Scalar Non-linear Conservation Laws with Integrable Boundary Data

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Abstract. We consider the initial value problem with boundary control for a scalar non-linear conservation law

$$u_t + [f(u)]_x = 0, \qquad u(0,x) = 0, \qquad u(\cdot,0) = \tilde{u} \in \mathcal{U}, \tag{$*$}$$

on the domain $\Omega = \{(t,x) \in \mathbb{R}^2 :\ t \geq 0,\ x \geq 0\}$. Here $u = u(t,x)$ is the state variable, $\mathcal{U}$ is a set of bounded boundary data regarded as controls and f is assumed to be strictly convex. We give a characterization of the set of attainable profiles at a fixed time $T > 0$ and at a fixed point $\bar{x} > 0$

$$\begin{aligned} \mathcal{A}(T,\mathcal{U}) &= \{u(T,\cdot) :\ u \text{ is a solution of } (*)\} \\ \mathcal{A}(\bar{x},\mathcal{U}) &= \{u(\cdot,\bar{x}) :\ u \text{ is a solution of } (*)\}, \end{aligned} \qquad \mathcal{U} = \mathbb{L}^\infty(\mathbb{R}^+).$$

Moreover we prove that $\mathcal{A}(T,\mathcal{U})$ and $\mathcal{A}(\bar{x},\mathcal{U})$ are compact subsets of $\mathbb{L}^1$ and $\mathbb{L}^1_{loc}$ respectively, whenever $\mathcal{U}$ is a set of controls which pointwise satisfy closed convex constraints, together with some additional integral inequalities.

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1. Introduction

The paper is concerned with the initial-boundary value problem for a scalar non-linear conservation law in one space dimension

$$u_t + [f(u)]_x = 0, \tag{1.1}$$
$$u(0,x) = 0, \qquad\qquad t, x \geq 0, \tag{1.2}$$
$$u(t,0) = \tilde{u}(t), \tag{1.3}$$

where $u = u(t,x)$ is the state variable, $\tilde{u}$ is a measurable bounded boundary data and f is assumed to be a strictly convex function. Following [13] we shall only consider weak entropic solutions of (1.1)-(1.2) which satisfy the boundary condition (1.3) in a weak sense.
Here we study the system (1.1)-(1.3) from the point of view of control theory, regarding the boundary data $\tilde{u}$ as a control. Given a set $\mathcal{U} \subset \mathbb{L}^\infty(\mathbb{R}^+)$ of admissible controls, we study the set of attainable profiles at a fixed time T

$$\mathcal{A}\,(T, \mathcal{U}) = \Big\{ u(T,\cdot) : \ u \text{ is a solution to (1.1)-(1.3) with } \tilde{u} \in \mathcal{U} \Big\}.$$

We will give a precise characterization of the attainable set when $\mathcal{U} = \mathbb{L}^\infty(\mathbb{R}^+)$ by using the theory of generalized characteristics developed by Dafermos [4]. Applications to calculus of variations and problems of optimization motivate the study of topological properties of $\mathcal{A}\,(T, \mathcal{U})$. Here closure and compactness of the attainable set will be established in connection with classes of boundary controls which are measurable selections of a bounded multifunction with closed convex values, and satisfy certain integral inequalities. In the proof of such results a key role will be played by the weak* compactness of the set of fluxes $\{f(\tilde{u}) : \ u \in \mathcal{U}\}$ of admissible boundary controls.
Results concerning the set of attainable profiles at a fixed point in space $\bar{x} > 0$

$$\mathcal{A}\,(\bar{x}, \mathcal{U}) = \Big\{ u(\cdot,\bar{x}) : \ u \text{ is a solution to (1.1)-(1.3) with } \tilde{u} \in \mathcal{U} \Big\}$$

can be derived by similar arguments.
The compactness of the attainable sets allows us to prove the existence of solutions for a class of optimization problems, where the cost functional depends on the profiles of the solutions at some time T or at a fixed point $\bar{x}$. In §5 we apply these results to a model of traffic flow where one wants to minimize the average time spent by cars travelling through a given stretch of highway. The controller acts by varying the density of cars entering the highway.

2. Preliminaries and Statements of Main Results

Formulation of the Problem

On the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t \geq 0, \ x \geq 0\}$ consider the mixed initial-boundary value hyperbolic problem,

$$u_t + [f(u)]_x = 0, \tag{2.1}$$

$$u(0,x) = \bar{u}(x), \qquad\qquad t, x \geq 0, \tag{2.2}$$

$$u(t,0) = \tilde{u}(t), \tag{2.3}$$

where $\tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+)$, $\bar{u} \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable strictly convex function. Denote $b(x) = (f')^{-1}(x)$ whenever $x \in$ Range (f') and $b(0) = -\infty$ if $0 \notin$ Range (f').
We recall that problems of this type do not possess classical solutions since discontinuities arise in finite time even if the initial and boundary data are smooth. Hence it is natural to consider weak solutions in the sense of distributions satisfying the usual *entropy conditions* ([10], [12])

$$u(t,x-) \geq u(t,x+), \qquad\qquad t,x > 0. \tag{2.4}$$

As pointed out in [2], [5] and [13], in general the Dirichlet condition (2.3) may not be fulfilled pointwise a.e., thus following [13] we require that an entropic solution u to (2.1)-(2.3) satisfies the above condition in a weaker sense which is motivated by the classical vanishing viscosity method (see [2], [13] and Definition 1). In [2] an entropic solution to (2.1)-(2.3) is obtained as the limit of solutions of suitable approximating parabolic problems, while in [13] Le Floch generalizes a result of Lax for the Cauchy problem for the scalar conservation law (see [11]), expressing a solution in terms of the pointwise minimum of a function $y \mapsto \Psi(t,x,y)$ for any $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ (see also Remark 2.1). Concerning uniqueness, in [13] an $\mathbb{L}^1$-semigroup property in the class of piecewise regular solutions is established (see Remark 2.2).
As observed in [13], any solution of (2.1)-(2.3) with boundary data $\tilde{u}$ such that $f'(\tilde{u}(t)) < 0$ on a subset I of $\mathbb{R}^+$ of positive measure, can be obtained with the boundary data

$$\tilde{u}'(t) = \begin{cases} b(0) & \text{if} \quad t \in I \\ \tilde{u}(t) & \text{otherwise.} \end{cases}$$

Hence it is not restrictive to assume that the characteristics at the boundary are always entering the domain, i.e. $f'(\tilde{u}(t)) \geq 0$ for a.e. t: this hypothesis will be adopted in the rest of the paper. We recall here the definition of solution to (2.1)-(2.3) as stated in [13].

Definition 1. A function $u \in \mathbb{L}^1(\Omega;\mathbb{R})$ is a solution of (2.1)-(2.3) if

i) it is a weak entropic solution of (2.1) in the interior of Ω;
ii) there exists a set $\mathcal{E} \subset \mathbb{R}^+$ with zero measure such that

$$\lim_{\substack{t\to 0+ \\ t \notin \mathcal{E}}} \int_0^x u(t,\xi)\, d\xi = \int_0^x \bar{u}(\xi)\, d\xi, \qquad x \geq 0; \tag{2.5}$$

iii) the boundary condition is satisfied in the following weak sense: there exist a set $\mathcal{F} \subset \mathbb{R}^+$ with zero measure and two functions $\Upsilon : \mathbb{R}^+ \to \mathbb{R}$ and $\mu : \mathbb{R}^+ \to \{-1, 0, 1\}$ such that

$$\lim_{\substack{x \to 0^+ \\ x \notin \mathcal{F}}} \int_0^t f(u(s,x))\ ds = \int_0^t \Upsilon(s)\ ds, \qquad t \geq 0, \tag{2.6}$$

$$\lim_{\substack{x \to 0^+ \\ x \notin \mathcal{F}}} \operatorname{sgn}\ f'(u(t,x)) = \mu(t), \qquad \text{a.e. } t \geq 0 \tag{2.7}$$

and

$$\begin{cases} \Upsilon(t) = f(\tilde{u}(t)) & \text{if } \mu(t) \geq 0 \\ \Upsilon(t) \geq f(\tilde{u}(t)) & \text{if } \mu(t) = -1 \end{cases} \qquad \text{a.e. } t > 0. \tag{2.8}$$

Remark 2.1. In [13] Le Floch proves that under the above assumptions there exists a solution u to (2.1)-(2.3), having right and left limits in t and x at every point in the interior of Ω and such that for any fixed $t \geq 0$ $u(t, \cdot)$ has at most countably many discontinuities. Moreover it satisfies the bounds

$$\begin{gathered} \|u(\cdot,\cdot)\|_\infty \leq \max\left\{\|\bar{u}(\cdot)\|_\infty, \|\tilde{u}(\cdot)\|_\infty\right\}, \\ \min\left\{f(u):\ |u| \leq \|\tilde{u}\|_\infty, \|\bar{u}(\cdot)\|_\infty\right\} \leq\ \Upsilon(t)\ \leq \max\left\{\|f(\bar{u}(\cdot))\|_\infty,\ \|f(\tilde{u})(\cdot)\|_\infty\right\} \end{gathered} \tag{2.9}$$

for a.e. $t > 0$. Such a solution admits the following explicit representation inside the domain

$$u(t,x) = b\left(\frac{x - y(t,x)}{t}\right) \qquad t > 0, \quad x > 0, \tag{2.10}$$

where $y(t,x)$ denotes a point of minimum value for the function

$$y \mapsto \Psi_\Upsilon(t,x,y) = \begin{cases} \displaystyle\int_0^y \bar{u}(s)\ ds + t\ g\left(\frac{x-y}{t}\right) & \text{if } y \geq 0 \\ \displaystyle -\int_0^\tau \Upsilon(s)\ ds + (t-\tau)\ g\left(\frac{x}{t-\tau}\right) & \text{if } y \leq 0 \end{cases} \tag{2.11}$$

with g denoting the Legendre transform of a superlinear convex map $\tilde{f}$ which coincides with f on the closed ball $\{u \in \mathbb{R} :\ |u| \leq \|\tilde{u}\|_\infty\}$ and τ satisfying

$$\frac{x-y}{t} = \frac{x}{t-\tau}, \qquad y \leq 0.$$

Notice that in [11] it is shown that for any given $t \in [0,T]$ the function $y \mapsto \Psi_\Upsilon(t,x,y)$ attains its minimum at a single point for all but at most countably many $x > 0$. Furthermore the existence of the traces at $x = 0$ in the sense of (2.6)-(2.7) for the functions $f(u)$, sgn $f'(u)$ holds in general for any map u admitting a representation as in (2.10) with Ψ_Υ defined by (2.11) in connection with some $\mathbb{L}^\infty$ function Υ.

Remark 2.2. Regarding uniqueness in [13] it is established the following $\mathbb{L}^1$-semigroup property: if u and v are piecewise continuously differentiable solutions of (2.1)-(2.3) associated with initial and boundary data $\bar{u}$, $\tilde{u}$ and $\bar{v}$, $\tilde{v}$ respectively ($\tilde{u}$, $\tilde{v} \geq b(0)$), then

$$\int_0^{+\infty} |u(t,x) - v(t,x)|\, dx \leq \int_0^{+\infty} |\bar{u}(x) - \bar{v}(x)|\, dx + \int_0^t |f(\tilde{u}(s)) - f(\tilde{v}(s))|\, ds \tag{2.12}$$

holds for any $t > 0$. This property can be extended to all the solutions associated with an $\mathbb{L}^\infty$ boundary condition (for details see the Appendix) and hence any solution to (2.1)-(2.3) admits a representation of the form (2.10) for a.e. $(t,x) \in \operatorname{int} \Omega$.

