Diffusive Relaxation Schemes for Discrete-Velocity Kinetic Equations

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Abstract

Many kinetic models of the Boltzmann equation have a diffusive scaling that leads to the Navier-Stokes type parabolic equations, such as the heat equation, the porous media equations, the advection-diffusion equation and the viscous Burgers equation. In such problems the diffusive relaxation parameter may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes, and it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter. Earlier approaches that work from the rarefied regimes to the Euler regimes do not directly apply to these problems since here, in addition to the stiff relaxation term, the convection term is also stiff. Our idea is to reformulate the problem in the form commonly used for the relaxation schemes to conservation laws by properly combining the stiff component of the convection terms into the relaxation term. This, however, introduces new difficulties due to the dependence of the stiff source term on the gradient. We show how to overcome this new difficulty with a adequately designed, economical discretization procedure for the relaxation term. These schemes are shown to have the correct diffusion limit. Several numerical results in one and two dimensions are presented, which show the robustness, as well as the uniform accuracy of our schemes.

1. Introduction

Many kinetic models of the Boltzmann equation have a diffusive scaling that leads to the Navier-Stokes type parabolic equations, such as the heat equation, the porous media equations, the advection-diffusion equation and the Burgers equation. In such problems the diffusive relaxation parameter may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes, and it is desirable to develop a class of numerical schemes that can work uniformly with respect to this relaxation parameter.

It is known at the formal level that, when the relaxation parameter ϵ is small, the Boltzmann equation can be approximated by the Euler equations to the leading order and the Navier-Stokes equations to $O(\epsilon)$ [Cer]. For many kinetic models, however, the Navier-Stokes or the diffusive limit is well established mathematically. For example, in some recent papers [PuTo, LiTo, GaPe], it was shown that the equations of slow and fast diffusion, the porous media equation and the Burgers equation can be obtained as the Navier-Stokes or diffusive limit of kinetic models of Boltzmann type.

Although we are aiming at developing numerical schemes for kinetic models, the study of the diffusive limit has a critical importance. In order for a scheme to be useful not only in the rarefied regime (where ϵ is big) but also in the hydrodynamic regime (where ϵ is small) allowing the use of $\Delta t, \Delta x >> \epsilon^2$, where Δt and Δx are the time steps and grid size respectively, the scheme should possess the *correct diffusion limit*, in the sense that the asymptotic limit that leads from the kinetic equations to the hydrodynamic (Euler, Navier-Stokes or diffusion) equations should be preserved at the discrete level. Past progress on developing robust numerical schemes for kinetic equations has been strongly guided by the study of the hydrodynamic limit [LMM, JL, CJR, GPT].

To understand such a diffusive limit, we consider the one dimensional Goldstein-Taylor model [Go, Ta] under the diffusive scaling

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{\epsilon}\frac{\partial u}{\partial x} &= \frac{1}{\epsilon^2}(v-u)\,, \\ \frac{\partial v}{\partial t} - \frac{1}{\epsilon}\frac{\partial v}{\partial x} &= \frac{1}{\epsilon^2}(u-v)\,, \end{aligned} \tag{1.1}$$

where $(x,t) \in \mathbf{R} \times \mathbf{R}^+$, with nonnegative initial data

$$u(x,0) = u_0(x)\,; \quad v(v,o) = v_0(x)\,.$$

In (1.1) ϵ is the positive relaxation parameter. The macroscopic (fluid) variables for this model are the mass density ρ and the flux j, defined by

$$\rho = u + v\,, \qquad j = \epsilon^{-1}(u-v)\,. \tag{1.2}$$

Since u and v can be expressed in terms of ρ and j, system (1.1) is equivalent to the following system for the mass density and the flux

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{1}{\epsilon^2}\frac{\partial \rho}{\partial x} &= -\frac{2}{\epsilon^2} j\,. \end{aligned} \tag{1.3}$$

In the zero relaxation (or diffusion) limit ($\epsilon \to 0^+$), system (1.3) can be approximated to leading order by

$$j = -\frac{1}{2}\frac{\partial \rho}{\partial x}\,, \tag{1.4a}$$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\frac{1}{2}\frac{\partial \rho}{\partial x} = 0\,. \tag{1.4b}$$

The state satisfying (1.4a) will be called the *local equilibrium,* while (1.4b) is the equation of continuum mechanics generated by the kinetic model, that in this case is the heat equation.

The diffusive limit for (1.3) has been first studied by Kurtz [Ku] and McKean [McK]. It was shown in [Ku] and [McK] that the solution $\rho^\epsilon(x,t), j^\epsilon(x,t)$ to system (1.3) corresponding to initial values $u^\epsilon(x,0) = u_0(x), v^\epsilon(x,0) = v_0(x)$ converges strongly in L^1_x to ρ for all $t \geq 0$, while ϵj^ϵ converges to zero in $L^2_{x,t}$. The limit density ρ satisfies the heat equation (1.4b), with $\rho_0 = u_0 + v_0$ as initial data. In addition the result is independent of the choice of u_0 and v_0, and one can choose initial data far from the local equilibrium ($u = v$), without creating any initial layer.

Jin and Levermore studied semi-discrete numerical schemes for hyperbolic systems with stiff relaxation terms that have a long time diffusive behavior [JL]. Fully discrete numerical discretizations for such problems were studied recently by Naldi and Pareschi [NaPa]. This problem, in general, is more challenging than the old approaches in this direction [CJR, Jin] for the Euler regimes, since in the latter only the source term is stiff. Special care must be taken to assure that the schemes possess the correct diffusive limit, in the sense that the asymptotic limit that leads from (1.3) to (1.4) should be preserved at the discrete level [LMM]. It is also necessary to overcome the difficulty of the stiff convection term. As observed in [NaPa] standard splitting schemes based on a separation of the convective and collisional parts fail to give the correct asymptotic limit. Moreover, since the characteristic speed of the hyperbolic part is of order $1/\epsilon$, the usual approach of keeping the convective part explicit and the collision part implicit for stiff source problem [CJR, Jin] leads to a Courant-Friedrichs-Levy (CFL) condition like $\Delta t \approx \epsilon \Delta x$. In the diffusive regime where $\epsilon < \Delta x$ this is unnecessary since a parabolic CFL condition $\Delta t \approx (\Delta x)^2$ should suffice.

Our key idea to attack this problem is to reformulate (1.3) as a (nonstiff) linear hyperbolic system with stiff relaxation term (hereafter called *the diffusive relaxation system*)

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \\ &\frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} = -\frac{2}{\epsilon^2}\left[j + \frac{1-\epsilon^2}{2}\frac{\partial \rho}{\partial x}\right]\,. \end{aligned} \tag{1.5}$$

It is immediately recognizable that system (1.5) has the form used to construct relaxation schemes for conservation laws by Jin and Xin (cfr. [JiXi]). There a given one dimensional hyperbolic scalar conservation law of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} F(\rho) = 0\,, \tag{1.6}$$

with initial data

$$\rho(x,0) = \rho_0(x) \geq 0\,,$$

is approximated by the following linear hyperbolic system with a stiff relaxation term

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,, \\ &\frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} = -\frac{1}{\epsilon}\left[j - F(\rho)\right]\,, \end{aligned} \tag{1.7}$$

with initial data

$$\rho^\epsilon(x,0) = \rho_0(x)\,, \quad j^\epsilon(x,0) = j_0(x) = F(\rho_0(x))\,.$$

Theoretical justification for the passage from (1.7) to (1.6) (the Euler limit) was made in [CLL, Na], while the numerical discretization for such problem was studied in [CJR, Jin]. In such a problem only the source term is stiff, thus a proper splitting of an explicit convection and an implicit source term suffices to give a scheme with a formal uniform accuracy in ϵ.

The new formulation (1.5), due to its close relation with (1.7), allows us to use some of the numerical techniques used to solve (1.7) [CJR, Jin, JiXi]. However, we face additional difficulties here. First, the aforementioned transition from hyperbolic (1.7) to hyperbolic (1.6) is obtained through the local equilibrium $j = F(\rho)$. Here, we have to consider more general local equilibria of the type

$$j = F\left(\rho, \frac{\partial \rho}{\partial x}\right)\,. \tag{1.8}$$

The dependence of F on the spatial derivative completely changes the character of the limiting equation and solution. Secondly, the flux j in (1.2) depends on the scale parameter ϵ.

In this paper we show how to develop a class of numerical schemes for (1.5) that work uniformly with respect to ϵ. While earlier approaches (cfr. [CP, CJR, GPT, Pa]) that concentrated on getting the correct Euler limit, where the limiting equation is hyperbolic, lead to a CFL condition of the type $\Delta t \approx \epsilon \Delta x$ when applied to (1.5) in the diffusive regime, our new approach here gives a class of schemes that have a parabolic CFL condition $\Delta t \approx (\Delta x)^2$, which is the only natural condition in the diffusive regime. In other regimes our schemes maintain all the nice properties of those constructed earlier, such as the ability to capture shocks and other types of discontinuities with high resolutions, and avoiding solving nonlinear algebraic systems. The key ingredients in our approach include second order nonlinear upwind discretization on the convection term, an implicit center difference for the source term near the parabolic regime, and a second order Runge-Kutta splitting method developed in [CJR].

The rest of the paper is organized as follows. The next section is devoted to a detailed description of the physical aspects related to the kinetic theory of gas dynamics. Subsequentely we introduce the kinetic models under consideration and the variety of limiting diffusion equations we can obtain. Some rigorous convergence results are also outlined. In section 3 we give the details of the numerical schemes solving the diffusive relaxation systems. Upwind schemes for the linear convection terms are combined with two different discretizations of the stiff diffusive terms: a fully implicit scheme based on central differences and a semi-implicit scheme based on second order upwinding on the kinetic variables. A second order Runge-Kutta splitting scheme proposed in [CJR] is used for the time integration. The diffusive limit of the schemes is derived in section 4, which shows that the new schemes developed here have the correct diffusive limit. The numerical results for 1-D and 2-D problems are presented in section 5. We conclude the paper with some remarks in section 6.

