e(V, S_2) - e_0 - \frac{1}{2}[P_0 + P(V, S_2)]\left(V_0 - V\right).$$

Evaluating at $V = V_2$ and substituting (2.20) to eliminate $e_2 - e_0$ yields

$$h_{S_2}(V_2) = \frac{1}{2}(P_1 + P_2)\left(V_1 - V_2\right) + \frac{1}{2}(P_0 + P_1)\left(V_0 - V_1\right) - \frac{1}{2}(P_0 + P_2)\left(V_0 - V\right).$$

Each term on the right hand side corresponds to the area of a trapezoid in the V–P plane. By the Bethe-Weyl theorem, the entropy increasing shocks are compressive. Hence, $V_2 < V_1 < V_0$. It then follows from the geometry (shown in figure 2) that

$$h_{S_2}(V_2) = \left[\begin{array}{c}\text{signed area of triangle connecting}\\ \text{points } (V_0, P_0),\ (V_1, P_1) \text{ and } (V_2, P_2)\end{array}\right], \tag{2.21}$$

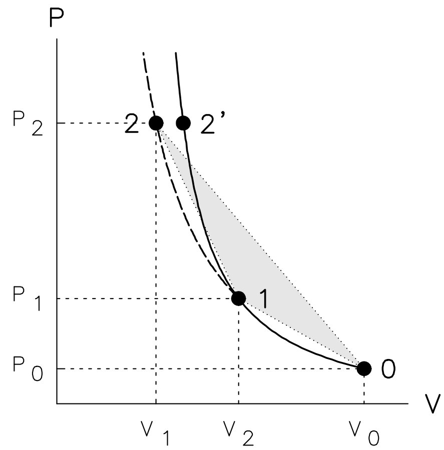

FIGURE 2. Geometry of Hugoniot loci in V–P plane used in proof of Triple Shock Theorem. The solid line and dashed lines are Hugoniot loci based on states 0 and 1 respectively. The dotted lines correspond to the Rayleigh lines connecting the end states of shocks 0-1, 1-2 and 0-2′. The shaded area is triangle formed by the states 0, 1, and 2.

where the area is positive if the point (V_2, P_2) lies below the line connecting (V_0, P_0), (V_1, P_1).

By Corollary 6 the state behind shock 0-1 is subsonic and the state ahead of shock 1-2 is supersonic. From (2.3) and $c^2 = -V^2 \frac{\partial P}{\partial V}|_S$, we obtain

$$
\begin{array}{ccccc}
[\rho_1(\sigma_{01} - u_1)]^2 & < & (\rho_1 c_1)^2 & < & [\rho_1(\sigma_{12} - u_1)]^2 \\
\dfrac{P_1 - P_0}{V_0 - V_1} & < & -V_1^2 \, \frac{\partial P}{\partial V}\big|_S \, (V_1, S_1) & < & \dfrac{P_2 - P_1}{V_1 - V_2}
\end{array}
$$

Therefore, in the V–P plane state 2 lies above the line connecting states 0 and state 1. It then follows from (2.21) that $h_{S_2}(V_2) < 0$. Hence, $\tilde{h}(P_2, S_2) < 0$. By (2.14), $\frac{\partial \tilde{h}}{\partial S}|_P > 0$. Therefore, the zero of $\tilde{h}(P_2, S)$ occurs for $S_{2'} > S_2$. □

COROLLARY 8. *A sequence of N shocks has less entropy then a single strong shock to the same pressure.*

Proof. The triple shock entropy theorem is a special case of the corollary for $N = 2$. The proof for N shocks is by induction. Suppose the corollary is true for $N - 1$ shocks, then we need to extend it to N shocks. Let the initial state be denoted 0 and the state behind the shocks be denoted 1, 2, $\cdots$, N. Let α denote the state behind a single shock from state 0 to a pressure P_N, and let β denote the state behind a single shock from state 1 to a pressure P_N. By the triple shock entropy theorem $S_\beta < S_\alpha$. Applying the corollary to the $N - 1$ shocks from state 1 to state N, we have $S_N < S_\beta$. Therefore, $S_N < S_\alpha$. By induction the corollary is true for any N. □

A thermodynamic state is uniquely determined by S and P. From thermodynamic identities, the following differential relations can be derived for other thermodynamic variables of interest, see for example, Appendix A in Menikoff and Plohr (1989):

$$dH = V dP + T dS \,, \tag{2.22}$$

$$\frac{PV}{T} dT = \frac{\Gamma}{\gamma} V dP + \frac{\gamma g - \Gamma^2}{\gamma} T dS \,, \tag{2.23}$$

$$\gamma de = VdP + (\gamma - \Gamma)TdS \,, \tag{2.24}$$

$$\gamma PdV = -VdP + \Gamma TdS \,, \tag{2.25}$$

where $g = \frac{PV}{C_V T} > 0$ is a dimensionless inverse specific heat. We also note that thermodynamic stability requires the entropy to be jointly concave in the variables V and e. This implies that $\gamma g \geqslant \Gamma^2$. These relations enable us to extend the Triple Shock Entropy Theorem to the other thermodynamic variables.

COROLLARY 9. *Consider the state behind a sequence of shocks to the same final pressure as a single shock. The multiple shocks have a lower enthalpy and lower temperature than a single shock. Moreover, if $\Gamma \leqslant \gamma$ then multiple shocks have a lower specific energy, and if $\Gamma > 0$ then multiple shocks have a lower specific volume.*

Proof. By Corollary 8 multiple shocks to the same P as a single shock have a lower entropy. From (2.22) and (2.23), $\frac{\partial H}{\partial S}|_P > 0$ and $\frac{\partial T}{\partial S}|_P > 0$. Hence multiple shocks have a lower enthalpy and temperature. With the additional condition $\Gamma < \gamma$, then by (2.24) $\frac{\partial e}{\partial S}|_P > 0$ and the specific energy is lower for multiple shocks. Finally, if $\Gamma > 0$ then by (2.25) $\frac{\partial V}{\partial S}|_P > 0$ and multiple shocks have a lower specific volume or higher density. □

The particle velocity is compared in the next section.

3. Shock interactions in one-dimension

In this section we derive a bound on the particle velocity behind a sequence of two shocks in terms of the particle velocity behind a single strong shock to the same pressure. The bound on the particle velocity will enable us to determine properties of 1-D shock interactions. Uniqueness of any shock interaction requires that all Hugoniot loci are monotonic in the u–P plane. The modified medium condition $\Gamma \leqslant \gamma$ is satisfied by all single phase materials we know of and is sufficient to guarantee the needed monotonicity. It is equivalent to Bethe's "shock splitting" condition (III) and the only known violations of it occur for a few solid/solid phase transitions.

THEOREM 10. *Suppose $\mathcal{G} > 0$ and $\Gamma \leqslant \gamma$ are satisfied everywhere. Consider a sequence of two entropy increasing shocks from state 0 to state 1 and from state 1 to state 2, and a third shock from state 0 to state $2'$ with $P_{2'} = P_2$. Then*

$$(u_2 - u_1)^2 + (u_1 - u_0)^2 < (u_{2'} - u_0)^2 \;. \tag{3.1}$$

Proof. Combining equations (2.2) and (2.4) a point on the Hugoniot locus based on state 0 satisfies

$$e - e_0 = \frac{1}{2}(u - u_0)^2 + P_0(V_0 - V) \,.$$

For the three shock 0-1, 1-2 and 0-$2'$ we have

$$e_1 - e_0 = \frac{1}{2}(u_1 - u_0)^2 + P_0(V_0 - V_1) \,,$$

$$e_2 - e_1 = \frac{1}{2}(u_2 - u_1)^2 + P_1(V_1 - V_2) \,,$$

$$e_{2'} - e_0 = \frac{1}{2}(u_{2'} - u_0)^2 + P_0(V_0 - V_{2'}) \,.$$

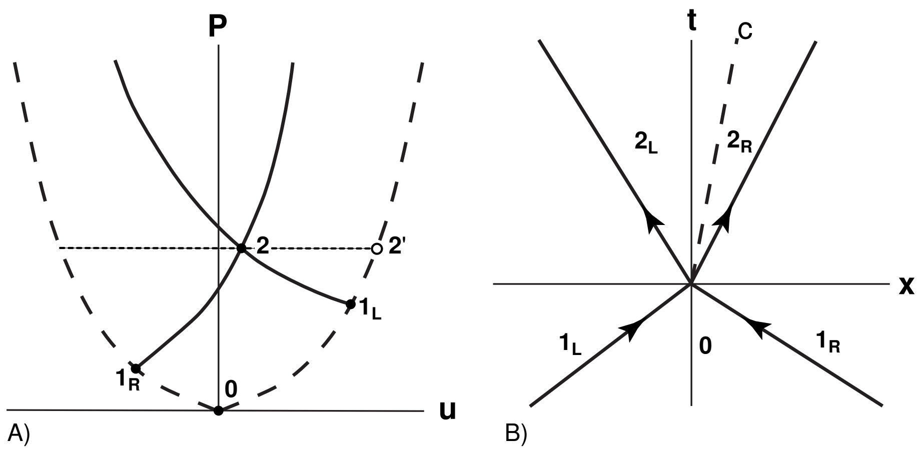

FIGURE 3. The collision of two shocks of the opposite family. (A) The projection of the Hugoniot loci in the u–P. The dashed lines are Hugoniot loci for the incoming shocks. The solid lines are the Hugoniot loci for the outgoing shocks. The outgoing waves correspond to the intersection of the the two solid curves. (B) Wave diagram in the x–t plane.

