Scalar Non-linear Conservation Laws with Integrable Boundary Data

Fabio Ancona

Dipartimento di Matematica and C.I.R.A.M.
Università di Bologna
Piazza P.S. Donato n. 5, Bologna 40127, Italy
ancona@ciram3.ing.unibo.it

Andrea Marson

Dipartimento di Elettronica per l'Automazione - Sez. Matematica
Università di Brescia
Via Branze n. 38, Brescia 25123, Italy
marson@bsing.ing.unibs.it

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Abstract. We consider the initial-boundary value problem for a scalar non-linear conservation law

$$u_t + [f(u)]_x = 0, \qquad u(0,x) = \bar{u}(x), \qquad u(\cdot,0) = \tilde{u}(t), \tag{$*$}$$

on the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t \geq 0, \ x \geq 0\}$. Here $u = u(t,x)$ is the state variable, $\bar{u}$, $\tilde{u}$ are integrable (possibly unbounded) initial and boundary data, and f is assumed to be strictly convex and superlinear. We first derive an explicit formula for a solution of $(*)$ which generalizes a result of Le Floch in the case where $\bar{u} \in \mathbb{L}^1$, $f(\tilde{u}) \in \mathbb{L}^1_{loc}$. Moreover, we extend a contractive property and estabilish a comparison principle for integrals of solutions to $(*)$.

Next we study problem $(*)$ taking $\bar{u} \equiv 0$ and letting $\tilde{u}$ vary into a given set $\mathcal{U} \subseteq \mathbb{L}^1_{loc}$ of boundary data regarded as admissible controls. In particular, we give a characterization of the set of attainable profiles at a fixed time $T > 0$

$$\mathcal{A}(T,\mathcal{U}) = \big\{u(T,\cdot) : \ u \text{ is a solution of } (*) \text{ with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U}\big\},$$

and at a fixed point $\bar{x} > 0$

$$\mathcal{A}(\bar{x},\mathcal{U}) = \big\{u(\cdot,\bar{x}) : \ u \text{ is a solution of } (*) \text{ with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U}\big\},$$

when

$$\mathcal{U} = \big\{\tilde{u} \in \mathbb{L}^1_{loc}(\mathbb{R}^+) : \ f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+), \ f'(\tilde{u}) \geq 0\big\}.$$

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1. Introduction

The paper deals with the initial-boundary value problem for a scalar non-linear conservation law in one space dimension

$$u_t + [f(u)]_x = 0, \tag{1.1}$$

$$u(0,x) = \bar{u}(x), \qquad t, x \geq 0, \tag{1.2}$$

$$u(t,0) = \tilde{u}(t), \tag{1.3}$$

where $u = u(t,x)$ is the state variable, $\bar{u}$, $\tilde{u}$ are integrable (possibly unbounded) initial and boundary data, and f is assumed to be a superlinear strictly convex function. For problems of this type, since classical solutions develop discontinuities in finite time, no matter how smooth their initial and boundary data, it is natural to consider weak solutions satisfying the usual *entropy conditions* ([13], [15])

$$u(t,x-) \geq u(t,x+), \qquad t,x > 0. \tag{1.4}$$

Moreover, as it is well known, in general the Dirichlet condition (1.3) may not be fulfilled pointwise a.e., thus following [16] we shall require that an entropy solution u to (1.1)-(1.3) satisfies the above condition in a weaker sense which is motivated by the classical vanishing viscosity method (see [4], [16] and Definition 1 here).

In [16] Le Floch derived an explicit formula for a solution of (1.1)-(1.3) when $\bar{u}$, $\tilde{u} \in \mathbb{L}^\infty$. Here we generalize this result in the case where $\bar{u} \in \mathbb{L}^1, f(\tilde{u}) \in \mathbb{L}^1{}_{loc}$. Moreover, we show that such a solution coincides with the trajectory of a semigroup-type map $(t, \bar{u}, \tilde{u}) \to S_t(\bar{u}, \tilde{u})$ which is Lipschitz continuous with respect to $\bar{u}, f(\tilde{u})$. We also derive a comparison principle for integrals of solutions to (1.1)-(1.3) as a consequence of the fact that if $u = u(t,x)$ denotes the weak-entropy solution to (1.1)-(1.3), then $\mathcal{U} = \mathcal{U}(t,x) = -\int_x^{+\infty} u(t,\xi)\, d\xi$ turns out to be the viscosity solution to a mixed problem for the Hamilton-Jacobi equation $\mathcal{U}_t + f(\mathcal{U}_x) = f(0)$ corresponding to (1.1).

