A SIMPLE APPROACH TO GLIMM'S INTERACTION ESTIMATES

WEN-AN YONG

ABSTRACT. This note presents a simple approach toward interaction estimates of elementary waves for hyperbolic conservation laws. The new approach uses neither the Taylor expansion nor induction. The same technique is used to simplify the different interaction estimates in studying nonhomogeneous systems and initial-boundary value problems with Glimm's scheme.

1. INTERACTION ESTIMATES

This short note presents a simple approach to discuss interaction estimates of elementary waves for hyperbolic conservation laws

$$u_t + f(u)_x = 0, \qquad x \in \mathbf{R},\ t > 0, \tag{1.1}$$

where $u, f \in \mathbf{R}^N$. This system is assumed to be strictly hyperbolic and genuinely nonlinear in each characteristic field (see [4]). The following arguments also apply to systems with linearly degenerate characteristic fields.

In the profound paper [3], Glimm established global existence of weak solutions to initial value problems of (1.1), provided that the total variation of initial data is sufficiently small. The smallness of initial data depends essentially on two quantities, one is from the solvability condition of Riemann problems for (1.1) and another is the constant in the interaction estimate.

In order to formulate the interaction problem, we define a transformation

$$T : \mathbf{R}^N \times \mathbf{R}^N \longrightarrow \mathbf{R}^N,$$

where $T(u_l, \epsilon)$ is the state u_N which is connected to $u_l (= u_0)$ on the right by the elementary wave curves of (1.1). More precisely, $u_k = T^{(k)}_{\epsilon_k} u_{k-1}$ and the one-parameter transformation $T^{(k)}_{\sigma} u_{k-1}$ describes the composite k-shock/rarefaction wave curve through u_{k-1}, $k = 1, 2, \cdots, N$. It is usually assumed that $T^{(k)}_{\sigma} u_{k-1}$ is a state connected to u_{k-1} on the right by a k-rarefaction (k-shock) wave if $\sigma \geq 0$ ($\sigma < 0$). See [5] for these terminology.

In [4], Lax proved that if u_l is close to u_r, then there is a unique $\epsilon \in \mathbf{R}^N$ such that $u_r = T(u_l, \epsilon)$. That is, the Riemann problem of (1.1) with initial data

$$u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases}$$

is uniquely solvable. Now let u_l, u_m and u_r be three states close to each other. Then there are unique $\gamma, \delta \in \mathbf{R}^N$ such that $u_m = T(u_l, \gamma)$ and $u_r = T(u_m, \delta)$.

1991 *Mathematics Subject Classification.* Primary 35L65.

Key words and phrases. Hyperbolic conservation laws, Glimm's method.

The interaction problem can be described as finding a little bit precise dependence relation of ϵ on γ and δ. Since $T(u_l, \epsilon) = T(T(u_l, \gamma), \delta)$, we can write ϵ as a function of γ, δ and u_l, namely, $\epsilon(u_l; \gamma, \delta)$.

We first consider the following simple cases, where

$$\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_i, 0, \cdots, 0), \quad \delta = (0, \cdots, 0, \delta_j, \delta_{j+1}, \cdots, \delta_N)$$

with $i < j$ or $i = j$ and $\min\{\gamma_i, \delta_i\} \geq 0$. Note that Riemann problems are uniquely solvable. For such a case, the γ-waves and the δ-waves fit together to give the unique solution. Thus, we have

$$\begin{aligned} & \epsilon(u_l; \gamma_1, \gamma_2, \cdots, \gamma_i, 0, \cdots, 0, \delta_j, \delta_{j+1}, \cdots, \delta_N) \\ = \ & (\gamma_1, \gamma_2, \cdots, \gamma_i, 0, \cdots, 0) + (0, \cdots, 0, \delta_j, \delta_{j+1}, \cdots, \delta_N) \end{aligned} \tag{1.2}$$

for $i, j = 1, 2, \cdots, N$. It is clear that this relation holds when $i = 0$ or $j = N + 1$.