Subtracting the third equation from the sum of the first two equations yields

$$\frac{1}{2}\left[(u_{2'}-u_0)^2-(u_2-u_1)^2-(u_1-u_0)^2\right]=(P_1-P_0)(V_1-V_2)+(e_{2'}-e_2)+P_0(V_{2'}-V_2)\,. \tag{3.2}$$

On the right hand side, the first term is positive since entropy increasing shocks are pressure increasing ($P_1 > P_0$ for shock 0-1) and compressive ($V_2 < V_1$ for shock 1-2). Since $P_2 = P_{2'}$, using (2.24) and (2.25) the next two terms can be expressed as

$$(e_{2'}-e_2)+P_0(V_{2'}-V_2)=\int_{S_2}^{S_{2'}}\left(1-\frac{\Gamma}{\gamma}\frac{P_2-P_0}{P_2}\right)T\,dS\bigg|_{P_2}\,. \tag{3.3}$$

The integrand is positive since $\frac{\Gamma}{\gamma}\leqslant 1$ and $0<\frac{P_2-P_0}{P_2}<1$. By the triple shock entropy theorem, $S_2 < S_{2'}$. Hence, the right hand side of (3.2) is positive and inequality (3.1) holds. □

Next we discuss the consequences of the Theorem 10 for 1-D shock interactions. First we consider the collision of two shocks of the opposite wave family and show that the two outgoing waves must be shocks. The moment of shock impact defines the two constant states of a Riemann problem. A Riemann problem can be solved graphically as the intersection in the u–P plane of the right wave curve based on the right state with the left wave curve based on the left state. For the shock interaction the Hugoniot loci are sketched in figure 3 in the u–P plane and the x–t plane. Before collision there is an incoming left facing shock 0–$1L$ and an incoming and right facing shock 0–$1R$. In order to prove that the outgoing waves are always shocks, we must show that the Hugoniot locus for an outgoing shock do not cross the Hugoniot locus for an incoming shock as shown in figure 4.

From figure 3, for right facing waves the condition that the Hugoniot loci do not cross is $u_{2R} < u_{2'}$. This can be expressed in terms of the change in velocity across the three shocks, 0–$1R$, $1R$–$2R$ and 0–$2'$,

$$|u_2-u_1|-|u_1-u_0|<|u_{2'}-u_0|\,, \tag{3.4}$$

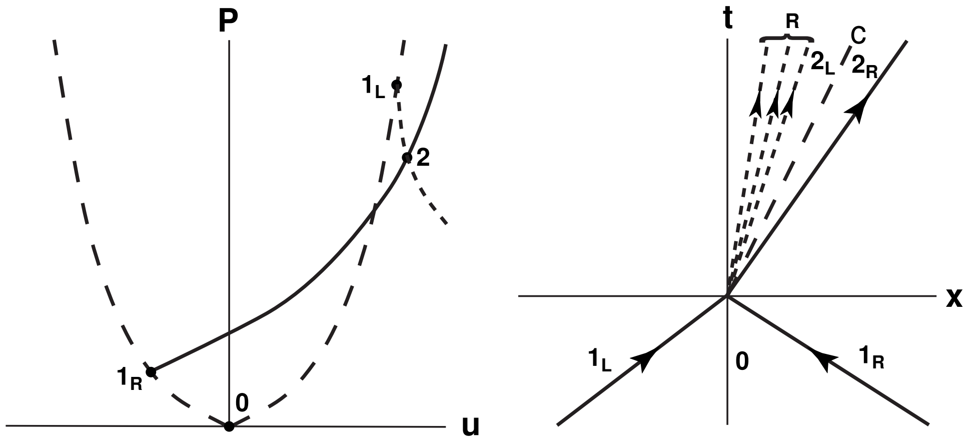

FIGURE 4. Excluded case in which the collision of two shocks of the opposite family would result in a reflected rarefaction.

where to simplify the notation we have denoted $1R$ by 1 and $2R$ by 2. Since $|u_2 - u_1| - |u_1 - u_0| < |u_2 - u_1|$, and inequality (3.1) implies that $|u_1 - u_0| < |u_{2'} - u_0|$ the condition (3.4) is satisfied. Hence the Hugoniot loci do not cross and the outgoing waves are always shocks.

Next we consider the collision of two shocks of the same wave family. The Hugoniot loci corresponding to the incoming shocks can cross in the u–P plane as illustrated in figure 5. In particular, there may be a region on the second Hugoniot loci for which

$$|u_{2'} - u_0| < |u_2 - u_1| + |u_1 - u_0| \ ,$$

even though by Theorem 10

$$(u_2 - u_1)^2 + (u_1 - u_0)^2 < (u_{2'} - u_0)^2 \ .$$

As a consequence, there are two possible outcomes when one shock overtakes another. There is always a stronger transmitted shock of the same wave family, but either a reflected rarefaction or a reflected shock of the opposite wave family. These two cases have been previously noted by von Neumann (1945) and Fritz (1996). Typically, the reflected wave is a rarefaction. In the case with two shocks, Theorem 10 implies that the reflected shock is very weak compared to the transmitted shock.

When the outgoing waves are both shocks, there are two categories of streamlines, see figure 5c. One set of streamlines cross the two incoming shocks and the reflected outgoing shock (opposite wave family to the incoming shocks). The other streamlines cross only the transmitted shock (same wave family as the incoming shocks). Since these streamlines are separated by a contact across which the pressure is continuous, Corollary 8 then applies with $N = 3$. Consequently, the state behind the transmitted shock has the highest entropy, and the shock interaction increases the rate of entropy production. Though Corollary 8 does not apply when one of the outgoing waves is a rarefaction, we conjecture that the entropy production also increases for this case.

4. Shock interactions in two-dimensions

In 2-D, sequences of oblique shocks commonly occur in wave patterns resulting from shock interactions. The simplest examples for which the Triple Shock Entropy theorem

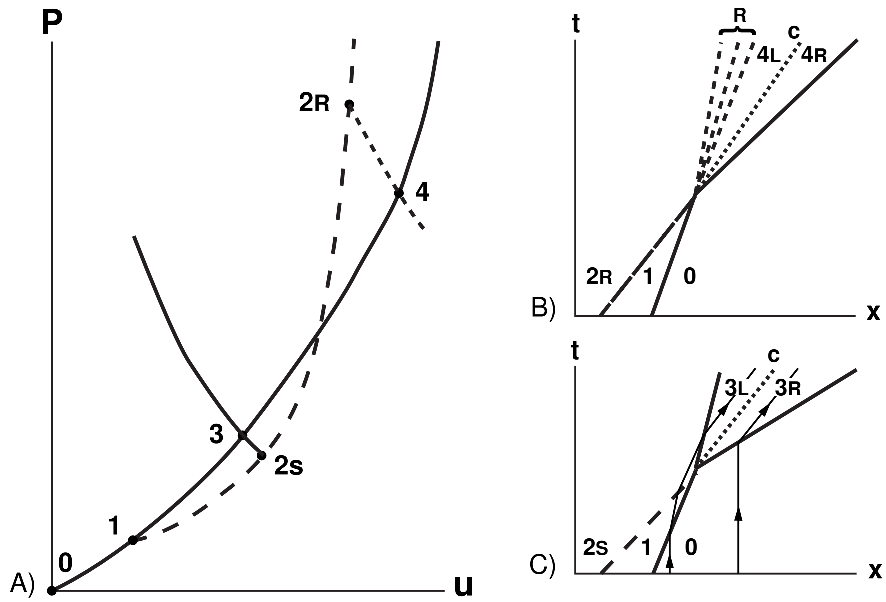

FIGURE 5. The interaction (overtake) of two shocks of the same wave family. (A) Hugoniot loci in the u–P plane. (B) x–t wave diagram when outgoing waves consist of reflected rarefaction and transmitted shock. (C) x–t wave diagram when outgoing waves are both shocks. Thin lines with arrows represent streamlines.

apply are illustrated in figure 6. The shock waves in these patterns can be classified as incoming or outgoing (Landau and Liftshitz (1959), chpt. X, § 102). The classification is according to the direction in which information flows. A wave is incoming if the component of the velocity in the direction of the front points towards the intersection point of the waves, which we denote the node, otherwise the wave is outgoing. Typically, two incoming shocks interact giving rise to two outgoing waves (shock or rarefaction) separated by a contact, see *e.g.*, Glimm *et al.* (1985). Degenerate cases, in which one of the outgoing waves has zero strength and disappears, are important when the flow across a wave pattern is transonic. In the figures, an arrow on a shock front pointing towards the node indicates an incoming wave while an arrow pointing away from the node indicates an outgoing wave. The same convention is used in figures 3 and 4 to indicate incoming and outgoing waves in 1-D. In fact, there is a strong similarity between 2-D wave patterns in the x–y plane and 1-D shock interactions in the x–t plane.

