Hyperbolic Conservation Laws with a Moving Source

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Abstract

The purpose of this paper is to investigate the wave behavior of hyperbolic conservation laws with a moving source. When the speed of the source is close to one of the characteristic speeds of the system, nonlinear resonance occurs and instability may result. We will study solutions with a single transonic shock wave for a general system $u_t + f(u)_x = g(x,u)$. Suppose that the i-th characteristic speed is close to zero. We propose the following stability criterion:

$l_i \frac{\partial g}{\partial u} r_i < 0$ for nonlinear stability, $l_i \frac{\partial g}{\partial u} r_i > 0$ for nonlinear instability

Here l_i and r_i are the i-th normalized left and right eigenvectors of $\frac{df}{du}$ respectively. By using a variation of the Glimm scheme and studying the evolution of the single transonic shock wave, we prove the existence of solutions and verify the asymptotic stability (or instability).

1 Introduction

In this paper, we study the time-asymptotic stability and instability of solutions to systems of conservation laws with a moving source:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= g(x-ct, u), \quad (x,t) \in \mathrm{R} \times \mathrm{R}_+, \\ u(x,0) &= u_0(x), \quad x \in \mathrm{R}. \end{aligned}$$

Here $u = u(x,t)$ takes on values in an open set $\Omega \subset \mathrm{R}^n$ and f is a non-linear smooth map from Ω to R^n. The function g represents external effects, c its speed. We assume that the systems are strictly hyperbolic and that each characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax [10]. By the change of variables $x - ct \to x$, $f(u) \to f(u) - cu$, the system is reduced to

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = g(x,u), \tag{1.1}$$
$$u(x,0) = u_0(x). \tag{1.2}$$

Thus, without loss of generality, we only consider systems of the form (1.1). Since (1.1) and (1.2) do not have smooth solutions in general (See Lax [11]), we look for weak solutons, that is, bounded measurable functions satisfying

$$\iint_{t\geq 0}\left(u\frac{\partial\varphi}{\partial t}+f(u)\frac{\partial\varphi}{\partial x}+g(x,u)\varphi\right)\,dxdt+\int_{-\infty}^{\infty}u_0(x)\varphi(x,0)\,dx \;=\; 0 \tag{1.3}$$

for any smooth function $\varphi(x,t)$ with compact support in $\mathrm{R}\times\mathrm{R}_+$.

It is known that with limiting values at $x=\pm\infty$, a general solution of hyperbolic conservation laws

$$\frac{\partial u}{\partial t}+\frac{\partial f(u)}{\partial x}=0 \tag{1.4}$$

tends to an asymptotic state consisting of shock waves, rarefaction waves and traveling waves, which can be found by solving the Riemann problem with the initial data at $x=\pm\infty$ (Liu [15]). The solution is asymptotically stable by which we mean that the asymptotic state depends smoothly on the initial data. As for system (1.1), when the characteristic speeds $\lambda_i(u)$ are non-zero and do not change signs for all u and $i=1,\ldots,n$, its solution is also asymptotically stable (Liu [14]). Although the evolution of the solution can be complicated, it tends to a simple non-interacting wave pattern time-asymptotically. In this case, the asymptotic state consists of the aforementioned elementary waves and a steady wave which is a solution of

$$\frac{df(u)}{dx}=g(x,u). \tag{1.5}$$

However, richer wave phenomena happen when one of the characteristic speeds, say, λ_i is around zero. The purpose of this paper is to investigate the wave behavior of system (1.1) when the i-th characteristic field is genuinely nonlinear and its characteristic speed λ_i changes signs, with particular attention given to a single transonic shock wave. Under such situations, a standing shock wave and the effects of the source term propagate at the same speed. Thus, the solution is not always asymptotically stable due to the interaction of the source term and transonic waves. Here, g plays a dominant role in the determination of the asymptotic stability. For instance, for the quasi-one-dimensional model of gas flows through a nozzle:

$$\begin{aligned}
\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x} &= -\frac{a'(x)}{a(x)}\rho u\\
\frac{\partial(\rho u)}{\partial t}+\frac{\partial}{\partial x}(\rho u^2+p) &= -\frac{a'(x)}{a(x)}\rho u^2\\
\frac{\partial(\rho E)}{\partial t}+\frac{\partial}{\partial x}(\rho uE+pu) &= -\frac{a'(x)}{a(x)}(\rho uE+pu),
\end{aligned}$$

where $a(x)$ is the cross sectional area of the duct, ρ,u,p and E are respectively the density, velocity, pressure and the total energy of the gas, and $x\in\mathrm{R}$, $t>0$, the stability does not depend on the equation of the state; nevertheless, the geometry of a duct (i.e. the sign of $a'(x)$) completely determines the stability of the flow: flows through an expanding duct are stable; transonic flows through a contracting duct are unstable(Liu [16]). A moving magnetic field for magnetohydrodynamics (MHD) can also be expressed as a moving source (Hoffman [9]). Motivated by the studies of physical models, we assume that the source has finite strength, that is, $g(x,u)$ tends to zero sufficiently fast as $|x| \to \infty$. For simplicity, $g(x,u)$ is assumed to be piecewise smooth with compact support in $x : g(x,u) = 0$ for $|x| \geq 1$, and $g(x,u) \neq 0$ for $|x| < 1$. We also assume that $g(x,u)$ is smooth with respect to u.

We propose the following stability criterion :

$$l_i \frac{\partial g}{\partial u} r_i \ < \ 0, \quad (\text{ nonlinear stability}) \tag{1.6}$$

$$l_i \frac{\partial g}{\partial u} r_i \ > \ 0, \quad (\text{ nonlinear instability}) \tag{1.7}$$

for all $|x| < 1$ and all $u \in \Omega$, where l_i and r_i are the i-th left and right eigenvectors of $\frac{df(u)}{du}$ and normalized by $l_i \cdot r_j = \delta_{ij}$. Note that this criterion is consistent with the cases of a single equation, [17], and the nozzle flows mentioned above, [16]. We shall concentrate on the solutions of (1.1) containing a single transonic shock wave. In particular, we show that if a standing shock wave is perturbed and when (1.7) holds, it accelerates and leaves the region $|x| \leq 1$ in finite time. Thus the time-asymptotic state differs greatly from the initial state. However, under the condition (1.6), it decelerates and tends to a nearby asymptotic state.

The plan of the paper is as follows. In Section 2, we briefly review some definitions and basic properties of system (1.1). The crucial estimates on the change of the speed of a transonic shock wave are studied in Section 3. The main tool for constructing solutions to system (1.1) is the random choice method of Liu[14], which generalizes the Glimm scheme[6] for conservation laws. The idea is to decompose the solution into nonlinear modes for conservation laws (1.4) and steady waves (1.5). We shall describe this in Section 4. Local wave interactions are established in Section 5. The most notable feature in the unstable case is that the speed of the single transonic shock wave is bigger than or equal to its initial speed, instead of monotone increasing as shown in Section 3. This is due to the involvement of nearby weak waves. It thus demands a global estimate on nonlinear wave interactions to establish the estimate of the shock speed. The existence of solutions in the stable case is also proved with the same spirit. We shall deal with the global estimates and verify the asymptotic stability (or instability) in Section 6.

For other works on nozzle flows, see Glimm et al.[8] and Embid et al.[5]. For conservation laws with a moving source, see Asakura [1] and Chen and Glimm [2].

2 Preliminaries

System (1.1) is assumed to be strictly hyperbolic, that is, $\frac{df(u)}{du}$ has real and distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ with right and left eigenvectors $r_i(u)$ and $l_i(u)$, $i = 1, \cdots, n$, which are normalized by $l_i \cdot r_j = \delta_{ij}$. Each characteristic field is either genuinely nonlinear or linearly degenerate in the sense of Lax [10], i.e.,

$$\begin{aligned} \nabla\lambda_i(u) \cdot r_i(u) &\neq 0 \quad \text{for all u , or} \\ \nabla\lambda_i(u) \cdot r_i(u) &= 0 \quad \text{for all u .} \end{aligned}$$

For the sake of transparency and simplicity in the presentation, we assume throughout this paper that λ_1 is close to zero and $\lambda_j > 0$ for $j \geq 2$.