2. Kinetic Models and Their Diffusion Approximations

2.1 Passage from the Mesoscopic to the Macroscopic Description

The passage from the mesoscopic description of kinetic theory to the macroscopic one of continuum theory is usually described by the asymptotic relations between solutions of the Boltzmann equation and solutions of Euler and Navier-Stokes equations. To understand this, consider the Boltzmann equation in dimensionless form [Ce, TrMu]

$$\nu \frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\epsilon} J(f,f)\,. \tag{2.1}$$

This equation describes the evolution of the one-particle distribution function $f = f(x,v,t)$, where x, v, t are respectively the space, velocity and time variables. $J(f,f)$ is the collision operator, whose details specify the type of binary collisions. The dimensionless parameters $\nu > 0$ and $\epsilon > 0$ are, respectively, the *Mach number* and the *Knudsen number*. It is well-known that the concept of continuum theory is valid only in the limit $\epsilon \to 0$. This limit, for constant ν, formally leads to the equation

$$J(f,f) = 0\,, \tag{2.2}$$

which gives a unique class of solutions to equation (2.2) known as the *local Maxwellians*

$$M^f(x,v,t) = \rho(x,t)\,[2\pi T(x,t)]^{-n/2} \exp\left\{ \frac{-|v-u(x,t)|^2}{2T(x,t)} \right\}\,,$$

where

$$\rho(x,t) = \int_{\mathbf{R}^n} f(x,v,t)\, dv$$

is the mass density,

$$u(x,t) = \frac{1}{\rho}\int_{\mathbf{R}^n} v f(x,v,t)\, dv$$

is the mean velocity, and T is the local temperature defined by

$$T(x,t) = \frac{1}{n\rho}\int_{\mathbf{R}^n} |v-u|^2 f(x,v,t)\, dv.$$

In the limit $\epsilon \to 0$ one expects that the solution to equation (2.1) should tend to the Maxwellian M whose fluid-dynamic moment variables ρ, u and T solve a system of equations of continuum theory. There are two classical and natural asymptotic regimes [Cer, BGL]. The first one is obtained for fixed ν, namely

$$\epsilon \to 0\,; \quad \nu = 1\,, \tag{2.3}$$

and leads to the Euler equations of compressible fluids (known as the Euler limit). The second one is obtained for ν of the order of ϵ, that is

$$\epsilon = \nu \to 0\,, \tag{2.4}$$

which leads to the Navier-Stokes equations for incompressible fluids, in which the viscosity and heat conduction terms are of the order of ϵ.

Unfortunately, the validity of this asymptotic expansion (known as the Hilbert expansion) depends upon an assumption of the existence of a smooth solution to the resulting continuum system, that is far to be verified in this general setting.

For this reason, in recent years the Boltzmann equation has been substituted by simpler models that have a finite number of allowed velocities, maintaining most of the properties of the complete equations (conservation of mass, momentum and total energy, entropy principle). In the next paragraphs we will introduce several of such models.

2.2. One-Dimensional Systems

Among the one-dimensional discrete velocity models of the Boltzmann equation, the simplest ones are certainly the models with two velocities. These models are supposed to describe the evolution of the velocity distribution of a fictitious gas composed of two kinds of particles that move parallel to the x-axis with constant and equal speeds, either in the positive x-direction with a density u, or in the negative x-direction with a density v. The Boltzmann equation corresponds to the system

$$\begin{aligned} \nu\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} &= \frac{1}{\epsilon} J_\epsilon(u,v)\,, \\ \nu\frac{\partial v}{\partial t} - c\frac{\partial v}{\partial x} &= -\frac{1}{\epsilon} J_\epsilon(u,v)\,, \end{aligned} \tag{2.5}$$

where c is the velocity of the molecules, and the *collision operator* J_ϵ has the form

$$J_\epsilon(u,v) = k(u,v)\left\{-\alpha_\epsilon u + \beta_\epsilon v + \gamma_\epsilon uv\right\}\,, \tag{2.6}$$

where $k(u,v), \alpha_\epsilon, \beta_\epsilon$ and γ_ϵ are nonnegative real constants. It is seen from (2.6) that there are three collision rules allowed, which can be described by $+- \to ++$ with a rate $k\gamma_\epsilon$, $+ \to -$ with a rate $k\alpha_\epsilon$, and $- \to +$ with a rate $k\beta_\epsilon$. When $\gamma_\epsilon > 0$, the presence of the process $+- \to ++$, together with the absence of $++ \to +-$, means the violation of the detailed balance, and time reversal invariance. This phenomenon obviously reflects also in the limit equation.

The macroscopic variables of model (2.5) are the local mass density ρ and the flux j defined by

$$\rho = u + v, \quad j = c(u - v)/\nu.$$

Adding and subtracting the two equations in (2.5) one obtains the following macroscopic equations for ρ and j

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \nu^2 \frac{\partial j}{\partial t} + c^2 \frac{\partial \rho}{\partial x} &= -\frac{2\nu}{\epsilon} \widehat{J}_\epsilon(\rho, j)\,, \end{aligned} \tag{2.7}$$

where

$$\widehat{J}_\epsilon(\rho, j) = -\frac{\nu}{c} J_\epsilon(u, v).$$

The scaling (2.3) leads to the equation

$$\widehat{J}_0(\rho, j) = 0\,, \tag{2.8}$$

that defines the local Maxwellian in this case. Whenever (2.7) can be solved for j in terms of ρ

$$j = F(\rho)\,, \tag{2.9}$$

system (2.7) relaxes towards the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} F(\rho) = 0\,. \tag{2.10}$$

As discussed in the introduction, the relaxation of such kinetic systems towards hyperbolic conservation laws has been object of several earlier investigations, and the mathematical theory is almost consolidated, thus is not our focus here.

The second scaling, which corresponds to $\nu = \epsilon$, leads to the equation

$$c^2 \frac{\partial \rho}{\partial x} = -2\widehat{J}_0(\rho, j)\,, \tag{2.11}$$

If (2.11) can be solved for j in terms of ρ to give

$$j = F\left(\rho, \frac{\partial \rho}{\partial x}\right)\,, \tag{2.12}$$

then system (2.7) will relax towards the second-order partial differential equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} F\left(\rho, \frac{\partial \rho}{\partial x}\right) = 0 \,. \tag{2.13}$$

We remark that, even for simple kinetic models like (2.5)-(2.6), the variety of equations of continuum theory we can obtain in the limit is very rich. In the sequel, we briefly discuss some of them giving a short review of the cases in which the passage to the limit can be justified rigorously. In particular, we emphasize those aspects that are important in connection with the numerical schemes we will introduce.

In general, we will consider the initial-boundary value problem for system (2.5), with periodic or reflecting boundary conditions. From a theoretical point of view, one can easily show that the presence of the boundaries does not affect the solution in any bounded set in the limit procedure, provided the boundaries themselves are suitably located. In fact given $\epsilon > 0$, the velocity of propagation of the hyperbolic system (2.5) is exactly $1/\epsilon$. This means that, given any time $T > 0$, for $t \leq T$ the solution $u^\epsilon(x,t), v^\epsilon(x,t)$ on the interval $\left[-\frac{1}{\epsilon}(1+T), \frac{1}{\epsilon}(1+T)\right]$ depends only on the initial values on the interval $\left[-\frac{1}{\epsilon}(1+2T), \frac{1}{\epsilon}(1+2T)\right]$, provided the boundaries are located at $[-1/\epsilon^\gamma, 1/\epsilon^\gamma]$, for γ large enough. In other words, for ϵ small enough, the presence of the boundaries does not affect the solution on the interval $\left[-\frac{1}{\epsilon}(1+T), \frac{1}{\epsilon}(1+T)\right]$.

2.2.1 The heat equation

From now on, let us set $c = 1$. The classical heat equation may be obtained as the limiting equation of (2.5) with the choice

$$k(u,v) = k_0 \,; \quad \alpha_\epsilon = \beta_\epsilon = 1 \,; \quad \gamma_\epsilon = 0 \tag{2.14}$$

in the collision operator (2.6). This kinetic model is the Goldstein-Taylor model [Go, Ta] given by (1.1) with the right hand side multiplied by k_0, which represents the forward equation for the density of a molecule moving with constant speed c along the x-axis, subject to spontaneous reversals of direction, at the jump times of a standard Poisson process of constant rate k_0. The corresponding equations for ρ and j are

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0 \,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -2k_0 j \,. \end{aligned} \tag{2.15}$$

The local equilibrium (Maxwellian) is given by

$$k_0 j = -\frac{1}{2}\frac{\partial \rho}{\partial x} \,, \tag{2.16}$$

while (2.13) becomes

$$\frac{\partial \rho}{\partial t} = \frac{1}{2} D \frac{\partial^2 \rho}{\partial x^2} \,, \tag{2.17}$$

where $D = 1/k_0$ is the *diffusion coefficient.*

System (2.15), or equivalently (1.3), being the prototype of models we will consider in the sequel, is important in itself. It is often called *hyperbolic heat equation.* It was Maxwell who first introduced the concept of a relaxation time, and for this reason these equations are sometimes referred to in the literature as the Maxwell-Cattaneo equations. Several authors have applied this concept to the problem of heat conduction (see [TaRa] and its references) and there has been a resurgence of interest in these equations in the last five years (see [Sh, TaNa, RoAr] and references therein). Characteristic based numerical schemes to solve (2.5) that are able to handle the small relaxation parameter ϵ were given by Roe and Arora [RoAr].

As discussed in the introduction, from a theoretical point of view, it is well known that the mass solution to the Goldstein-Taylor model converges towards the solution to the heat equation (2.17). Recently this problem has been studied in [LiTo] as a particular case (the linear one) of a more general situation leading to the porous medium equation. Due to the linearity, one obtains uniform bounds on the kinetic densities in $W^{m,p}$, for $m \geq 0, 1 \leq p \leq \infty$, and thus convergence in the same spaces when the initial data satisfy a suitable decay at infinity.

2.2.2 The advection-diffusion equation

Let A be a real constant such that $|A| \leq 1/\epsilon$. If we choose

$$k(u,v) = k_0\,; \quad \alpha_\epsilon = 1 + A\epsilon\,, \quad \beta_\epsilon = 1 - A\epsilon\,, \quad \gamma_\epsilon = 0\,, \tag{2.18}$$

then the collision operator (2.6) takes the form

$$J_\epsilon(u,v) = k_0\left[v - u + A\epsilon(u+v)\right]\,. \tag{2.19}$$

The kinetic model we obtain is a particular linear case of the Ruijgrook-Wu model [RuWu]. Thus, under the diffusive scaling (2.4), the equations for ρ and j read

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -2k_0\left(j - A\rho\right)\,. \end{aligned} \tag{2.20}$$

In the limit $\epsilon \to 0$, the second equation relaxes to the local equilibrium

$$j = A\rho - \frac{1}{2k_0}\frac{\partial \rho}{\partial x}\,, \tag{2.21}$$

and (2.20a) gives the advection-diffusion equation

$$\frac{\partial \rho}{\partial t} + A\frac{\partial \rho}{\partial x} = \frac{1}{2}D\frac{\partial^2 \rho}{\partial x^2}\,, \tag{2.22}$$

where $D = 1/k_0$.