An exceptional case is the Mach reflection (MR) shown in figure 6a. It consists of one incoming wave splitting into two outgoing waves and a contact. As discussed in a later subsection, this wave splitting does not cause a shock wave to be unstable. A consequence of the 2-D stability is that a Mach reflection with a reflected centered rarefaction does not occur.

A degenerate cross node (DCN) is illustrated in figure 6b. Superficially, this system looks like a Mach reflection and is sometimes refered to as an inverted Mach reflection (Courant & Friedrichs, 1948) and Hornung (1986); but there are significant differences. Firstly, it has two incoming shocks while the MR has only one. The extra incoming shock belongs to the opposite wave family than the Mach stem in a MR. Secondly, this reflection requires an extra boundary condition to generate the second incoming shock, e.g., a downstream blunt body as sketched or a downstream sonic choke. Thirdly, the contact is directed away from the wall while in MR it impacts on the wall. A cross

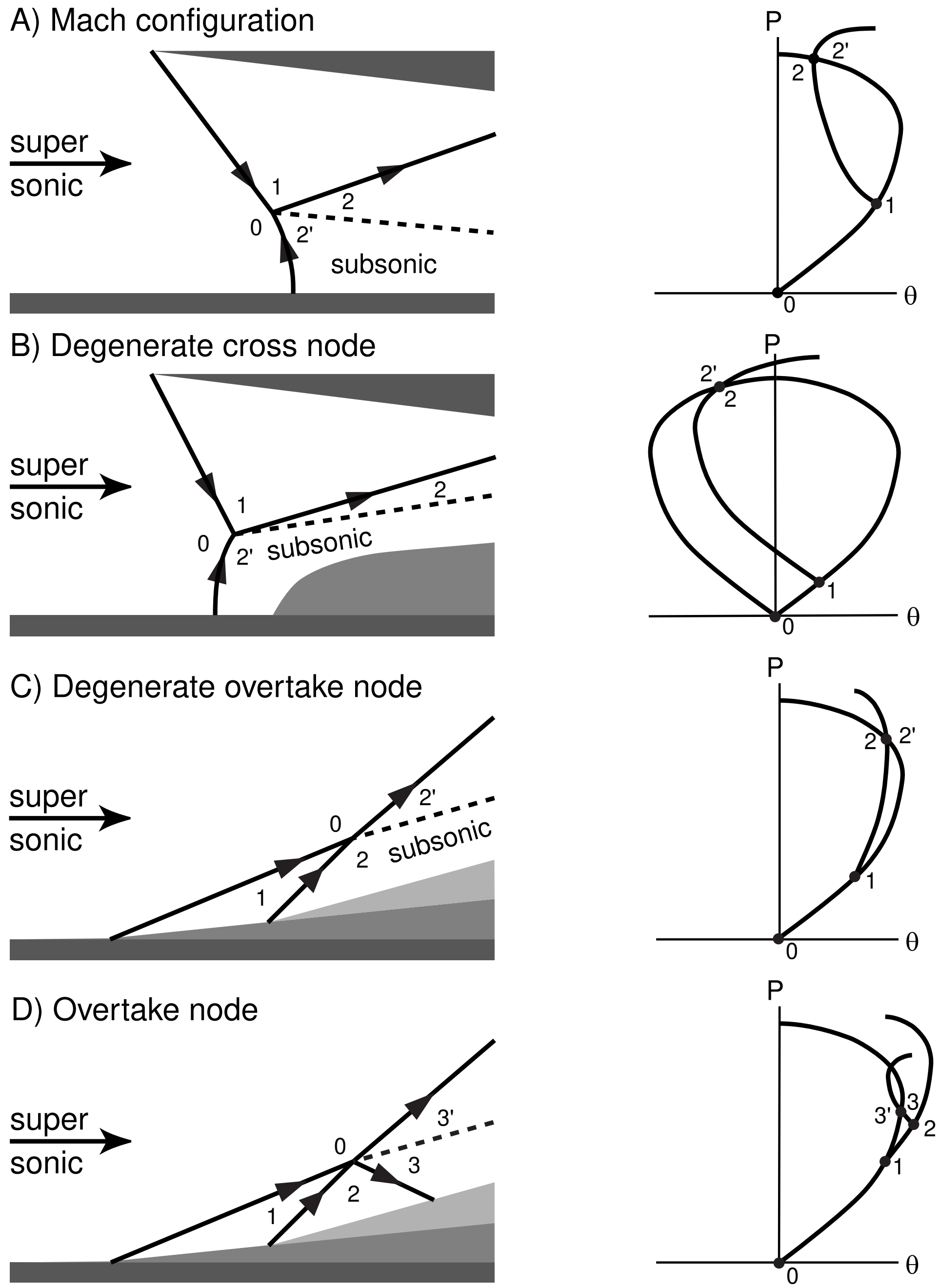

FIGURE 6. Examples of 2-D wave patterns: (a) Mach reflection, (b) degenerate cross node, (c) degenerate overtake node, (d) overtake node with two outgoing shocks.

node usually has two outgoing waves. In the degenerate case, one outgoing wave has zero strength. Despite the degeneracy, this wave pattern can generically occur because the flow behind the second incoming wave is subsonic. Consequently, the shock front can curve in response to downstream boundary conditions and compensate for the lower dimensionality of the degenerate wave pattern.

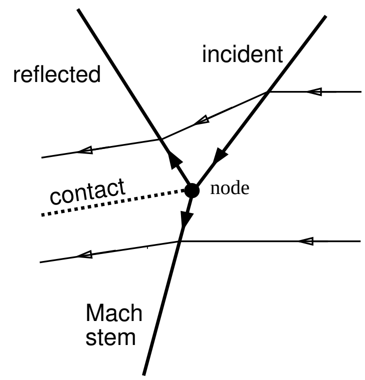

FIGURE 7. Stream lines for a Mach configuration in the rest frame of the node.

An overtake node also has two incoming shocks, but now they belong to the same wave family rather than the opposite family as in the cross node. A degenerate overtake node (DON) is illustrated in figure 6c. The outgoing waves consist of a transmitted shock and a contact. This only occurs when the flow downstream of the second incoming shock is subsonic. In the general case, the downstream flow is supersonic and the overtake node gives rise to two outgoing waves separated by a contact; the extra wave may be either a shock, as shown in figure 6d, or an expansion (not illustrated). The overtake node with two outgoing shocks is analogous to the 1-D shock interaction shown in figure 5c.

We consider in detail a Mach configuration. As shown in figure 7, streamlines either go through the incident and reflected shocks, or through the Mach stem. Downstream of the node (triple point), pressure is continuous. The Triple Shock Entropy Theorem implies that the entropy behind the Mach stem is higher than the entropy behind the reflected shock. Therefore, the intersection of three shocks *must* generate a contact. Weak reflections sometimes have the appearance, due to limited resolution, of 3 shocks intersecting without a contact. These patterns actually correspond to a von Neumann reflection Colella and Henderson (1990) in which the node is smeared out and not a singular point.

For a stationary wave pattern, the total enthalpy $\frac{1}{2}|\vec{u}|^2 + H$ is constant. By Corollary 9 the enthalpy is higher behind the Mach stem. Therefore, in the frame of the node, the velocity is lower behind the Mach stem than behind the reflected wave. Thus, at the contact in a Mach configuration there must be both an entropy jump and a velocity jump. Since the velocity jump occurs in the tangential component of the velocity, the contact is a velocity shear layer and a Mach configuration generates vorticity. The same considerations apply to the other examples shown in figure 6.