We first recall the classical definitions of elementary waves for (1.4). For each genuinely nonlinear field, there are shock waves and rarefaction waves, and for each linearly degenerate field, there are contact discontinuities. An i-shock wave (u_-, u_+) is a discontinuity satisfying the Rankine-Hugoniot(R-H) condition

$$(u_+ - u_-)\sigma(u_-, u_+) = f(u_+) - f(u_-),$$

where $u_\pm = u(x(t) \pm 0, t), \sigma(u_-, u_+) = x'(t)$, and the Lax entropy condition

$$\lambda_i(u_+) < \sigma(u_-, u_+) < \lambda_i(u_-).$$

If $\sigma(u_-, u_+) = \lambda_i(u_-) = \lambda_i(u_+)$, (u_-, u_+) is called a contact discontinuity. An i-rarefaction wave (u_-, u_+) connecting u_- on the left and u_+ on the right is a smooth solution of (1.4) with speed $\lambda_i(u_+) \geq \lambda_i(u_-)$, which takes values on the rarefaction curve $R_i(u_-)$: the integral curve of $r_i(u)$ through u_-. Let $S_i(u_-)$ denote the R-H curve, i.e.,

$$S_i(u_-) = \{u : (u - u_-)\sigma(u_-, u) = f(u) - f(u_-) \quad \text{for some scalar } \sigma(u_-, u)\}.$$

And set

$$\begin{aligned} R_i^+(u_-) &= \{u : u \in R_i(u_-), \text{ and } \lambda_i(u) \geq \lambda_i(u_-)\} \\ S_i^-(u_-) &= \{u : u \in S_i(u_-), \text{ and } \lambda_i(u) < \sigma(u_-, u) < \lambda_i(u_-)\} \\ T_i(u_-) &= R_i^+(u_-) \cup S_i^-(u_-), \quad \text{if the i-th field is genuinely nonlinear,} \\ T_i(u_-) &= R_i(u_-)(= S_i(u_-)), \quad \text{if the i-th field is linearly degenerate.} \end{aligned}$$

It can be shown by the implicit function theorem that $S_i(u_-)$ is a smooth curve in a small neighborhood of u_- and is tangent to $R_i(u_-)$ at u_- up to second order derivatives. In other words, if $R_i(u_-(s))$ and $S_i(u_-(s))$ are parametrized by the arclength s with $u_-(0) = u_-$, we have

$$|R_i(u_-(s)) - S_i(u_-(s))| = O(|u_- - R_i(u_-(s))|^3). \tag{2.1}$$

Moreover,

$$\sigma(u_-, u) = \frac{1}{2}(\lambda_i(u_-) + \lambda_i(u)) + O(|u - u_-|^2). \tag{2.2}$$

The Riemann problem (u_-, u_+) is an initial value problem for (1.4) with two constant initial states:

$$u(x,0) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0 \end{cases}$$

To solve it, we find states $u_i, i = 0, 1, \cdots, n, u_0 = u_-, u_n = u_+, u_i \in T_i(u_{i-1})$ so that the solution consists of elementary i-waves (u_{i-1}, u_i) (See Lax [10]). The strength of the i-wave in (u_-, u_+) is measured by

$$\begin{aligned} (u_-, u_+)_i &= \lambda_i(u_i) - \lambda_i(u_{i-1}), \quad \text{for genuinely nonlinear } i\text{-field} \\ (u_-, u_+)_i &= \tau_i(u_i) - \tau_i(u_{i-1}), \quad \text{for linearly degenerate } i\text{-field} \end{aligned}$$

where τ_i is any nonsingular parameter along T_i. It is noted that an i-shock wave has negative strength while an i-rarefaction wave has positive strength.

We now turn to the study of steady waves. A solution to (1.5) is called a steady state solution or a steady wave, which does not depend on t. An i-standing shock wave (u_l, u_r) is a solution for (1.3) of the form

$$u(x,t) = \begin{cases} u_l(x), & x < 0 \\ u_r(x), & x > 0 \end{cases}$$

such that $u_l(x)$ and $u_r(x)$ are steady state solutions, $f(u_l(0)) - f(u_r(0)) = 0$ (the R-H condition), and $\lambda_i(u_r(0)) < 0 < \lambda_i(u_l(0))$ (the Lax entropy condition). When $\lambda_i(u) \neq 0$ for all i, we can express the system of ordinary differential equations (1.5) as

$$\frac{du}{dx} = \sum_{i=1}^{n} \frac{l_i(u) \cdot g(x,u)}{\lambda_i(u)} r_i(u), \quad x \in \mathrm{R}. \tag{2.3}$$

3 A Basic Lemma

Consider a 1-shock wave (u_l, u_r) which is a perturbation of a standing shock wave, that is, $\lambda_1(u_r) < \sigma(u_l, u_r) < \lambda_1(u_l)$, and its speed $\sigma \equiv \sigma(u_i, u_r)$ is small compared to its strength $\alpha \equiv |(u_l, u_r)|$. We will establish the key estimate on the evolution of (u_l, u_r) in this section. We introduce the following notations. $x = x_0$ and $x = x_1$ are two fixed space positions. Given a state u, $(u, \hat{u})$ represents the steady wave $u(x)$, (1.5), such that $u(x_0) = u$ and $u(x_1) = \hat{u}$. $\tilde{u}$ stands for the state in $S_1^-(u)$ so that $\sigma(u, \tilde{u}) = \sigma$. We also use $O(1)$ throughout this paper to denote a constant depending only on system (1.4).

Define

$$G(x) \equiv \sup\{|g(x,u)| + \left|\frac{\partial g}{\partial u}(x,u)\right| : u \in \Omega\},$$

which is assumed to be small compared to α. The following lemma shows that when a standing shock wave is perturbed under the condition (1.7), it accelerates; however, it decelerates under the condition (1.6).

Lemma 3.1 *Let (u_l, u_r) be a 1-shock wave which is a perturbation of a standing 1-shock wave. Suppose that its speed σ and $G(x)$ are sufficiently small compared to its strength α.*

Let $(u_l, \hat{u}_l)$ and $(u_r, \hat{u}_r)$ be two steady waves and the 1-wave in the solution of the Riemann problem $(\hat{u}_l, \hat{u}_r)$ be denoted by $(\hat{u}_l, u_1)$. Then there exists a positive $|O(1)|$ such that

(1)

$$|u_1 - \hat{u}_r| = |O(1)|\alpha \int_{x_0}^{x_1} G(x)dx.$$

(2) *when (1.6) holds,*

$$\frac{1}{2}|O(1)| \int_{x_0}^{x_1} G(x)dx \leq \sigma(u_l, u_r) - \sigma(\hat{u}_l, u_1) \leq 2|O(1)| \int_{x_0}^{x_1} G(x)dx,$$

when (1.7) holds,

$$\frac{1}{2}|O(1)| \int_{x_0}^{x_1} G(x)dx \leq \sigma(\hat{u}_l, u_1) - \sigma(u_l, u_r) \leq 2|O(1)| \int_{x_0}^{x_1} G(x)dx.$$

Proof. Without loss of generality, we assume that $\lambda_1(u_l) > 0$, σ is positive, and $x_0 < x_1$. First by the R-H condition, we have

$$\begin{aligned} f(u_l) - f(\tilde{u}_l) &= \sigma(u_l - \tilde{u}_l), \quad \text{and} \\ f(\hat{u}_l) - f(\tilde{\hat{u}}_l) &= \sigma(\hat{u}_l - \tilde{\hat{u}}_l). \end{aligned} \tag{3.1}$$

Note that $\tilde{u}_l = u_r$ and $\hat{\tilde{u}}_l = \hat{u}_r$. Since $u(x) \equiv (u_l, \hat{u}_l)$ and $\bar{u}(x) \equiv (\tilde{u}_l, \hat{\tilde{u}}_l)$ are steady state solutions, by integrating (1.5),

$$f(\hat{u}_l) - f(u_l) = \int_{x_0}^{x_1} g(x, u(x))dx \tag{3.2}$$

$$f(\hat{\tilde{u}}_l) - f(\tilde{u}_l) = \int_{x_0}^{x_1} g(x, \bar{u}(x))dx \tag{3.3}$$