It is interesting to remark that in this and other cases the choice of the relaxing system is not unique. In fact, if we consider the following system

$$\begin{aligned} \frac{\partial u}{\partial t} + \left(\frac{1}{\epsilon} + a\right) \frac{\partial u}{\partial x} &= \frac{1}{\epsilon^2} k_0 (v - u)\,, \\ \frac{\partial v}{\partial t} - \left(\frac{1}{\epsilon} + b\right) \frac{\partial v}{\partial x} &= \frac{1}{\epsilon^2} k_0 (u - v)\,, \end{aligned} \tag{2.23}$$

where $a, b \in \mathbf{R}$. The equivalent system for ρ and j is then

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} + \frac{a-b}{2}\frac{\partial \rho}{\partial x} + \epsilon \frac{a+b}{2}\frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} + \frac{a+b}{2}\frac{\partial \rho}{\partial x} + \epsilon^2 \frac{a-b}{2}\frac{\partial j}{\partial x} &= -2k_0 j\,. \end{aligned} \tag{2.24}$$

For this system, the limiting equation as $\epsilon \to 0$ is

$$\frac{\partial \rho}{\partial t} - \frac{1}{2} D \frac{\partial^2 \rho}{\partial x^2} + \frac{a-b}{2k_0}\frac{\partial \rho}{\partial x} = 0\,. \tag{2.25}$$

However system (2.23) does not belong to the class of kinetic models defined by (2.5)-(2.6).

2.2.3 The porous media equation

Nonlinear degenerate parabolic equations can be obtained as the diffusion limit of (2.4) by choosing $k(u, v)$ varying with u and v. If one sets

$$k(u, v) = \rho^m\,; \quad \alpha_\epsilon = \beta_\epsilon = 1\,; \quad \gamma_\epsilon = 0\,, \tag{2.26}$$

where $m \leq 1$, then (2.7) becomes

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0 \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -2\rho^m j \end{aligned} \tag{2.27}$$

When $\epsilon \to 0$, the local equilibrium is given by

$$j = -\frac{1}{2\rho^m}\frac{\partial \rho}{\partial x} = -\frac{1}{2(1-m)}\frac{\partial \rho^{1-m}}{\partial x}\,. \tag{2.28}$$

Thus, the system relaxes to the nonlinear parabolic equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2(1-m)}\frac{\partial^2 \rho^{1-m}}{\partial x^2}\,. \tag{2.29}$$

If $0 < m < 1$, we obtain in the limit the *fast diffusion equation.* The case $-1 < m < 0$ corresponds to the *slow diffusion equation.* The porous media equation is achieved for $m \leq -1$. Note that the choice (2.26) is a particular case of the assumption

$$k(u,v) = k(\rho) = \left[\frac{d}{d\rho}P(\rho)\right]^{-1} \tag{2.30}$$

where the function $P(\rho)$ is increasing from $P(0) \geq 0$. This choice leads to the equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial x^2}P(\rho)\,. \tag{2.31}$$

It is interesting to remark that the first model for which the asymptotic problem (2.27) awakened the interest of the researchers was Carleman's one. This model is included in (2.26), and corresponds to the choice $m = 1$. This asymptotic problem was first investigated by Kurtz [Ku] and McKean[McK] by means of the theory of nonlinear semigroups. They proved that, for initial data $u_0(x) = v_0(x) \in L^1(\mathbf{R})$ the mass density $\rho_\epsilon(x,t)$ satisfying (2.27) converges in L^1_x for all $t \geq 0$ to $\rho(x,t)$ satisfying the nonlinear diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{4}\frac{\partial^2}{\partial x^2}\left(\log \rho\right)\,,$$

while $\epsilon j_\epsilon(x,t)$ converges to zero.

McKean's result [McK] has been recently extended by Toscani and Pulvirenti [PuTo] to the case $0 \leq m < 1$. A complete treatment of the case $m \leq 1$ is due to Lions and Toscani [LiTo].

2.2.4 The Burgers equation

The last one-dimensional example we mention is the nonlinear Ruijgrok-Wu model [RuWu]. It corresponds to the parameters

$$k(u,v) = k_0; \quad \alpha_\epsilon = \beta_\epsilon = 1\,; \quad \gamma_\epsilon = \epsilon C\,, \tag{2.32}$$

where C is a real constant. In this case (2.7) becomes

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,, \\ &\epsilon^2\frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} = -2k_0\left[j - \frac{C}{2}\left(\rho^2 - \epsilon^2 j^2\right)\right]\,. \end{aligned} \tag{2.33}$$

As $\epsilon \to 0$, (2.33b) relaxes to

$$J = \frac{C}{2}\rho^2 - \frac{1}{2k_0}\frac{\partial \rho}{\partial x}\,,$$

and from (2.33a) we get

$$\frac{\partial \rho}{\partial t} + \frac{C}{4}\frac{\partial \rho^2}{\partial x} = \frac{D}{2}\frac{\partial^2 \rho}{\partial x^2}\,, \tag{2.34}$$

namely the Burgers equation.

In a recent paper [GaPe], Gabetta and Perthame studied the asymptotic problem for the Ruijgrok-Wu model in both scalings, leading to a conservation law and to Burgers equation (with or without viscosity). They showed that the solution to the initial value problem for system (2.33) corresponding to initial data of bounded variation is of bounded variation, and the family $\{\rho_\epsilon\}$ of mass densities is uniformly bounded in $BV(\mathbf{R})$ in any finite interval of time, while the family j_ϵ of the fluxes is uniformly bounded in $L^2(\mathbf{R}\times[0,T])$. These conditions insure the passage to the limit for the density in strong L^1-norm.

2.3 Multi-Dimensional Systems

In this subsection, we shall introduce the multi-dimensional version of the model (2.5). In three dimensional space, we will study the system

$$\nu\frac{\partial u_i}{\partial t} + c_i \cdot \nabla_x u_i = \frac{1}{\epsilon} J_{i,\epsilon}(u,u)\,, \tag{2.35}$$

where $u = (u_1, u_2, u_3, u_4, u_5, u_6)$ and the collision operator $J_{i,\epsilon}$ has the form

$$J_{i,\epsilon}(u,u) = k(u,u)\left\{ -\alpha_\epsilon^i u_i + \sum_{\substack{h=0\\ h\neq i}}^{6} (\beta_\epsilon^h u_h + \gamma_\epsilon^{ih} u_i u_h) \right\}\,, \tag{2.36}$$

with $k(u,u), \alpha_\epsilon^i, \beta_\epsilon^i, \gamma_\epsilon^{ih}$ nonnegative real constants such that $\alpha_\epsilon^i = 5\,\beta_\epsilon^i$ and $\gamma_\epsilon^{ih} = -\gamma_\epsilon^{hi}$, $\forall\, i$.

In (2.35) the velocity vectors c_i are given by

$$c_i = \begin{cases} ((-1)^{i+1}c, 0, 0), & i = 1,2\,; \\ (0, (-1)^{i+1}c, 0), & i = 3,4\,; \\ (0, 0, (-1)^{i+1}c) & i = 5,6\,. \end{cases}$$

By choosing $c = 1$ for simplicity and $\nu = \epsilon$, the local mass density ρ and the flux vector j are given by

$$\begin{aligned} \rho &= \sum_{i=1}^{6} u_i\,, \\ j &= \left(\frac{u_1 - u_2}{\epsilon}, \frac{u_3 - u_4}{\epsilon}, \frac{u_5 - u_6}{\epsilon}\right)\,, \end{aligned} \tag{2.37}$$

which satisfies the equations

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t}+\sum_{i=0}^{3} \frac{\partial j_i}{\partial x_i}=0\,, \\
&\epsilon^2 \frac{\partial j_i}{\partial t}+\frac{\partial \rho}{\partial x_i}=-\widehat{J}_{i,\epsilon}(\rho, j)+\frac{\partial \omega_i}{\partial x_i}\,, \quad i=1,2,3\,,
\end{aligned}
\tag{2.38}
$$

where $\widehat{J}_\epsilon$ and ω denote the vectors

$$
\begin{aligned}
&\widehat{J}_\epsilon=\left(\frac{J_{1,\epsilon}-J_{2,\epsilon}}{\epsilon}, \frac{J_{3,\epsilon}-J_{4,\epsilon}}{\epsilon}, \frac{J_{5,\epsilon}-J_{6,\epsilon}}{\epsilon}\right)\,, \\
&\omega=\left(\rho-\left(u_1+u_2\right), \rho-\left(u_3+u_4\right), \rho-\left(u_5+u_6\right)\right).
\end{aligned}
\tag{2.39}
$$

The presence of the vector ω is the main difference between the one-dimensional system (2.7) and the three-dimensional system (2.35). This is a consequence of the fact that in more than one-dimension the system of the macroscopic equations can not be expressed only in terms of the mean quantities. Clearly when $u_{1,2}=u_{1,2}(x)$, $u_{3,4}=u_{3,4}(y)$ and $u_{5,6}=u_{5,6}(z)$,

$$
\frac{\partial \omega_i}{\partial x_i}=0, \quad \forall i. \tag{2.40}
$$

It is easy to see that condition (2.40) is satisfied by every choice of $\alpha_\epsilon^i, \beta_\epsilon^i$ and γ_ϵ^{ih} such that $J_{1,2,\epsilon}=J_{1,2,\epsilon}(x)$, $J_{3,4,\epsilon}=J_{3,4,\epsilon}(y)$ and $J_{5,6,\epsilon}=J_{5,6,\epsilon}(z)$. That is when system (2.35) describes the evolution of three one-dimensional models which do not interact one each other.

Under this assumption the limit $\epsilon \rightarrow 0$ leads to the equations

$$
\frac{\partial \rho}{\partial x_i}=-\widehat{J}_{i,0}(\rho, j). \tag{2.41}
$$

If we suppose that (2.41) can be solved for j as

$$
j=F\left(\rho, \nabla_{\mathrm{x}} \rho\right)\,, \tag{2.42}
$$

we obtain the parabolic conservation law

$$
\frac{\partial \rho}{\partial t}+\operatorname{div} F\left(\rho, \nabla_{\mathrm{x}} \rho\right)=0. \tag{2.43}
$$

For example, if we consider only the cases that have a limit equation of the form (2.30) we will study system (2.35) for $\beta_\epsilon^i=1/6$, $\forall i$, $\gamma_\epsilon^{hi}=0$, $\forall h, i$ and $k(\rho, j)=k(\rho)$. Note that with this choice of parameters condition (2.40) is not satisfied and the model is really three-dimensional.

If the collision operator is given by

$$J_{i,\epsilon}(u,u) = \frac{k(\rho)}{6}(\rho - 6u_i)\,, \tag{2.44}$$

then, as $\epsilon \to 0$, we get the local equilibrium

$$j_i = -\frac{1}{3k(\rho)}\frac{\partial \rho}{\partial x_i}.$$

Thus system (2.38) relaxes to the parabolic equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{3}\Delta P(\rho). \tag{2.45}$$

We remark that from a theoretical point of view, the case $k(\rho) = \rho^{-m}, m > 0$, has been recently studied by Lions and Toscani [LiTo].