Next, following [1] we study problem (1.1)-(1.3) taking $\bar{u} \equiv 0$ and letting $\tilde{u}$ vary into a given set $\mathcal{U} \subset \mathbb{L}^1{}_{loc}$ of integrable boundary data regarded as admissible controls. We consider the set of attainable profiles at a fixed time T

$$\mathcal{A}\,(T,\mathcal{U}) = \Big\{ u(T,\cdot) :\ u \text{ is a solution to (1.1)-(1.3) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U} \Big\},$$

and at a fixed point in space $\bar{x} > 0$

$$\mathcal{A}\,(\bar{x},\mathcal{U}) = \Big\{ u(\cdot,\bar{x}) :\ u \text{ is a solution to (1.1)-(1.3) with } \bar{u} \equiv 0 \text{ and } \tilde{u} \in \mathcal{U} \Big\}.$$

By using similar techniques to the ones in [1] we derive a precise characterization of $\mathcal{A}\,(T,\mathcal{U})$, $\mathcal{A}\,(\bar{x},\mathcal{U})$ when

$$\mathcal{U} = \Big\{ \tilde{u} \in \mathbb{L}^1_{loc}(\mathbb{R}^+) :\ f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+),\ f'(\tilde{u}) \geq 0 \Big\}.$$

Moreover, we estabilish the compacteness of the attainable sets $\mathcal{A}(T,\mathcal{U}) \subseteq \mathbb{L}^1$, $\mathcal{A}(\bar{x},\mathcal{U}) \subseteq \mathbb{L}^1_{loc}$, in connection with classes of boundary controls which are measurable selections of a uniformly integrable multifunction with closed convex values, and satisfy certain integral inequalities. In the proof of such results a key role is played by the weak compactness in $\mathbb{L}^1$ of the set of fluxes $\{f(\tilde{u}): \ u \in \mathcal{U}\}$ of admissible boundary controls.

Finally we apply the comparison principle estabilished in the first part of the paper to construct the optimal boundary control for a optimization problem of traffic flow where one is interested in minimizing the average time spent by cars travelling through a given stretch of highway and the controller acts by varying the density of cars entering the highway.

2. Preliminaries and Statements of Main Results

2.1 Formulation of the Problem

On the domain $\Omega = \{(t,x) \in \mathbb{R}^2 : \ t \geq 0, \ x \geq 0\}$ consider the mixed initial-boundary value hyperbolic problem (1.1)-(1.3) where $f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$, $\bar{u} \in \mathbb{L}^1(\mathbb{R}^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is a twice continuously differentiable stricly convex function satisfying $\lim_{u\to\infty} f(u)/|u| = +\infty$. Denote $b = (f')^{-1}$. Throughout the paper we shall use f^{-1} to denote the inverse map of the restriction of f to the interval $[b(0), +\infty)$.

As observed in the introduction, we shall only consider weak entropy solutions of (1.1)-(1.2) which satisfy the boundary condition (1.3) in a weaker sense which is precised in Definition 1 below. Notice that, as remarked in [16], any solution of (1.1)-(1.3) with boundary data $\tilde{u}$ such that $f'(\tilde{u}(t)) < 0$ on a subset I of $\mathbb{R}^+$ of positive measure, can be obtained with the boundary data

$$\tilde{u}'(t) = \begin{cases} b(0) & \text{if} \quad t \in I \\ \tilde{u}(t) & \text{otherwise.} \end{cases}$$

Hence it is not restrictive to assume that the characteristics at the boundary are always entering the domain, i.e. $f'(\tilde{u}(t)) \geq 0$ for a.e. t: this hypothesis will be adopted in the rest of the paper. We recall here the definition of solution to (1.1)-(1.3) as stated in [16].