It follows from (3.1), (3.2) and (3.3) that

$$\begin{aligned} f(\tilde{\hat{u}}_l) - f(\hat{\tilde{u}}_l) = & \int_{x_0}^{x_1} [g(x, u(x)) - g(x, \bar{u}(x))]dx \\ & + \sigma[(u_l - \tilde{u}_l) - (\hat{u}_l - \tilde{\hat{u}}_l)] \end{aligned}$$

By the mean value theorem, there exists a state u_* in a neighborhood around $\tilde{\hat{u}}_l$ and $\hat{\hat{u}}_l$ so that

$$
\begin{aligned}
df(u_*)\cdot(\tilde{\hat{u}}_l-\hat{\hat{u}}_l) &= \int_{x_0}^{x_1}[g(x,u(x))-g(x,\bar{u}(x))]dx \\
&\quad +\sigma[(u_l-\tilde{u}_l)-(\hat{u}_l-\tilde{\hat{u}}_l)]
\end{aligned}
$$

Multiplying the above equality by $l_i(u_*)$, we obtain

$$
\begin{aligned}
\lambda_i(u_*)\cdot l_i(u_*)\cdot(\tilde{\hat{u}}_l-\hat{\hat{u}}_l) &= \int_{x_0}^{x_1} l_i(u_*)[g(x,u(x))-g(x,\bar{u}(x))]dx \\
&\quad +\sigma l_i(u_*)[(u_l-\tilde{u}_l)-(\hat{u}_l-\tilde{\hat{u}}_l)] \qquad (3.4)
\end{aligned}
$$

Since $\lambda_1(u_l)>0$ and by (2.2), it is obvious that $\lambda_1(u(x))>0, \quad \lambda_1(\bar{u}(x))<0$ and

$$
\lambda_1(u_*)<0. \qquad (3.5)
$$

$\tilde{\hat{u}}_l-\hat{\hat{u}}_l$ can be expressed as

$$
\tilde{\hat{u}}_l-\hat{\hat{u}}_l = d_1 r_1(u_*)+\cdots+d_n r_n(u_*), \qquad (3.6)
$$

for some constants d_i, $i=1,\cdots,n$. By (3.4) and (3.5),

$$
\begin{aligned}
d_i\lambda_i(u_*) &= \int_{x_0}^{x_1} l_i(u_*)[g(x,u(x))-g(x,\bar{u}(x))]dx \\
&\quad +\sigma l_i(u_*)[(u_l-\tilde{u}_l)-(\hat{u}_l-\tilde{\hat{u}}_l)]
\end{aligned}
$$

Since λ_1 is close to zero, and λ_i's are bounded away from zero for $i\neq 1$, it follows that

$$
\left|\frac{d_1}{d_i}\right| = O(1)\left|\frac{\lambda_i(u_*)}{\lambda_1(u_*)}\right| \gg 1, \quad \text{for } i\neq 1. \qquad (3.7)
$$

By the mean value theorem again, there exist states $u_*(x)$ for each $x\in(x_0,x_1)$ so that

$$
\begin{aligned}
d_1\lambda_1(u_*) &= \int_{x_0}^{x_1} l_1(u_*)\frac{\partial g}{\partial u}(x,u_*(x))(u(x)-\bar{u}(x))dx \\
&\quad +\sigma l_1(u_*)[(u_l-\tilde{u}_l)-(\hat{u}_l-\tilde{\hat{u}}_l)] \qquad (3.8)
\end{aligned}
$$

Expand $u(x)-\bar{u}(x)$ as

$$
u(x)-\bar{u}(x)=c_1(x)r_1(u_*)+\cdots+c_n(x)r_n(u_*). \qquad (3.9)
$$

By (2.3), we know that $u(x)$ and $\bar{u}(x)$ take values almost along R_1 , thus

$$
\begin{aligned}
c_1(x) &> 0 \ , \quad \text{and} \\
c_1(x) &\gg |c_i(x)| \ , \quad i\neq 1, \qquad (3.10)
\end{aligned}
$$

for all $x \in (x_0, x_1)$. By (3.8) and (3.9),

$$\begin{aligned} d_1 &= \frac{1}{\lambda_1(u_*)}\{\int_{x_0}^{x_1} c_1(x)[l_1(u_*)\frac{\partial g}{\partial u}(x, u_*(x))r_1(u_*)]dx \\ &+\sum_{i=2}^{n}\int_{x_0}^{x_1} c_i(x)[l_1(u_*)\frac{\partial g}{\partial u}(x, u_*(x))r_i(u_*)]dx \\ &+\sigma l_1(u_*)[(u_l - \tilde{u}_l) - (\hat{u}_l - \tilde{\hat{u}}_l)]\} \end{aligned} \tag{3.11}$$

Hence by choosing σ sufficiently small, we obtain that

$$d_1 > 0 \qquad \text{if } l_1\frac{\partial g}{\partial u}r_1 < 0 \tag{3.12}$$

$$d_1 < 0 \qquad \text{if } l_1\frac{\partial g}{\partial u}r_1 > 0. \tag{3.13}$$

The above inequalities and (3.4)-(3.7) thus imply that

$$\lambda_1(\hat{\tilde{u}}_l) < \lambda_1(\tilde{\hat{u}}_l) \qquad \text{if } l_1\frac{\partial g}{\partial u}r_1 < 0 \tag{3.14}$$

$$\lambda_1(\hat{\tilde{u}}_l) > \lambda_1(\tilde{\hat{u}}_l) \qquad \text{if } l_1\frac{\partial g}{\partial u}r_1 > 0. \tag{3.15}$$

We now study the case when (1.7) holds. The other case can be proved by the same argument. First by (3.15), we know $\lambda_1(\hat{\tilde{u}}_l) > \lambda_1(\tilde{\hat{u}}_l)$. Since S_1 is tangent to R_1 up to second order derivatives, when α is chosen sufficiently small, depending only on system (1.4), we have

$$\lambda_1(u_1) > \lambda_1(\tilde{\hat{u}}_l). \tag{3.16}$$

By (3.6), (3.7) and (3.11), we get

$$\tilde{\hat{u}}_l - \hat{\tilde{u}}_l = d_1 r_1(\tilde{\hat{u}}_l) + \cdots + d_n r_n(\tilde{\hat{u}}_l) + O(1)d_1\int_{x_0}^{x_1} G(x)dx, \tag{3.17}$$

$$d_1 = -|O(1)|\frac{\alpha}{\lambda_*}\int_{x_0}^{x_1} G(x)dx, \tag{3.18}$$

$$d_i = O(1)\alpha d_1, \text{ for } i \neq 1, \tag{3.19}$$

where λ_* is the minimum of $|\lambda_1(u)|$ for all u under consideration. Note that λ_* is of the same order of α. It follows from the last three estimates and (2.1) that

$$|u_1 - \hat{\tilde{u}}_l| = O(1)\alpha\int_{x_0}^{x_1} G(x)dx.$$

Also, by (2.2) and (3.16)-(3.19),

$$\begin{aligned} \sigma(\hat{u}_l, u_1) - \sigma(u_l, u_r) &= \sigma(\hat{u}_l, u_1) - \sigma(\hat{u}_l, \tilde{\hat{u}}_l) \\ &= |O(1)||u_1 - \tilde{\hat{u}}_l| \\ &= |O(1)|\frac{\alpha}{\lambda_*}\int_{x_0}^{x_1} G(x)dx. \end{aligned}$$

This completes the proof of Lemma 3.1.

4 Construction of Approximate Solutions

We now describe the difference scheme for constructing the solution of system(1.1). We basically adopt the difference scheme introduced in Liu [14]. When the single relatively strong shock wave is involved, the front tracking scheme (Chern[3]) is applied to trace the evolution of the relatively strong shock wave.

Choose an equidistributed sequence $\{a_k\}_{k=0}^{\infty}$ in $(0,1)$ and mesh lengths $\Delta x = r, \Delta t = s$ satisfying the Courant-Friedrichs-Lewy condition:

$$\frac{r}{s} > 2 \sup_{i,u} |\lambda_i(u)|.$$

Let h,k be integers, $a_{hk} = ((h+a_k)r, ks)$ denote the sample points on the line $t = ks$. In order to obtain the desired estimates, we consider space-like curves which are piecewise linear curves consisting of line segments joining a_{hk} to $a_{h+1,k+1}$ or to $a_{h+1,k-1}$. The upper-half plane $t \geq 0$ are covered by "diamonds," the corners of which are the sample points. Given two space-like curves I and J, we recall that I is an immediate successor to J if I lies toward a larger time than J and they pass through the same sample points except one.