4.1. *Nonuniqueness*

For 2-D wave interactions, the outgoing waves are found from a graphical construction analogous to the one used in section 3 for 1-D shock interactions. Instead of the 1-D wave curves in the u–P plane, the 2-D construction uses shock polars, i.e., the pressure P as a function of the turning angle θ of a streamline through a stationary oblique shock. In the rest frame of the node, the normal velocity of the contact behind the node vanishes. The shock polar analysis corresponds to the 2-D "jump conditions" for the conservation laws around the singular point (the node) about which shock waves intersect. The rest frame of the node is essential for the shock polar analysis since the turning angle of a streamline is not a Galilean invariant.

In contrast to 1-D wave curves, shock polars are bounded and not monotonic. Consequently, the shock polars in the graphical construction may have no intersection or more than one intersection. A similar situation occurs in 1-D when the EOS is non-convex. Analogies with the non-convex 1-D case are noted later. When there is no intersection, the problem of nonexistence is resolved with more complex wave patterns; either with transonic curved shocks or with composite wave patterns. Examples of the first type are Von Neumann reflection Colella and Henderson (1990), and anomalous reflection Grove and Menikoff (1990). These wave patterns are not steady but may be pseudo-steady. Thus, resolution of nonexistence may require time dependent solutions. Composite wave patterns consist of multiple simple nodes separated by smoothly varying flow. The multiple nodes are a result of internal shocks generated by the incoming waves. Since the nodes have different velocities the analysis of each simple node requires a different set of incoming and outgoing shock polars. Examples of composite wave patterns include complex Mach reflection, double Mach reflection and the precursor wave pattern studied by Henderson *et al.* (1991). An analogous situation occurs in 1-D for a non-convex EOS. A rarefaction curve ends when $\mathcal{G} = 0$, and in order for all Riemann problems, *i.e.*, shock interactions, to have a solution it is necessary to continue the rarefaction curve with a composite consisting of a rarefaction followed by an anomalous (entropy increasing expansive) shock and then a second rarefaction, see Zel'dovich and Raizer (1967), chpt. XI, § 20. In this article we only consider the problem of multiple intersections of the shock polars and the question it raises for nonuniqueness of shock interactions.

For time dependent problems, the incoming wave types are assumed to be known at the initial time. When the solution changes slowly in time the wave pattern is determined by continuity of the polar solution. Some solutions have wave patterns in which the flow behind an outgoing wave is subsonic. In these cases, downstream boundary conditions are needed to determine the rest frame of the node. Time dependent boundary conditions can cause a wave pattern to bifurcate or change form. Bifurcations can be triggered by acoustic waves impacting a node (the limiting case in which a weak node collides and scatters off another node) or forced by a sudden change in geometry, *e.g.*, a shock impacting a wedge. Thus, for time dependent problems the nonuniqueness of shock interactions is a consequence of the ambiguity of when a wave pattern bifurcates and the nonuniqueness of the possible wave patterns into which another can bifurcate. Both shock tube experiments and numerical experiments have shown that when a shock impacts a wedge leading to a Mach reflection, the path of the triple point can be greatly affected by a boundary layer due to either viscosity or heat conduction Henderson *et al.* (1997). Thus, dissipative mechanisms at small scales can lead to local downstream boundary conditions which affects the bifurcation process.

Additional complications arise when determining the wave patterns that occur in steady state flows. In steady state, the wave pattern must be compatible with the global flow. Moreover, the identification of incoming and outgoing waves may depend on the downstream flow; e.g, figure 6b indicates how a downstream obstacle can lead to a degenerate cross node in place of a more common regular reflection. Furthermore, the number of boundary conditions for the spatial region of interest depends on the nature of the flow, in particular, the number of families of characteristics that are directed into the region. Frequently, the downstream boundary conditions are not completely known ahead of time. For example, a vacuum pump may determine only the pressure or average flow rate in the downstream portion of a wind tunnel. As a local condition, this may not be sufficient to determine whether the out flow is supersonic or subsonic. Hence the number of incoming characteristics and the corresponding value of the Riemann invari-

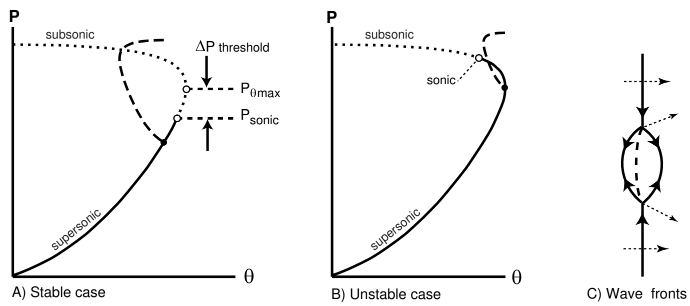

FIGURE 8. Two-dimensional shock splitting. (A) Pressure threshold when 2-D stability condition holds. (B) Shock polars when stability condition is not satisfied. (C) Shock front with pair of Mach nodes that spontaneously form when 2-D stability condition is violated. Dotted lines with arrows indicate the direction of propagation of the shock fronts and of each node.

ants must be determined as part of the global solution. Consequently, the wave patterns are not always known a priori.

Hysteresis observed in experiments leading to steady state flow shows that more than one global solution is compatible with the applied boundary conditions. In principle, the transients at the start of an experiment and the small scale dissipative mechanisms (such as viscosity or heat conduction which can lead to boundary layers) determine the asymptotic steady state flow. Without knowing the time history or the details of the dissipative mechanisms, the local stability of possible wave patterns is an important means for selecting which steady wave pattern is most likely to occur. Next we show that stability of a planar shock in multi-dimensions is an example of the considerations that can be used to resolve the nonuniqueness of shock polar solutions.

4.2. *Stability*

One example of nonuniqueness is the possibility of shock splitting, *i.e.*, a Mach configuration represents an incoming shock splitting into two outgoing shocks, see figure 6a. If this splitting were to occur spontaneously then a shock wave would be unstable. Linear stability analysis Kontorovich (1959) can be used to determine the conditions for which a planar shock wave is stable to 2-D perturbations. Fowles (1981) showed that the shock stability condition can be expressed simply as an algebraic condition on the equation of state

$$(\Gamma + 1)\frac{\Delta P}{P} \leqslant \gamma \,. \tag{4.1}$$

When $\Gamma + 1 \leqslant \gamma$, all pressure increasing shocks are 2-D stable. Moreover, as derived in Appendix B, Fowles showed that the stability inequality is equivalent to a geometric condition on a shock polar, namely, the sonic point occurs at a lower pressure than the maximum turning angle. As a consequence there is a minimum strength for the reflected wave in a Mach configuration since on the shock polar, see figure 8a, the ahead state lies below the sonic point and the behind state lies above the maximum turning angle. We next show that the minimum reflected wave strength can be interpreted as giving rise to a nonzero threshold for a perturbation to cause a shock to split.

Let us consider a shock wave for which the stability condition is violated. A continuous path can be constructed by which a shock can split into two shocks with a higher entropy as follows. We choose the reference frame in which the shock front is stationary and the incoming velocity is given by (B 2). In this frame the shock corresponds to the detachment point on the shock polar. By theorem 15 the detachment point is supersonic. If the tangential component of the incoming velocity is increased slightly then the shock state remains supersonic but lies slightly below the detachment point on the new shock polar as shown in figure 8b. Consequently, the shock can split into a pair of Mach configurations, see figure 8c, in which the Mach stem has only a slightly greater shock pressure and the reflected wave is arbitrarily weak. By slowly increasing the tangential component of the incoming velocity, there is a continuous path in which the reflected shock increases in strength. It follows from the triple shock theorem that the shock splitting raises the entropy of the flow. This simple picture underlies the weakly non-linear analysis of Majda and Rosales (1983).†

Both Kontorovich (1959) and Fowles (1981) presented an alternative way of viewing the instability. Consider a shock interaction in which one of the incoming waves is weak, *i.e.,* an acoustic wave. An important parameter characterizing the interaction is the reflection coefficient; the ratio of the pressure jump across the reflected wave to the pressure jump across the weak incoming wave. If the stability condition is violated, then there is an angle of incidence between the shock and the weak wave such that the reflection coefficient is infinite. This angle of incidence corresponds to the reference frame in which the shock state coincides with the detachment point on the shock polar. The infinite reflection coefficient implies that an arbitrarily weak perturbation leads to a finite outgoing wave. Consequently, the outgoing waves are spontaneously formed and the shock front is unstable leading to transverse waves propagating along the shock front. A similar transverse instability has been observed experimentally and numerically for detonation waves; see *e.g.,* Fickett and Davis (1979) and references therein and Bourlioux and Majda (1992). However, the instability of the detonation wave depends on the reaction zone dynamics and is not merely the result of violating the stability condition for the equation of state of the reaction products; see also Mond and Rutekvich (1994) for the derivation of a modified stability condition applicable to a partly dispersed shock in which the relaxation is due to ionization.