In this paper we are interested only in solution of (2.1)-(2.3) with null initial data $\bar{u}$. From now on we will adopt the semigroup notation $S_t\tilde{u}$ for the unique solution of (1.1)-(1.3) at time t. We shall be concerned with basic properties of the attainable sets for (1.1)-(1.2)

$$\mathcal{A}\,(T,\mathcal{U}) \doteq \{S_T\tilde{u} : \ \tilde{u} \in \mathcal{U}\}, \tag{2.13}$$

$$\mathcal{A}\,(\bar{x},\mathcal{U}) \doteq \left\{S_{(\cdot)}\tilde{u}(\bar{x}) : \ \tilde{u} \in \mathcal{U}\right\}, \tag{2.14}$$

which consist of all profiles that can be attained at a fixed time $T > 0$ and at a fixed point $\bar{x} > 0$ by solutions of (1.1)-(1.2) with boundary data that varies inside a given class $\mathcal{U} \subseteq \mathbb{L}^\infty$ of admissible boundary controls. In particular we give a characterization of

$$\mathcal{A}(T) \doteq \left\{S_T\tilde{u} : \ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+), \ \tilde{u} \geq b(0)\right\}, \tag{2.15}$$

$$\mathcal{A}(\bar{x}) \doteq \left\{S_{(\cdot)}\tilde{u}(\bar{x}) : \ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+), \ \tilde{u} \geq b(0)\right\}, \tag{2.16}$$

and we establish the compactness of (2.13),(2.14) in connection with a special class of admissible boundary controls.

Statements of the Main Results

We present here the statements of the main results. Throughout the following

$$D^- w(x) = \liminf_{h\to 0} \frac{w(x+h) - w(x)}{h}, \qquad D^+ w(x) = \limsup_{h\to 0} \frac{w(x+h) - w(x)}{h},$$

will denote respectively the lower and upper Dini derivatives of a function w at x.

Theorem 1. *In connection with problem (1.1)-(1.2), for any fixed $T > 0$, $\mathcal{A}(T)$ is the set of all bounded functions w which satisfy the following conditions*

$$w(x) \neq 0 \Longrightarrow f'(w(x)) \geq \frac{x}{T}, \tag{2.17}$$

$$w(x-) \neq 0 \quad \textit{and} \quad w(y) = 0 \ \forall\, y > x \Longrightarrow f'(w(x-)) > \frac{x}{T}, \tag{2.18}$$

$$D^+ w(x) \leq \frac{f'(w(x))}{x f''(w(x))}, \tag{2.19}$$

for every $x > 0$.

Remark 2.3. By definition an element $\tilde{w} \in \mathcal{A}(T) \subseteq \mathbb{L}^\infty(\mathbb{R}^+)$ is an equivalence class of essentially bounded measurable functions. Hence the above characterization must be interpreted in the sense that $\tilde{w} \in \mathcal{A}(T)$ iff there exists a representative w in the class $\tilde{w}$ satisfying (2.17)-(2.19).
Notice that if a bounded function w satisfies (2.17), then there exists $a > 0$ such that $w(x) = 0$ if $x \geq a$. Therefore the boundedness of w toghether with (2.17), (2.19) imply that w has finite total increasing variation (and hence finite total variation as well) on subsets of $\mathbb{R}^+$ bounded away from the origin. Thus we may assume that w admits left limit in any point and (2.18) makes sense. Moreover from (2.19) it follows that $w(x-) > w(x+)$ at every point of discontinuity.

Remark 2.4. Having in mind the extension of the above result to attainable sets for classes of admissible boundary controls in $\mathbb{L}^1(\mathbb{R}^+)$, it is useful to rewrite condition (2.19) in the following form

$$w(y) \leq w(x) + \int_x^y \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi, \qquad \forall\, x, y > 0, \quad y \geq x. \tag{2.19$'$}$$

which is shown to be equivalent to (2.19) at the end of § 3).

Theorem 2. *In connection with problem (1.1)-(1.2), for any fixed $\bar{x} > 0$, $\mathcal{A}(\bar{x})$ is the set of all bounded functions ρ which satisfy the following conditions*

$$\rho(t) \neq 0 \Longrightarrow f'(\rho(t)) \geq \frac{\bar{x}}{t}, \tag{2.20}$$

$$\rho(\tau+) \neq 0 \quad \text{and} \quad \rho(t) = 0\ \forall\, t < \tau \Longrightarrow f'(\rho(\tau+)) > \frac{\bar{x}}{\tau}, \tag{2.21}$$

$$D^-\rho(t) \geq \frac{f'(\rho(t))}{t f''(\rho(t))}, \tag{2.22}$$

for every $t > 0$.

The proof of Theorem 1 is given in §3, the proof of Theorem 2 is entirely similar so it is omitted.

In order to achieve the closure of the attainable sets for (1.1)-(1.2) we need to restrict the class of admissible boundary controls by means of a suitable multifunction G.

Theorem 3. *Let $G : \mathbb{R}^+ \hookrightarrow [b(0), +\infty)$ be a measurable uniformly bounded multifunction with convex closed values, $q_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, N$, measurable maps convex w.r.t. the second variable, $g_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \dots, N$, measurable maps and let J be a possibly empty subset of $\mathbb{R}^+$. Denote*

$$\mathcal{U} = \Big\{ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+) : \tilde{u}(t) \in G(t), \quad \text{for a.e. } t, \\ \int_0^t q_i\big(s, f(\tilde{u}(s))\big)\, ds \leq g_i(t) \quad \forall\, t \in J,\ \forall\, i = 1, \dots, N \Big\}. \tag{2.23}$$

Then $\mathcal{A}(T, \mathcal{U})$, $T > 0$, and $\mathcal{A}(\bar{x}, \mathcal{U})$, $\bar{x} > 0$, are compact subsets of $\mathbb{L}^1(\mathbb{R}^+)$ and $\mathbb{L}^1_{loc}(\mathbb{R}^+)$ respectively.

The proof of Theorem 3 is given in §4.

Remark 2.5. The convexity assumption on the multifunction G cannot be relaxed in order to ensure the closure of the attainable set, as shown by the following

Example. Consider the problem (1.1)-(1.2) associated with the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0 \tag{2.24}$$

and assume that the admissible boundary controls are all the measurable functions taking values in $\{0, 2\}$. We claim that the corresponding attainable set at time $T = 1$ is not closed in the topology of $\mathbb{L}^1$. Indeed, define

$$\tilde{u}^\nu(t) = \begin{cases} 2 & \text{if} \quad \dfrac{k}{2^\nu} \le t \le \dfrac{k+1}{2^\nu} \quad k \text{ even} \\ \\ 0 & \text{if} \quad \dfrac{k}{2^\nu} \le t \le \dfrac{k+1}{2^\nu} \quad k \text{ odd} \end{cases} \qquad 0 \le k \le 2^\nu - 1. \tag{2.25}$$

Observe that $f(\tilde{u}^\nu)$ converges weakly in $\mathbb{L}^1$ to $f(\tilde{u})$, with $\tilde{u}(t) \equiv \sqrt{2}$. Hence by the same arguments of § 4 it can be shown that $S_{(\cdot)}\tilde{u}^\nu(\cdot)$ converges in the $\mathbb{L}^1$ norm to a solution of (2.24), (1.2), (1.3) with boundary data $\tilde{u}$: then

$$S_1\tilde{u}(x) = \begin{cases} \sqrt{2} & \text{if } 0 < x < \sqrt{2}/2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.26}$$

It can be easily seen that such a profile cannot be obtained with a boundary data $\tilde{u}'$ which takes values in $\{0, 2\}$. Indeed, by tracing the backward generalized characteristics [4] and recalling (2.8), one gets

$$\tilde{u}'(t) = \sqrt{2}, \quad \forall\, t \in [1/2, 1]. \tag{2.27}$$

Remark 2.6. The convexity assumption on the functions q_i cannot be relaxed too. Indeed, consider Burger's equation (2.24) with admissible boundary data $\tilde{u}$ taking values in $[0, 2]$ and satisfying the inequality

$$\int_{1/2}^{1} \tilde{u}(s)\; ds \le \frac{1}{2} \tag{2.28}$$

which is an integral constraint of the type given in (2.23) with

$$q(s, v) \doteq \begin{cases} 0 & \text{if} \quad 0 \le s < 1/2 \\ \operatorname{sgn}(v)\sqrt{2|v|} & \text{otherwise.} \end{cases}$$

Observe that the same sequence defined by (2.25) fulfills such a constraint. On the other hand, from (2.27) it follows that the profile in (2.26) cannot be attained by using any boundary control satisfying (2.28).

As stated in the introduction, the compactness of the attainable sets guarantees the existence of optimal controls for a class of minimization problems.

Corollary 1. *Let $F_1 : \mathbb{L}^1(\mathbb{R}^+) \to \mathbb{R}$, $F_2 : \mathbb{L}^1([0,\tau]) \to \mathbb{R}$, $\tau > 0$, be lower semicontinuous functionals and let $\mathcal{U}$ be defined as in (2.23). Then for every fixed $T, \bar{x} > 0$ the optimal control problems*

$$\min_{\tilde{u}\in\mathcal{U}} F_1\left(S_T\tilde{u}(\cdot)\right), \qquad \min_{\tilde{u}\in\mathcal{U}} F_2\left(S_{(\cdot)}\tilde{u}(\bar{x})\right),$$

admit a solution.

3. Proof of Theorem 1

The proof will be divided into two steps:

Step 1. Show that any element $S_T\tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+)$ of the attainable set satisfies (2.17)-(2.19).

Step 2. Show that if $w \in \mathrm{BV}([\alpha, +\infty))$, $\forall\ \alpha > 0$, is a bounded function satisfying (2.17)-(2.19), then there exists $\tilde{u} \in \mathbb{L}^\infty([0,T])$, $\tilde{u} \geq b(0)$, such that $S_T\tilde{u} = w$.

Step 1

A technical result will be proved first.

Lemma 3.1. *Let $w : \mathbb{R} \to \mathbb{R}$, $x > 0$, be a bounded right continuous function having right and left limits in any point. Then $\varphi : x \mapsto \dfrac{f'(w(x))}{x}$ is non-increasing iff (2.19) holds.*

Proof. Observe that non-increasing monotonicity of φ is equivalent to

$$D^+\varphi(x) \leq 0 \qquad \forall\ x > 0. \tag{3.1}$$

Suppose first that $x > 0$ is a point of continuity for w. Hence, being $f'' > 0$

$$\limsup_{h\to 0} \frac{\varphi(x+h) - \varphi(x)}{h} = \limsup_{h\to 0}\left[\frac{f'(w(x+h)) - f'(w(x))}{(w(x+h) - w(x))}\ \frac{w(x+h) - w(x)}{(x+h)h} - \frac{f'(w(x))}{x(x+h)}\right]$$

$$= \frac{f''(w(x))}{x}\limsup_{h\to 0}\frac{w(x+h) - w(x)}{h} - \frac{f'(w(x))}{x^2}, \tag{3.2}$$

which shows that (3.1) and (2.19) are equivalent.
In the case w is not continuous at x, assume (3.1) holds: then $w(x-) > w(x)$. Indeed, if it is false, then $f'(w(x-)) < f'(w(x))$ by convexity of f; hence there exists $y < x$ such that $\varphi(y) < \varphi(x)$ which contradicts the monotonicity assumption on φ. There follows that

$$D^+w(x) = \limsup_{h\to 0^+}\frac{w(x+h) - w(x)}{h};$$

thus (2.19) follows taking in (3.2) the $\limsup$ as $h \to 0^+$. Conversely, if (2.19) holds then still

$w(x-) > w(x)$. Since w and hence φ are right continuous it follows that $\varphi(x-) > \varphi(x)$, due to the monotonicity of f'. Thus it is sufficient to prove (3.1) for $h \to 0^+$. This follows immediatly from (3.2) using the same arguments as before. $\square$