The approximate solution $u_\Delta(x,t) \equiv u_\Delta(x,t;\{a_k\})$ is defined inductively. Suppose that $u_\Delta(x,t)$ has been constructed for $0 \leq t < ks$, then $u_\Delta(x,ks+0)$ is defined as follows:

$$u_\Delta(x,ks+0) \text{is the solution of (1.5), for } \; hr < x < (h+1)r,$$

with the initial value

$$u_\Delta(hr + a_k r, ks+0) \equiv u_\Delta(hr + a_k r, ks-0).$$

Hence, $u_\Delta(x,ks+0)$ consists of standing waves with possible discontinuities at $x = hr$, h integers. We next resolve the discontinuity, for example, considering (hr,ks) as the origin, by solving the Riemann problem for (1.4) with initial data:

$$u(x,0) = \begin{cases} u_\Delta(hr-0,ks+0), & x<0 \\ u_\Delta(hr+0,ks+0), & x>0 \end{cases}$$

The solution consists of $n+1$ constant states $v_0 = u_\Delta(hr-0,ks+0), v_1, \cdots, v_n = u_\Delta(hr+0,ks+0)$ connected by rarefaction waves and shock waves. Let $v_j(x), 0 \leq j \leq n$, denote the steady wave with the initial value $v_j(0) = v_j$. If (v_{j-1}, v_j) is a shock wave with speed $\sigma_j = \sigma(v_{j-1}, v_j)$, we simply set

$$u(x,t) = \begin{cases} v_{j-1}(x), & \text{for } \; x < \sigma_j t \\ v_j(x), & \text{for } \; x > \sigma_j t. \end{cases}$$

When $(v_{j-1}, v_j) = \psi_j(\frac{x}{t})$ is a rarefaction wave, we solve the singular boundary value problem:

$$\begin{aligned}
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} &= g(x,u), \text{ in the region } x_{j-1}(t) < x < x_j(t), \\
u(x_{j-1}(t), t) &= v_{j-1}(x_{j-1}(t)) \\
\lim_{t \to 0} u(\eta t, t) &= \psi_j(\eta),
\end{aligned}$$

where $x_{j-1}(t)$ and $x_j(t)$ are curves satisfying

$$\begin{aligned}
x_{j-1}(t) \ &: \ \frac{dx}{dt} = \lambda_j(v_{j-1}(x)), \quad x(0) = 0, \\
x_j(t) \ &: \ \frac{dx}{dt} = \lambda_j(v_j(x)), \quad x(0) = 0.
\end{aligned}$$

Details are found in [14] . In this way we have constructed the approximate solution involving only weak waves.

When the single relatively strong 1-shock wave is involved, we use the following variation. Suppose that the approximate solution is constructed for $0 \leq t < ks$. We denote by $x = x_f(t)$ the locus of the front of the single relatively strong 1-shock wave at time t and

$$j_f(k) \equiv \left[\frac{x_f(ks)}{r} \right].$$

We call the interval $((j_f - 1)r, (j_f + 2)r)$ the front region at $t = ks$. Inside the front region, we first solve the steady wave equation (1.5) in the intervals $(j_f - 1)r < x < x_f$ and $x_f < x < (j_f + 2)r$ with the initial values:

$$\begin{aligned}
u((j_f - 1 + a_k)r) &= u_\Delta((j_f - 1 + a_k)r, ks - 0) \\
u((j_f + 1 + a_k)r) &= u_\Delta((j_f + 1 + a_k)r, ks - 0)
\end{aligned}$$

respectively and denote these solutions by $u^-(x)$ and $u^+(x)$ respectively. Next we solve the Riemann problem(1.4) so that

$$u(x, ks) = \begin{cases} u^-(x_f(ks)), & \text{for } x < x_f(ks) \\ u^+(x_f(ks)), & \text{for } x > x_f(ks). \end{cases}$$

The solution $u(x,t)$ thus contains a relatively strong 1-shock wave, say, (v_0, v_1) with speed σ_1. Solve again the steady wave equation (1.5) in the intervals $(j_f - 1)r < x < x_f$ and $x_f < x < (j_f + 2)r$ with the initial values

$$\begin{aligned}
v(x_f(ks)) &= v_0 \\
v(x_f(ks)) &= v_1
\end{aligned}$$

respectively and denote their solutions by $v^-(x)$ and $v^+(x)$ respectively. Now, we can define the approximate solution in the front region as follows:

$$
\begin{aligned}
u_\Delta(x,t) &= \begin{cases} v^-(x), & \text{for } (j_f-1)r < x < x_f \\ v^+(x), & \text{for } x_f < x < (j_f+2)r, \end{cases} \\
x_f(t) &= \sigma_1(t-ks) + x_f(ks)
\end{aligned}
$$

for $ks \leq t < (k+1)s$. And the discontinuities at $x=(j_f-1)r$ and $x=(j_f+2)r$ are treated by the same construction described in the last paragraph.

5 The Local Interaction Estimate

We first study the interaction among weak waves which are outside the front region. Let Δ be any diamond outside the front region. Suppose that the waves entering Δ are the solutions of the Riemann problems (u_1,u_2) and (u_3,u_4) centered at $(h_1 r, ks)$ and $(h_2 r, ks)$ respectively, $h_1 = h_2 - 1$. The center of Δ is $(hr,(k+1)s)$, h equal to h_1 or h_2 . We use the notation $(\bar{u}_4, u_4)$ to represent the steady wave $u(x)$ in the interval $(hr,(h+1)r)$ so that $u(hr)=\bar{u}_4$ and $u((h+1)r)=u_4$. When h equals h_1, the waves leaving Δ are the solution of the Riemann problem $(u_1,\bar{u}_4)$. In this case, waves in (u_3,u_4) have negative speed, and we set

$$
Q^1(\Delta) \equiv \frac{1}{\lambda_*^2}\sum_i |(u_3,u_4)_i| \cdot \int_{hr}^{(h+1)r} G(x)dx.
$$

When h equals h_2, the waves leaving Δ are the solution of the Riemann problem $(\hat{u}_1,u_4)$ and waves in (u_1,u_2) have positive speed; we set

$$
Q^1(\Delta) \equiv \frac{1}{\lambda_*^2}\sum_i |(u_1,u_2)_i| \cdot \int_{(h-1)r}^{hr} G(x)dx.
$$

Recall that a j-wave and a k-wave are approaching if either the one on the left belongs to the larger characterisic family, or both waves come from the same family and at least one wave is a shock. $Q^0(\Delta)$ is defined as

$$
\begin{aligned}
Q^0(\Delta) &= Q^0((u_1,u_2),(u_3,u_4)) \\
&\equiv \sum \{|(u_1,u_2)_i(u_3,u_4)_j| : (u_1,u_2)_i \text{ and } (u_3,u_4)_j \text{ are approaching .}\}
\end{aligned}
$$

And

$$
Q(\Delta) \equiv Q^0(\Delta) + Q^1(\Delta).
$$

Let δ denote the set of waves issuing from $(hr,(k+1)s)$ and δ_i the strength of the i-wave in δ. Our interaction estimate is as follows:

Lemma 5.1 *Soppose that there exists $\lambda_* > 0$ so that $|\lambda_i(u)| > \lambda_*$ for all u under consideration, $1 \le i \le n$. Then,*

$$\delta_i = (u_1, u_2)_i + (u_3, u_4)_i + O(1)Q(\Delta),$$

where $O(1)$ is a constant depending only on system(1.4).

Proof. We only consider the case when h equals h_1, $\alpha = (u_1, \bar{u}_4)$. The other case can be proved by the same argument. Owing to the interaction estimates of elementary waves for conservation laws [6],

$$\delta_i = (u_1, \bar{u}_4)_i = (u_1, u_2)_i + (u_2, \bar{u}_4)_i + O(1)Q^0((u_1, u_2), (u_2, \bar{u}_4)). \tag{5.1}$$

Since $(\bar{u}_4, u_4)$ and (u_2, u_3) are steady waves and the solution of the Riemann problem depends continuously on the end states, we have

$$(u_2, \bar{u}_4)_i = (u_3, u_4)_i + O(1)\frac{1}{\lambda_*^2}|u_3 - u_4| \int_{hr}^{(h+1)r} G(x)dx. \tag{5.2}$$

By (5.1) and (5.2), we obtain

$$\delta_i = (u_1, u_2)_i + (u_3, u_4)_i + O(1)Q^0(\Delta) + O(1)Q^1(\Delta).$$

This completes the proof of Lemma 5.1.