3. Numerical schemes

In this section we discuss the discretization of the kinetic systems introduced in section 2. We derive accurate and stable discretizations that work with a uniform accuracy for all range of ϵ. In the derivation we will focus first on 1-D systems. Later we will discuss the discretization of the multidimensional systems, which is just a dimension-by-dimension extension of the 1-D case.

3.1 Previous Methods

The numerical solution of this kind of problems has been studied by several authors [RoAr, JL, NaPa]. These earlier works focus on developing schemes that work effectively when ϵ is small, while our goal is to develop schemes that work for all range of ϵ.

First it is interesting to point out, as observed recently in [NaPa], that on the contrary to the Euler scaling (1.7), in the diffusive scaling the popular operator splitting method does not provide the correct diffusive behavior as $\epsilon \to 0$. In fact, the usual splitting applied to system (1.3) consists in solving first the homogeneous problem

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= 0\,, \end{aligned} \tag{3.1}$$

and then the ordinary differential equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} &= -J\,.n \end{aligned} \tag{3.2}$$

Hence when $\epsilon \to 0$ we obtain the equilibrium equations

$$J = 0, \qquad \frac{\partial \rho}{\partial x} = 0, \tag{3.3}$$

that clearly are inconsistent with (1.4a,b). This explains why the usual operator splitting method gives very poor results near the parabolic regime.

On the other hand a natural splitting of such problems is to separate the non-stiff part

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} &= 0\,, \end{aligned} \tag{3.4}$$

from the stiff relaxation part

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= 0\,, \\ \epsilon^2 \frac{\partial j}{\partial t} &= -(j + \frac{\partial \rho}{\partial x})\,. \end{aligned} \tag{3.5}$$

The method now leads to the right asymptotic behavior for small values of ϵ. Unfortunately this splitting does not provide any simplification to the numerical solution of problem (1.3). Moreover the understanding of basic wave patterns for (3.4)-(3.5) is harder then that of the corresponding problem (1.3).

In [RoAr] some characteristic-based methods for the hyperbolic heat equation were analyzed. The authors show that for smooth solutions the schemes perform better than the usual fractional step method (3.1)-(3.2) and yield accurate solutions even when the relaxation parameter ϵ is much less then the time step Δt. While it is a step in the right direction, in the diffusive scaling the method in [RoAr] suffers of several deficiencies. In fact the geometry of the characteristic mesh does not allow to obtain a limit stability constraint independent of ϵ.

A detailed study of semidiscrete approximation methods for hyperbolic systems in the scaling (1.7) that lead to a long time behavior governed by reduced system which are parabolic in nature has been carried on in [JL]. The authors show that the standard high resolution methods for hyperbolic conservation laws fail to capture the parabolic behavior unless the small relaxation rate is resolved by a fine spatial grid.

Applying their arguments to system (1.3) we obtain that the classical first order upwinding on the characteristic variables $\rho \pm \epsilon j$ exhibit an asymptotic behavior for small values of ϵ with leading truncation error

$$\frac{\Delta x}{2\epsilon} \frac{\partial^2 \rho}{\partial x^2}. \tag{3.6}$$

When $\Delta x/\epsilon \approx 1$ or $\Delta x >> \epsilon$ the numerical dissipation rate will be comparable or dominate the physical dissipation and the numerical solution will dissipate faster then the physical one. Similar arguments hold for second order upwind schemes.

For this reason the authors [JL] introduced a modified high order Godunov scheme that possess the correct asymptotic behavior allowing the use of coarse grids. Roughly speaking their idea is to build in the numerical schemes the asymptotics that lead from the hyperbolic system to the parabolic equation. However they did not take into account the time discretization, which is the central issue in this paper.

The development of fully discrete high order schemes for problem (1.3) has been studied recently in [NaPa]. They put a particular attention to the development of underresolved numerical schemes ($\Delta x > \epsilon$, $\Delta t > \epsilon^2$) that work in the stiff regime. Their idea is to use a modified upwind method for (1.3a) combined with a fully implicit centered based method for (1.3b). In the limit $\epsilon \to 0$ the authors show that the schemes are equivalent to second order discretizations of the equilibrium diffusion equations under a stability constraint independent of ϵ. Numerical examples on one-dimensional systems verify their analysis.

3.2 Diffusive Relaxation Schemes

Consider the 1-D system in macroscopic variables

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{1}{\epsilon^2}\frac{\partial \rho}{\partial x} &= -\frac{1}{\epsilon^2}\widehat{J}_\epsilon(\rho, j)\,. \end{aligned} \tag{3.7}$$

As discussed in the introduction, the key idea is to rewrite (3.7) in the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \phi(\epsilon)\frac{\partial \rho}{\partial x} &= -\frac{1}{\epsilon^2}\left[\widehat{J}_\epsilon(\rho, j) + (1-\epsilon^2\phi(\epsilon))\frac{\partial \rho}{\partial x}\right]\,, \end{aligned} \tag{3.8}$$

where $\phi(\epsilon)$ is such that $0 \leq \phi(\epsilon) \leq 1/\epsilon^2$. This restriction on ϕ guarantees the positivity of $\phi(\epsilon)$ and $(1-\epsilon^2\phi(\epsilon))$. The simplest choice of ϕ is given by

$$\phi(\epsilon) = \begin{cases} 1\,; & \epsilon \leq 1 \\ \dfrac{1}{\epsilon^2}, & \epsilon > 1\,, \end{cases} \tag{3.9}$$

for which problem (3.8) takes the form (1.5) when $\epsilon \leq 1$.

As was done normally for a kinetic equation or a hyperbolic system with relaxation, (3.8) can be splitted into two subproblems

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \phi(\epsilon)\frac{\partial \rho}{\partial x} &= 0\,, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned}
\frac{\partial \rho}{\partial t} &= 0\,, \\
\frac{\partial j}{\partial t} &= -\frac{1}{\epsilon^2}\left[\widehat{J}_\epsilon(\rho,j) + (1-\epsilon^2\phi(\epsilon))\frac{\partial \rho}{\partial x}\right]\,.
\end{aligned} \tag{3.11}$$

Hence we have a pure convective step (3.10) and a stiff source step (3.11). Note that for $\phi(\epsilon) = 0$ the splitting (3.10)-(3.11) take the form (3.4)-(3.5) whereas for $\phi(\epsilon) = 1/\epsilon^2$ it take the form (3.1)-(3.2).

The main reason for introducing this splitting is given by the fact that, on the contrary of the standard splitting we have seen in the previous section, when $\epsilon \to 0$, the relaxation step (3.11) always projects the solution to the correct local equilibrium

$$\frac{\partial \rho}{\partial x} = -\widehat{J}_0(\rho,j)\,, \tag{3.12}$$

which is a sufficient condition to guarantee that the splitting preserves the asymptotic limit from the kinetic equations to the diffusion equations.

The most natural way to solve (3.10)-(3.11) is to use an explicit scheme for the convection step and to treat the stiff source term implicitly. The basic idea is to solve (3.10) by means of upwind approximations and (3.11) using numerical methods that posses the correct diffusive limit.

3.2.1 Convection step

In this section we shall discuss the discretization of the linear convection terms. Since this step is non stiff and hyperbolic, it is natural to use explicit upwind schemes.

By introducing the characteristic variables $U = (\rho + \phi^{-1/2}j)/2$ and $V = (\rho - \phi^{-1/2}j)/2$, (3.10) can be written in a diagonal form as

$$\begin{aligned}
\frac{\partial U}{\partial t} + \phi^{1/2}\frac{\partial U}{\partial x} &= 0\,, \\
\frac{\partial V}{\partial t} - \phi^{1/2}\frac{\partial V}{\partial x} &= 0\,,
\end{aligned} \tag{3.13}$$

where U and V travel with the characteristic speeds $\pm\phi^{1/2}$. Note that when $\phi(\epsilon) \neq 1/\epsilon^2$, U and V do not coincide with the characteristic variables of the unsplit problem.

Let the spatial grid points be $x_{i+1/2}$, $i = \ldots,-1,0,1,\ldots$ with uniform mesh width $\Delta x = x_{i+1/2} - x_{i-1/2}$. The discrete time level t^n are also spaced uniformly with time step $\Delta t = t^{n+1} - t^n$ for $n = 0,1,2,\ldots$. Here the assumption of a uniform grid is only for simplicity. As usual we denote by U_i^n the cell averages of U in the cell $[x_{i-1/2}, x_{i+1/2}]$ at time t^n

$$U_j^n = \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t^n)\,dx,$$

and by $U_{i+1/2}^n = U(x_{i+1/2},t^n)$ its nodal values.

Since (3.13) is a linear hyperbolic system in its diagonal form, it is straightforward to apply the upwind schemes. After done this, we convert them into the system for ρ and j, so to get

$$\begin{aligned} \rho_i^{n+1} &= (1-\lambda)\rho_i^n + \frac{\lambda}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\lambda}{2\phi^{1/2}}(j_{i+1}^n - j_{i-1}^n)\,, \\ j_i^{n+1} &= (1-\lambda)j_i^n + \frac{\lambda}{2}(j_{i+1}^n + j_{i-1}^n) - \frac{\lambda\phi^{1/2}}{2}(\rho_{i+1}^n - \rho_{i-1}^n)\,, \end{aligned} \tag{3.14}$$

where $\lambda = \phi^{1/2}\Delta t/\Delta x$ is the CFL number satisfying

$$\lambda \le 1\,.$$

With the choice of $\phi(\epsilon)$ in (3.9), this CFL condition has the upper bound $\frac{\Delta t}{\Delta x} \le 1$.

To achieve an accuracy of $O(\Delta x^2)$ we use the second order upwind scheme of van Leer [VL] that utilizing a slope limiter:

$$\begin{aligned} \rho_i^{n+1} &= (1-\lambda)\rho_i^n + \frac{\lambda}{2}(\rho_{i+1}^n + \rho_{i-1}^n) - \frac{\lambda}{2\phi^{1/2}}(j_{i+1}^n - j_{i-1}^n) \\ &\quad + \frac{1}{4}\lambda\left[\Delta x(\sigma_{i-1}^+ - \sigma_i^+ - \sigma_{i+1}^- + \sigma_i^-)\right]\,, \\ j_i^{n+1} &= (1-\lambda)j_i^n + \frac{\lambda}{2}(j_{i+1}^n + j_{i-1}^n) - \frac{\lambda\phi^{1/2}}{2}(\rho_{i+1}^n - \rho_{i-1}^n) \\ &\quad + \frac{\phi^{1/2}}{4}\lambda\left[\Delta x(\sigma_{i+1}^- - \sigma_i^- + \sigma_{i-1}^+ - \sigma_i^+)\right]\,, \end{aligned} \tag{3.15}$$

where $\sigma_i^\pm$, the slope of $(\rho \pm \phi^{-1/2}j)/2$ on the i-th cell, is given by

$$\begin{aligned} \sigma_i^\pm &= \frac{1}{\Delta x}\left(\pm\rho_{i\pm1}^n - \phi^{-1/2}j_i^n \mp \rho_i^n + \phi^{-1/2}j_{i\pm1}^n\right)\psi(\theta_i^\pm), \\ \theta_i^\pm &= \left[\frac{\rho_i^n \pm \phi^{-1/2}j_i^n - \rho_{i-1}^n \mp \phi^{-1/2}j_{i-1}^n}{\rho_{i+1}^n \pm \phi^{-1/2}j_{i+1}^n - \rho_i^n \mp \phi^{-1/2}j_i^n}\right]^{\pm1}, \end{aligned} \tag{3.16}$$

and ψ is the particular slope limiter function.