Recalling Remark 2.1 we can choose a representative function w of $S_T\tilde{u}$ which is right continuous. Assume that $f'(w(x)) < x/T$ and let $\xi(\cdot)$ denote the maximal backward generalized characteristic through (T, x). Observe that $\xi(\cdot)$ is a genuine characteristic (see [4] Theorem 3.2) and hence, by Theorem 3.3 in [4] , $S_{(\cdot)}\tilde{u}(\xi(\cdot)) = v$ a.e. on $[0, T]$ for some constant v such that $\dot{\xi} = f'(v)$. Since Theorem 4.1 in [4] implies $v(0) = w(x)$, it follows that $\xi(t) = x + f'(w(x))(t - T)$ for all $t \in [0, T]$. Hence $\xi(0) = x - Tf'(w(x)) > 0$ which implies $w(x) = S_0\tilde{u}(\xi(0)) = 0$ thus proving (2.17).
Next, suppose that there exists $x > 0$ such that $f'(w(x-)) \leq x/T$. If $w(x-) = 0$ there's nothing to prove. Otherwise $f'(w(x-)) = x/T$. If $w(x+) = w(x-)$, again there's nothing to prove, otherwise, from arguments similar to the previous ones and since genuine characteristics do not intersect in the interior of Ω, it follows that $w(y) = 0 \ \forall y > x$ and hence $w(x-) > 0$. Observe now that the values of the solution in the interior of the funnel confined between minimal and maximal backward characteristics through (T, x) depend only on the values of the solution at $t = 0$. Thus $S_t\tilde{u}(x) = 0$ for any $0 < t < T$ and $x > f'(w(x-))t$. There follows that the minimal characteristic is not genuine, which gives a contradiction, proving (2.18).
To prove (2.19) by Lemma 3.1 it is sufficient to show that the function $\varphi : x \mapsto f'(w(x))/x$ is non-increasing. Let $0 < x_1 < x_2$ be given and trace the maximal backward characteristics $\xi_1(\cdot)$, $\xi_2(\cdot)$ through (T, x_1) and (T, x_2) respectively. By the same arguments as above they have the form

$$\xi_i(t) = x_i + f'(w(x_i))(t - T) \qquad i = 1, 2 \tag{3.3}$$

as long as they exist. Assume that $f'(w(x_1)) < f'(w(x_2))$ (otherwise the result is obvious) and let $\tau \in \mathbb{R}$ be such that $\xi_1(\tau) = \xi_2(\tau)$ where, with an abuse of notation, $\xi_i(\cdot)$ denote the functions in (3.3) defined for all $t \in \mathbb{R}$. Since ξ_1 and ξ_2 are genuine characteristics and hence do not intersect in the interior of Ω (see [4]), we deduce that $\xi_i(\tau) \leq 0$. Otherwise it should be $\tau < 0$ which implies, by arguments as above, $f'(w(x_1)) = f'(w(x_2)) = f'(0)$. Therefore

$$1 + \frac{f'(w(x_1))}{x_1}(\tau - T) = \frac{\xi_1(\tau)}{x_1} \leq \frac{\xi_2(\tau)}{x_2} = 1 + \frac{f'(w(x_2))}{x_2}(\tau - T)$$

showing $\varphi(x_1) \geq \varphi(x_2)$.

Step 2

Choose $w \in \mathbb{L}^\infty(\mathbb{R}^+)$ satisfying (2.17)-(2.19). By Remark 2.3 we can assume that w is right continuous. Observe first that if $w \equiv 0$ then the boundary control

$$\tilde{u} \equiv \begin{cases} 0 & \text{if} \quad f'(0) \geq 0 \\ b(0) & \text{if} \quad f'(0) < 0 \end{cases}$$

clearly produces the null solution. Next we prove the result in the case w is made up of two constant states.

Proposition 3.1. *Let $\omega, r > 0$ be given with $f'(\omega) > r/T$. Then there exists $\tilde{u} \in \mathbb{L}^\infty([0,T])$, $\tilde{u} \geq b(0)$, such that*

$$S_T \tilde{u}(x) = \begin{cases} \omega & \text{if } x < r \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Proof. If $c \doteq [f(\omega) - f(0)]/\omega \geq r/T$, set $t_1 = T - r\,\omega/[f(\omega) - f(0)] \geq 0$. Then

$$\tilde{u}(t) = \begin{cases} \omega & \text{if} \quad t_1 < t < T \\ 0 & \text{if} \quad 0 < t < t_1 \text{ and } f'(0) \geq 0 \\ b(0) & \text{if} \quad 0 < t < t_1 \text{ and } f'(0) < 0 \end{cases}$$

produces the solution

$$S_t \tilde{u}(x) = \begin{cases} \omega & \text{if} \quad 0 < x < r + \dfrac{f(\omega) - f(0)}{\omega}(t - T) \\ \\ 0 & \text{otherwise} \end{cases}$$

which satisfy (3.4).

Now assume $c < r/T$ and call $t_2 = T - r/f'(\omega) > 0$. For any $\bar{t} \in [0, t_2)$ and $v \geq \omega$ define the function $\phi_{\bar{t},v} : [0,T] \to [\omega, +\infty)$ by setting

$$\phi_{\bar{t},v}(t) = \begin{cases} v & \text{if } 0 \leq t < \bar{t} \\ b\left[f'(v) + \dfrac{t - \bar{t}}{t_2 - \bar{t}}\left(f'(\omega) - f'(v)\right)\right] & \text{if } \bar{t} \leq t < t_2 \\ \omega & \text{if } t \geq t_2. \end{cases} \tag{3.5}$$

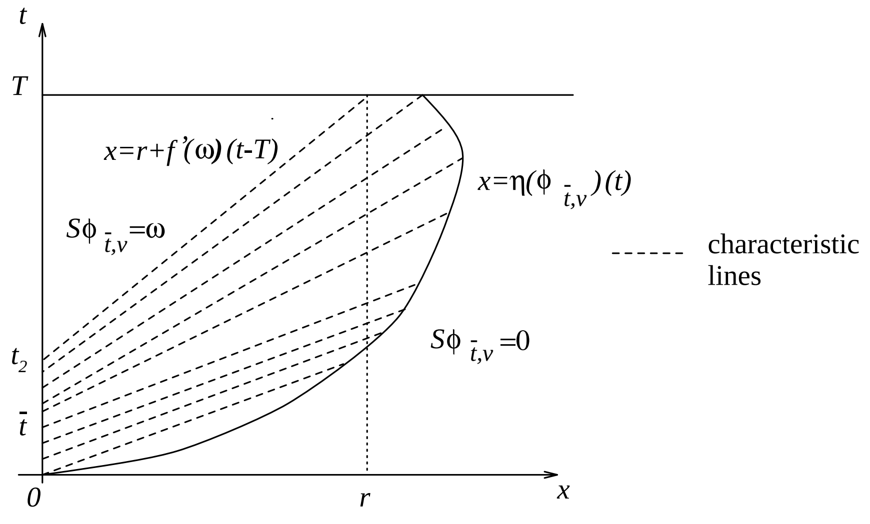

Figure 1

If $v \geq \omega$ satisfies $f(v) > f(0)$, since $t \mapsto \phi_{\bar{t},v}(t)$ is decreasing on $[0, t_2]$ it can be easily seen that $S_{(\cdot)}\phi_{\bar{t},v}$ has a single shock curve $t \mapsto \eta(\phi_{\bar{t},v})(t)$ departing from the origin such that $S_t\phi_{\bar{t},v}(x) = 0$ for $x > \eta(\phi_{\bar{t},v})(t)$ as long as $\eta(\phi_{\bar{t},v})(\cdot)$ exists (see fig. 1).

We claim that there exist ω_0, $\omega_1 > \omega$ and $0 \le \tau_0$, $\tau_1 < t_2$ such that $\eta(\phi_{\tau_0,\omega_0})(\cdot)$ and $\eta(\phi_{\tau_1,\omega_1})(\cdot)$ are defined on $[0,T]$ and

$$\eta(\phi_{\tau_0,\omega_0})(T) < r \le \eta(\phi_{\tau_1,\omega_1})(T) \tag{3.6}$$

First we prove the existence of τ_1 and ω_1. To this end we show that there exist $v > \omega$ and $s \in (0,T)$ such that

$$\frac{f(v)-f(0)}{v}\, s > r + |c|(T-s), \tag{3.7}$$

$$0 < s - \frac{1}{f'(v)}\,\frac{f(v)-f(0)}{v}\, s < t_2. \tag{3.8}$$

Indeed, if $\lim_{v\to+\infty} f'(v) = +\infty$ then choose $s = t_2/2$ and $v > \omega$ satisfying (3.7). Otherwise, being $f'(\omega) > r/T$ and hence

$$\frac{r}{T} < \lim_{v\to+\infty} f'(v) = \lim_{v\to+\infty} \frac{f(v)-f(0)}{v}, \tag{3.9}$$

there exists $\bar v > \omega$ such that $T\left[f(\bar v) - f(0)\right]/\bar v > r$. Then, using the continuity of the map

$$t \mapsto \frac{f(\bar v)-f(0)}{\bar v}\, t - r - |c|(T-t),$$

we find some $s \in (0,T)$ satisfying (3.7) with $v = \bar v$. But (3.9) and the convexity of f guarantee that there exists $v \ge \bar v$ satisfying (3.7)-(3.8) as well. Now set

$$\omega_1 = v \qquad\qquad \tau_1 = s - \frac{1}{f'(v)}\,\frac{f(v)-f(0)}{v}\, s. \tag{3.10}$$

It follows that

$$\begin{aligned}
\eta(\phi_{\tau_1,\omega_1})(T) &= \int_0^s \dot\eta(\phi_{\tau_1,\omega_1})(t)\; dt + \int_s^T \dot\eta(\phi_{\tau_1,\omega_1})(t)\; dt \\
&\ge \frac{f(\omega_1)-f(0)}{\omega_1}\, s + c(T-s) \\
&> r + (|c|+c)(T-s) \ge r.
\end{aligned} \tag{3.11}$$

Now we set $\tau_0 = 0$ and prove the existence of ω_0. If $c > 0$, take $\omega_0 = \omega$. Otherwise set

$$\bar v = \sup\left\{v \ge \omega :\ S_T\phi_{0,v} \equiv 0\right\}. \tag{3.12}$$

By the previous analysis, $\bar v < +\infty$. Moreover, since the map $v \mapsto \phi_{0,v}$ is continuous from $[\omega,+\infty)$ into $\mathbb{L}^\infty([0,T])$ w.r.t. the $\mathbb{L}^1$ norm, from Remark 2.2 it follows that $S_T\phi_{0,\bar v} \equiv 0$. If $v > \bar v$, then $\eta(\phi_{0,v})(\cdot)$ is defined on $[0,T]$ and $\eta(\phi_{0,v})(T) > 0$. Indeed, if not, then there exists $\tau < T$ such that $\eta(\phi_{0,v})(\tau) = 0$. There follows that $S_\tau\phi_{0,v} \equiv 0$ and that $f(\phi_{0,v}(t)) \le f(\phi_{0,v}(\tau)) < f(0)$ $\forall t \ge \tau$. Hence $S_t\phi_{0,v} \equiv 0$ $\forall t \ge \tau$, which contradicts (3.12). Moreover, if $0 < x < \eta(\phi_{0,v})(T)$, then $S_T\phi_{0,v}(x) \ge \omega$. In fact, due to (2.18), the minimal backward characteristic through $(T, \eta(\phi_{0,v})(T))$ reaches the t-axis in positive time. Since genuine characteristics do not intersect, all maximal backward characteristics through (T,x), $0 < x < \eta(\phi_{0,v})(T)$, intersect the t-axis. Since $\phi_{0,v}(t) \ge \omega$

for any $t \in [0,T]$, by arguments similar to the ones used in Step 1 we deduce that $S_T\phi_{0,v}(x) \geq \omega$. There exists $\delta > 0$ such that if $\bar{v} < v < \bar{v} + \delta$ then $\eta(\phi_{0,v})(T) < r$. Indeed assume by contradiction that there exists a decreasing sequence $(v_n)_{n\in\mathbb{N}}$ converging to $\bar{v}$ such that $\eta(\phi_{0,v_n})(T) \geq r \;\forall n$. Then

$$\left\|S_T\phi_{0,\bar{v}} - S_T\phi_{0,v_n}\right\|_{\mathbb{L}^1} \geq \int_0^r |S_T\phi_{0,v_n}(x)| \; dx \geq \omega r$$

which contradicts the continuity of the map $v \mapsto S_T\phi_{0,v}$, proving the existence of ω_0 with the required property. Consider now the continuous map $\phi : [0,1] \to \mathbb{L}^\infty([0,T])$ defined by

$$\phi(\lambda) = \lambda\phi_{\tau_1,\omega_1} + (1-\lambda)\phi_{\tau_0,\omega_0}. \tag{3.13}$$