Now we turn to the case involving the single relatively strong 1-shock wave. The estimate is similar to the above lemma except that instead of advancing one diamond, we need to advance three diamonds in the front region simultaneously. We still denote these three diamonds by Δ. Let $\Delta_{h,k}$ represent the diamond whose center is (hr, ks). Assume $a_k \in [\frac{1}{2}, 1)$. Then,

$$\Delta = \Delta_{j_f(k),k+1} \cup \Delta_{j_f(k)+1,k+1} \cup \Delta_{j_f(k)+2,k+1}.$$

The case for $a_k \in (0, \frac{1}{2})$ can be treated by the same analysis. Let β stand for the relatively strong 1-shock wave issuing from $(x_f(k), ks)$. We denote by α the set of waves issuing from $((j_f(k)+2)r, ks)$. The waves in α entering $\Delta_{j_f(k)+1,k+1}$ are denoted by α^l and α^r are the waves entering $\Delta_{j_f(k)+2,k+1}$. Since $\lambda_j > 0$ for $j \ge 2$, α^l only consists of 1-waves. Let γ be the set of waves issuing from $((j_f(k)+3)r, ks)$ and entering $\Delta_{j_f(k)+2,k+1}$. Set $u_l(x), u_m(x)$, and $u_r(x)$ to be the steady waves such that β connects $u_l(x)$ and $u_m(x)$, and α^l connects $u_m(x)$ and $u_r(x)$ at $t = ks$.

Set

$$\begin{array}{ll}
w_- \equiv u_l(x_f(k)), & w_+ \equiv u_m(x_f(k)) \\
\hat{w}_- \equiv u_l(x_f(k+1)), & \hat{w}_+ \equiv u_m(x_f(k+1)) \\
v_- \equiv u_m((j_f(k)+2)r), & v_+ \equiv u_r((j_f(k)+2)r) \\
u_- \equiv u_l(x_f(k+1))(= \hat{w}_-), & u_+ \equiv u_r(x_f(k+1)).
\end{array}$$

Thus at the next time step, the strong 1-shock wave issuing from $(x_f(k+1),(k+1)s)$ has strength $(u_-,u_+)_1$. Let (z_-,z_+) stand for the wave issuing from $((j_f(k)+2)r,(k+1)s)$; we have the following estimate:

Lemma 5.2 *Suppose that there exists $\lambda_*>0$ so that $|\lambda_i(u)|>\lambda_*$ for all u under consideration, $1\le i\le n$. Then,*

$$
\begin{aligned}
(u_-,u_+)_1 &= \beta+\alpha_1^l+O(1)Q^0(\beta,\alpha^l) \\
&\quad +O(1)\frac{1}{\lambda_*^2}\left\{|\beta|\int_{x_f(k)}^{x_f(k+1)}G(x)dx+|\alpha^l|\int_{x_f(k+1)}^{(j_f(k)+2)r}G(x)dx\right\},
\end{aligned}
$$

$$
\begin{aligned}
(z_-,z_+)_j &= \alpha_j^r+\gamma_j+O(1)\left\{Q^0(\beta,\alpha^l)+Q^0(\alpha^r,\gamma)\right\} \\
&\quad +O(1)|\beta|\int_{x_f(k)}^{x_f(k+1)}G(x)dx+O(1)\frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r}G(x)dx \\
&\quad +O(1)\frac{|\gamma|}{\lambda_*^2}\int_{(j_f(k)+2)r}^{(j_f(k)+3)r}G(x)dx\ ,
\end{aligned}
$$

for $1\le j\le n$.

Proof. First by the steady wave equation (1.5), we have

$$
\begin{aligned}
\hat{w}_+-\hat{w}_- &= w_+-w_-+O(1)\frac{1}{\lambda_*^2}|w_+-w_-|\int_{x_f(k)}^{x_f(k+1)}G(x)dx && (5.3)\\
u_+-\hat{w}_+ &= v_+-v_-+O(1)\frac{1}{\lambda_*^2}|v_+-v_-|\int_{x_f(k+1)}^{(j_f(k)+2)r}G(x)dx && (5.4)
\end{aligned}
$$

By Lemma 3.1, we have

$$
(\hat{w}_+,\hat{w}_-)_j=O(1)|w_--w_+|\int_{x_f(k)}^{x_f(k+1)}G(x)dx\ ,\ \text{for } j\neq 1\ . \tag{5.5}
$$

And

$$
\begin{aligned}
(u_-,u_+)_j &= (\hat{w}_-,u_+)_j \\
&= (\hat{w}_-,\hat{w}_+)_j+(\hat{w}_+,u_+)_j+O(1)Q^0((\hat{w}_-,\hat{w}_+),(\hat{w}_+,u_+)). && (5.6)
\end{aligned}
$$

The solution of the Riemann problem (u_-,u_+) consists of $n+1$ constant states, say, $v_0=u_-,v_1,\cdots,v_n=u_+$. By (5.3) - (5.6), we obtain

$$
\begin{aligned}
(u_-,u_+)_j &= O(1)Q^0(\beta,\alpha^l)+O(1)|\beta|\int_{x_f(k)}^{x_f(k+1)}G(x)dx \\
&\quad +O(1)\frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r}G(x)dx,\ \text{for } j\neq 1. && (5.7)
\end{aligned}
$$

Note that $\beta_j = 0$ for $j \neq 1$. In particular, the single strong 1-shock wave (v_0, v_1) at $t = (k+1)s$ has strength

$$\begin{aligned}(u_-, u_+)_1 &= \beta + \alpha_1^l + O(1)Q^0(\beta, \alpha^l) \\ &\quad + O(1)\frac{|\beta|}{\lambda_*^2}\int_{x_f(k)}^{x_f(k+1)} G(x)dx + O(1)\frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r} G(x)dx.\end{aligned}$$

Set $z(x)$ to be the steady wave such that

$$z(x_f(k+1)) = v_1.$$

Since (z_-, z_+) is the wave issuing from $((j_f(k)+2)r, (k+1)s)$, we know by the construction of the approximate solution $u_\Delta(x,t)$ that

$$\begin{aligned}z_- &= z((j_f(k)+2)r), \\ z_+ &= u_\Delta((j_f(k)+2)r + 0, (k+1)s).\end{aligned}$$

By the steady wave equation (1.5) again,

$$(z_-, v_+)_j = (v_1, u_+)_j + O(1)\frac{1}{\lambda_*^2}|v_1 - u_+|\int_{x_f(k+1)}^{(j_f(k)+2)r} G(x)dx. \tag{5.8}$$

And applying Lemma 5.1, we obtain

$$(v_+, z_+)_j = \alpha_j^r + \gamma_j + O(1)Q^0(\alpha^r, \gamma) + O(1)\frac{|\gamma|}{\lambda_*^2}\int_{(j_f(k)+2)r}^{(j_f(k)+3)r} G(x)dx. \tag{5.9}$$

Thus, from (5.7) - (5.9),

$$\begin{aligned}(z_-, z_+)_j &= (z_-, v_+)_j + (v_+, z_+)_j + O(1)Q^0((z_-, v_+), (v_+, z_+)) \\ &= \alpha_j^r + \gamma_j + O(1)\left\{Q^0(\beta, \alpha^l) + Q^0(\alpha^r, \gamma)\right\} \\ &\quad + O(1)|\beta|\int_{x_f(k)}^{x_f(k+1)} G(x)dx + \frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r} G(x)dx \\ &\quad + O(1)\frac{|\gamma|}{\lambda_*^2}\int_{(j_f(k)+2)r}^{(j_f(k)+3)r} G(x)dx\ .\end{aligned}$$

This completes the proof of Lemma 5.2.