For scheme (3.15), in order to be total variation diminishing (TVD) [Har], one simple choice of the slope is

$$\psi(\theta) = \max\{0, \min\{1, \theta\}\}.$$

Sharper limiter functions are given by the superbee slope of Roe

$$\psi(\theta) = \max\{0, \min\{1, 2\theta\}, \min\{\theta, 2\}\}\,,$$

or by van Leer slope-limiter function

$$\psi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}\,.$$

A more general condition for ψ to assure the TVD property of scheme (3.15) is [Swe]

$$0 \leq \frac{\psi(\theta)}{\theta} \leq 2.$$

3.2.2 Relaxation step

Since our goal is to develop a scheme that works for all range of ϵ, clearly the stiff source term should be treated implicitly for a better numerical stability. Unfortunately, unlike the classical relaxation step in the Euler scaling, in the diffuse relaxation step the stiff source term depends also on the spatial derivative of ρ. Note that the usage of implicit upwind schemes originate a system of equations in j_i^{n+1} that in general cannot be solved explicitly. Moreover the usage of high resolution TVD schemes leads to a nonlinear system of equations in j_i^{n+1}. Previous numerical results [NaPa] show that a robust, economical discretization for the correct numerical behavior near the parabolic regime should be based on implicit centered schemes.

We first develop the schemes in the linear case (i.e. $\gamma_\epsilon = 0$ and $k(\rho, j) = k_0$). We will then generalize the schemes to nonlinear systems at the end of the section. Let $B = \alpha_\epsilon + \beta_\epsilon$ and assume

$$\alpha_\epsilon - \beta_\epsilon = A\epsilon\,, \tag{3.17}$$

where A is a real constant. Note that this assumption is satisfied by all the kinetic models we have considered in Section 2. For simplicity we will set $k_0 = 1$. In this situation the collision operator $J_\epsilon(u, v)$ becomes

$$J_\epsilon(u, v) = J_\epsilon^L(u, v) = [-\alpha_\epsilon u + \beta_\epsilon v]\ , \tag{3.18}$$

which corresponds to the relaxation term $\widehat{J}_\epsilon(\rho, j)$ for the macroscopic variables given by

$$\widehat{J}_\epsilon(\rho, j) = \widehat{J}_\epsilon^L(\rho, j) = \frac{1}{2}\left[A\rho + Bj\right]\ , \tag{3.19a}$$

and the diffusion approximation is given by

$$\frac{\partial \rho}{\partial t} - \frac{A}{B}\frac{\partial \rho}{\partial x} = \frac{2}{B}\frac{\partial^2 u}{\partial x^2}\,. \tag{3.19b}$$

The stiff source step for the kinetic variables u and v then reads

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{1}{2\epsilon^2}\left[J_\epsilon(u, v) + \epsilon(1 - \epsilon^2\phi(\epsilon))\frac{\partial(u + v)}{\partial x}\right]\ , \\ \frac{\partial v}{\partial t} &= -\frac{1}{2\epsilon^2}\left[J_\epsilon(u, v) + \epsilon(1 - \epsilon^2\phi(\epsilon))\frac{\partial(u + v)}{\partial x}\right]\ . \end{aligned} \tag{3.20}$$

As studied in [Jin, CJR], a fully implicit backward Euler method in time guarantees that the numerical relaxation step projects the solution into the local Maxwellian, a necessary condition for the correct asymptotic limit. Thus we should take

$$\begin{aligned}
\frac{u_i^{n+1}-u_i^n}{\Delta t} &= \frac{1}{2\epsilon^2}\Big[J_\epsilon(u_i^{n+1},v_i^{n+1})+\\
&\quad +\epsilon(1-\epsilon^2\phi(\epsilon))\frac{(u_{i+1/2}^{n+1}-u_{i-1/2}^{n+1}+v_{i+1/2}^{n+1}-v_{i-1/2}^{n+1})}{\Delta x}\Big],\\
\frac{v_i^{n+1}-v_i^n}{\Delta t} &= -\frac{1}{2\epsilon^2}\Big[J_\epsilon(u_i^{n+1},v_i^{n+1})+\\
&\quad +\epsilon(1-\epsilon^2\phi(\epsilon))\frac{(u_{i+1/2}^{n+1}-u_{i-1/2}^{n+1}+v_{i+1/2}^{n+1}-v_{i-1/2}^{n+1})}{\Delta x}\Big],
\end{aligned} \tag{3.21}$$

or equivalently in macroscopic variables,

$$\begin{aligned}
\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} &= 0\,,\\
\frac{j_i^{n+1}-j_i^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[\widehat{J}_\epsilon(\rho_i^{n+1},j_i^{n+1})+(1-\epsilon^2\phi(\epsilon))\frac{(\rho_{i+1/2}^{n+1}-\rho_{i-1/2}^{n+1})}{2\Delta x}\right].
\end{aligned} \tag{3.22}$$

where the numerical fluxes $u_{i\pm1/2}^{n+1}$, $v_{i\pm1/2}^{n+1}$ and $\rho_{i\pm1/2}^{n+1}$ have to be determined in terms of u_i^n, v_i^n, ρ_i^n in order to have an economical discretization procedure.

To determine the nodal values we observe that applying the first order upwind scheme on the kinetic variables u and v at the generic time t^n we obtain

$$\begin{aligned}
u_{i+1/2}^n &= u_i^n, \quad v_{i+1/2}^n = v_{i+1}^n,\\
\rho_{i+1/2}^n &= \frac{1}{2}(\rho_i^n+\rho_{i+1}^n)+\frac{\epsilon}{2}(j_i^n-j_{i-1}^n)\,.
\end{aligned} \tag{3.23}$$

The values of $u_{i-1/2}^n$, $v_{i-1/2}^n$ and $\rho_{i-1/2}^n$ are obtained simply by translating i to $i-1$ in (3.23). The second order extension using slope limiters, however, yield a system of nonlinear equations that have to be solved using iterative methods, thus is not our choice here.

In order to have a second order implicit scheme that avoids solving nonlinear algebraic equations, we select the nodal values by centered schemes. Specifically,

$$\begin{aligned}
u_{i+1/2}^n &= \frac{1}{2}(u_i^n+u_{i+1}^n), \quad v_{i+1/2}^n = \frac{1}{2}(v_{i+1}^n+v_i^n)\,,\\
\rho_{i+1/2}^n &= \frac{1}{2}(\rho_i^n+\rho_{i+1}^n)\,.
\end{aligned} \tag{3.24}$$

Consequently, taking into account that ρ does not change in time i.e. $\rho_i^{n+1}=\rho_i^n$ during this step, substituting (3.24) into system (3.22) we obtain

$$\begin{aligned}
\rho^{n+1} &= \rho^n\,,\\
j_i^{n+1} &= \lambda_1 j_i^n - \lambda_2\left[A\rho_i^n+(1-\epsilon^2\phi(\epsilon))\frac{(\rho_{i+1}^n-\rho_{i-1}^n)}{2\Delta x}\right],
\end{aligned} \tag{3.25}$$

where

$$\lambda_1 = \frac{2\epsilon^2}{2\epsilon^2 + B\Delta t}, \quad \lambda_2 = \frac{\Delta t}{2\epsilon^2 + B\Delta t}.$$

Thus, although we have an implicit relaxation term, the new values ρ^{n+1} and j^{n+1} can be updated *explicitly*. This is the major reason that we use a linear scheme in this step.

In the same fashion, solving (3.21) for u_i^{n+1} and v_i^{n+1} gives

$$\begin{aligned} u_i^{n+1} &= \mu_1 u_i^n + (1-\mu_2) v_i^n + \\ &+ \lambda_2 \left[\epsilon(1-\epsilon^2\phi(\epsilon)) \frac{(u_{i+1}^n - u_{i-1}^n + v_{i+1}^n - v_{i-1}^n)}{2\Delta x} \right], \\ v_i^{n+1} &= \mu_2 v_i^n + (1-\mu_1) u_i^n - \\ &- \lambda_2 \left[\epsilon(1-\epsilon^2\phi(\epsilon)) \frac{(u_{i+1}^n - u_{i-1}^n + v_{i+1}^n - v_{i-1}^n)}{2\Delta x} \right], \end{aligned} \tag{3.26}$$

where

$$\mu_1 = \frac{2\epsilon^2 + \beta_\epsilon \Delta t}{2\epsilon^2 + B\Delta t}, \quad \mu_2 = \frac{2\epsilon^2 + \alpha_\epsilon \Delta t}{2\epsilon^2 + B\Delta t}.$$

For most of kinetic equations one can usually solve an implicit collision operator explicitly, due to the fact that the macroscopic variables remain unchanged in the collision step [CP, Jin, CJR, JX].

Another way to approximate system (3.20) is to use the upwind selection (3.23) into (3.21)-(3.22) but in *explicit* form. The main motivation is given by the fact that the first order selection (3.23b) leads to, as $\epsilon \to 0$, a second order approximation to the diffusion equation. In fact, using (3.23) in (3.22b) one gets

$$j_i^{n+1} = \lambda_1 j_i^n - \lambda_2 A \rho_i^n + \lambda_2 (1-\epsilon^2\phi(\epsilon)) \left[\frac{(\rho_{i+1}^n - \rho_{i-1}^n)}{2\Delta x} + \epsilon \frac{(2j_i^n - j_{i+1}^n - j_{i-1}^n)}{2\Delta x} \right] \tag{3.27}$$

As $\epsilon \to 0$, this gives the same local equilibrium

$$j_i^{n+1} = \lambda_1 j_i^n - \lambda_2 A \rho_i^n + \lambda_2 \frac{(\rho_{i+1}^n - \rho_{i-1}^n)}{2\Delta x}$$

as the implicit center difference (3.25b), despite that (3.23) is only first order. However, when ϵ is large, (3.27) is only a first order approximation to (3.20). In order to have a scheme with uniformly second order accuracy, we use van Leer's second order upwind (3.15), i.e.,

$$\rho_{i+1/2}^n = \frac{1}{2}(\rho_i^n + \rho_{i+1}^n) + \frac{\epsilon}{2}(j_i^n - j_{i+1}^n) + \frac{\Delta x}{4}\lambda(\sigma_i^+ - \sigma_{i+1}^-), \tag{3.28}$$

where $\sigma_i^\pm$ is the slope of the i-cell defines by (3.16) when $\phi^{-1/2} = \epsilon$. Clearly, as $\epsilon \to 0$, (3.28) is at least as good as (3.27).