Definition 1. A continuous map $u : \mathbb{R}^+ \to \mathbb{L}^1(\mathbb{R}^+)$ is a solution of (1.1)-(1.3) if

i) it is a weak entropy solution of (1.1) in the interior of Ω, i.e. for any nonnegative function $\phi \in \mathcal{C}^1_c(\mathbb{R}^+ \times \mathbb{R}^+)$ and any $k \in \mathbb{R}$,

$$\iint_{\mathbb{R}^+ \times \mathbb{R}^+} \Big[|u-k| \ \phi_t + \operatorname{sgn}(u-k) \ (f(u) - f(k)) \ \phi_x \Big] \ dx \ dt \geq 0.$$

ii)

$$\lim_{t\to 0^+} \int_0^x u(t,\xi) \ d\xi = \int_0^x \bar{u}(\xi) \ d\xi, \qquad x \geq 0; \tag{2.1}$$

iii) the boundary condition is satisfied in the following weak sense: there exist a set $\mathcal{F} \subset \mathbb{R}^+$ with zero measure and two functions $\Upsilon : \mathbb{R}^+ \to \mathbb{R}$ and $\mu : \mathbb{R}^+ \to \{-1, 0, 1\}$ such that

$$\lim_{\substack{x\to 0+\\ x\notin\mathcal{F}}} \int_0^t f(u(s,x))\ ds = \int_0^t \Upsilon(s)\ ds, \qquad t \geq 0, \tag{2.2}$$

$$\lim_{\substack{x\to 0+\\ x\notin\mathcal{F}}} \operatorname{sgn}\ f'(u(t,x)) = \mu(t), \qquad \text{a.e. } t \geq 0 \tag{2.3}$$

and

$$\begin{cases} \Upsilon(t) = f(\tilde{u}(t)) & \text{if } \mu(t) \geq 0 \\ \Upsilon(t) \geq f(\tilde{u}(t)) & \text{if } \mu(t) = -1 \end{cases} \qquad \text{a.e. } t > 0. \tag{2.4}$$

Remark 2.1. Notice that a continuous map $u : \mathbb{R}^+ \to \mathbb{L}^1(\mathbb{R}^+)$ is a solution of (1.1)-(1.3) in the sense of Definition 1 if and only if it is a weak entropy solution obtained by the vanishing viscosity method as in the formulation given by Bardos, Leroux and Nedelec ([4]), i.e.

$$\iint\limits_{\mathbb{R}^+\times\mathbb{R}^+} \Big[|u-k|\ \phi_t + \operatorname{sgn}(u-k)\ (f(u)-f(k))\ \phi_x\Big]\ dx\ dt$$
$$+ \int_{\mathbb{R}^+} \operatorname{sgn}(\tilde{u}-k)\ (f(u(t,0))-f(k))\ \phi(t,0)\ dt + \int_{\mathbb{R}^+} |\bar{u}-k|\ \phi(0,x)\ dx \geq 0,$$

for any nonnegative function $\phi \in \mathcal{C}^1_c(\mathbb{R}^+\times\mathbb{R}^+)$ and any $k \in \mathbb{R}$.

2.2 Explicit Representation of Solutions and Well-Posedness of the Problem

In [16] Le Floch gives an explicit representation formula of the solution to (1.1)-(1.3) in the case where $\bar{u},\ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+)$. Here we extend this result as follows

Theorem 1. *Let $\bar{u} \in \mathbb{L}^1$, $f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$. Then the problem (1.1)-(1.3) admits a solution $u = u(t,x)$ which can be computed with the following procedure.*
For any positive number t, let $y = y(t)$ be a point which minimizes the function

$$[0,+\infty) \ni y \mapsto \int_0^y \bar{u}(\xi)\ d\xi + tg\left(-\frac{y}{t}\right) \doteq \Psi_{\bar{u}}(t,y). \tag{2.5}$$