In this case, we set

$$\begin{aligned}Q^0(\Delta) &= Q^0(\beta, \alpha^l) + Q^0(\alpha^r, \gamma) \\ Q^1(\Delta) &= \begin{cases} \frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r} G(x)dx + \frac{|\gamma|}{\lambda_*^2}\int_{(j_f(k)+2)r}^{(j_f(k)+3)r} G(x)dx, \text{ if (1.6) holds,} \\ \\ |\beta|\int_{x_f(k)}^{x_f(k+1)} G(x)dx + \frac{|\alpha^l|}{\lambda_*^2}\int_{x_f(k+1)}^{(j_f(k)+2)r} G(x)dx + \frac{|\gamma|}{\lambda_*^2}\int_{(j_f(k)+2)r}^{(j_f(k)+3)r} G(x)dx, \\ \text{if (1.7) holds.}\end{cases}\end{aligned}$$

6 Global Existence And Stability

The asymptotic stability criterion (1.6) and (1.7) will be verified in this section. We first recall that a noninteracting wave pattern means a solution of system (1.1) consisting of elementary waves for system (1.4) in the region $x \notin (0,1)$ and of steady waves and a possible standing shock wave in the region $x \in (0,1)$; moreover, these waves do not interact in the positive time direction. In [14], Liu shows that when the characteristic speeds are away from zero and the L^1 norm of $G(x)$ is sufficiently small, the solutions of system (1.1) are stable. We state this result as follows.

Theorem 6.1 *Suppose that $u(x,t)$ is a noninteracting wave pattern for (1.1) with the property that $u(x,t)$ does not contain any standing shock wave and all waves in $u(x,t)$ are weak. Assume that $\lambda_1(u(x,t)) \neq 0$.*

Then $u(x,t)$ is asymptotically stable in the sense that if an initial data $\bar{u}(x,0)$ is such that the total variation of $(u(x,0) - \bar{u}(x,0))$ is small, then the solution $\bar{u}(x,t)$ of (1.1) tends to an asymptotic state consisting of elementary waves which are close to those in $u(x,t)$ both in position and strength.

Proof. See Liu [14].

In contrast to the above theorem, when a standing shock wave is perturbed, complicated wave interactions occur and the solutions of system (1.1) are not always stable. We will study its stability and instability in the following theorems.

Let $u_*(x,t) = u_*(x)$ be a solution of (1.1), which consists of steady waves and a standing 1-shock wave at $x = x_*$, $0 \leq x_* \leq 1$:

$$u_*(x,t) = \begin{cases} u_*(-\infty), & x < 0 \\ u_1(x), & 0 \leq x < x_* \\ u_2(x), & x_* < x \leq 1 \\ u_*(+\infty), & x > 1 \end{cases}$$

where $u_1(x)$ and $u_2(x)$ are solutions of (1.5), $(u_1(x_*), u_2(x_*))$ is a 1-shock wave with strength α_0 and speed

$$\sigma(u_1(x_*), u_2(x_*)) = 0.$$

$u_*(-\infty) \equiv u_1(0)$, $u_*(+\infty) \equiv u_2(1)$. Note that $\lambda_1(u_2(x_*)) < 0 < \lambda_1(u_1(x_*))$. $G(x)$ is assumed to be small compared to α_0 and $|\lambda_i(u)| > \lambda_* > 0$, $1 \leq i \leq n$, for all u under consideration.

Theorem 6.2 *The wave pattern $u_*(x,t)$ is unstable provided that the source term $g(x,u)$ satisfies*

$$l_1 \cdot \frac{\partial g(x,u)}{\partial u} \cdot r_1 > 0, \quad 0 < x < 1.$$

To be precise, there exists $u(x,t)$ whose initial data is a perturbation of $u_*(x)$ and the total variation of $(u(x,0)-u_*(x))$ is sufficiently small so that as $t\to\infty$, $u(x,t)$ tends to one of the stable asymptotic states without any standing shock wave as studied in Theorem 6.1.

Proof. For the sake of definiteness, we assume that $0<x_*<1$. We choose a perturbation $u(x,0)$ consisting of steady waves and a 1-shock wave at $x=x_*$ with positive speed $\sigma(x_*)$ as follows:

$$u(x,0)\equiv\begin{cases} u_1(0), & x<0\\ u_1(x), & 0\le x<x_*\\ \bar u_2(x), & x_*<x\le 1\\ \bar u_2(1), & x>1,\end{cases}$$

where $\bar u_2(x_*)\in S_1^-(u_1(x_*))$ is chosen arbitrarily close to $u_2(x_*)$ so that

$$\sigma(x_*)=\sigma(u_1(x_*),\bar u_2(x_*))>0,$$

and $\bar u_2(x)$ represents the steady wave with the initial value $\bar u_2(x_*)$ at $x=x_*$. Let β_* stand for the strength of $(u_1(x_*),\bar u_2(x_*))$.

We will adopt the difference scheme described in Section 4 to prove the existence of the solution $u(x,t)$ with the initial data $u(x,0)$. The proof requires two series of estimates. One is to find a uniform bound on the total variations of the approximate solutions $u_\Delta(x,t)$. The other is to study the speed of the relatively strong 1-shock. Our strategy is to use induction on certain nonlinear functionals constructed to detect global wave interactions. Once this uniform bound is established, with the aid of Helly's theorem, we can extract a convergent subsequence of $u_\Delta(x,t)$ in $L^1_{loc}(R^2)$, and by the consistency theorem (Liu [13,14]), this subsequence converges to a weak solution $u(x,t)$ of (1.1) and (1.2). Moreover, we simultaneously show that the speed of the relatively strong 1-shock wave is bigger than or equal to the initial speed $\sigma(x_*)$. Hence, this strong shock leaves the region $0\le x\le 1$ in finite time, which differs greatly from the original state $u_*(x,t)$.

We now give the global interaction estimates. Let J be a space-like curve. Suppose that the relatively strong 1-shock wave crosses J at the k-th time step. The locus of the shock front at time $t=ks+0$ is denoted by x_J, its strength β_J, and its speed σ_J. To establish the uniform bound, we define a nonlinear functional $F(J)$ as follows:

$$\begin{aligned} F(J) &\equiv L(J)+KQ(J)\\ L(J) &\equiv \sum\{|\alpha| : \alpha \text{ is the strength of any elementary waves crossing } J\}\\ \bar L(J) &\equiv L(J)-|\beta_J|\\ Q(J) &\equiv Q_0(J)+Q_1(J)+Q_2(J)\\ Q_0(J) &\equiv \sum\{|\alpha\beta| : \alpha \text{ and } \beta \text{ are strengths of elementary waves which are approaching}\\ &\qquad \text{and cross } J.\}\end{aligned}$$

$$
\begin{aligned}
Q_1(J) \;\; \equiv \;\; & |\beta_J| \int_{x_J}^{\infty} G(x)dx \\
& + \sum \{ \frac{|\alpha|}{\lambda_*^2} \int_{hr}^{\infty} G(x)dx : \alpha \text{ is the strength of 1-wave which crosses } J \text{ and issues} \\
& \text{from } (hr, ks), hr < x_J \} \\
& + \sum \{ \frac{|\alpha|}{\lambda_*^2} \int_{x_J}^{\infty} G(x)dx + \frac{|\alpha|}{\lambda_*^2} \int_{x_J}^{hr} G(x)dx : \alpha \text{ is the strength of 1-wave which} \\
& \text{crosses } J \text{ and issues from } (hr, ks), hr > x_J . \} \\
Q_2(J) \;\; \equiv \;\; & \sum \{ \frac{|\alpha|}{\lambda_*^2} \int_{hr}^{\infty} G(x)dx : \alpha \text{ is the strength of i-wave which crosses } J \text{ and issues} \\
& \text{from } (hr, ks), i \neq 1. \} \quad ,
\end{aligned}
$$

where K is some large number to be determined later.

We will prove by induction that for any suitably chosen space-like curve J,

$$
\begin{aligned}
F(J) \;\; & \leq \;\; F(0) + C|\beta_*| \int_{x_*}^{x_J} G(x)dx && (6.1) \\
Q(\Lambda) \;\; & \leq \;\; 2[Q(0) - Q(J)] + 4|\beta_*| \int_{x_*}^{x_J} G(x)dx && (6.2) \\
\sigma_J \;\; & \geq \;\; \sigma(x_*) && (6.3)
\end{aligned}
$$

where 0 is the space-like curve in the strip $0 \leq t \leq s$, Λ is the region between 0 and J, and $Q(\Lambda)$ is the sum over all $Q(\Delta)$, Δ any diamond in Λ. C is a constant depending only on system (1.4).