Set $\eta(\phi(\lambda))(T) = 0$ if $S_T\phi(\lambda) \equiv 0$. Then from the continuity of $\lambda \mapsto S_T\phi(\lambda)$, it follows that the map $\lambda \mapsto \eta(\phi(\lambda))(T)$ is continuous. Indeed, by the previous analysis, $S_T\phi(\lambda)(x) \geq \omega$ whenever $x < \eta(\phi(\lambda))(T)$. Hence

$$\begin{aligned}
|\eta(\phi(\lambda_1))(T) - \eta(\phi(\lambda_2))(T)| &\leq \frac{1}{\omega}\left|\int_{\eta(\phi(\lambda_1))(T)}^{\eta(\phi(\lambda_2))(T)} |S_T\phi(\lambda_1)(x) - S_T\phi(\lambda_2)(x)| \; dx\right| \\
&\leq \frac{1}{\omega}\left\|S_T\phi(\lambda_1) - S_T\phi(\lambda_2)\right\|_{\mathbb{L}^1},
\end{aligned}$$

which approaches zero as $\lambda_1 - \lambda_2 \to 0$. It follows that there exists $\bar{\lambda} \in [0,1]$ such that $\eta(\phi(\bar{\lambda}))(T) = r$. We claim that $S_T\phi(\bar{\lambda})$ satisfies (3.4). Indeed if $x < r$ let $t \mapsto \theta(t)$ be the maximal backward characteristic through (T,x). Then by (2.17) there exists $\tau \geq 0$ such that $\theta(\tau) = 0$. Actually $\tau \geq t_2$. If not, then

$$\dot{\theta}(t) = f'(S_T\phi(\bar{\lambda})(x)) = \frac{x}{T-\tau} < \frac{r}{T-t_2} = f'(\omega)$$

which gives a contradiction since f' is increasing and $S_T\phi(\bar{\lambda})(x) \geq \omega$. Thus $\tau \geq t_2$, from which there follows that $S_T\phi(\bar{\lambda})(x) = \omega$. □

Throughout the following we denote by $\psi(\omega, r) \in \mathbb{L}^\infty([0,T])$ a boundary control such that $S_T\psi(\omega,r)$ satisfies (3.4). In order to prove Step 2 in the general case we shall adopt the following procedure.

1. For every $x > 0$ we trace the lines θ_x^-, θ_x^+ through (T,x) with slope $f'(w(x-))$ and $f'(w(x+))$ respectively. These will be the minimal and maximal backward characteristics through (T,x) of the candidate solution. Due to (2.17), if $w(x) \neq 0$ they reach the t-axis in positive time. The assumption (2.19) guarantees that the lines $\{\theta_x^\pm : \; x > 0\}$ do not intersect each other in the interior of Ω.
2. Since a solution is constant along minimal and maximal backward characteristics [4], for every $t \in [0,T]$ for which there exists $x > 0$ such that $\theta_x^\pm(t) = 0$, we define $\tilde{u}(t) = w(x)$. The set of the remaining t is a disjoint union of open intervals. On any of such intervals $\tilde{u}$ is defined so as to produce a compression wave which generates a discontinuty at time T.
3. By using the fact that a solution is constant along genuine characteristics, we define a function $u : (0,T) \times \mathbb{R}^+ \to \mathbb{R}$ which is candidate to be $S_{(\cdot)}\tilde{u}$ and we prove that u is a weak entropic solution of (1.1)-(1.2) in the interior of Ω.

4. We show that u satisfies the boundary condition related to the boundary control $\tilde{u}$ in the sense of iii) of Definition 1 and that $u(T-,\cdot) = w$.

1. For each $x > 0$ consider the lines

$$\theta_x^- : t \mapsto x + f'(w(x-))(t-T) \tag{3.14}$$

$$\theta_x^+ : t \mapsto x + f'(w(x))(t-T), \tag{3.15}$$

defined for $t \leq T$. By Remark 2.3 and convexity of f one has $\theta_x^-(t) \leq \theta_x^+(t)\ \forall\, t$. We claim that for any $0 < x < y$ the lines $\theta_x^\pm$ and $\theta_y^\pm$ do not intersect in the interior of Ω. By the previous argument it suffices to prove that $\theta_x^+(t) > \theta_y^-(t)$ in the interior of Ω. If $f'(w(x)) \geq f'(w(y-))$ the claim is obvious. Otherwise since $w(x) \neq w(y-)$, one of the two is nonzero. Hence due to (2.17) one of the two holds: $f'(w(x)) \geq x/T$ or $f'(w(y-)) \geq x/T$. Let $\tau < T$ be such that $\theta_x^+(\tau) = \theta_y^-(\tau) \doteq \xi$. Then $\tau \geq 0$ or $\xi \leq 0$. Actually $\xi \leq 0$. Indeed, let φ be as in Lemma 3.1. Then $\varphi(y-) \leq \varphi(x)$. Hence

$$\frac{\xi}{x} = 1 + \varphi(x)(\tau - T) \leq 1 + \varphi(y-)(\tau - T) = \frac{\xi}{y}$$

and since $x < y$ it follows $\xi \leq 0$, proving the claim.

2. Define

$$x_0 \doteq \inf\left\{x > 0 :\ w(y) = 0\ \forall\ y \geq x\right\}. \tag{3.16}$$

To get a boundary control $\tilde{u}$ that produces a solution of (1.1)-(1.3) that attains w, we consider the following partition of the interval $[0,T]$ (see fig. 2):

$$I_1 \doteq \left\{t \in [0,T] :\ \exists!\ x > 0 :\ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0\right\}; \tag{3.17}$$

$$I_2 \doteq \left\{t \in [0,T] :\ \exists\ 0 < x < y :\ \theta_x^+(t) = \theta_y^-(t) = 0\right\}; \tag{3.18}$$

$$\begin{aligned} I_3 \doteq \big\{t \in [0,T] :\ \nexists\ x > 0 :\ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0,& \\ \exists\ t' \in (0,t) \cap [I_1 \cup I_2],\ \exists\ t'' \in (t,T) \cap [I_1 \cup I_2]\big\};& \end{aligned} \tag{3.19}$$

$$I_4 \doteq \left\{t \in [0,T] :\ \forall t' \geq t\ \nexists\ x > 0 :\ \theta_x^-(t') = 0 \text{ or } \theta_x^+(t') = 0\right\}; \tag{3.20}$$

$$I_5 \doteq \left\{t \in [0,T] :\ \forall t' \leq t\ \nexists\ x > 0 :\ \theta_x^-(t') = 0 \text{ or } \theta_x^+(t') = 0\right\}. \tag{3.21}$$

Here any of these sets could be empty. The above sets, whenever nonempty, satisfy the following properties:

i) I_2 contains at most countably many points;

ii) I_3 is the disjoint union of at most countably many open intervals $(\mathcal{I}_\nu)_{\nu\in\mathbb{N}}$ of the form

$$\mathcal{I}_\nu = (\tau_\nu^1, \tau_\nu^2), \qquad \theta_{x_\nu}^+(\tau_\nu^1) = \theta_{x_\nu}^-(\tau_\nu^2) = 0 \qquad \exists\ x_\nu > 0, \tag{3.22}$$

where x_ν is a point of discontinuity for w.

iii) I_4 is an interval of the form $I_4 = (\tau^4, T]$ with $\tau^4 \in I_1 \cup I_2$.

iv) I_5 is an interval of the form $I_5 = [0, \tau^5)$ with $\theta_{x_0}^-(\tau^5) = 0$.

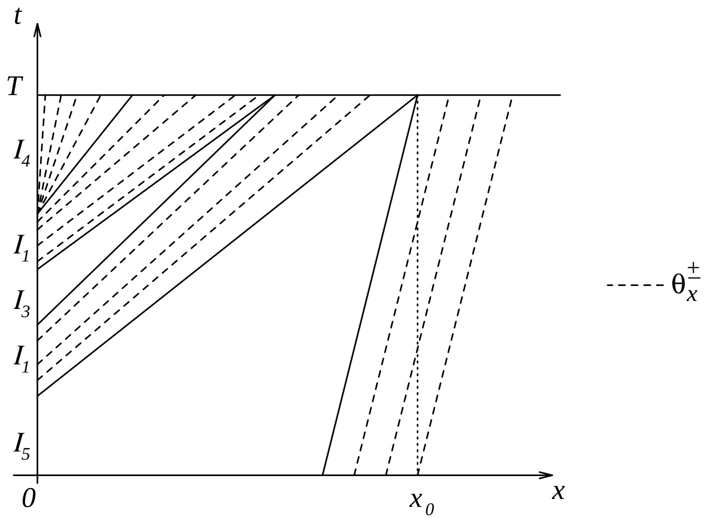

Figure 2

To show *i)* it is sufficient to observe that, since the lines $\{\theta_x^\pm\}_{x>0}$ do not intersect in the interior of Ω, for each $t \in I_2$ the set

$$J_t \doteq \left\{x > 0 : \ \theta_x^-(t) = 0 \text{ or } \theta_x^+(t) = 0\right\} \tag{3.23}$$

is an interval and $J_s \cap J_t = \emptyset$ for any $s, t \in I_2$, $s \neq t$.
Regarding *ii)-iv)*, we first show that $I_3 \cup I_4 \cup I_5$ is open in $[0, T]$. Indeed, let $t \in I_3 \cup I_4 \cup I_5$ and assume by contradiction that $(t_\nu)_{\nu \in \mathbb{N}} \subseteq I_1 \cup I_2$ is a sequence converging to t. Then there exists a sequence $(y_\nu)_{\nu \in \mathbb{N}} \subseteq \mathbb{R}^+$ such that $\theta_{y_\nu}^\pm(t_\nu) = 0$. By eventually taking a subsequence, we shall assume $\theta_{y_\nu}^+(t_\nu) = 0$, the other case being entirely similar. Since w is bounded, from (2.17) it follows that $(y_\nu)_{\nu \in \mathbb{N}}$ is bounded, so it admits a converging subsequence which is still denoted by $(y_\nu)_{\nu \in \mathbb{N}}$. Call $\bar{y}$ its limit point. Again, up to a subsequence we can assume that $f'(w(y_\nu)) \to f'(w(\bar{y}))$. Then $0 = \theta_{y_\nu}^+(t_\nu) \to \theta_{\bar{y}}^+(t)$, which gives a contradiction. Observe now that $\inf I_4 \in I_1 \cup I_2$. Indeed, if $\inf I_4 = 0$, then it belongs to $I_1 \cup I_2$ by (2.17) since $w \not\equiv 0$. Otherwise, since $I_3 \cup I_4 \cup I_5$ is open, if $\inf I_4 \notin I_1 \cup I_2$, then there exists $t' < \inf I_4$ such that $(t', \inf I_4) \subseteq I_3 \cup I_4 \cup I_5$ which clearly gives a contradiction. Since by definition I_4 is an interval and $\sup I_4 = T$, this suffices to prove *iii)*.
Concerning *iv)*, in a similar way it can be proved that $\tau^5 = \sup I_5 \in I_1 \cup I_2$. Set

$$z \doteq \sup\left\{x > 0 : \ \theta_x^-(\tau^5) = 0 \text{ or } \theta_x^+(\tau^5) = 0\right\}. \tag{3.24}$$