In order to study other complex cases, we recall that $\epsilon(u_l; \gamma, \delta)$ is a C^2-function of its arguments and sufficiently smooth in each quadrant of (γ, δ)-space. We also need the following fact

$$T_\epsilon(u_l; 0) = R(u_l) \equiv (r_1, r_2, \cdots, r_N), \tag{1.3}$$

where r_i is the i-th right eigenvector of the Jacobian of f. These facts are shown in [4] and [5].

In addition, we need the following two identities:

$$a_{nn} - a_{0n} - a_{n0} + a_{00} = \sum_{i,j=1}^{n} (a_{ij} - a_{i-1,j} - a_{i,j-1} + a_{i-1,j-1}), \tag{1.4}$$

where $a_{ij} (i, j = 0, 1, \cdots, n)$ are numbers, vectors or whatever; and

$$\phi(1,1) = \phi(1,0) + \phi(0,1) - \phi(0,0) + \int_0^1 \int_0^1 \phi_{\sigma\tau}(\sigma, \tau) d\sigma d\tau, \tag{1.5}$$

where $\phi(\sigma, \tau)$ is a smooth function of two variables.

Now we are in a position to prove Glimm's interaction theorem ([3]).

Theorem. Let (γ, δ) be in a compact subset of the domain of $\epsilon(u_l; \gamma, \delta)$ for fixed u_l. Then

$$\epsilon(u_l; \gamma, \delta) = \gamma + \delta + \sum_{i>j} \gamma_i \delta_j R^{-1}(R_i r_j - R_j r_i) + D(\gamma, \delta) O(|\gamma| + |\delta|),$$

where $D(\gamma, \delta) = \sum_{i>j} |\gamma_i||\delta_j| + \sum_{\min\{\gamma_i, \delta_i\}<0} |\gamma_i||\delta_i|$ and $R_i = r_i \cdot \nabla_u$.

Proof. Let $a_{ij} = \epsilon(u_l; \gamma_1, \gamma_2, \cdots, \gamma_i, 0, \cdots, 0, \delta_{j+1}, \delta_{j+2}, \cdots, \delta_N)$. Then we use the identities in (1.4), (1.2) and (1.5) to obtain

$$\epsilon(u_l; \gamma, \delta) = \gamma + \delta + \sum_{i,j=1}^{N} \gamma_i \delta_j \int_0^1 \int_0^1 \epsilon_{ij}(\sigma, \tau) d\sigma d\tau, \tag{1.6}$$

where $\epsilon_{ij}(\sigma, \tau) = \epsilon_{\gamma_i \delta_j}(u_l; \gamma_1, \gamma_2, \cdots, \gamma_{i-1}, \sigma\gamma_i, 0, \cdots, 0, \tau\delta_j, \delta_{j+1}, \cdots, \delta_N)$.

The relation in (1.6) already implies Glimm's estimate. In fact, if $i < j$ or if $i = j$ and $\min\{\gamma_i, \delta_i\} \geq 0$, then $\epsilon_{ij}(\sigma, \tau) = 0$ due to (1.2).

Next, we calculate $\epsilon_{\gamma_i\delta_j}(u_l;0,0)(1 \le j \le i \le N)$. From $T(u_l,\epsilon) = T(T(u_l,\gamma),\delta)$, (1.2) and (1.3), we deduce that $T_\epsilon(u_l,\gamma)\epsilon_{\delta_j}(u_l;\gamma,0) = r_j(T(u_l,\gamma))$. Moreover, using $\epsilon_{\delta_j}(u_l;0,0) = e_j$ due to (1.2) we get that

$$T_{\epsilon\gamma_i}(u_l,0)e_j + T_\epsilon(u_l,0)\epsilon_{\delta_j\gamma_i}(u_l;0,0) = R_i r_j(u_l).$$

On the other hand, since $T_{\epsilon\gamma_i}(u_l,0) = (R_1r_i, R_2r_i, \cdots, R_ir_i, R_ir_{i+1}, \cdots, R_ir_N)$, we have

$$\epsilon_{\delta_j\gamma_i}(u_l;0,0) = \begin{cases} R^{-1}(R_ir_j - R_jr_i)(u_l), & i>j \\ 0, & \text{otherwise.}\end{cases}$$

Thus,

$$\sum_{i,j=1}^{N}\gamma_i\delta_j\int_0^1\int_0^1\epsilon_{ij}(\sigma,\tau)d\sigma d\tau = \sum_{i>j}\gamma_i\delta_jR^{-1}(R_ir_j - R_jr_i) + D(\gamma,\delta)O(|\gamma|+|\delta|)$$

and, by using (1.6), hence the proof is complete. □

Remark 1.1. The above proof depends only on the relations in (1.2) (and (1.3)), which follows from the unique solvability of Riemann problems for (1.1). See [4].