Let $m = m(t)$ be the unique positive absolutely continuous function such that $m(0+) = 0$ and

$$\left\{\frac{d}{dt}m(t) - f(\tilde{u}(t)) + f\left(b\left(-\frac{y(t)}{t}\right)\right)\right\}\{\varphi(t) - m(t)\} \geq 0, \quad \text{a.e. } t > 0, \tag{2.6}$$

for any nonnegative $\varphi \in \mathbb{L}^\infty$. Set

$$\Upsilon(t) = \frac{d}{dt}m(t) + f\left(b\left(-\frac{y(t)}{t}\right)\right), \qquad \text{a.e. } t > 0. \tag{2.7}$$

Then

$$u(t,x) = b\left(\frac{x - y(t,x)}{t}\right) \qquad t > 0, \quad x > 0, \tag{2.8}$$

where $y(t,x)$ denotes a point of minimum value for the function

$$y \mapsto \Psi_\Upsilon(t,x,y) = \begin{cases} \displaystyle\int_0^y \bar{u}(s)\ ds + t\ g\left(\frac{x-y}{t}\right) & \text{if } y \geq 0 \\ \displaystyle -\int_0^\tau \Upsilon(s)\ ds + (t-\tau)\ g\left(\frac{x}{t-\tau}\right) & \text{if } y \leq 0 \end{cases} \tag{2.9}$$

with g denoting the Legendre transform of f and τ satisfying

$$\frac{x-y}{t} = \frac{x}{t-\tau}, \qquad y \leq 0.$$

Moreover Υ represents the trace of $f(u(t,x))$ at $x = 0$ in the sense of (2.2), and the following bounds are satisfied

$$f\left(\tilde{u}(t)\right) \leq \Upsilon(t) \leq \max\left\{ f\left(\tilde{u}(t)\right), f\left(b\left(-\frac{y(t)}{t}\right)\right)\right\} \qquad \text{a.e. } t. \tag{2.10}$$

Remark 2.2. Notice that the function $(t,x) \mapsto y(t,x)$ is locally bounded, and hence, if (2.8) holds, $u \in \mathbb{L}^\infty_{loc}(\mathbb{R}^{>0} \times \mathbb{R}^{>0})$.

Remark 2.3. The assumption that the flux $f(\tilde{u})$ of the boundary data is in $\mathbb{L}^1_{loc}$ cannot be relaxed in order to obtain a solution in $\mathbb{L}^1$ to the mixed problem (1.1)-(1.3), as it is clear from the following

Example. Consider the mixed problem (1.1)-(1.3) for the Burger's equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

with initial condition $\bar{u} \equiv 0$ and boundary data

$$\tilde{u}(t) = \begin{cases} \dfrac{1}{\sqrt{1-t}} & \text{if} \quad 0 < t < 1 \\ 0 & \text{if} \quad t \geq 1. \end{cases} \tag{2.11}$$

If this problem admits a solution $u = u(t,x)$, then such a solution must be computed by considering the increasing sequence of boundary data $(\tilde{u}_\nu)_{\nu\in\mathbb{N}}$,

$$\tilde{u}_\nu(x) = \begin{cases} \dfrac{1}{\sqrt{1-t}} & \text{if} \quad 0 < t < 1 - \dfrac{1}{\nu} \\ \dfrac{1}{\sqrt{1/\nu}} & \text{if} \quad 1 - \dfrac{1}{\nu} \leq t < 1 \\ 0 & \text{if} \quad t \geq 1, \end{cases} \tag{2.12}$$

and setting $u = \sup_\nu u_\nu$, where $u_\nu = u_\nu(t,x)$ is the solution to (1.1)-(1.3) with initial condition $\bar{u} \equiv 0$ and boundary data $\tilde{u}_\nu$. We claim that

$$u(t,x) = \begin{cases} \dfrac{2}{x + \sqrt{x^2 - 4t + 4}} & \text{if} \quad t < 1,\ 0 < x < \xi(t) \\ 0 & \text{if} \quad t < 1,\ x > \xi(t) \\ \dfrac{x}{t-1} & \text{if} \quad t > 1, \end{cases} \tag{2.13}$$