Let $y > z$ and suppose that $w(y) \neq 0$. Then by (2.17) and (3.21) $\theta_y(\tau^5) = 0$, which contradicts (3.24). Thus it must be $z \geq x_0$. If $z > x_0$, then $0 = \theta_z^\pm(\tau^5) = z + f'(0)(\tau^5 - T)$. Hence there exists $y > z$ and $t \in (0, \tau^5)$ such that $\theta_y^\pm(t) = y + f'(0)(t - T) = 0$, which gives a contradiction by the definition of I_5. Thus $z = x_0$ and hence $\theta_{x_0}^-(\tau^5) = 0$ proving *iv)*.
Regarding *ii)*, since $\inf I_4, \sup I_5 \notin I_3$, I_3 is open, hence it is a disjoint union of at most countably many open intervals $\mathcal{I}_\nu = (\tau_\nu^1, \tau_\nu^2)$. Moreover $\tau_\nu^1, \tau_\nu^2 \in I_1 \cup I_2$ since $I_3 \cup I_4 \cup I_5$ is open. Call

$$x_\nu^1 \doteq \inf\left\{x > 0 : \ \theta_x^-(\tau_\nu^1) = 0 \text{ or } \theta_x^+(\tau_\nu^1) = 0\right\}$$
$$x_\nu^2 \doteq \sup\left\{x > 0 : \ \theta_x^-(\tau_\nu^2) = 0 \text{ or } \theta_x^+(\tau_\nu^2) = 0\right\}.$$

Then $x_\nu^1 = x_\nu^2 \doteq x_\nu$. In fact $x_\nu^2 \le x_\nu^1$ since the lines $\{\theta_x^\pm\}_{x>0}$ do not intersect in the interior of Ω. If $x_\nu^2 < x_\nu^1$, then choose $y \in (x_\nu^2, x_\nu^1)$. Then there exists $\tau \in (\tau_\nu^1, \tau_\nu^2)$ such that $\theta_y^\pm(\tau) = 0$, which is a contradiction. Since by (2.19) w satisfies (2.8), the conclusion of *ii)* follows immediatly.
Now we are ready to define the boundary data which produces the given profile:

$$\tilde u(t) = \begin{cases} w(x-) & \text{if} \quad t \in I_1,\ \theta_x^-(t) = 0 \\ w(x) & \text{if} \quad t \in I_1,\ \theta_x^+(t) = 0 \\ w\left((\sup J_t)-\right) & \text{if} \quad t \in I_2 \\ b\left(\dfrac{x_\nu}{T-t}\right) & \text{if} \quad t \in \mathcal{I}_\nu \subseteq I_3 \\ b(0) & \text{if} \quad t \in I_4 \\ \psi(w(x_0-), x_0)(t) & \text{if} \quad t \in I_5. \end{cases} \tag{3.25}$$

Notice that if $t \in \mathcal{I}_\nu \subseteq I_3$, then

$$f'(w(x_\nu)) < \frac{x_\nu}{T-t} < f'(w(x_\nu-)),$$

and hence $x_\nu/(T-t) \in \text{Range } f'$. Moreover, if $I_4 \ne \emptyset$, then $b(0) > -\infty$. Indeed, fix $\varepsilon > 0$. Then for any $x \in (0, \varepsilon(t - \tau^4))$ we have $0 < f'(w(x)) \le \varepsilon$. In fact let $\xi > 0$ be such that $\theta_\xi^\pm(\tau^4) = 0$. If $f'(w(x)) > \varepsilon$, then there exists $\tau > \tau^4$ such that $\theta_\zeta^\pm(\tau) = 0$, thus contradicting (3.20). If $f'(w(x)) \le 0$, then $\theta_\zeta^\pm$ and $\theta_\xi^\pm$ would intersect in the interior of Ω. Hence $\lim\limits_{x\to 0^+} f'(w(x)) = 0$. Due to the boundedness of w, this implies $0 \in \text{Range } f'$. Thus (3.25) is well defined.

3. For each $s \in \mathcal{I}_\nu \subseteq I_3$ define the line

$$\theta_s : t \mapsto f'(\tilde u(s))(t-s) = \frac{x_\nu}{T-s}(t-s), \qquad s < t < T, \tag{3.26}$$

which is entirely contained in the open set $\left\{(t,x) :\ s < t < T,\ \theta_{x_\nu}^-(t) < x < \theta_{x_\nu}^+(t)\right\}$. Observe that any of the θ_s cannot intersect one of the $\theta_x^\pm$ in the interior of Ω, otherwise $\theta_x^\pm$ would intersect $\theta_{x_\nu}^-$ or $\theta_{x_\nu}^+$ too. Denote (see fig. 3)

$$\begin{aligned} \mathcal{A}_1 &\doteq \left\{(\tau,\xi) \in \text{int }\Omega :\ \xi \le \theta_{x_0}^-(\tau)\right\} \\ \mathcal{A}_2 &\doteq \left\{(\tau,\xi) \in \text{int }\Omega :\ \xi > \theta_{x_0}^-(\tau))\right\}. \end{aligned} \tag{3.27}$$

We claim that for any $(\tau,\xi) \in \mathcal{A}_1$ there exists a unique line through (τ,ξ) belonging to the family $\Theta \doteq \{\theta_x^\pm :\ x > 0\} \cup \{\theta_s :\ s \in I_3\}$. The uniqueness of such a line follows from the previous remark and from the fact that the lines of each family $\{\theta_x^\pm :\ x > 0\}$ and $\{\theta_s :\ s \in I_3\}$ do not intersect in the interior of Ω. Regarding the existence observe that if $\xi \ne \theta_x^\pm(\tau)$ for any $x > 0$, then there exists $s \in I_3$ such that $\theta_s(\tau) = \xi$. Indeed, the set

$$\mathcal{B}(\tau) \doteq \left\{0 < x < \theta_{x_0}^-(\tau) :\ \nexists\, y > 0 :\ \theta_y^\pm(\tau) = x\right\} \tag{3.28}$$

is open. In fact, let $x \in \mathcal{B}(\tau)$ and assume by contradiction that there exists in $(0, \theta_{x_0}^-(\tau))$ a sequence $x_\nu = \theta_{y_\nu}^\pm(\tau)$, $y_\nu > 0$, converging to x. By eventually taking a subsequence, we shall assume that

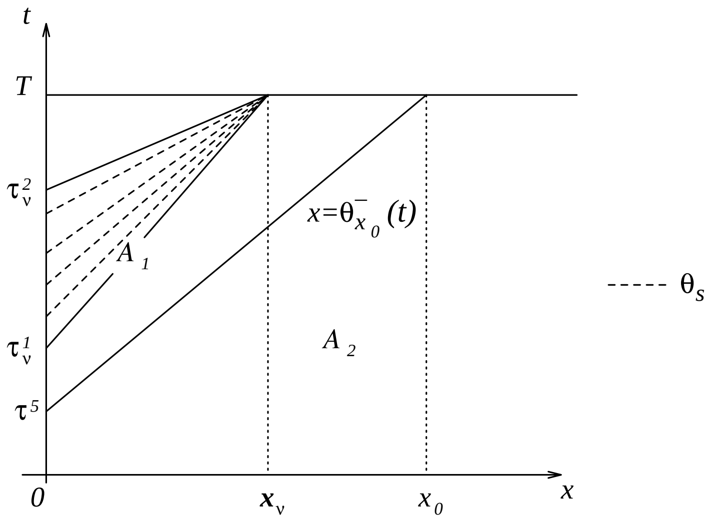

Figure 3

$x_\nu = \theta^+_{y_\nu}(\tau)$, the other case being entirely similar. Since w is bounded, from (2.17) it follows that $(y_\nu)_{\nu\in\mathbb{N}}$ is bounded. Therefore there exists a subsequence, which we still denote by $(y_\nu)_{\nu\in\mathbb{N}}$, converging to some $\bar y > 0$ and such that that $f'(w(y_\nu)) \to f'(w(\bar y))$. Then $\theta^+_{y_\nu}(\tau) \to \theta^+_{\bar y}(\tau)$ and hence $x = \theta^+_{\bar y}(\tau)$ which gives a contradiction.

Now, let (ξ_1, ξ_2) be the connected component of $\mathcal{B}(\tau)$ containing ξ. Then as above there exists $y > 0$ such that $\theta^-_y(\tau) = \xi_1$ and $\theta^+_y(\tau) = \xi_2$. Let $t_1 > t_2$ be such that $\theta^-_y(t_1) = \theta^+_y(t_2) = 0$. Then clearly it must be $(t_2, t_1) = \mathcal{I}_\nu$, $y = x_\nu$ and

$$\frac{x_\nu - \xi}{T - \tau} = \frac{x_\nu}{T - s} = \dot\theta_s,$$

for some $\nu \in \mathbb{N}$ and $s \in (t_2, t_1)$. Thus by (3.26) one has $\theta_s(\tau) = \xi$.

Consider now the function $u : (0, T) \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$u(\tau,\xi) = \begin{cases} w(x) & \text{if} \quad (\tau,\xi) \in \mathcal{A}_1,\ \theta^+_x(\tau) = \xi \quad \exists\ x > 0, \\ w(x-) & \text{if} \quad (\tau,\xi) \in \mathcal{A}_1,\ \theta^-_x(\tau) = \xi \quad \exists\ x > 0, \\ \tilde u(s) & \text{if} \quad (\tau,\xi) \in \mathcal{A}_1,\ \theta_s(\tau) = \xi \quad \exists\ s \in I_3, \\ S_\tau\psi(w(x_0-), x_0)(\xi) & \text{if} \quad (\tau,\xi) \in \mathcal{A}_2,\ w(x_0-) > 0, \\ 0 & \text{if} \quad (\tau,\xi) \in \mathcal{A}_2,\ w(x_0-) = 0. \end{cases} \tag{3.29}$$

We claim that, for every $(\tau,\xi) \in \mathcal{A}_1$, $u(\tau,\cdot)$ *is continuous on* $(0, \theta^-_{x_0}(\tau)]$ and $u(\cdot,\xi)$ *is continuous on* $[\tau, T)$. We only give the proof of the first property, the second one being derived in an entirely similar way. To this end we first show that $u(\tau,\cdot)$ satisfies the following properties on $(0, \theta^-_{x_0}(\tau)]$:

a) if there exists $x > 0$ such that $\theta^-_x(\tau) = \xi$, then $u(\tau,\cdot)$ is left continuous at ξ;

b) if there exists $x > 0$ such that $\theta^+_x(\tau) = \xi$, then $u(\tau,\cdot)$ is right continuous at ξ;

c) if $\xi \in \mathcal{B}(\tau)$, then $u(\tau,\cdot)$ is continuous at ξ.

Observe first that if $\zeta \in \mathcal{B}(\tau)$, so that $\theta^-_{x_\nu}(\tau) < \zeta < \theta^+_{x_\nu}(\tau)$ for some $\nu \in \mathbb{N}$ and $\zeta = \theta_s(\tau)$ for some

$s \in \mathcal{I}_\nu$, then

$$f'(w(x_\nu)) = \dot\theta^+_{x_\nu} = \frac{x_\nu}{T-\tau^1_\nu} < \frac{x_\nu}{T-s} < \frac{x_\nu}{T-\tau^2_\nu} = \dot\theta^-_{x_\nu} = f'(w(x_\nu-)).$$

Hence, since f' is strictly increasing,

$$w(x_\nu) < u(\tau,\zeta) < w(x_\nu-). \tag{3.30}$$

We now prove *a)*. Let $x, \xi > 0$ be such that $\theta^-_x(\tau) = \xi$. Then, by (3.29) $u(\tau,\xi) = w(x-)$. Fix $\varepsilon > 0$ and choose $\delta > 0$ such that

$$|w(y) - w(x-)| \le \varepsilon \qquad \forall y \in (x-\delta, x). \tag{3.31}$$

Let $\xi_\delta = \theta^+_{x-\delta}(\tau)$. By point 1, $\xi_\delta < \xi$. Then, for every $\zeta \in (\xi_\delta, \xi)$,

$$|u(\tau,\zeta) - u(\tau,\xi)| \le \varepsilon. \tag{3.32}$$

Indeed, using again point 1, if $\zeta = \theta^\pm_y(\tau)$ for some $y > 0$ then $y \in (x-\delta, x)$ and hence (3.32) follows from (3.31). Otherwise $\zeta \in \mathcal{B}(\tau)$ and (3.30) holds for some $x_\nu \in (x-\delta, x)$. Again (3.32) follows from (3.31).