Remark 1.2. Since $\epsilon(u_l;\gamma,\delta)$ is sufficiently smooth in each quadrant of (γ,δ)-space, higher order expansions can be also obtained by using (1.6). See [7].

2. Nonhomogeneous Systems

In studying nonhomogeneous systems with the fractional step Glimm scheme, we face to solve ϵ, in terms of γ, δ and Δt, from the following relation (see, e.g., [1] and [6])

$$T(S(u,\Delta t),\epsilon) = S(T(T(u,\gamma),\delta),\Delta t).$$

Here Δt is a positive constant and $S(u,t)$ is the solution or its approximation to

$$\frac{dv}{dt} = s(v), \quad v(0) = u \tag{2.1}$$

with $s(v)$ being the source term of the nonhomogeneous system under consideration. By Glimm's interaction theorem (see Section 1), there is an α such that

$$T(u,\alpha) = T(T(u,\gamma),\delta),$$

where $\alpha = \gamma + \delta + O(1)D(\gamma,\delta)$. Thus, we only need to deal with the relation

$$T(S(u,\Delta t),\epsilon) = S(T(u,\alpha),\Delta t). \tag{2.2}$$

This is a commute problem of operators. It is worthwhile to point out that $S(\cdot,t)$ is an operator entirely similar to $T_\sigma^{(k)}$. Moreover, in the cases of rarefaction waves or contact discontinuities, $T_\sigma^{(k)}u$ solves exactly the problem in (2.1) with the k-th eigenvector $r_k(u)$ replacing $s(v)$ there.

As in the previous section, we write $\epsilon = \epsilon(u;\alpha,t)$ due to the relation in (2.2). Since $S(u,0) = u$ and $T(u,0) = u$, it immediately follows from the unique solvability of Riemann problems that

$$\epsilon(u;\alpha,0) = \alpha, \qquad \epsilon(u;0,t) = 0. \tag{2.3}$$

Consequently, we use the identity in (1.5) to obtain

$$\begin{aligned}\epsilon(u;\alpha,\Delta t) =&\alpha + \alpha\Delta t\int_0^1\int_0^1 \epsilon_{\alpha t}(u;\sigma\alpha,\tau\Delta t)d\sigma d\tau \\ =&\alpha + \alpha\Delta t\epsilon_{\alpha t}(u;0,0) + \alpha\Delta t O(|\alpha| + \Delta t).\end{aligned} \tag{2.4}$$

Now we turn to calculate $\epsilon_{\alpha t}(u;0,0)$. From the relation in (2.2) we deduce that

$$\begin{aligned}T_\epsilon(S(u,t),\epsilon)\epsilon_{t\alpha} =&S_{ut}(T(u,\alpha),t)T_\epsilon(u,\alpha) \\ &- T_{u\epsilon}(S(u,t),\epsilon)S_t(u,t)\epsilon_\alpha - T_{\epsilon\epsilon}(S(u,t),\epsilon)\epsilon_t\epsilon_\alpha.\end{aligned} \tag{2.5}$$

Thanks to (2.3), we have $\epsilon_\alpha(u;\alpha,0) = I$ and $\epsilon_t(u;0,t) = 0$. Thus, we obtain from (2.5), (2.3), (1.3) and (2.1) that

$$R(u)\epsilon_{t\alpha}(u;0,0) = s_u(u)R(u) - R_sR(u), \tag{2.6}$$

where $R_s = s(u)\cdot\nabla_u$.