Conversely, when the stability condition holds there is no continuous path along which a shock wave can split into a higher entropy state. The instability condition can be interpreted as giving rise to a threshold. Waves with an amplitude below the threshold are scattered from a shock front (the analog of elastic scattering) while waves above the threshold cause a bifurcation of the shock front in which a pair of Mach nodes are formed (the analog of inelastic scattering).

Next we discuss the transition between regular and Mach reflection. There is a range in parameter space (shock strength and incident shock angle) for which a symmetric pair of incoming shocks can give rise, based on shock polar analysis, to either regular reflection (cross node) or Mach reflection (pair of Mach nodes). A transition criterion is frequently used to resolve the nonuniquess in the wave pattern. Many steady state wind tunnel and shock tube experiments have shown that if the flow is two-dimensional then the transition criterion for strong incident waves corresponds to the von Neumann or mechanical equilibrium point on the shock polar Mölder (1971), Pantazapol *et al.* (1972), Henderson and Lozzi (1975), Hornung *et al.* (1979), Hornung and Robinson (1982).

In the wind tunnel experiments, see figure 9, the incoming flow velocity is fixed and

† Majda and Rosales (1983) attempted to explain the transverse instability observed experimentally in propagating detonation waves. Their analysis neglected the dynamics of the reaction zone by treating the detonation wave as a discontinuity. Consequently, their analysis really applies to shock waves rather than detonation waves.

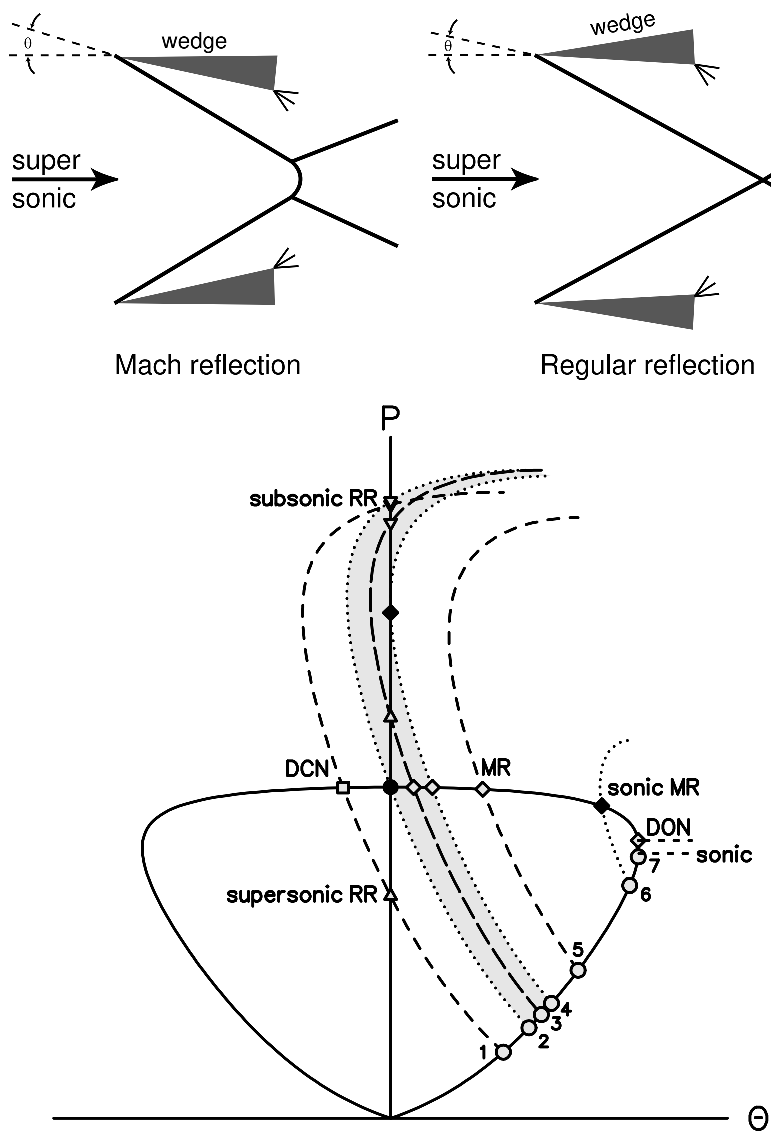

FIGURE 9. Hysteresis effect for transition between regular reflection and Mach reflection. (a) Wind tunnel experiment in which hysteresis is observed. The wedges can be rotated about upstream corner to vary incoming shock waves. (b) Shock polar diagram for wave pattern generated by experiment. The polar for the incoming wave is shown as a solid line and the polar for the reflected wave as a dotted or dashed line. The symbols denote the following wave states: open circle is an incoming shock state, upward pointing triangle is a supersonic regular reflection (RR) state, downward pointing triangle is a subsonic RR state, open diamond is a Mach reflection (MR) state, open square is a degenerate cross node (DCN) state, solid circle is the von Neumann (mechanical equilibrium) transition point, solid diamond is either the detachment transition point or the point at which the reflected wave in Mach configuration becomes subsonic. Horizontal dashed lines bound the small region for degenerate overtake node (DON) states. The reflected polars with both RR and MR comprise the shaded region.

the shock waves are generated by wedges placed in the flow stream. As the wedge angle is varied the state behind the lead (upstream) shock moves along a fixed shock polar. At some angle the reflected shock polar intersects the incident shock polar at its maximum pressure. This intersection point is known as the von Neumann or mechanical equilibrium point. For smaller wedge angles the incoming shock pressure is lower and only regular reflection is possible, while at larger angles both regular and Mach reflection are possible. At the von Neumann point, the pressure immediately downstream of the shock intersection point is the same for both regular and Mach reflection. The triple shock theorem implies that the Mach configuration has a higher entropy than that of the regular reflection. By continuity, in the neighborhood of the mechanical equilibrium point, the Mach configuration still has the higher entropy. For a continuous transition one expects the higher entropy configuration would be favored based on thermodynamic stability. This provides a heuristic motivation for the von Neumann point as the transition criterion.

A similar nonuniquess occurs in 1-D when the equation of state is non-convex. In some limited regimes in phase space, both a single shock wave and a composite wave, consisting of a sonic shock followed by a compressive wave, can have the same final pressure. We note that a compressive wave can spreads out rather than focusing into a shock in regions of phase space with $\mathcal{G} < 0$. The nonuniquess is resolved with the Oleĭnik-Liu extended entropy condition Liu (1976); shock waves on the Hugoniot locus for which there exists a shock with a lower pressure and a larger shock speed are excluded and replaced by composite waves. The admissible waves can be justified with a stability argument; perturb the shock by smearing it into a smooth profile and then analyze the evolution of the profile by examining how the characteristics focus to regenerate the shock wave. The extended entropy condition is equivalent to selecting the wave, single shock or composite, which for a given pressure has the higher entropy.

4.3. *Hysteresis*

In the region in which the solution to the shock polars is not unique, hysteresis has been observed both experimentally and numerically for the regular to Mach transition Chpoun *et al.* (1995), Vuillon *et al.* (1995), Ivanov *et al.* (1995a), Ivanov *et al.* (1995b), Henderson *et al.* (1997). To set the stage for our discussion of the transition we first describe the possible wave patterns derived from shock polar analysis. The shock polar diagram corresponding to the quasi-steady wind tunnel experiments is shown in figure 9. This particular diagram is for an ideal gas EOS with $\gamma = 1.4$ and an incident flow Mach number of 3.5. (The structure would be qualitatively different only for a low incident Mach number.) There are several distinct regimes:

(I) Regular reflection only (state 1)

At small wedge angles the incident shock has low pressure and there are three possible wave patterns: (i) supersonic regular reflection, (ii) degenerate cross node, (iii) subsonic regular reflection. The degenerate cross node is not admissible since it has two incoming waves and the flow can only support one. This is because the contacts behind the DCN would form a diverging nozzle. Since the flow is subsonic the pressure would have to increase. But the downstream boundary condition does not support the pressure increase. Hence the degenerate cross node can not occur in the steady flow. The subsonic regular reflection is unstable to small perturbations Teshukov (1989). Therefore, it can not occur in the asymptotic steady state flow. Consequently, in this regime only the supersonic regular reflection occurs.

(II) Von Neumann (mechanical equilibrium) point (state 2)

For a particular incoming shock strength, the supersonic RR coincides with the maximum pressure on the incident shock polar. This is the transition between the DCN in the previous case and the MR in the next case.

(III) Regular reflection and Mach reflection (state 3)
Immediately above the von Neumann point (slightly stronger incident shock wave) both supersonic regular reflection and Mach reflection are possible. The reflected wave in the Mach configuration is supersonic. Again the subsonic regular reflection is inadmissible based on stability.