Suppose that (6.1) - (6.3) have been shown for $J = J_1$. Let J_2 be an immediate successor to J_1, Δ the diamond between them. If Δ meets the front region, we choose J_2 so that Δ contains exactly the three diamonds in the front region, which are needed for the estimate of local interactions in Section 5. We now establish (6.1)-(6.3) for $J = J_2$. Since the only wave crossing 0 is the 1-shock wave issuing from $(x_*, 0)$ with strength β_*, it follows from Lemma 5.1 and 5.2 that the strength of this 1-shock wave is

$$
\beta_{J_1} = \beta_* + O(1)Q(\Lambda_1) + O(1)\frac{|\beta_*|}{\lambda_*^2} \int_{x_*}^{x_{J_1}} G(x)dx \qquad (6.4)
$$

as it crosses J_1. Here Λ_1 is the region between 0 and J_1. By the induction hypothesis (6.2),

$$
\begin{aligned}
Q(\Lambda_1) \;\; & \leq \;\; 2[Q(0) - Q(J_1)] + 4|\beta_*| \int_{x_*}^{x_{J_1}} G(x)dx \\
& \leq \;\; 2 \left[|\beta_*| \int_{x_*}^{\infty} G(x)dx - |\beta_{J_1}| \int_{x_{J_1}}^{\infty} G(x)dx \right] + 4|\beta_*| \int_{x_*}^{x_{J_1}} G(x)dx
\end{aligned}
$$

With $||G||_1$ sufficiently small, the above estimate yields

$$
Q(\Lambda_1) \leq 16|\beta_*| \int_{x_*}^{x_{J_1}} G(x)dx \; . \qquad (6.5)
$$

It also leads to

$$\frac{1}{2}|\beta_*| \leq |\beta_{J_1}| \leq 2|\beta_*|. \tag{6.6}$$

Since (1.7) holds, by Lemma 3.1, Lemma 5.1, 5.2, (6.5) and (6.6), the speed of this shock wave satisfies

$$\begin{aligned}
\sigma_{J_2} - \sigma(x_*) &= |O(1)| \int_{x_*}^{x_{J_2}} G(x)dx + O(1)Q(\Lambda_1) \\
&= |O(1)| \int_{x_*}^{x_{J_2}} G(x)dx + O(1)|\beta_*| \int_{x_*}^{x_{J_1}} G(x)dx \qquad (6.7) \\
&\geq 0 \qquad (6.8)
\end{aligned}$$

Here, instead of only steady waves as in Lemma 3.1, the nearby weak waves are also involved in the evolution of the relatively strong 1-shock wave; therefore, besides the estimate $\sigma_{J_2} - \sigma_{J_1}$, we need to investigate $\sigma_{J_2} - \sigma(x_*)$. Since 1-waves to the right(left) of this shock have negative(positive) speed and σ_{J_1} is positive by (6.3), we have from Lemma 5.1 and 5.2

$$\begin{aligned}
L(J_2) - L(J_1) &\leq O(1)[Q(\Delta) + \frac{|\beta_{J_1}|}{\lambda_*^2} \int_{x_{J_1}}^{x_{J_2}} G(x)dx] \qquad (6.9) \\
Q_0(J_2) - Q_0(J_1) &\leq O(1)\bar{L}(J_1)[Q(\Delta) + \frac{|\beta_{J_1}|}{\lambda_*^2} \int_{x_{J_1}}^{x_{J_2}} G(x)dx] - Q^0(\Delta) \qquad (6.10) \\
Q_1(J_2) - Q_1(J_1) &+ Q_2(J_2) - Q_2(J_1) \\
&\leq O(1)\frac{1}{\lambda_*^2} \|G\|_1 [Q(\Delta) + \frac{|\beta_{J_1}|}{\lambda_*^2} \int_{x_{J_1}}^{x_{J_2}} G(x)dx] - Q^1(\Delta) \qquad (6.11)
\end{aligned}$$

Hence,

$$\begin{aligned}
Q(J_2) - Q(J_1) &\leq Q(\Delta)[O(1)(\bar{L}(J_1) + \frac{\|G\|_1}{\lambda_*^2}) - 1] \\
&\quad + O(1)(\bar{L}(J_1) + \frac{\|G\|_1}{\lambda_*^2}) \frac{|\beta_{J_1}|}{\lambda_*^2} \int_{x_{J_1}}^{x_{J_2}} G(x)dx.
\end{aligned}$$

Note that if $x_{J_1} = x_{J_2}$, $\bar{L}(J_1)$ should be replaced by $L(J_1)$ in (6.10). By choosing G small enough, the above estimate and (6.5) imply that

$$Q(J_2) - Q(J_1) \leq -\frac{1}{2}Q(\Delta) + |\beta_{J_1}| \int_{x_{J_1}}^{x_{J_2}} G(x)dx.$$

By (6.6),

$$Q(\Delta) \leq 2(Q(J_1) - Q(J_2)) + 4|\beta_*| \int_{x_{J_1}}^{x_{J_2}} G(x)dx.$$

Using (6.2), we obtain

$$Q(\Lambda_2) \leq 2(Q(0) - Q(J_2)) + 4|\beta_*| \int_{x_*}^{x_{J_2}} G(x)dx,$$

where Λ_2 is the region between 0 and J_2. Also,

$$\begin{aligned}
F(J_2) - F(J_1) &= L(J_2) - L(J_1) + K(Q(J_2) - Q(J_1)) \\
&\leq (O(1) - \frac{K}{2})Q(\Delta) + (\frac{O(1)}{\lambda_*^2} + K)|\beta_{J_1}| \int_{x_{J_1}}^{x_{J_2}} G(x)dx \\
&\leq -\frac{1}{2}Q(\Delta) + C|\beta_*| \int_{x_{J_1}}^{x_{J_2}} G(x)dx,
\end{aligned}$$

for K chosen sufficiently large. Note that C depends only on system (1.4) as β_* depends only on system (1.4). In virtue of (6.1) again,

$$F(J_2) \leq F(0) + C|\beta_*| \int_{x_*}^{x_{J_2}} G(x)dx.$$

Therefore, (6.1)-(6.3) hold for $J = J_2$. Since the total variation and L are equivalent metrics, the compactness of the approximate solutions follows as a consequence. We omit the details.

Hence, by the construction of the approximate solutions and (6.3), we observe that the relatively strong shock wave must leave the region $0 \leq x \leq 1$ in finite time. The proof of Theorem 6.2 is completed.

Theorem 6.3 *Suppose that $u(x,0)$ is a small perturbation of the standing wave $u_*(x)$ so that the total variation of $(u(x,0) - u_*(x))$ is sufficiently small and $u(x,0)$ contains a 1-shock wave (u_-, u_+) at $x = x_*$ with $\lambda_1(u_+) < 0 < \lambda_1(u_-)$. Assume that $G(x)$ and the shock speed $\sigma(u_-, u_+)$ are small compared to the strength β_* of the shock wave (u_-, u_+).*

Then there exists a solution $u(x,t)$ of (1.1) which tends to a nearby noninteracting wave pattern as $t \to \infty$ provided that

$$l_1 \cdot \frac{\partial g(x,u)}{\partial u} \cdot r_1 < 0 \ , \ 0 < x < 1 \ .$$

Proof. Similar to Theorem 6.2, there still exists a relatively strong 1-shock wave for all time; however, this shock wave, distinct from the corresponding one in Theorem 6.2, stays close to the unperturbed standing shock wave $(u_1(x_*), u_2(x_*))$. Eventhough the shock wave may oscillate, it tends to decelerate and will not cause strong nonlinear interactions eventually. Hence, the existence of global solutions can be guaranteed.