The proof of *b)* is entirely similar and *c)* follows with an analogous argument by using the continuity of $\tilde u$ on I_3 instead of the existence of right and left limits of w.

From *a)*, *b)* and *c)* we now derive the continuty of $u(\tau,\cdot)$ on $(0, \theta^-_{x_0}(\tau)]$. Indeed if $\xi = \theta^-_x(\tau) = \theta^+_x(\tau)$ for some $x > 0$ or $\xi \in \mathcal{B}(\tau)$ the conclusion is obvious. Otherwise, assume $\xi = \theta^-_{x_\nu}(\tau) < \theta^+_{x_\nu}(\tau)$ for some $\nu \in \mathbb{N}$. Since $\zeta \in \mathcal{B}(\tau)$ for any $\zeta \in (\xi, \theta^+_{x_\nu}(\tau))$ it follows

$$\lim_{\zeta\to\xi^+} u(\tau,\zeta) = \lim_{\zeta\to\xi^+} b\left(\frac{x_\nu - \zeta}{T-\tau}\right) = b\left(\frac{x_\nu - \xi}{T-\tau}\right) = b(f'(w(x_\nu-))) = u(\tau,\xi),$$

i.e. $u(\tau,\cdot)$ is right continuous at ξ, and hence continuous as well by *a)*. In a similar way it can be shown that if $\xi = \theta^+_x(\tau) > \theta^-_x(\tau)$, then $u(\tau,\cdot)$ is continuous at ξ.

In order to prove that u is a weak entropic solution of (1.1) in the region $\mathcal{A}_1$, we now show that *u is locally Lipschitz continuous.* As above we only prove that, for every $\tau \in (0,T)$, $u(\tau,\cdot)$ is locally Lipschitz continuous on $(0, \theta^-_{x_0}(\tau))$. Observe first that, with the same arguments of Step 1, from the definition of u it follows

$$D^+_\xi u(\tau,\xi) \le \frac{f'(u(\tau,\xi))}{\xi f''(u(\tau,\xi))}$$

for any $0 < \xi < \theta^-_{x_0}(\tau)$. Hence, to derive the Lipschitz continuity of $u(\tau,\cdot)$ it suffices to show that locally there exists a constant $C_1 \le 0$ such that

$$D^-_\xi u(\tau,\xi) \ge C_1 \qquad \forall\, \xi \in (0, \theta^-_{x_0}(\tau)). \tag{3.33}$$

If $D^-_\xi u(\tau,\xi) \ge 0$ there is nothing to prove. Otherwise let $\tau < T' < T$ be fixed. Since by construction

$$u(t, \xi + f'(u(\tau,\xi))(t-\tau) = u(\tau,\xi) \qquad \forall\, t \in [\tau, T],\ (\tau,\xi) \in \mathcal{A}_1, \tag{3.34}$$

for every $\zeta \in (0, \theta^-_{x_0}(\tau))$ there exists a unique $z = z(\zeta) \in (0, \theta^-_{x_0}(T'))$ such that

$$\zeta = z + f'(u(T', z))(\tau - T'), \qquad u(T', z) = u(\tau, \zeta).$$

Observe that

$$\begin{aligned} D^-_\xi u(\tau, \xi) &= \liminf_{z \to z(\xi)} \frac{u(T', z) - u(T', z(\xi))}{(z - z(\xi)) + [f'(u(T', z)) - f'(u(T', z(\xi)))](\tau - T')} \\ &= \liminf_{z \to z(\xi)} \left(\frac{z - z(\xi)}{u(T', z) - u(T', z(\xi))} + \frac{f'(u(T', z)) - f'(u(T', z(\xi)))}{u(T', z) - u(T', z(\xi))} (\tau - T') \right)^{-1}. \end{aligned} \tag{3.35}$$

Choose a sequence $(z_\nu)_{\nu \in \mathbb{N}}$ converging to $z(\xi)$ such that

$$D^-_\xi u(\tau, \xi) = \lim_{\nu \to +\infty} \left(\frac{z_\nu - z(\xi)}{u(T', z_\nu) - u(T', z(\xi))} + \frac{f'(u(T', z_\nu)) - f'(u(T', z(\xi)))}{u(T', z_\nu) - u(T', z(\xi))} (\tau - T') \right)^{-1}. \tag{3.36}$$

By the continuity of $u(T', \cdot)$

$$\lim_{\nu \to +\infty} \frac{f'(u(T', z_\nu)) - f'(u(T', z(\xi)))}{u(T', z_\nu) - u(T', z(\xi))} = f''(u(T', z(\xi))$$

and hence

$$\lim_{\nu \to +\infty} \frac{z_\nu - z(\xi)}{u(T', z_\nu) - u(T', z(\xi))}$$

does exist. Call ℓ its value. We observe that $\ell \leq 0$. In fact, assume by contradiction that $\ell > 0$. For ν sufficiently large

$$\frac{u(T', z_\nu) - u(T', z(\xi))}{z_\nu - z(\xi)} > 0. \tag{3.37}$$

Let $\xi_\nu = z_\nu + f'(u(T', z_\nu))(\tau - T')$. Hence $\xi_\nu \to \xi$ as $\nu \to +\infty$. Since f' is increasing, (3.34) and (3.37) imply

$$\frac{u(\tau, \xi_\nu) - u(\tau, \xi)}{\xi_\nu - \xi} = \frac{u(T', z_\nu) - u(T', z(\xi))}{\xi_\nu - \xi} > 0,$$

which contradicts the assumption on $D^-_\xi u(\tau, \xi)$. By (3.36) $D^-_\xi u(\tau, \xi) \geq 1/f''(u(T', z(\xi))(\tau - T')$, proving (3.33).

Since u is locally Lipschitz continuous, then it is a.e. differentiable on $\mathcal{A}_1$ and by construction it satisfies $u_t + f'(u)u_x = 0$ a.e.. Moreover by definition it is a weak entropic solution to (1.1) in $\mathcal{A}_2$. Now observe that, for any $t \in [0, T]$, $u(t, \theta^-_{x_0}(t)-) = w(x_0-)$ since $u(t, \cdot)$ is left continuous at $\theta^-_{x_0}(t)$. On the other hand, if $w(x_0-) > 0$ then one has $w(x_0-) = S_t\psi(w(x_0-), x_0)(\theta^-_{x_0}(t)-) = S_t\psi(w(x_0-), x_0)(\theta^-_{x_0}(t)+)$ since $\theta_{x_0^-}$ is a minimal backward characteristic of $S_{(\cdot)}\psi(w(x_0-), x_0)$. If $w(x_0-) = 0$ then $u(t, \theta^-_{x_0}(t)+) = 0$. Thus $u(t, \theta^-_{x_0}(t)-) = u(t, \theta^-_{x_0}(t)+)$ for any $t \in (0, T)$. It follows that u is a weak entropic solution to (1.1) in the interior of Ω. Furthermore it clearly fulfills (1.2) in the sense of ii) in Definition 1.

4. We claim that for any $t \in I_1 \cup I_3 \cup I_4$

$$\lim_{x \to 0^+} u(t,x) = \tilde{u}(t). \tag{3.38}$$

If $t \in I_1 \cup I_3$ (3.38) follows by using the same arguments at point 3. Let $t \in I_4$ and fix $\varepsilon > 0$. For any $x \in (0, \varepsilon(t - \tau^4))$ we have $0 < f'(u(t,x)) \leq \varepsilon$. Indeed fix $\xi > 0$ such that $\theta^{\pm}_{\xi}(\tau^4) = 0$. By construction it does not exist $s \in I_3$ such that $\theta_s(t) = x$. Hence $x = \theta^{\pm}_{\zeta}(t)$ for some $\zeta > 0$ and $f'(u(t,x)) = f'(w(\zeta\pm))$. If $f'(u(t,x)) > \varepsilon$, then there exists $\tau > \tau^4$ such that $\theta^{\pm}_{\zeta}(\tau) = 0$, thus contradicting (3.20). If $f'(u(t,x)) \leq 0$, then $\theta^{\pm}_{\zeta}$ and $\theta^{\pm}_{\xi}$ would intersect in the interior of Ω. Hence $\lim_{x \to 0^+} f'(u(t,x)) = 0$, so that (3.38) holds. Moreover since $f'(\tilde{u}(t)) > 0$ for every $t \in I_1 \cup I_3$, it follows that

$$\lim_{x \to 0^+} \operatorname{sgn} f'(u(t,x)) = 1 \qquad \forall\, t \in I_1 \cup I_3 \cup I_4. \tag{3.39}$$

Thus if $t \in I_1 \cup I_3 \cup I_4$ u satisfies the boundary condition related to $\tilde{u}$ in the sense of Definition 1. If $t \in I_5$ such a boundary condition is fulfilled by construction. Hence u solves (1.1)-(1.3) with $\tilde{u}$ as in (3.25). Now we show that

$$\lim_{t \to T^-} \int_0^{+\infty} |u(t,x) - w(x)| \; dx = 0. \tag{3.40}$$

Let $(t_\nu)_{\nu \in \mathbb{N}}$ be an arbitrary increasing sequence converging to T. Then

$$\int_0^{+\infty} |u(t_\nu,x) - w(x)| \; dx = \int_0^{x_0} |u(t_\nu,x) - w(x)| \; dx + \int_{x_0}^{+\infty} |u(t_\nu,x)| \; dx. \tag{3.41}$$

Let us estimate each term in the right hand side of (3.41). Concerning the first term we show that

$$\lim_{\nu \to +\infty} u(t_\nu, x) = w(x) \qquad \forall\, x \in (0, x_0). \tag{3.42}$$

In fact, let $\varepsilon > 0$ be given and fix $\delta > 0$ such that $|w(y) - w(x)| \leq \varepsilon$ whenever $x \leq y < x + \delta$. Let $\tau < T$ be such that $\theta^-_{x+\delta}(\tau) = x$ (such a τ do exists since $f'(w(x)) \geq x/T$). We claim that if $t_\nu > \tau$ then $|u(t_\nu,x) - w(x)| \leq \varepsilon$. Assume first $x \in \mathcal{B}(t_\nu)$. Then $\theta^-_{x_{k(\nu)}}(t_\nu) < x < \theta^+_{x_{k(\nu)}}(t_\nu)$ for some $k(\nu) \in \mathbb{N}$, with $x \leq x_{k(\nu)} < x + \delta$ since $\theta^+_x, \theta^{\pm}_{x_{k(\nu)}}$ and $\theta^-_{x+\delta}$ do not intersect each other in the interior of Ω. Hence from the above remark and (3.30) it follows $|u(t_\nu,x) - w(x)| \leq \varepsilon$. Suppose now $x \notin \mathcal{B}(t_\nu)$. Then with arguments similar to the previous ones we get that $x = \theta^{\pm}_y(t_\nu)$ with $x \leq y < x + \delta$ and $u(t_\nu,x) = w(y\pm)$. The conclusion follows easily.