(IV) Sonic regular reflection
As the incident shock strength increases a point is reached at which the regular reflection becomes sonic. For stronger incident shocks both regular reflection solutions are subsonic.

(V) Detachment point (state 4)
There is a distinguished incident shock strength at which the two possible regular reflections coalesce. The regular reflection is subsonic. For stronger incident shocks there are no regular reflection shock polar solutions.

(VI) Mach reflection only (state 5)
The shock polar diagram has a unique solution corresponding to Mach reflection. The reflected wave in the Mach configuration is supersonic.

(VII) Mach reflection with sonic reflected wave (state 6)
As the incident shock strength increases a point is reached at which the reflected wave in the Mach configuration becomes sonic. For stronger incident shocks the reflected wave is subsonic.

(VIII) Termination point for Mach reflection (state 7)
As the incident shock strength is further increased a point is reached at which the deflection angle vanishes for the reflected wave in the Mach configuration. This is the termination point of the Mach wave patterns.

(IX) Degenerate overtake node
For still larger incident shock strength, the shock polar diagram has a unique solution corresponding to a degenerate overtake node. This wave pattern is not admissible since it has 2 incoming waves and the flow can support only one incoming wave. Most likely, the flow pattern will have a single curved shock wave which begins at the tip of one wedge and ends at the tip of the other wedge. A similar curved shock occurs in the pseudo-steady case; Hornung (1986), figure 13. For weak incident shocks this becomes the von Neumann reflection Colella and Henderson (1990).

(X) Beyond maximum turning angle
When the wedge angle is greater than the maximum turning angle on the incident shock polar, no shocks in the channel between the wedges are possible. A bow wave will develop in front of the wedges leading to subsonic flow in the channel.

Down stream boundary conditions and stability are determining factors in selecting admissible steady state wave patterns. However, in regime III there still remains two possibilities; either regular reflection or Mach reflection. We note that Hornung (1986) presented a similar catalog of wave patterns for the pseudo-steady flow that occurs in shock tube experiments studying the impact of a shock on a wedge. In the shock tube experiments, the shock strength is fixed and the wedge angle is varied. In this case the incident wave speed in the frame of the node is not fixed. Consequently, the different regimes can not be pictured on a single shock polar diagram.

For the wind tunnel experiments the incoming flow velocity is fixed and the wedge angle is slowly varied. The wave pattern is quasi-steady but changes with the wedge angle. Some of the experiments show the following hysteresis effect. At small wedge angles there is only one allowed wave pattern and regular reflection is observed. As the wedge angle is increased and the incident pressure rises above the von Neumann point, the regular reflection is maintained. Beyond the detachment point regular reflection is not possible and the pattern abruptly changes to Mach reflection. Then as the wedge angle decreases the Mach reflection continues to occur until the von Neumann point. Thus, with increasing wedge angle the transition from regular to Mach reflect occurs near the detachment point while with decreasing wedge angle the transition from Mach to regular reflection occurs at the von Neumann point. Other experiments do not show a hysteresis effect and both transitions occur at the von Neumann point.

The hysteresis effect implies that both configurations are locally stable. Several experiments suggest that the transition is affected by the flow behind the node: (i) When a shock reflects from a curved wall, the transition angle varies with the shape of the wall; Henderson and Lozzi (1979), Ben-Dor *et al.* (1980), Takayama and Sasaki (1983). (ii) Both regular and Mach reflection occur simultaneously when a planar shock impacts a cylindrical cone whose axis is not aligned with the shock normal. The angle between the shock and the cone wall varies with the azimuthal angle of the cone, and at some angle the wave pattern changes from regular to Mach reflection. The transition angle along the cone is not the same as for a planar wedge; Suzuki *et al.* (1995), Yang *et al.* (1996). (iii) Cylindrically or spherically converging shocks appear to become unstable to transverse waves (*i.e.,* Mach configurations spontaneously form) at large convergence ratios when the flow gradients behind the shock front becomes large; Wallus and Demmig (1995).

To understand this phenomenon, we suggest an analogy with a gas/liquid phase transitions. The following are the relevant aspects of a phase transition. The constraints of the system require that in equilibrium the two phases have the same pressure and temperature. Long lived meta-stable states can exist, such as a super-cooled gas. The life time of the meta-stable state can be greatly diminished by impurities such as nucleation centers. This is because there is a potential barrier between a super-cooled gas and a liquid. Surface energy at a gas/liquid interface causes small drops formed by thermodynamic fluctuations to be thermodynamically unfavorable and to evaporate before they can grow into the stable equilibrium state. A nucleation site decreases the potential barrier and increase the rate at which large drops form.

For the analogy we have in mind the two wave patterns, Mach reflection and regular reflection, play the role of the two phases. In steady state flow, the total enthalpy is a constant. The equilibrium configuration is determined by maximizing the entropy subject to the constraint that both the pressure and the total enthalpy are fixed. By the triple shock theorem, the Mach reflection corresponds to the equilibrium state and the regular reflection to a meta-stable state.

The hysteresis effect for the regular to Mach transition may be explained by the hypothesis that there is a transition threshold. Acoustic noise sets the background perturbation amplitude and is the analog of thermodynamic fluctuations. A flow gradient or a boundary layer Henderson *et al.* (1997) along the wedge wall affects the transition threshold and plays a role similar to that of a nucleation center on the potential barrier. The life time of the meta-stable regular reflection configuration can be decreased causing a transition to the maximum entropy Mach reflection either by lowering the threshold or by raising the background noise. Conversely, the transition can be delayed by either raising the threshold or by lowering the background noise. Only when perturbations are below the threshold would hysteresis be observed Ivanov *et al.* (1995b). Without flow gradients to raise the threshold, there is usually sufficient noise in a wind tunnel for the most stable local configuration to occur. This is compatible with the many experiments that measure the transition between regular and Mach reflection to occur at the von Neumann point.

An important source of flow gradients are from 3-D effects; for example, when a shock impacting a cone at an angle of attack, the transverse gradient is from the asymmetry of the conical flow and as noted previously this affects the transition. In recent experiments with open sided models Chpoun *et al.* (1995), Fomin *et al.* (1996), Skews (1996) there are transverse gradients that have a decisive influence on the hysteresis effect. In particular, if the aspect ratio, height to width of the model inlet, is greater than 2.5 then transition occurs at the von Neumann point. In this case the transverse gradient are small and have no affect on transition and there is no hysteresis. However, if the aspect ratio is reduced to 1 or less then there is a significant hysteresis effect.

In summary, the triple shock entropy theorem helps to determine the most stable equilibrium wave pattern. For neutral stability, non-linear effects give rise to a transition threshold that allows for meta-stable or long lived non-equilibrium states. Hysteresis results from non-uniqueness due to the existence of meta-stable states. When the background noise exceeds the transition threshold, then after the transient subsides, the system stabilizes in the equilibrium configuration with the maximum entropy.

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Appendix A. Implicit Value Theorem

Our proof that the Hugoniot locus for a convex equation of state can be parameterized by the entropy, Corollary 5, depended critically on the implicit value theorem. This theorem is a corollary of the Inverse Function theorem. Here we state a simple version of the theorem sufficient for our needs.

THEOREM 11 (INVERSE FUNCTION THEOREM). *Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Let $f'(a)$ denote the derivative of f at a; i.e., the matrix of partial derivatives. If $\det f'(a) \neq 0$ then there are open sets V and W such that $a \in V$ and the restriction $f : V \to W$ has a continuous inverse $f^{-1} : W \to V$ that is differentiable.*

For an elementary proof, see for example, Spivak (1965), p. 35.

For our application to the Hugoniot locus we define the function from $\mathbb{R}^2 \to \mathbb{R}^2$ by $f(P,S) = (\tilde{h}(P,S), S)$. The derivative matrix is given by

$$f' = \begin{pmatrix} \partial_P \tilde{h} & 0 \\ \partial_S \tilde{h} & 1 \end{pmatrix} .$$

It follows that $\det f' = \partial_P \tilde{h}$. Therefore, the inverse function theorem applies when $\partial_P \tilde{h} \neq 0$, and then locally $P = P(\tilde{h}, S)$. Since the Hugoniot locus is defined by $\tilde{h} = 0$, there is then a neighborhood of a point on the Hugoniot locus in which $P_h = P(0,S)$. Because P and S are sufficient to define the state behind the shock front, the Hugoniot locus is locally parameterized by S about any point at which $\partial_P \tilde{h} \neq 0$. This is an example of how the equation $\tilde{h}(P,S) = 0$ implicitly defines a function $P_h(S)$ when one of the partial derivatives (in this case $\partial_P \tilde{h}$) is non-zero.