We refer to $R^{-1}(u)s_u(u)R(u) - R^{-1}(u)R_sR(u)$ as the *growth matrix* for the corresponding nonhomogeneous system. The matrix $R^{-1}(u)R_sR(u)$ characterizes the interaction of source terms and flux terms.

3. Initial-Boundary Value Problems

In this section, we consider initial-boundary value problems for systems of hyperbolic conservation laws, that is, (1.1) in $x<0$ with boundary conditions posed at $x=0$

$$B(u,t) = 0, \tag{3.1}$$

where $B(u,t)$ takes values in $\mathbf{R}^\nu$ and ν is the number of negative eigenvalues of $f_u(u)$. We do not assume that the boundary is non-characteristic.

Lemma 3.1. *Let (u,t) satisfy*

$$\det B_u(u,t)(r_1, r_2, \cdots, r_\nu)(u) \neq 0. \tag{3.2}$$

Then there exists $\eta(u,t) > 0$ such that for each $\delta\in\mathbf{R}^\nu$ with $|\delta|\le\eta(u,t)$, there is a unique $\epsilon = (\overset{\nu}{\hat\epsilon}, \overset{N-\nu}{0})$, with $|\hat\epsilon|$ sufficiently small, satisfying

$$B(T(u,\epsilon),t) = B(u,t) - \delta.$$

Proof. For $\hat\epsilon,\delta\in\mathbf{R}^\nu$, define

$$F(u,t;\hat\epsilon,\delta) = B(T(u,\hat\epsilon,\overset{N-\nu}{0}),t) - B(u,t) + \delta.$$

Then $F(u,t;0,0) = 0$. Moreover, $F_{\hat\epsilon}(u,t;0,0) = B_u(u,t)(r_1,r_2,\cdots,r_\nu)(u)$ is invertible due to (3.2). By applying the implicit function theorem to $F(u,t;\hat\epsilon,\delta)$ at $(\hat\epsilon,\delta)=0$ for fixed (u,t), we deduce that there exists $\eta=\eta(u,t)>0$ such that for each δ with $|\delta|\le\eta(u,t)$, there is a unique $\hat\epsilon = \hat\epsilon(u,t;\delta)$, with $|\hat\epsilon|$ sufficiently small, such that $F(u,t;\hat\epsilon,\delta) = 0$. Namely,

$$B(T(u,\epsilon),t) = B(u,t) - \delta.$$

This completes the proof. □

If (u,t) satisfies $|B(u,t)|\le\eta(u,t)$ as well as the relation in (3.2), then by taking $\delta = B(u,t)$ in the above lemma, we have

Corollary 3.2. *Let (u,t) satisfy the relation in (3.2) and $|B(u,t)| \leq \eta(u,t)$. Then there is a unique $\epsilon = (\overset{\nu}{\hat{\epsilon}}, \overset{N-\nu}{0})$, with $|\hat{\epsilon}|$ sufficiently small, satisfying $B(T(u,\epsilon),t) = 0$.*

Having this corollary, we turn to consider the *boundary Riemann problem*, that is, solving the system (1.1) under the boundary condition (3.1) and the initial condition $u(x,0) = u_l$ (a constant state) for $x < 0$. The above corollary claims that if $|B(u_l,t)| \leq \eta(u_l,t)$ and (3.2) holds for $u = u_l$, then the boundary Riemann problem has a unique solution consisting of $\nu + 1$ constant states

$$u_0 = u_l, u_1 = T^{(1)}_{\epsilon_1} u_0, u_2 = T^{(2)}_{\epsilon_2} u_1, \cdots, u_\nu = T^{(\nu)}_{\epsilon_\nu} u_{\nu-1},$$

so that u_j is connected to u_{j-1} on the right by a j-elementary wave $(j = 1, 2, \cdots, \nu)$.

In studying initial-boundary value problems, we have to deal with interaction problems of elementary waves near the boundary (see [2]). Such problems can be described as solving ϵ, in terms of γ, δ and u, from

$$B(T(u,\epsilon),t) = 0 = B(T(T(u,\gamma),\delta),t').$$

Here δ's and ϵ's last $(N - \nu)$ components vanish, and so do γ's first ν components. We point out the following fact. For simplicity, we denote $\hat{\epsilon}$ by ϵ and assume that B does not depend on t later on.