Furthermore there exists $C_2 > 0$ such that $|u(t_\nu,x) - w(x)| \leq C_2$ for any $x \in (0, x_0)$. Hence by the Dominated Convergence Theorem we get

$$\lim_{\nu \to +\infty} \int_0^{x_0} |u(t_\nu,x) - w(x)| \; dx = 0. \tag{3.43}$$

Concerning the second term in the right hand side of (3.41), observe first that if $w(x_0-) = 0$, then $f'(0) \geq x/T$, due to (2.17). Hence $u(t_\nu, x) = 0$ for any $x \geq x_0$ since $x_0 + f'(0)(t_\nu - T) \leq x_0$. Otherwise, $t \mapsto S_t\psi(w(x_0-), x_0)$ is continuous as a map from $[0,T]$ into $\mathbb{L}^1(\mathbb{R}^+)$ and

$S_T\psi(w(x_0-), x_0)(y) = 0$ whenever $y \geq x_0$. By combining this with (3.41) and (3.43) and by the arbitrary choice of $(t_\nu)_{\nu\in\mathbb{N}}$, we obtain (3.40).

Proof of Remark 2.4

As in Remark 2.3 the boundedness of w together with $(2.19)'$ imply that w has finite total increasing variation (and hence total increasing variation as well) on sets bounded away from the origin. Thus we can assume that w has left and right limits at every point and is right continuous. Moreover $(2.19)'$ imply that $w(x-) \geq w(x)$. Next observe that $(2.19)'$ holds iff the function

$$\gamma : x \mapsto w(x) - \int_c^x \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi, \qquad c > 0, \tag{3.44}$$

is non-increasing on $\mathbb{R}^+$ and hence iff

$$D^+\gamma(x) \leq 0 \qquad \forall\ x > 0. \tag{3.45}$$

Now we show that

$$D^+\gamma(x) = D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}. \tag{3.46}$$

If $x > 0$ is a point of continuity for w then

$$D^+\gamma(x) = \limsup_{h\to 0} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}.$$

Otherwise since w is right continuous and $w(x-) > w(x)$,

$$\limsup_{h\to 0^+} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = D^+w(x) - \frac{f'(w(x))}{xf''(w(x))}$$

$$\limsup_{h\to 0^-} \left[\frac{w(x+h) - w(x)}{h} - \frac{1}{h}\int_x^{x+h} \frac{f'(w(\xi))}{\xi f''(w(\xi))}\, d\xi\right] = -\infty,$$

which imply (3.46).

4. Proof of Theorem 3

We will give the proof of the statement concerning $\mathcal{A}(T, \mathcal{U})$, the one concerning $\mathcal{A}(\bar{x}, \mathcal{U})$ being entirely similar. Let $(\tilde{u}_\nu)_{\nu\in\mathbb{N}} \subset \mathcal{U}$. Then, being G bounded, by (2.9) and (2.17) there exist $C, \alpha > 0$ such that

$$|S_t\tilde{u}_\nu(x)| \leq \begin{cases} C & \text{if} \quad x < \alpha \\ 0 & \text{if} \quad x \geq \alpha \end{cases} \qquad \forall\ t \in [0, T], \quad \forall\ \nu \in \mathbb{N}. \tag{4.1}$$

Hence $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$, $\big(S_{(\cdot)}\tilde{u}_\nu\big)_{\nu\in\mathbb{N}}$ are weak* relatively compact in $\mathbb{L}^\infty(\mathbb{R}^+)$, $\mathbb{L}^\infty(\Omega)$ respectively so that we can assume

$$S_T\tilde{u}_\nu \overset{\star}{\rightharpoonup} w \quad \text{in} \quad \mathbb{L}^\infty(\mathbb{R}^+), \tag{4.2}$$

$$S_{(\cdot)}\tilde{u}_\nu \overset{\star}{\rightharpoonup} u \quad \text{in} \quad \mathbb{L}^\infty(\Omega), \tag{4.3}$$

for some functions $w \in \mathbb{L}^\infty(\mathbb{R}^+)$, $u \in \mathbb{L}^\infty(\Omega)$. We shall prove that $w \in \mathcal{A}(T,\mathcal{U})$ and that there exists a subsequence of $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ converging to w in $\mathbb{L}^1(\mathbb{R}^+)$. By (4.1) and Remark 2.3 for every $a>0$ there exists $C_a>0$ such that

$$\text{T.V.}\,\{S_t\tilde{u}_\nu;[a,+\infty)\} \le C_a \qquad \forall\ t\in[0,T], \quad \forall\ \nu. \tag{4.4}$$

Moreover there exists $L>0$ such that if $0<a'<a$, then

$$\int_a^{+\infty} |S_t\tilde{u}_\nu(x) - S_s\tilde{u}_\nu(x)|\ dx \le L|t-s|C_{a'} \qquad \forall\ t,s>0, \quad \forall\ \nu. \tag{4.5}$$

By Helly's Theorem for any fixed $a>0$ there exists a subsequence $\big(S_t\tilde{u}_{\nu_j}\big)_{j\in\mathbb{N}}$ which converges to some function $v_a(t,\cdot)$ in $\mathbb{L}^1_{loc}([a,+\infty))$ for every $t\in[0,T]$. But (4.3) implies that such a function must coincide with u and hence by using (4.1), for every $t\in[0,T]$, the original sequence $(S_t\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ converges to $u(t,\cdot)$ in $\mathbb{L}^1(\mathbb{R}^+)$. In particular, from the convergence of $(S_T\tilde{u}_\nu)_{\nu\in\mathbb{N}}$ to $u(T,\cdot)$ and (4.2) it follows that $u(T,\cdot)=w$. Thus to complete the proof it remains to show that u is a solution of (1.1)-(1.3) corresponding to a boundary data $\tilde{u}\in\mathcal{U}$.

By (4.1) and the regularity of f it can be assumed that, for every $t\in[0,T]$, $(f(S_t\tilde{u}_\nu))_{\nu\in\mathbb{N}}$ converges in $\mathbb{L}^1(\mathbb{R}^+)$ to $f(u(t,\cdot))$. It follows that, for any nonnegative $\mathcal{C}^1$ function ϕ with compact support in $[0,T]\times(0,+\infty)$ and for any $k\in\mathbb{R}$, we obtain

$$\begin{aligned} &\iint \big\{|u-k|\,\phi_t + \big(f(u)-f(k)\big)\text{sgn}\,(u-k)\phi_x\big\}\ dxdt \\ &\quad = \lim_{\nu\to+\infty} \iint \big\{|S_t\tilde{u}_\nu - k|\,\phi_t + \big(f(S_t\tilde{u}_\nu)-f(k)\big)\text{sgn}\,(S_t\tilde{u}_\nu - k)\phi_x\big\}\ dxdt \\ &\quad \ge 0. \end{aligned} \tag{4.6}$$

Hence u is a weak entropic solution of (1.1)-(1.2) in the interior of Ω.

Next we show that the traces of the functions $f(u)$, sgn $f'(u)$ at $x=0$ exist in the sense of (2.6)-(2.7). By Remark 2.1 it is sufficient to prove that u admits in the interior of Ω the representation (2.10). Let Υ_ν, $\nu\in\mathbb{N}$, be the traces of $f(S_{(\cdot)}\tilde{u}_\nu)$, $\nu\in\mathbb{N}$. By Remarks 1.1-1.2, for every given $t\in[0,T]$ and for any $\nu\in\mathbb{N}$, $S_t\tilde{u}_\nu(x) = b\left((x-y_\nu(t,x))/t\right)$ for a.e. $x>0$ with $y_\nu(t,x)$ denoting the unique point where the function $y\mapsto\Psi_{\Upsilon_\nu}(t,x,y)$ defined by (2.11) attains its minimum. Since by (2.9) and (4.1) Υ_ν are uniformly bounded, there exists a subsequence still denoted $(\Upsilon_\nu)_{\nu\in\mathbb{N}}$ which converges weak* in $\mathbb{L}^\infty$ to some function $\Upsilon\in\mathbb{L}^\infty([0,T])$. Thus for every $(t,x)\in\text{int}\,\Omega$ the sequence of maps $(\Psi_{\Upsilon_\nu}(t,x,\cdot))_{\nu\in\mathbb{N}}$ converges uniformly to $\Psi_\Upsilon(t,x,\cdot)$ and hence for all $t\in[0,T]$ and for a.e. $x>0$ the corresponding minimum points $y_\nu(t,x)$ being unique (see Remark 2.1) converge to the minimum point $y(t,x)$ of $\Psi_\Upsilon(t,x,\cdot)$ proving that u satisfies (2.10).

Observe now that $f(\tilde{u}_\nu)$ are uniformly bounded and hence it can be assumed that

$$f(\tilde{u}_\nu) \stackrel{\star}{\rightharpoonup} \Phi \quad \text{in} \quad \mathbb{L}^\infty([0,T])$$

for some function $\Phi \in \mathbb{L}^\infty([0,T])$. Since $f(\tilde{u}_\nu(t)) \in f(G(t))$ and by (2.8) $f(\tilde{u}_\nu(t)) \le \Upsilon_\nu(t)$ for a.e. t, being f convex and G convex closed valued it follows that $\Phi(t) \in f(G(t))$ and $\Phi(t) \le \Upsilon(t)$ for a.e. t. Hence there exists a measurable selection $\tilde{u}$ from G such that

$$\Phi(t) = f(\tilde{u}(t)), \quad f(\tilde{u}(t)) \in f(G(t)), \quad f(\tilde{u}(t)) \le \Upsilon(t), \quad \text{for a.e. } t > 0.$$

Since, for any $t \in J$, on bounded subsets of $\mathbb{L}^\infty$ the functionals $y \mapsto \int_0^t q_i(s, y(s))\, ds$, $i = 1, \ldots, N$, are sequentially lower semicontinuous w.r.t. weak convergence on $\mathbb{L}^1$ (see Theorem 3 in [9]), it follows that $\tilde{u} \in \mathcal{U}$. Therefore to prove that u fulfills iii) in Definition 1, it remains to show that $\Upsilon(t) = f(\tilde{u}(t))$ whenever $\mu(t) \ge 0$, with μ denoting the trace of sgn $f'(u)$ at $x = 0$ as defined in (2.7). Assume that $\mu(t) = 0$. Then there exists $\delta > 0$ such that $f'(u(t,x)) = 0$ whenever $x \in (0,\delta) \setminus \mathcal{F}$, so that $\Upsilon(t) = f(b(0)) = f(\tilde{u}(t))$.
Now consider the set

$$\mathcal{P} \doteq \{t \in [0,T] : \ \mu(t) = 1\}, \tag{4.7}$$

and assume that $\mathcal{P}$ has positive measure. Let μ_ν be the trace of sgn $f'(S_{(\cdot)}\tilde{u}_\nu)$ as defined in (2.7). We claim that

$$\liminf_{\nu \to +\infty} \mu_\nu(t) \ge 0 \qquad \text{for a.e. } t \in \mathcal{P}. \tag{4.8}$$

Indeed, suppose that (4.8) does not hold. Then there exists $\mathcal{P}' \subseteq \mathcal{P}$ with positive measure such that for every $t \in \mathcal{P}'$ there is a subsequence $(\mu_{\nu_k}(t))_{k\in\mathbb{N}}$ of $(\mu_\nu(t))_{\nu\in\mathbb{N}}$ such that $\mu_{\nu_k}(t) = -1$ for all k. This means that, for any such t, $f'(S_t\tilde{u}_{\nu_k}(x)) < 0$ for x sufficiently close to zero. Hence by (2.17), since genuine characteristics do not intersect in the interior of the domain, it follows that $S_t\tilde{u}_{\nu_k}(x) = 0$ for every $x > 0$ and hence $f'(0) < 0$. Fix $R > 0$ and define

$$\mathcal{R} \doteq \{(t,x) \in \mathcal{P}' \times [0,R] : \ f'(u(t,x)) > 0\}. \tag{4.9}$$

Clearly $meas(\mathcal{R}) > 0$. Let $0 < \varepsilon < meas(\mathcal{R})/2$. By Egoroff's Theorem there exists $\mathcal{R}' \subset \mathcal{R}$ such that $meas(\mathcal{R} \setminus \mathcal{R}') < \varepsilon$ and $S_{(\cdot)}\tilde{u}_\nu$ converges uniformly to u on $\mathcal{R}'$. Therefore, if $(t,x) \in \mathcal{R}'$, for ν sufficiently large $S_t\tilde{u}_\nu(x) \ge b(0)$ which gives a contradiction since $f'(0) < 0$ implies $0 < b(0)$ by the convexity of f. Hence $\lim_{\nu\to\infty}(f(\tilde{u}_\nu)(t) - \Upsilon_\nu(t)) = 0$ for a.e. $t \in \mathcal{P}$. Since $f(\tilde{u}_\nu) \stackrel{\star}{\rightharpoonup} f(\tilde{u})$ and $\Upsilon_\nu \stackrel{\star}{\rightharpoonup} \Upsilon$ in $\mathbb{L}^\infty$, we get $f(\tilde{u})(t) = \Upsilon(t)$ for a.e. $t \in \mathcal{P}$.