Appendix B. Detachment and sonic points on a shock polar

A shock polar describes the properties of stationary oblique shocks used in the analysis of 2-D wave patterns. For a given upstream state, the shock polar relates the downstream pressure to the deflection angle of a streamline through the shock front. It plays a similar role for determining the outcome of 2-D shock interactions as the projection of the Hugoniot locus in the u–P plane does for the solution of a 1-D Riemann problem.

In this section, the properties of a shock polar are derived which we use in our analysis of 2-D shock stability. First, we relate the shock polar to the Hugoniot locus. It is convenient to choose a reference frame for 1-D shock waves in which the flow ahead of the shock front is at rest, see figure 10a. We denote by u_s and u_p the velocity of the shock front and the particle velocity behind the front, respectively. When $\mathcal{G} > 0$ and $\Gamma < \gamma$, either u_s or u_p may be used to parameterize the 1-D Hugoniot locus. Moreover, we consider only the physical entropy increasing shocks. These are compressive and pressure increasing. For an oblique 2-D shock wave, we use the reference frame in which the shock front is at rest, see figure 10b. We denote by q_0 the velocity ahead of the front, and β_0 and β_1 angle between the shock front and the streamline ahead and behind the front respectively. Since the flow ahead of shock is supersonic, the shock polar is only defined for $q_0 > c_0$ and the angle of the front is restricted to the range $\sin^{-1}(c_0/q_0) \leqslant \beta_0 \leqslant \frac{\pi}{2}$.

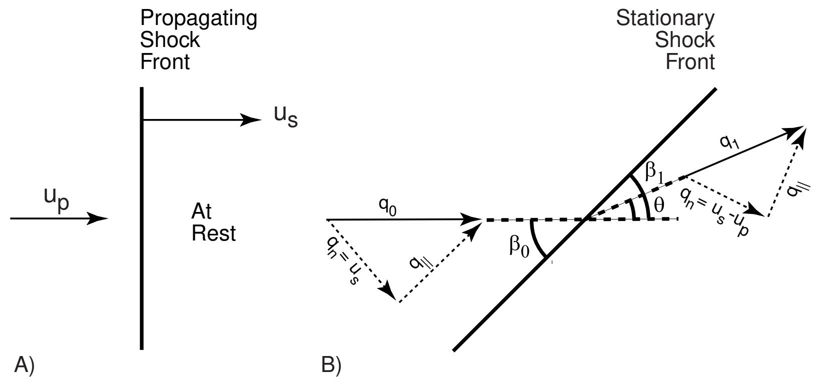

FIGURE 10. Notation used to describe velocities across shock front. (A) 1-D (normal) shock wave in the rest frame of the state ahead of the shock front. (B) Oblique 2-D shock wave in the rest frame of the shock front.

The component of the velocity parallel to the front is denoted $q_{\shortparallel}$. It is continuous across the front. Hence, $q_{\shortparallel} = q_0 \cos(\beta_0) = q_1 \cos(\beta_1)$. The deflection angle of the streamlines is given by $\theta = \beta_0 - \beta_1$. As in 1-D, there are two non-degenerate wave families; oblique shocks with positive and negative deflection angle ($\theta > 0$ and $\theta < 0$) are the analog of the left and right facing 1-D waves.

From the geometry in figure 10, the angles of the streamlines for an oblique shock are related to the 1-D shock velocities as follows

$$\tan\beta_0 = \frac{u_s}{q_{\shortparallel}}\,,$$
$$\tan\beta_1 = \frac{u_s - u_p}{q_{\shortparallel}}\,.$$

Using the trigonometric identity for the tangent of the difference of two angles, we obtain

$$\begin{aligned}\tan\theta &= \frac{\tan\beta_0 - \tan\beta_1}{1 + \tan\beta_0 \tan\beta_1} \\ &= \frac{u_s u_p}{q_0^2 - u_s u_p} \cdot \frac{q_{\shortparallel}}{u_s}\end{aligned} \tag{B 1}$$

At the detachment point of a shock polar, the deflection angle is maximum. It is determined by the condition

$$\frac{d}{du_p} \log\tan\theta = 0\,.$$

Straight forward algebra then reduces to the equation

$$q_0^2 = u_s^2 + (u_s - u_p) u_p \frac{du_s}{du_p}\,. \tag{B 2}$$

From this we can deduce

THEOREM 12. *Suppose $\mathcal{G} > 0$ and $\Gamma \leqslant \gamma$ are satisfied everywhere. Consider the 1-parameter family of shock polars in which the upstream thermodynamic state (V_0, S_0) is fixed and q_0 is varied. Then the following conditions are equivalent:*

(i) *On the Hugoniot locus based on the state (V_0, S_0) the quantity $u_s^2 + (u_s - u_p) u_p \frac{du_s}{du_p}$ is monotonically increasing with increasing shock strength.*

(ii) *Every shock coincides with the detachment point on a shock polar for a unique value of* q_0.

(iii) *Every shock polar has a unique local extremum in* θ, *and hence one and only one detachment point.*

COROLLARY 13. *Suppose* $\mathcal{G} > 0$ *and* $\Gamma \leqslant \gamma$ *are satisfied everywhere. Consider the Hugoniot locus based on the thermodynamic state* (V_0, S_0). *If on the Hugoniot locus*

$$V_0 \frac{d^2 P}{du_p^2} + \frac{du_s}{du_p} > 0 \tag{B 3}$$

is satisfied, then every shock polar with the initial thermodynamic state (V_0, S_0) *has a unique detachment point.*

Proof. By differentiation of (B 2), the monotonicity condition in theorem 12 is equivalent to

$$u_p\Big(1 + \frac{du_s}{du_p}\Big)\frac{du_s}{du_p} + (u_s - u_p)\Big(u_p \frac{d^2 u_s}{du_p^2} + 3\frac{du_s}{du_p}\Big) \geqslant 0 \,. \tag{B 4}$$

From the Hugoniot jump conditions

$$\Delta P = \rho_0 u_p u_s \,.$$

The second derivative of P is given by

$$V_0 \frac{d^2 P}{du_p^2} = u_p \frac{d^2 u_s}{du_p^2} + 2\frac{du_s}{du_p} \,. \tag{B 5}$$

Substituting (B 5) into (B 4) we obtain

$$(u_s - u_p)\left(V_0 \frac{d^2 P}{du_p^2} + \frac{du_s}{du_p}\right) + u_p\Big(1 + \frac{du_s}{du_p}\Big)\frac{du_s}{du_p} \geqslant 0 \,.$$

Since $u_s > u_p$ and $du_s/du_p > 0$, (B 3) is a sufficient condition for (B 4). □

When the EOS satisfies $\mathcal{G} > 0$ and $\Gamma \leqslant \gamma$, the shock velocity is monotonically increasing, *i.e.*, $du_s/du_p > 0$. Consequently, convexity of $P(u_p)$ is a sufficient condition for a unique detachment point. We note that convexity of wave curves, *i.e.*, $d^2P/du_p^2 > 0$, is also a sufficient condition to guarantee convergence when solving 1-D Riemann problems with a second order Newton iteration algorithm.

The stability of shock waves in multi-dimensions is determined by the position on the shock polar of the sonic point relative to the detachment point. First, we determine the sonic point. In the rest frame of the shock front, the total enthalpy is a constant. Hence,

$$q^2 - c^2 = q_0^2 + 2H_0 - (2H + c^2) \,, \tag{B 6}$$

where $H = e + PV$ is the specific enthalpy. From this we can deduce

LEMMA 14 (TESHUKOV (1986)). *Suppose* $\mathcal{G} > 0$ *is satisfied everywhere. Consider the 1-parameter family of shock polars in which the upstream thermodynamic state* (V_0, S_0) *is fixed and* q_0 *is varied. For all* $q_0 > c_0$ *the shock polars have a unique sonic point if and only if along the Hugoniot locus based on the state* (V_0, S_0) *the quantity* $H + \frac{1}{2}c^2$ *is monotonically increasing with shock strength.*

Proof. On the shock polar, the flow behind weak shocks, $\sin\beta_0 \gtrsim c_0/q_0$, is supersonic while the flow behind nearly normal shocks, $\sin\beta_0 \lesssim \frac{\pi}{2}$, is subsonic. By continuity there

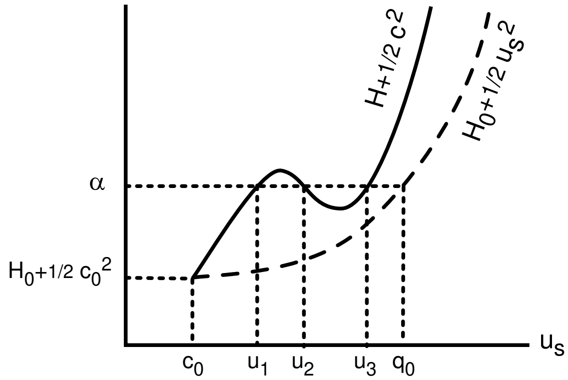

FIGURE 11. Sketch of functions used to show when a shock polar has nonunique sonic point. Hugoniot locus is parameterized by u_s. The solid line is the function $H + \frac{1}{2}c^2$ and the dashed line is the function $H_0 + \frac{1}{2}u_s^2$. A value of q_0 and the corresponding shocks for multiple sonic points are indicated.

is at least one sonic point on every shock polar. If $H + \frac{1}{2}c^2$ is monotonic then by (B 6) $q^2 - c^2$ is monotonic and there is only one sonic point.