Theorem 3.3. *Suppose u satisfies the relation in (3.2) and the inequality*

$$|B(u) - B(T(T(u,\gamma),\delta))| \leq \eta(u)$$

holds for all (γ,δ) with $|\gamma|, |\delta|$ sufficiently small.

Then there is a unique $\epsilon = \epsilon(u;\gamma,\delta)$, with $|\epsilon|$ sufficiently small, satisfying

$$B(T(u,\epsilon)) = B(T(T(u,\gamma),\delta)). \tag{3.3}$$

Furthermore, $\epsilon(u;\gamma,\delta)$ can be expanded as

$$\epsilon(u;\gamma,\delta) = \delta + [B_u(r_1, r_2, \cdots, r_\nu)]^{-1} B_u(r_{\nu+1}, r_{\nu+2}, \cdots, r_n)(u)\gamma + O(1)|\gamma|(|\delta| + |\gamma|).$$

Proof. By taking δ to be $B(u) - B(T(T(u,\gamma),\delta))$ in Lemma 3.1, we immediately conclude the existence of ϵ satisfying the relation in (3.3).

For proving the expansion, we use the uniqueness in Lemma 3.1 to obtain $\epsilon(u;0,\delta) = \delta$. Thus, it follows from the identity in (1.5) that

$$\begin{aligned} \epsilon(u;\gamma,\delta) = &\ \epsilon(u;0,\delta) + \epsilon(u;\gamma,0) - \epsilon(u;0,0) + \gamma\delta \int_0^1 \int_0^1 \epsilon_{\gamma\delta}(u;\sigma\gamma,\tau\delta) d\sigma d\tau \\ = &\ \delta + \epsilon(u;\gamma,0) + O(1)|\gamma||\delta|. \end{aligned}$$

On the other hand, we deduce from the relation in (3.3) and Taylor's theorem that

$$\epsilon(u;\gamma,0) = F_\epsilon^{-1}(u,0) B_u(u)(r_{\nu+1}, r_{\nu+2}, \cdots, r_n)(u)\gamma + O(|\gamma|^2). \tag{3.4}$$

Since $F_\epsilon(u,0) = B_u(u)(r_1, r_2, \cdots, r_\nu)(u)$, the proof is complete. $\square$

Remark 3.1. The relation in (3.4) describes the reflection of the γ's waves. From the expansion we see that, by choosing the eigenvectors carefully,

$$|\epsilon(u;\gamma,0)| \leq |\gamma| \tag{3.5}$$

for $|\gamma|$ small. This inequality is usually referred to as the dissipativity of boundary conditions. See [2].

Acknowledgment. This work was supported by the Deutsche Forschungsgemeinschaft through SFB 359 at the University of Heidelberg.

References

[1] C. M. Dafermos & L. Hsiao, Hyperbolic Systems of Balance Laws with Inhomogeneity and Dissipation, *Indiana U. Math. J.*, 31(1982), 471-491.
[2] H. Frid, Initial-Boundary Value Problems for Conservation Laws, *J. Diff. Eqns.*, 128(1996), 1-45.
[3] J. Glimm, Solutions in the Large for Nonlinear Hyperbolic Systems of Equations, *Comm. Pure Appl. Math.*, 18(1965), 697-715.
[4] P. D. Lax, Hyperbolic Systems of Conservation Laws II, *Comm. Pure Appl. Math.*, 10(1957), 537-566.
[5] J. Smoller, Shock Waves and Reaction-Diffusion Equations, *Springer, New York*, 1983.
[6] L. Ying & C. Wang, Solutions in the Large for Nonhomogeneous Quasilinear Hyperbolic Systems of Equations, *J. Math. Anal. Appl.*, 78(1980), 440-454.
[7] R. Young, Sup-Norm Stability for Glimm's Scheme, *Comm. Pure Appl. Math.*, 46(1993), 903-948.

Institut für Angewandte Mathematik, Universität Heidelberg, Im Neuenheimer Feld 294, 69120 Heidelberg, Germany
E-mail address: `yong@esprit.iwr.uni-heidelberg.de`