For the generalization to nonlinear systems we will restrict our analysis to the case $\gamma_\epsilon = C\epsilon^h$, with $h \geq 1$ and C a real constant, and $k(\rho, j) = k(\rho)$. As a matter of fact, these restrictions are satisfied by all the nonlinear kinetic models considered in Section 2. Under these assumptions we can write the nonlinear term as

$$J_\epsilon^N(u, v) = k(u+v)C\epsilon^h uv\,, \tag{3.29}$$

and

$$\widehat{J}_\epsilon^N(\rho, j) = k(\rho)\frac{C\epsilon^{h-1}}{4}(\rho^2 - \epsilon^2 j^2). \tag{3.30}$$

For $h \geq 1$ the nonlinear term in j does not create problem of stability and can be evaluated again explicitly since ρ does not change in this step. We can extend the schemes (3.22) to the nonlinear situation simply plugging on the right end side the term $\widehat{J}_\epsilon^L(\rho_i^{n+1}, j_i^{n+1}) + \widehat{J}_\epsilon^N(\rho_i^{n+1}, j_i^n)$ instead of $\widehat{J}_\epsilon^L(\rho_i^{n+1}, j_i^{n+1})$.

3.2.3 Second order Runge-Kutta splitting

In last section we only use the backward Euler method, which is only first order. In this section we describe the second order Runge-Kutta splitting scheme which combines two explicit steps for the convection terms and two implicit steps for the source terms. This splitting was introduced in [CJR] for the Euler scaling, but it can be directly applied here. Let us rewrite the diffusive relaxation system (3.8) in the form

$$\frac{\partial U}{\partial t} + \phi(\epsilon)\frac{\partial U}{\partial x} = -\frac{1}{\epsilon^2}R_\epsilon\left(U, \frac{\partial U}{\partial x}\right). \tag{3.31}$$

With these notations we can write the splitting (3.10)-(3.11) as

$$\begin{aligned} U^{n+\frac{1}{2}} &= U^n - \frac{\Delta t}{\epsilon^2}R_\epsilon\left(U^{n+\frac{1}{2}}, D_r(U^{n+\frac{1}{2}})\right)\,, \\ U^{n+1} &= U^{n+\frac{1}{2}} - \Delta t\phi(\epsilon)D_c(U^{n+\frac{1}{2}})\,, \end{aligned} \tag{3.32}$$

where D_c and D_r denote respectively the approximations of the spatial derivatives used for the convection and the relaxation step.

Applying the second order Runge-Kutta splitting developed in [CJR] to (3.32) gives

$$\begin{aligned} U^{n+\frac{1}{6}} &= U^n - \eta_1\frac{\Delta t}{\epsilon^2}R_\epsilon\left(U^{n+\frac{1}{6}}, D_r(U^{n+\frac{1}{6}})\right)\,, \\ U^{n+\frac{2}{6}} &= U^{n+\frac{1}{6}} - \zeta_1\Delta t\phi(\epsilon)D_c(U^{n+\frac{1}{6}})\,, \\ U^{n+\frac{3}{6}} &= U^{n+\frac{2}{6}} - \eta_2\frac{\Delta t}{\epsilon^2}R_\epsilon\left(U^{n+\frac{3}{6}}, D_r(U^{n+\frac{3}{6}})\right) \\ &\quad - \eta_3\frac{\Delta t}{\epsilon^2}R_\epsilon\left(U^{n+\frac{1}{6}}, D_r(U^{n+\frac{1}{6}})\right)\,, \\ U^{n+\frac{4}{6}} &= U^{n+\frac{3}{6}} - \zeta_2\Delta t\phi(\epsilon)D_c(U^{\frac{3}{6}})\,, \\ U^{n+\frac{5}{6}} &= \zeta_3 U^n + \zeta_4 U^{n+\frac{4}{6}}\,, \\ U^{n+1} &= U^{n+\frac{5}{6}} - \tau\frac{\Delta t}{\epsilon^2}R_\epsilon\left(U^{n+1}, D_r(U^{n+1})\right)\,. \end{aligned} \tag{3.33}$$

To achieve second order accuracy the eight parameters η_i, ζ_i, τ must satisfy the relations

$$\eta_1 = 0, \quad \eta_2 = \frac{2\tau - 1}{2(\tau - 1)}, \quad \eta_3 = -\frac{2\tau^2 - 2\tau + 1}{2\tau(\tau - 1)}, \tag{3.34}$$

$$\zeta_1 = \frac{1}{2\tau}, \quad \zeta_2 = -\frac{1}{2(\tau - 1)}, \quad \zeta_3 = 2\tau^2 - 2\tau + 1, \quad \zeta_4 = -2\tau(\tau - 1).$$

A set of parameters satisfying the previous equations and the stability conditions

$$\eta_i \geq 0, \forall i \quad \tau \geq 0 \tag{3.35}$$

is obtained choosing $\tau = 1/3$ in (3.34) [CJR].

As demonstrated in [CJR], similar to the backward Euler method, this time integrator has the correct diffusion limit in the sense that it always project the solution into the local equilibrium at two intermediate time steps, $t^{n+3/6}$ and t^{n+1}. Due to these projections the convection steps at time $t^{n+1/6}$ and $t^{n+4/6}$ will relax to a limiting equilibrium scheme for the corresponding equilibrium system. Furthermore, the implicit relaxation term can be solved explicitly, for the same reason as the backward Euler method. Since $\eta_1 = 0$ the first step does not project into the local equilibrium. As observed in [CJR] a simple way to overcome this burden and to avoid the initial layer error is to use the first order splitting scheme (3.32) for the first time step along with the new splitting scheme.

3.2.4 Higher dimensional case

We now generalize the previous scheme to systems of kinetic equations in several space variables. In the case of multidimensions a similar discretization can be applied to each space dimension. For the sake of simplicity, and to avoid unessential difficulties, we will describe the schemes in the simplified situation of

$$J_{i,\epsilon}(u, u) = \frac{k(\rho)}{6}(\rho - 6u_i).$$

Following the same arguments used in section 3.2.2 the schemes can be easily generalized to more complex collision operators.

The corresponding three-dimensional diffusive relaxation system can be written as

$$\begin{aligned} \frac{\partial \rho_i}{\partial t} + \frac{\partial j_i}{\partial x_i} &= \frac{k(\rho)}{3\epsilon^2}\left[\rho - 3\rho_i\right], \quad i = 1, 2, 3\,, \\ \frac{\partial j_i}{\partial t} + \phi(\epsilon)\frac{\partial \rho_i}{\partial x_i} &= -\frac{1}{\epsilon^2}\left[k(\rho) j_i + (1 - \epsilon^2\phi(\epsilon))\frac{\partial \rho_i}{\partial x_i}\right], \quad i = 1, 2, 3\,, \end{aligned} \tag{3.36}$$

where $\rho_i = u_{2i-1} + u_{2i}$ with $\sum_i \rho_i = \rho$.

For time integration, the second order Runge-Kutta splitting scheme (3.33) applies here, except that D_c and D_r now become multidimensional discrete derivative operators. The second order upwind discretizations (3.15) and (3.16) can be applied to the characteristic variables $\rho_i \pm \phi^{-1/2} j_i$ in the i-th space dimension for $i = 1, 2, 3$. In the same way the stiff source solver can be applied directly dimension by dimension to the right hand side of (3.36).

For reader's convenience we work out explicitly the 2-D scheme. Let us denote the vector $j(x,y) = (p(x,y), q(x,y))$ and $\rho_1(x,y) = r(x,y)$, $\rho_2(x,y) = s(x,y)$ where $r(x,y) + s(x,y) = \rho(x,y)$. Consider the 2-D kinetic equations

$$\begin{aligned} \frac{\partial r}{\partial t} + \frac{\partial p}{\partial x} &= \frac{k(\rho)}{2\epsilon^2}\left[s - r\right], \\ \frac{\partial s}{\partial t} + \frac{\partial q}{\partial y} &= \frac{k(\rho)}{2\epsilon^2}\left[r - s\right], \\ \frac{\partial p}{\partial t} + \phi(\epsilon)\frac{\partial r}{\partial x} &= -\frac{1}{\epsilon^2}\left[k(\rho)p + (1 - \epsilon^2\phi(\epsilon))\frac{\partial r}{\partial x}\right], \\ \frac{\partial q}{\partial t} + \phi(\epsilon)\frac{\partial s}{\partial y} &= -\frac{1}{\epsilon^2}\left[k(\rho)q + (1 - \epsilon^2\phi(\epsilon))\frac{\partial s}{\partial y}\right]. \end{aligned} \tag{3.37}$$

Suppose the spatial grid points are located at $(x_{i+1/2}, y_{j+1/2})$ with uniform mesh widths $\Delta x = x_{i+1/2} - x_{i-1/2}$ and $\Delta y = y_{j+1/2} - y_{j-1/2}$ respectively. For a vector $U(x,y)$ the point value is $U_{i+1/2,j+1/2} = U(x_{i+1/2}, y_{j+1/2})$. We use U^n_{ij} to denote the cell average

$$U^n_{ij} = \frac{1}{\Delta x \Delta y}\int_{x_{i-1/2}}^{x_{i+1/2}}\int_{y_{j-1/2}}^{y_{j+1/2}} U(x,y)\,dx\,dy.$$