Since $\mathcal{G} > 0$, the Hugoniot locus may be parameterized by u_s. Figure 11 illustrates the case when $H+\frac{1}{2}c^2$ is not monotonic. Suppose the constant function $\alpha(u_s) = \alpha$ intersects the curve $H+\frac{1}{2}c^2$ more than once. We can choose q_0 such that $\alpha = H_0 + \frac{1}{2}q_0^2$. Since the flow behind a normal shock $(u_s = q_0)$ is subsonic, the curve $H_0 + \frac{1}{2}u_s^2$ lies below and to the right of the curve $H + \frac{1}{2}c^2$. Therefore, the values of $(u_s)_i$ at which $H + \frac{1}{2}c^2 = \alpha$ are all less than q_0, and the shock polar for q_0 has a sonic point at every intersection of α with the curve $H + \frac{1}{2}c^2$. Thus, if $H + \frac{1}{2}c^2$ is not monotonic then there is a q_0 such that the shock polar has more than one sonic point. □

For weak shocks it can be shown from a Taylor expansion of the Hugoniot jump conditions that

$$c = c_0 + (\mathcal{G}_0 - 1)u_p + O(u_p^2) \; ,$$
$$u_s = c_0 + \frac{1}{2}\mathcal{G}_0 u_p + O(u_p^2) \; .$$

Hence,

$$u_s^2 + (u_s - u_p)u_p\frac{du_s}{du_p} = c_0^2 + \frac{3}{2}\mathcal{G}_0 c_0 u_p + O(u_p^2) \; ,$$
$$H + \frac{1}{2}c^2 = H_0 + \frac{1}{2}c_0^2 + \mathcal{G}_0 u_p + O(u_p^2) \; .$$

Therefore, for convex equations of state when the Mach number ahead of the shock is near 1, *i.e.*, $0 < q_0/c_0 - 1 \ll 1$, shock polars have a unique detachment point and a unique sonic point.

More generally, after some algebra, it can be shown that along the Hugoniot locus

$$\frac{1}{T}\frac{d}{dS}\Big(H + \frac{1}{2}c^2\Big)_h = 1 + \Gamma + \gamma\Big[\mathcal{H} + \mathcal{G}\Big(\frac{-P}{T}\frac{dV}{dS}\Big)_h\Big] \; , \qquad \text{(B 7)}$$

where $\mathcal{H}$ is another dimensionless parameter describing the equation of state

$$\mathcal{H} = \frac{1}{2\gamma}\left.\frac{\partial c^2}{\partial E}\right|_V \; ,$$

and

$$\left(\frac{P}{T}\frac{dV}{dS}\right)_h = \frac{2V}{\Delta V}\left(\frac{1+\frac{1}{2}\Gamma\Delta V/V}{\gamma+(V/P)(\Delta P/\Delta V)}\right)$$

measures the variation of V with entropy along the Hugoniot locus. (Teshukov (1986) obtained an alternate formulae but it is harder to interpret.) We note that $\mathcal{H}$ is proportional to $\partial c^2/\partial T|_V$ and is typically positive. Moreover, V typically decreases with shock strength, *i.e.*, $-dV/dS|_h > 0$, except when the Hugoniot locus goes through regions of phase space in which dissociation or ionization are dominant effects. Consequently, the right hand side of (B 7) is typically positive and the Hugoniot locus typically satisfies the condition required for a shock polar to have a unique sonic point. Also, (B 4) is typically satisfied and shock polars typically have a unique detachment point.

Next we determine the relative positions of the sonic and detachment points. First we note that substituting the Hugoniot equation

$$2(H - H_0) = (V + V_0)\Delta P$$

into (B 6) we obtain for the sonic condition

$$q^2 - c^2 = q_0^2 - [\gamma PV + (V_0 + V)\Delta P]\,. \tag{B 8}$$

In order to evaluate (B 2) for the detachment point we need an expression for du_s/du_p. In their review article, Menikoff and Plohr (1989) have systematically determined the differential of all quantities along the Hugoniot locus. From their equations (4.26) and (4.27) we obtain

$$\frac{du_s}{du_p} = \frac{\left(\gamma + \frac{V}{P}\frac{\Delta P}{\Delta V}\right)\cdot\frac{V_0 m^2}{\Delta P}}{\gamma - \Gamma\frac{\Delta P}{P} - \frac{V}{P}\frac{\Delta P}{\Delta V}}\,, \tag{B 9}$$

where $m^2 = -\frac{\Delta P}{\Delta V} = (\rho_0 u_s)^2$ is the square of the mass flux through the shock front. Substituting (B 9) along with the equations for the velocities

$$\begin{aligned} u_s &= V_0 m\,, \\ u_p &= (V_0 - V)m\,, \\ u_s - u_p &= Vm\,, \end{aligned}$$

into (B 2) and subtracting $\gamma PV + (V_0 + V)\Delta P$ from both sides of the equation we obtain

$$q^2 - c^2 = V_0^2 m^2 - \gamma PV + (V_0 + V)\Delta P + \frac{(V_0 - V)Vm^2\left(\gamma + \frac{V}{P}\frac{\Delta P}{\Delta V}\right)\frac{V_0 m^2}{\Delta P}}{\gamma - \Gamma\frac{\Delta P}{P} - \frac{V}{P}\frac{\Delta P}{\Delta V}}\,.$$

Simple algebraic manipulations reduces the equation for the detachment point to

$$(q^2 - c^2)\left(\gamma - \frac{\Delta P}{P}\Gamma + \frac{V}{P}m^2\right) = -PV\left(\gamma - \frac{V}{P}m^2\right)\left[\gamma - \frac{\Delta P}{P}(\Gamma + 1)\right]. \tag{B 10}$$

We can now derive an important property of a shock polar.

THEOREM 15 (FOWLES (1981)). *Suppose $\mathcal{G} > 0$ and $\Gamma \leqslant \gamma$ are satisfied everywhere. On a shock polar the detachment point is subsonic if and only if*

$$\frac{\Delta P}{P}(\Gamma + 1) < \gamma\,. \tag{B 11}$$

Proof. We consider the sign of the factors on the left and right hand sides of (B 10). Since the physical shocks are pressure increasing, $0 < \Delta P/P < 1$. From the condition

$\Gamma \leqslant \gamma$, it follows that $\gamma - \frac{\Delta P}{P}\Gamma + \frac{V}{P}m^2 > 0$. Moreover, the state behind a physical shock is subsonic. Hence, $PV[\gamma - \frac{V}{P}m^2] = c^2 - (u_s - u_p)^2 > 0$. Therefore, the sign of $c^2 - q^2$ is the same as the sign of $\gamma - \frac{\Delta P}{P}(\Gamma + 1)$. Consequently, the detachment point is subsonic, $c > q$, if and only if (B 11) is satisfied. □

COROLLARY 16. *Suppose* $\mathcal{G} > 0$ *and* $\Gamma + 1 \leqslant \gamma$ *are satisfied everywhere. If a shock polar has a unique sonic point and a unique detachment point then in the* θ*–*P *plane the detachment point lies above the sonic point; i.e., the detachment point has a higher pressure than the sonic point.*

Proof. Points on the shock polar corresponding to weak shocks ($u_s \gtrsim c_0$) are supersonic while those corresponding to normal shocks ($u_s \lesssim q_0$) are subsonic. Since there is a unique sonic point on the shock polar, only those shocks with a pressure above the pressure at the sonic point are subsonic. By the previous theorem, the detachment point is subsonic. Therefore, it has a higher pressure than the sonic point. □

As discussed in section 4.2, inequality (B 11) is necessary for multi-dimensional shock stability. We note that since $\gamma > 0$, the inequality (B 11) is true for sufficiently weak shocks. Hence, weak shocks are always stable. For every shock to satisfy the multi-dimensional stability condition requires $\Gamma + 1 \leqslant \gamma$. This is stronger than the medium condition needed for uniqueness of all 1-D Riemann problems.