We will first establish a uniform bound on the wave interactions. We consider a different functional $F(J)$ defined as follows:

$$\begin{aligned}
F(J) &\equiv L(J) + KQ(J) \\
L(J) &\equiv |\sigma_J| + \bar{L}(J) \\
Q(J) &\equiv Q_0(J) + Q_1(J) + Q_2(J)
\end{aligned}$$

$$
\begin{aligned}
Q_1(J) \quad \equiv \quad & \sum\{\frac{|\alpha|}{\lambda_*^2}\int_{hr}^{\infty} G(x)dx : \alpha \text{ is the strength of 1-wave which crosses } J \text{ and issues} \\
& \text{from } (hr, ks), hr < x_J\} \\
& + \sum\{\frac{|\alpha|}{\lambda_*^2}\int_{x_J}^{hr} G(x)dx + \frac{|\alpha|}{\lambda_*^2}\int_{x_J}^{\infty} G(x)dx : \alpha \text{ is the strength of 1-wave which} \\
& \text{crosses } J \text{ and issues from } (hr, ks), hr > x_J\}
\end{aligned}
$$

Here $\bar{L}$, Q_0, and Q_2 are the same as those in Theorem 6.2 and the constant K will be determined later. Since we do not know a priori the sign of the speed of the relatively strong shock wave, we introduce its speed σ_J in place of the wave interactions involved with this strong shock in the functional $F(J)$. We assume that the total variation of $(u(x,0) - u_*(x))$ is so small that

$$
\bar{L}(0) < |\beta_*|^3.
$$

Thus,

$$
\begin{aligned}
F(0) \quad &= \quad |\sigma(x_*)| + \bar{L}(0) + KQ(0) \\
&\leq \quad |\sigma(x_*)| + \bar{L}(0)(1 + \frac{K\,\|G\|_1}{\lambda_*^2}) + K(T.V.(u(x,0)))^2,
\end{aligned}
$$

where $T.V.$ is the total variation. We further require that

$$
F(0) < \frac{C_1}{4}|\beta_*|^2,
$$

where C_1 depends only on system (1.4).

Suppose that the relatively strong 1-shock wave crosses J at the k_J-th time step. The locus of the shock front at time $t = ks + 0$ is denoted by $x(ks)$. We will prove by induction that

$$
|\beta_J| \leq C_2|\beta_*| \tag{6.12}
$$

$$
|Q(\Lambda)| \leq C_1|\beta_*|^2 \tag{6.13}
$$

$$
Q(J) \leq Q(0) - \frac{1}{2}Q(\Lambda) + C_2|\beta_*|\sum_{k=1}^{k_J}\left|\int_{x((k-1)s)}^{x(ks)} G(x)dx\right| \tag{6.14}
$$

$$
F(J) \leq F(0) - \frac{1}{2}Q(\Lambda) - \frac{1}{2}\sum_{k=1}^{k_J}\left|\int_{x((k-1)s)}^{x(ks)} G(x)dx\right| + O(1)\bar{L}(0), \tag{6.15}
$$

where C_2 is a constant depending only on system (1.4).

Suppose that (6.12)-(6.15) have been shown for $J = J_1$. Let J_2 be an immediate successor to J_1. With the help of Lemma 5.1 and 5.2, we get

$$
L(J_2) - L(J_1) = |\sigma_{J_2}| - |\sigma_{J_1}| + O(1)\left(Q(\Delta) + |\beta_{J_1}|\left|\int_{x(k_{J_1}s)}^{x(k_{J_2}s)} G(x)dx\right|\right) \tag{6.16}
$$

$$
\begin{aligned}
Q(J_2) - Q(J_1) &\le O(1)(\bar{L}(J_1) + \frac{\|G\|_1}{\lambda_*^2}) \left(Q(\Delta) + \frac{|\beta_{J_1}|}{\lambda_*^2} \left| \int_{x(k_{J_1}s)}^{x(k_{J_2}s)} G(x)dx \right| \right) \\
&\quad -Q(\Delta) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (6.17) \\
|\beta_{J_2}| &\le |\beta_*| + \bar{L}(0) + O(1) \left(Q(\Lambda_1) + \frac{|\beta_*|}{\lambda_*^2} \sum_{k=1}^{k_{J_1}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right| \right).
\end{aligned}
$$

If $k_{J_1} = k_{J_2}$, $\bar{L}(J_1)$ should be replaced by $(\bar{L}(J_1) + |\beta_{J_1}|)$ in (6.17). By (6.15),

$$
\sum_{k=1}^{k_{J_1}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right| < C_1 |\beta_*|^2.
$$

From the above estimate, (6.12), and (6.13), we obtain

$$
\begin{aligned}
|\beta_{J_2}| &\le |\beta_*| + |\beta_*|^3 + O(1)C_1|\beta_*|^2 + O(1)C_1|\beta_*| \\
&\le C_2|\beta_*|.
\end{aligned}
$$

Also, by (6.12), (6.15) and (6.17),

$$
Q(J_2) - Q(J_1) \le -\frac{1}{2}Q(\Delta) + C_2|\beta_*| \left| \int_{x(k_{J_1}s)}^{x(k_{J_2}s)} G(x)dx \right|, \tag{6.18}
$$

when $\|G\|_1$ is sufficiently small. By the induction hypothesis (6.14), it thus follows that

$$
Q(J_2) \le Q(0) - \frac{1}{2}Q(\Lambda_2) + C_2|\beta_*| \sum_{k=1}^{k_{J_2}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right|.
$$

Here Λ_2 is the region between 0 and J_2. Combining (6.16) and (6.18), we have

$$
F(J_2) - F(J_1) \le (O(1) - \frac{K}{2})Q(\Delta) + |\sigma_{J_2}| - |\sigma_{J_1}| + (O(1) + K)C_2|\beta_*| \left| \int_{x(k_{J_1}s)}^{x(k_{J_2}s)} G(x)dx \right|.
$$

By Lemma 3.1 and by telegraphing the above estimates for every time step between J_2 and 0,

$$
\begin{aligned}
F(J_2) - F(0) &\le (O(1) - \frac{K}{2})Q(\Lambda_2) - |O(1)| \sum_{k=1}^{k_{J_2}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right| \\
&\quad + O(1)(Q(\Lambda_2) + \bar{L}(0)) + (O(1) + K)C_2|\beta_*| \sum_{k=1}^{k_{J_2}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right|.
\end{aligned}
$$

Choosing K sufficiently large, we see that

$$
F(J_2) - F(0) \le -\frac{1}{2}Q(\Lambda_2) - \frac{1}{2} \sum_{k=1}^{k_{J_2}} \left| \int_{x((k-1)s)}^{x(ks)} G(x)dx \right| + O(1)\bar{L}(0).
$$

In particular,

$$Q(\Lambda_2) \le 2F(0) + 2O(1)\bar{L}(0) < C_1|\beta_*|^2.$$

Thus, (6.12)-(6.15) hold for $J = J_2$.

We now turn to studying the asymptotic behavior of the solution. Given $\epsilon > 0$, it follows from (6.13) that there exists $T = T(\epsilon)$ so that

$$Q(\{t \ge T\}) \le \epsilon \quad . \tag{6.19}$$

By choosing $M = M(\epsilon)$ sufficiently large, we have from (6.15) that

$$\text{total variation}_x\{u(x,t) : |x| \ge M\} \le \epsilon \quad . \tag{6.20}$$

Through (M,T) and $(-M,T)$ we draw generalized i-characteristic curves χ_i^+ aand χ_i^- respectively, $i = 1, \cdots, n$.([7]) The region between χ_i^+ and χ_i^- is denoted by Ω_i. Due to the strict hyperbolicity of the system, $\chi_i^\pm$ do not intersect $\chi_j^\pm$ for $i \ne j$ after some finite time $T_0 > T$. The estimates (6.15), (6.19) and (6.20) imply that the total amount of i-waves outside Ω_i after T_0 is $O(1)\epsilon$. Hence, in Ω_i, i-waves behave (mod $O(1)\epsilon$) like waves for scalar conservation law [15]. Consequently, modulo the error $O(1)\epsilon$, i-waves tend to an i-shock wave or an i-rarefaction wave if i-characteristic field is genuinely nonlinear, or a traveling wave if i-characteristic field is linearly degenerate, for $i \ne 1$. And 1-waves tend to a shock wave due to the presence of the relatively strong shock wave. We refer readers to [14,15] for more details. By (6.15) again, the speed of the 1-shock wave staying in the region $0 < x < 1$ is close to zero. This completes the proof of Theorem 6.3.

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