Applying the second order discretization to $r \pm \phi^{-1/2}p$ in the x-direction and $s \pm \phi^{-1/2}q$ in the y-direction to the convection step

$$\begin{aligned} \frac{\partial r}{\partial t} + \frac{\partial p}{\partial x} &= 0, \\ \frac{\partial s}{\partial t} + \frac{\partial q}{\partial y} &= 0, \\ \frac{\partial p}{\partial t} + \phi(\epsilon)\frac{\partial r}{\partial x} &= 0, \\ \frac{\partial q}{\partial t} + \phi(\epsilon)\frac{\partial s}{\partial y} &= 0, \end{aligned} \tag{3.38}$$

we obtain, similar to (3.15),

$$\begin{aligned} r^{n+1}_{ij} &= r^n_{ij}(1-\lambda) + \frac{\lambda}{2}(r^n_{i+1,j} + r^n_{i-1,j}) - \frac{\lambda}{2\phi^{1/2}}(p^n_{i+1,j} - p^n_{i-1,j}) \\ &+ \frac{1}{4}\lambda\left[\Delta x(\sigma^{x,+}_{i-1,j} - \sigma^{x,+}_{ij} - \sigma^{x,-}_{i+1,j} + \sigma^{x,-}_{ij})\right], \end{aligned} \tag{3.39}$$

$$\begin{aligned} s^{n+1}_{ij} &= s^n_{ij}(1-\mu) + \frac{\mu}{2}(s^n_{i,j+1} + s^n_{i,j-1}) - \frac{\mu}{2\phi^{1/2}}(q^n_{i,j+1} - q^n_{i,j-1}) \\ &+ \frac{1}{4}\mu\left[\Delta y(\sigma^{y,+}_{i,j-1} - \sigma^{y,+}_{ij} - \sigma^{y,-}_{i,j+1} + \sigma^{y,-}_{ij})\right], \end{aligned} \tag{3.40}$$

$$
\begin{aligned}
p_{ij}^{n+1} &= (1-\lambda)p_{ij}^n + \frac{\lambda}{2}(p_{i+1,j}^n + p_{i-1,j}^n) - \frac{\lambda\phi^{1/2}}{2}(r_{i+1,j}^n - r_{i-1,j}^n) \\
&\quad + \frac{\phi^{1/2}}{4}\lambda\left[\Delta x(\sigma_{i+1,j}^{x,-} - \sigma_{ij}^{x,-} + \sigma_{i-1,j}^{x,+} - \sigma_{ij}^{x,+})\right] ,
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
q_{ij}^{n+1} &= (1-\mu)q_{ij}^n + \frac{\mu}{2}(q_{i,j+1}^n + q_{i,j-1}^n) - \frac{\mu\phi^{1/2}}{2}(s_{i,j+1}^n - s_{i,j-1}^n) \\
&\quad + \frac{\phi^{1/2}}{4}\mu\left[\Delta y(\sigma_{i,j+1}^{y,-} - \sigma_{ij}^{y,-} + \sigma_{i,j-1}^{y,+} - \sigma_{ij}^{y,+})\right] ,
\end{aligned}
\tag{3.42}
$$

where $\mu = \phi^{1/2}\Delta t/\Delta y$ and the CFL condition becomes $\max\{\lambda,\mu\} \leq 1$.

The slope limiters are defined as

$$
\begin{aligned}
\sigma_{ij}^{x,\pm} &= \frac{1}{\Delta x}\left(\pm r_{i\pm1,j}^n - \phi^{-1/2}p_{ij}^n \mp r_{ij}^n + \phi^{-1/2}p_{i\pm1,j}^n\right)\psi(\theta_{ij}^{x,\pm}), \\
\theta_{ij}^{x,\pm} &= \left[\frac{r_{ij}^n \pm \phi^{-1/2}p_{ij}^n - r_{i-1,j}^n \mp \phi^{-1/2}p_{i-1,j}^n}{r_{i+1,j}^n \pm \phi^{-1/2}p_{i+1,j}^n - r_{ij}^n \mp \phi^{-1/2}p_{ij}^n}\right]^{\pm1} ,
\end{aligned}
\tag{3.43}
$$

$$
\begin{aligned}
\sigma_{ij}^{y,\pm} &= \frac{1}{\Delta y}\left(\pm s_{i,j\pm1}^n - \phi^{-1/2}q_{ij}^n \mp s_{ij}^n + \phi^{-1/2}q_{i,j\pm1}^n\right)\psi(\theta_{ij}^{y,\pm}), \\
\theta_{ij}^{y,\pm} &= \left[\frac{s_{ij}^n \pm \phi^{-1/2}q_{ij}^n - s_{i,j-1}^n \mp \phi^{-1/2}q_{i,j-1}^n}{s_{i,j+1}^n \pm \phi^{-1/2}q_{i,j+1}^n - s_{ij}^n \mp \phi^{-1/2}q_{ij}^n}\right]^{\pm1} .
\end{aligned}
\tag{3.44}
$$

In the same way the stiff source step

$$
\begin{aligned}
\frac{\partial r}{\partial t} &= \frac{k(\rho)}{2\epsilon^2}\left[s - r\right] , \\
\frac{\partial s}{\partial t} &= \frac{k(\rho)}{2\epsilon^2}\left[r - s\right] , \\
\frac{\partial p}{\partial t} &= -\frac{1}{\epsilon^2}\left[k(\rho)p + (1-\epsilon^2\phi(\epsilon))\frac{\partial r}{\partial x}\right] , \\
\frac{\partial q}{\partial t} &= -\frac{1}{\epsilon^2}\left[k(\rho)q + (1-\epsilon^2\phi(\epsilon))\frac{\partial s}{\partial y}\right] ,
\end{aligned}
\tag{3.45}
$$

can be approximated by fully implicit center differences or by semi-implicit upwind based schemes on the kinetic variables $(r \pm \epsilon p)$ and $(s \pm \epsilon q)$ in the x and y directions. In the first case we obtain

$$
\begin{aligned}
\frac{r_{ij}^{n+1} - r_{ij}^n}{\Delta t} &= \frac{k(\rho_{ij}^{n+1})}{2\epsilon^2}\left[s_{ij}^{n+1} - r_{ij}^{n+1}\right] , \\
\frac{s_{ij}^{n+1} - s_{ij}^n}{\Delta t} &= \frac{k(\rho_{ij}^{n+1})}{2\epsilon^2}\left[r_{ij}^{n+1} - s_{ij}^{n+1}\right] , \\
\frac{p_{ij}^{n+1} - p_{ij}^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[k(\rho_{ij}^{n+1})p_{ij}^{n+1} + (1-\epsilon^2\phi(\epsilon))\frac{(r_{i+1,j}^{n+1} - r_{i-1,j}^{n+1})}{2\Delta x}\right] , \\
\frac{q_{ij}^{n+1} - q_{ij}^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[k(\rho_{ij}^{n+1})q_{ij}^{n+1} + (1-\epsilon^2\phi(\epsilon))\frac{(s_{i,j+1}^{n+1} - s_{i,j-1}^{n+1})}{2\Delta y}\right] .
\end{aligned}
\tag{3.46}
$$

Here the situation is slightly different from the one-dimensional situation. Although we still have $\rho_{ij}^{n+1} = \rho_{ij}^{n}$, we need to solve the first two equations in order to put system (3.46) in explicit form.

It is important to note that in this situation, like all the kinetic models considered in section 2, the generalization to higher space dimensions can be easily done. Moreover, as $\epsilon \to 0$, we also get $s_{ij}^{n+1} = \rho_{ij}^{n}/2$ and $r_{ij}^{n+1} = \rho_{ij}^{n}/2$. Upwind discretizations, as in 1-D, can also be used in this step, resulting a scheme with the same good properties as that in 1-D.

Since the structure of the multidimensional diffusive relaxation system is similar to the 1-D system the numerical implementation for higher dimensional problems based on the dimension-by-dimension discretizations are formally not much harder than for 1-D problems.

4 Numerical Diffusion Limit

Unfortunately, for a problem possessing different length scales, the development of numerical analysis theory has been lagged behind the development of robust numerical schemes. However, the diffusion approximation can be used to study the numerical behavior near the diffusive regime, as was done in [LMM, JL, JL1] for related works. the idea is to analyze the behavior of the numerical scheme for $\epsilon^2 << \Delta t$ by deriving the diffusion limit of the scheme and comparing it with the continuous diffusion equation. If the former is a consistent and stable discretization of the latter we say we scheme has the *correct diffusion limit*. Schemes with the correct diffusion limit usually perform well in the diffusive regime where $\epsilon^2 << \Delta t$. Rigorous theoretical justification of such an approach was recently made in [GJL] for the boundary value problem of the linear transport equation in diffusive regimes.

We now study the numerical diffusion limit of our schemes (3.22). As pointed out in section 3.2.2, as $\epsilon \to 0$, both the implicit center difference and the semi-implicit upwind schemes we developed in section 3 collapse to

$$\widehat{J}_\epsilon(\rho_i^{n+1}, j_i^{n+1}) = -\frac{(\rho_{i+1}^{n+1} - \rho_{i-1}^{n+1})}{2\Delta x}\,. \tag{4.1}$$

With $\widehat{J}_\epsilon$ defined by (3.19), (4.1) gives

$$j_i^{n+1} = -\frac{1}{B}\left(A\rho_i^{n+1} + \frac{(\rho_{i+1}^{n+1} - \rho_{i-1}^{n+1})}{\Delta x}\right)\,. \tag{4.2}$$

Applying (4.2) into the convection step (3.14) gives

$$\begin{aligned}\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t} - \frac{A}{B}\frac{\rho_{i+1}^n - \rho_{i-1}^n}{2\Delta x} =& \frac{2}{B}\frac{\rho_{i+2}^n - 2\rho_i^n + \rho_{i-2}^n}{(2\Delta x)^2} \\ &+ \frac{\Delta x}{2}\frac{\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n}{(\Delta x)^2}\,.\end{aligned} \tag{4.3}$$

Comparing (4.3) with (3.19b) we find that (4.3) is a first order consistent and stable discretization of (2.22), with leading truncation error

$$\frac{\Delta x}{2}\frac{\partial^2 \rho}{\partial x^2}.$$

Thus this scheme has the correct diffusion limit, and is expected to capture the correct diffusive behavior in the regime when $\epsilon << 1$ and $\Delta t >> \epsilon^2$, with a first order accuracy in Δx.

the same procedure can be applied to show that our higher order scheme also has the correct diffusion limit. Substituting (4.2) into the second order upwind scheme (3.15), one gets

$$\begin{aligned}\frac{\rho_i^{n+1}-\rho_i^n}{\Delta t} - \frac{A}{B}\frac{\rho_{i+1}^n-\rho_{i-1}^n}{2\Delta x} =& \frac{2}{B}\frac{\rho_{i+2}^n-2\rho_i^n+\rho_{i-2}^n}{(2\Delta x)^2}+\\ &+\frac{\Delta x}{2}\frac{\rho_{i+1}^n-2\rho_i^n+\rho_{i-1}^n}{(\Delta x)^2}+\\ &+\frac{\Delta x}{4}\lambda(1-\lambda)\left[\sigma_{i-1}^{E,+}-\sigma_i^{E,+}-\sigma_{i+1}^{E,-}+\sigma_i^{E,-}\right],\end{aligned} \tag{4.4}$$

where $\sigma_i^{E,\pm}$ is $\sigma_i^{\pm}$ evaluated at the local equlibrium (4.2), i.e.,

$$\sigma_i^{E,\pm} = \frac{1}{\Delta x}\left(\pm\rho_{i\pm1}^n - j_i^n \mp \rho_i^n + j_{i\pm1}^n\right)\psi(\theta_i^{E,\pm})\Bigg|_{j_i^{n+1}=-\frac{1}{B}\left(A\rho_i^{n+1}+\frac{(\rho_{i+1}^{n+1}-\rho_{i-1}^{n+1})}{2\Delta x}\right)}$$

$$\theta_i^{E,\pm} = \left[\frac{\rho_i^n \pm j_i^n - \rho_{i-1}^n \mp j_{i-1}^n}{\rho_{i+1}^n \pm j_{i+1}^n - \rho_i^n \mp j_i^n}\right]^{\pm1}\Bigg|_{j_i^{n+1}=-\frac{1}{B}\left(A\rho_i^{n+1}+\frac{(\rho_{i+1}^{n+1}-\rho_{i-1}^{n+1})}{2\Delta x}\right)}.$$

It is straightforward to show that, if the solution is smooth, in which case the slope limiter is not applied, (4.4) is second order (spatially) discretization of (3.19b) since the last term in (4.4) serves as a antidiffusion term that cancels the first order error appeared in the second term on the right hand side of (4.4). Moreover, the scheme is stable under the diffusive stability condition.