5. An Application

Modelling traffic phenomena in first approximation it is reasonable to treat a flow of traffic on an highway as a continuum with an observable density $u(t,x)$ equal to the number of cars per unit length and a flux $f(t,x)$ equal to the number of cars crossing the point x per unit time. Making the assumption that at each point x the flux f is a function only of the density u at x, leads to the conservation law (see [8])

$$u_t + [uv(u)]_x = 0 \tag{5.1}$$

where $v(u)$ represents the velocity of the cars as a function of their density. In practice one often takes $v(u) = a_1 \ln(a_2/u)$ for suitable constants a_1 and a_2. Consider the problem of minimizing the mean time which occurs in driving through a stretch of the highway between an entry at a point $x = 0$ and an exit at a point $x = \bar{x}$ by controlling the density $\tilde{u}(t)$ of cars entering the highway at time t equal to the value of u at the boundary $x = 0$. Suppose that at time $t = 0$ no cars are on the stretch of highway $[0, \bar{x}]$. Let $g(t)$ be the number of cars arriving at $x = 0$ per unit time. We may assume that g is a continuous function with compact support. Let u_m be the maximum density i.e. the value for which the cars are bumper to bumper. Then there are quite natural assumptions that can be made on the boundary data $\tilde{u}$:

i) the net flux of cars entering the stretch of highway must be equal to the total number of cars arriving at the entry:

$$\int_0^{+\infty} \tilde{u}(s)v(\tilde{u}(s))\ ds = \int_0^{+\infty} g(s)\ ds; \tag{5.2}$$

ii) at any time $t > 0$ the total number of cars which have entered the highway untill that moment must be less or equal to the total number of cars arrived at the entry in the same period of time:

$$\int_0^{t} \tilde{u}(s)v(\tilde{u}(s))\ ds \le \int_0^{t} g(s)\ ds; \tag{5.3}$$

iii) the maximum number of cars entering the highway must be less or equal to the maximum density of cars allowed on the highway:

$$\tilde{u}(t) \in [0, u_m]; \tag{5.4}$$

iv) after a period of time sufficiently large no cars enter the highway:

$$\tilde{u}(t) = 0, \qquad t > \tau, \qquad \exists \tau > 0. \tag{5.5}$$

Then if $(t,x) \mapsto S_t\tilde{u}(x)$ denotes the solution to (5.1), (1.2), (1.3), we will be interested in minimizing the difference between the average incoming time of cars at $x = \bar{x}$ and at $x = 0$:

$$\left(\int_0^{+\infty} t\ S_t\tilde{u}(\bar{x})v\left(S_t\tilde{u}(\bar{x})\right)\ dt - \int_0^{+\infty} t\ g(t)\ dt\right)\left(\int_0^{+\infty} g(t)\ dt\right)^{-1}. \tag{5.6}$$

which clearly is equivalent to the minimization problem

$$\min_{\tilde{u}\in\mathcal{U}} \int_0^{+\infty} t\ S_t\tilde{u}(\bar{x})v\left(S_t\tilde{u}(\bar{x})\right)\ dt, \tag{5.7}$$

where the admissible set $\mathcal{U}$ consists of all $\mathbb{L}^\infty$ functions $\tilde{u}$ satisfying (5.2)-(5.5) for a.e. $t > 0$. Here we have a strictly concave flux $f(u) = uv(u)$. Since it is not restrictive to consider boundary data with characteristics entering the domain $\mathbb{R}^+ \times \mathbb{R}^+$, one can assume that $\tilde{u} \in [0, b(0)] \subseteq [0, u_m]$ for a.e. $t > 0$ and for any admissible boundary data $\tilde{u}$. Moreover by the basic structure of a solution to (1.1)-(1.3), from (5.5) it follows that $S_t\tilde{u}(\bar{x}) = 0$ for a.e. $t > \tau + \bar{x}\ b(0)/f(b(0)) \doteq \tau'$. Therefore problem (5.7) can be restated

$$\min_{\tilde{u}\in\mathcal{U}} \int_0^{\tau'} t\ S_t\tilde{u}(\bar{x})v\big(S_t\tilde{u}(\bar{x})\big)\ dt, \tag{5.8}$$

where $\mathcal{U}$ is a set of the form (2.23), q being the identity map and G the multifunction

$$G(t) = \begin{cases} [0, b(0)] & \text{if} \quad t \le \tau' \\ \{0\} & \text{otherwise,} \end{cases}$$

with an additional constraint given by (5.2). Observe that the compacteness of the attainable set $\mathcal{A}(\bar{x}, \mathcal{U})$ still holds in connection with such admissible set of boundary controls as it follows from the proof of Theorem 3. Thus, since the map $u \mapsto \int_0^{\tau'} t\ u(t)v(u(t))\ dt$ is continuous as a functional from $\{\, u \in \mathbb{L}^\infty([0,\tau']) :\ ||u||_\infty \le b(0)\,\}$ into $\mathbb{R}$ w.r.t. the $\mathbb{L}^1$ norm, by Corollary 1 problem (5.8) admits a solution.

6. Appendix

Here we extend the $\mathbb{L}^1$-contraction property (2.12) extabilished in [13] for piecewise continuously differentiable solutions of the mixed initial-boundary value problem (2.1)-(2.3), to the class of all solutions associated with every initial and boundary data in the domain

$$\mathcal{D} \doteq \big\{\, (\bar{u}, \tilde{u}) \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^\infty(\mathbb{R}^+) :\ \tilde{u}(t) \ge b(0) \quad \text{a.e.} \quad t\,\big\}.$$

In the following we denote $\mathcal{T}_t : \mathbb{L}^\infty \to \mathbb{L}^\infty$, $t > 0$, the translation operator i.e. $\mathcal{T}_t\tilde{u}(s) \doteq \tilde{u}(t + s)$, $\forall s > 0$.

Theorem 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable strictly convex function. Then there exists a continuous map $S : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{L}^\infty(\mathbb{R}^+)$ with the following properties:*

i) $S_0(\bar{u}, \tilde{u}) = \bar{u}, \qquad S_{s+t}(\bar{u}, \tilde{u}) = S_s\big(S_t(\bar{u}, \tilde{u}), \mathcal{T}_t\tilde{u}\big), \qquad \forall s, t > 0;$

ii) $\|S_t(\bar{u}, \tilde{u}) - S_t(\bar{v}, \tilde{v})\|_{\mathbb{L}^1(\mathbb{R}^+)} \le \|\bar{u} - \bar{v}\|_{\mathbb{L}^1(\mathbb{R}^+)} + \|f(\tilde{u}) - f(\tilde{v})\|_{\mathbb{L}^1([0,t])}, \qquad \forall t > 0;$

iii) each trajectory $t \to S_t(\bar{u}, \tilde{u})$ yields the unique solution (in the sense of Definition 1) to the initial-boundary value problem (2.1)-(2.3).

Proof. For any given $R > 0$ consider the set

$$\mathcal{D}_R \doteq \big\{\, (\bar{u}, \tilde{u}) \in \mathcal{D} :\ \|\tilde{u}\|_\infty \le R\big\}$$

endowed with the product topology of $\mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^1_{loc}(\mathbb{R}^+)$. Then to prove Theorem 4 it suffices to show that for any $R > 0$ there exists a continuous map $S : \mathbb{R}^+ \times \mathcal{D}_R \to \mathbb{L}^\infty(\mathbb{R}^+)$ satisfying *i), ii), iii)*.

Let $\widehat{\mathcal{D}}_R$ be the set of couples $(\bar{u}, \tilde{u}) \in \mathcal{D}_R$ of piecewise constant functions (with finite number of discontinuities). Observe first that any solution of (2.1)-(2.3) associated with initial and boundary data in $\widehat{\mathcal{D}}_R$ is piecewise continuously differentiable. Then for every $(\bar{u}, \tilde{u}) \in \widehat{\mathcal{D}}_R$ let $\widehat{S}_t(\bar{u}, \tilde{u})$ be the value at time t of the solution to (2.1)-(2.3) which, by Remark 2.2, is unique, admits a representation of the form (2.10) and satisfies the $\mathbb{L}^1$-contraction property *ii)*. Since $\widehat{\mathcal{D}}_R$ is a dense subset of $\mathcal{D}_R$ the continuous flow $\widehat{S} : \mathbb{R}^+ \times \widehat{\mathcal{D}}_R \to \mathcal{D}_R$ can be uniquely extended by continuity to a continuous map $S : \mathbb{R}^+ \times \mathcal{D}_R \to \mathcal{D}_R$ satisfying *ii)* as well. Thus the proof will be completed if we show that $t \to S_t(\bar{u}, \tilde{u})$ admits a representation of the form (2.10) for every $(\bar{u}, \tilde{u}) \in \mathcal{D}_R$.

Let $(\bar{u}_\nu)_{\nu\in\mathbb{N}}$, $(\tilde{u}_\nu)_{\nu\in\mathbb{N}}$, $(\bar{u}_\nu, \tilde{u}_\nu) \in \mathcal{D}_R$, be two sequences of piecewise constant functions such that

$$\begin{aligned} \bar{u}_\nu \to \bar{u} \qquad &\text{in} \quad \mathbb{L}^1(\mathbb{R}^+), \\ f(\tilde{u}_\nu) \to f(\tilde{u}) \quad &\text{in} \quad \mathbb{L}^1_{loc}(\mathbb{R}^+). \end{aligned}$$

Then by previous arguments, for every fixed $t > 0$, one has $S_t(\bar{u}_\nu, \tilde{u}_\nu)(x) = b\left((x - y_\nu(t,x))/t\right)$ for a.e. $x > 0$, $y_\nu(t,x)$ denoting the unique miminum point for the function $y \mapsto \Psi_{\Upsilon_\nu}(t,x,y)$ defined by (2.11) in connection with the trace Υ_ν at $x = 0$ of $f\big(S_{(\cdot)}(\bar{u}_\nu, \tilde{u}_\nu)\big)$. Observe that by (2.9) Υ_ν are uniformly bounded. Thus there exists a subsequence still denoted $(\Upsilon_\nu)_{\nu\in\mathbb{N}}$ which converges weak* in $\mathbb{L}^\infty$ to some function $\Upsilon \in \mathbb{L}^\infty(\mathbb{R}^+)$. Therefore for every $x > 0$ the sequence of maps $(\Psi_{\Upsilon_\nu}(t,x,\cdot))_{\nu\in\mathbb{N}}$ converges uniformly to $\Psi_\Upsilon(t,x,\cdot)$. This implies that for a.e. $x > 0$ the corresponding minimum points $y_\nu(t,x)$ being unique (see Remark 2.1) converge to the minimum point $y(t,x)$ of $\Psi_\Upsilon(t,x,\cdot)$ and hence $\big(b\left((x - y_\nu(t,x))/t\right)\big)_{\nu\in\mathbb{N}}$ converges to $b\left((x - y(t,x))/t\right)$ for a.e. $x > 0$ proving that $S_t(\bar{u}, \tilde{u})$ satisfies (2.10). $\square$

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