It has been demonstrated in [CJL] that the splitting scheme introduced in section 3 is of second order uniformly in ϵ and has the correct limit as $\epsilon \to 0$. Thus our scheme introduced in section 3 has the correct diffusion limit both spatially and temporally.

In conclusion the new schemes derived here have the correct diffusion limit, thus will yield good results in the diffusive regime, even with $\Delta x, \Delta t >> \epsilon^2$.

5 Numerical Results

In this section we present numerical results that demonstrate the performance of the schemes for the kinetic equations presented in section 2 in both 1-D and 2-D cases.

All the numerical examples presented here refer to the second order time splitting (3.33). The spatial discretizations for the convection step are the second order upwind scheme (3.15), or in 2D, (3.39-42). In the relaxation step we will compare the two different discretizations, i.e., the implicit center difference (abbreviated as FIC) and the semi-implicit second order upwind (SIU). In all the numerical tests we assume no physical boundaries are present. The second order accuracy at the boundaries of the computational domain is achieved using numerical reflecting boundary conditions. Note that on the contrary of classical splitting methods here we must apply the boundary conditions in both the convection and the relaxation steps.

In all our numerical results we take $\Delta t = O(\epsilon \Delta x)$ in the rarefied ($\epsilon >> \Delta x$) regime and $\Delta t = O(\Delta x^2)$ in the intermediate ($\epsilon \approx \Delta x$) and diffusive ($\epsilon << \Delta x$) regimes. The precise choices of Δt are reported in the figure captions. The slope limiters we use are the superbee limiter of Roe in the convection step and van Leer limiter for the SIU.

5.1. 1-D Tests

5.1.1 Linear systems

First we consider a pure diffusive linear problem. We solve the Goldstein-Taylor model in macroscopic form (2.15) with the following initial data

$$\begin{aligned} \rho_L &= 2.0, \quad j_L = 0, \qquad -1 < x < 0, \\ \rho_R &= 1.0, \quad j_R = 0, \qquad 0 < x < 1. \end{aligned} \tag{5.1}$$

As ϵ goes to zero this problem becomes a classical Riemann problem for the heat equation. That is the problem of two semi-infinite rods having temperature ρ_L and ρ_R brought into contact at $t = 0$. Numerical results for this problem has been recently obtained in [RoAr].

The numerical solutions for ρ and j in the rarefied and diffusive regime are depicted with the 'exact' solution, obtained using fine grids with $\Delta x = 0.0001$, in Figure 1a,b,c,d. In the rarefied regime ($\epsilon = 0.7$) both schemes provide accurate description of the shock without oscillations near the discontinuities. In particular in the diffusive regime ($\epsilon = 10^{-6}$), in both cases, the numerical solutions match the exact solution very well.

Next we consider the the linear Ruijgrok-Wu model in the form (2.20) which, as $\epsilon \to 0$, leads to the advection-diffusion equation (2.22). We take as initial data

$$\begin{aligned} \rho_L &= 4.0, \quad j_L = 0, \qquad -10 < x < 0, \\ \rho_R &= 2.0, \quad j_R = 0, \qquad 0 < x < 10. \end{aligned} \tag{5.2}$$

The exact solution for the diffusion equation (2.22) is well known, being

$$\rho(x,t) = \frac{1}{2}(\rho_L + \rho_R) + \frac{1}{2}(\rho_L - \rho_R)\,\mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right),$$

where erf() denotes the error function.

We take $\Delta x = 0.2$ and a final time of $t = 3.0$. In figure 2a we compare the numerical solution for the mass density in the intermediate regime $\epsilon = 0.5$ with the 'exact' solution obtained with $\Delta x = 0.001$. The computed solution for the mass density versus the analytical solution in the diffusive regime is plotted in figure 2b. As it can be seen the numerical results for both schemes describe the exact motion of the shock and in the small relaxation limit are in excellent agreement with the analytical ones.

5.1.2 Nonlinear systems

Next we test the schemes for a nonlinear diffusion problem. We take the generalized Carlemann model that we get from (2.27) with $m = -1$. This model relaxes to the porous media equation. We compare the numerical solution for $\epsilon << 1$ with the exact Barenblatt[Ba] solution for the porous media equation,

$$\begin{array}{lll} r(x,t) = \dfrac{1}{R(t)}\left[1-\left(\dfrac{x}{R(t)}\right)^2\right], & j(x,t) = \rho(x,t)\dfrac{2x}{R(t)^3}, & |x| \leq R(t)\,, \\ r(x,t) = 0, & j(x,t) = 0, & |x| > R(t)\,, \end{array} \tag{5.3}$$

where $R(t) = [12(t+1)]^{1/3}$, $t \geq 0$.

We take $\Delta x = 0.2$ and $x \in]-10, 10[$. In particular it has been shown recently [NaPa] that standard high resolution schemes fail to capture the sharp shock profiles of the analytical solution. In this problem there are two thin viscous shock layer moving to the right and to the left respectively. Both schemes we developed here correctly resolve the physical viscous layer (Figure 3a,b) and are competitive with those obtained in [NaPa].

Finally we consider the nonlinear Ruijgrok-Wu model (2.33). When $\epsilon \to 0$ this model leads to the Burgers equation (2.34). The exact solution to the shock-wave problem has been given in [RuWu].

The initial conditions is given by two local Maxwellian characterized by

$$\begin{array}{ll} \rho_L = 1.0, & -10 < x < 0, \\ \rho_R = 2.0, & 0 < x < 10. \end{array} \tag{5.4}$$

with $j = [(1+\rho^2\epsilon^2/2)^{1/2} - 1]/(2\epsilon^2)$.

Note that for this particular model the local Maxwellian equilibrium depends on the choice of ϵ. Figure 4a,b show for $\Delta x = 0.2$ the computed solution for the mass density in the intermediate ($\epsilon = 0.4$) and parabolic ($\epsilon = 10^{-8}$) regimes of both schemes. As it can be seen once again both schemes are comparable and give a very good description of the viscous shock profiles.

5.2. 2-D Tests

To test the schemes in two-dimensions we have considered the generalized Carlemann equations characterized by (3.37) with $k(\rho) = \rho^{-1}$ that relax towards the porous media equation (2.45) with $P(\rho) = \rho^2$. For brevity we present the results only for the FIC scheme. The SIU scheme provides very similar results. In all the computations we have taken $r(x,y) = s(x,y)$ and $p(x,y) = q(x,y) = 0$ at $t = 0$.

We consider two different problems. First we take a radially simmetric distribution characterized by

$$\rho(x,y) = \begin{cases} \cos\left(\dfrac{\pi\sqrt{(x^2+y^2)}}{2}\right), & x \in]-1,1[\\ 0, & |x| \geq 1. \end{cases} \tag{5.5}$$

We take $\Delta x = 0.05$ and integrate over the square $[-2,2] \times [-2,2]$. The behaviour of the computed solution at time $t = 0.25$ for the mass density and for the modulus of the flux vector in the diffusive regime are plotted in figure 5a,b. The results for the mass density and the flux describes correctly the viscous shock wave with the front moving radially. We refer to [Ar] for a detailed discussion of this problem.

Next we consider a 'oblique' shock wave problem defined by

$$\rho(x,y) = \begin{cases} 0.0, & x < 0, y > 0, \\ 0.8, & x < 0, y < 0, \\ 1.5, & x > 0, y < 0, \\ 1.0, & x > 0, y > 0. \end{cases} \tag{5.6}$$

We begin taking $\Delta x = 0.025$ and computing the solution in the rarefied regime $\epsilon = 0.1$ at time $t = 0.12$ on the box $[-1,1] \times [-1,1]$. Then we take $\Delta x = 0.25$ and consider the solution in the diffusive regime $\epsilon = 10^{-6}$ at time $t = 0.5$ on a larger square $[-10,10] \times [-10,10]$. The behaviour of the numerical solution for the mass density in the rarefied regime is presented in figure 6a. In figure 6b the viscous shock wave of the parabolic regime is shown. In both situations the method resolve well the dynamic of the moving shocks.

6 Conclusions

A class of numerical schemes based on a reformulation of the diffusive (hydrodynamic) scaling problem for discrete-velocity kinetic equations have been introduced and analyzed. The main feature of these schemes is that they work with uniform accuracy with respect to the relaxation parameter, with almost the same cost as an explicit scheme with a much better stability property.

Here we concentrate on developing the basic method. Other ideas, such as different choices of the function ϕ that depend also on the discretization parameters, or adaptive mesh refinement, could be incorporated to give more delicate results. Further theoretical studies such as entropy inequalities and TV bounds of the relaxation part are also under consideration.

We would like to emphasize that this approach can also be used for the study of the transition to Navier-Stokes type problems where the coefficient of viscosity approaches zero or for the study of the transition to Euler type problems where the Mach number approaches zero. For example in the first situation the diffuse relaxation system takes the form

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \phi(\epsilon)\frac{\partial \rho}{\partial x} &= -\frac{1}{\epsilon^2}\left[\widehat{J}_\epsilon(\rho, j) + (\mu - \epsilon^2\phi(\epsilon))\frac{\partial \rho}{\partial x}\right]\,, \end{aligned} \tag{6.1}$$

where μ is the viscosity coefficient. A simple choice of the function ϕ is then given by

$$\phi(\epsilon, \mu) = \begin{cases} 1, & \epsilon \leq \sqrt{\mu}\,, \\ \dfrac{\mu}{\epsilon^2}, & \epsilon > \sqrt{\mu}\,. \end{cases} \tag{6.2}$$

Finally we would also like to attempt to use the new formulation for more complex multi-dimensional kinetic models that relax to the full Navier-Stokes equations. The prospects in all these directions are very encouraging and we hope to present more challenging results in the future.

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