where $t \mapsto \xi(t)$ is a curve of discontinuity defined in the all interval $[0,1]$ and having the expansion $\xi(t) = (1/2)\ t + (1/16)\ t^2 + (1/48)\ t^3 + (29/3072)\ t^4 + (307/61440)\ t^5 + O(t^6)$.
Indeed, since $\tilde{u}_\nu \in \mathbb{L}^\infty$ we can use the explicit representation formula in [16] to compute the corresponding solution u_ν. We obtain

$$u_\nu(t,x) = \begin{cases} \dfrac{2}{x + \sqrt{x^2 - 4t + 4}} & \text{if} \quad t \leq 1 + \dfrac{x}{\sqrt{\nu}},\ 0 < x < \xi_\nu(t) \\ \dfrac{x}{t-1} & \text{if} \quad 1 + \dfrac{x}{\sqrt{\nu}} < t,\ 0 < x < \xi_\nu(t) \\ 0 & \text{if} \quad x > \xi_\nu(t), \end{cases} \tag{2.14}$$

where $t \mapsto \xi_\nu(t)$ is a curve of discontinuity defined in the all interval $[0,+\infty[$ and such that $\xi_\nu(t) = \xi(t)$ whenever $t < 1 - 1/\nu$. Moreover

$$\lim_{\nu\to+\infty} \xi_\nu(t) = +\infty \qquad \forall\ t > 1.$$

Indeed let t_ν be such that

$$\xi_\nu(t_\nu) = \sqrt{\nu}\,(t_\nu - 1)\,.$$

Then

$$\dot{\xi}_\nu(t) = \frac{\xi(t)}{2(t-1)}, \qquad \forall\ t > t_\nu.$$

Hence

$$\xi_\nu(t) = \xi_\nu(t_\nu)\sqrt{\frac{t-1}{t_\nu - 1}},$$

from which the claim follows being $\xi_\nu(t_\nu) \geq \xi(1) > 0$ and $\lim_{\nu\to+\infty} t_\nu = 1$.
Therefore, for any fixed $x, t > 0$, for ν sufficiently large $u_\nu(t,x) = u(t,x)$.
Observe that for $t < 1$ the function u defined in (2.13) is a weak entropy solution of problem (1.1)-(1.3) with initial condition $\bar{u} \equiv 0$ and boundary data $\tilde{u}$ defined in (2.11). On the other hand u is not locally integrable on $[1,+\infty) \times \mathbb{R}^{>0}$, hence it isn't a weak solution of the Burger's equation on the all space $\mathbb{R}^{>0} \times \mathbb{R}^{>0}$.

Regarding well-posedness of the mixed problem (1.1)-(1.3) we extend the $\mathbb{L}^1$-contraction property established in [1], Theorem 4 as follows. Set

$$\mathcal{D} \doteq \left\{(\bar{u},\tilde{u}) :\ \bar{u} \in \mathbb{L}^1(\mathbb{R}^+), f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+),\ \tilde{u}(t) \geq b(0) \text{ a.e. } t\right\} \tag{2.15}$$

and denote $\mathcal{T}_t : \mathbb{L}^1_{loc} \to \mathbb{L}^1_{loc}$, $t > 0$, the translation operator, i.e. $\mathcal{T}_t\tilde{u}(s) \doteq \tilde{u}(t+s)$, $\forall s > 0$.

Theorem 2. *There exists a map $S : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{L}^1(\mathbb{R}^+)$ with the following properties:*

i) $S_0(\bar{u}, \tilde{u}) = \bar{u}, \qquad S_{s+t}(\bar{u}, \tilde{u}) = S_s\big(S_t(\bar{u}, \tilde{u}), \mathcal{T}_t \tilde{u}\big), \qquad \forall s, t > 0;$

ii) $\|S_t(\bar{u}, \tilde{u}) - S_t(\bar{v}, \tilde{v})\|_{\mathbb{L}^1(\mathbb{R}^+)} \leq \|\bar{u} - \bar{v}\|_{\mathbb{L}^1(\mathbb{R}^+)} + \|f(\tilde{u}) - f(\tilde{v})\|_{\mathbb{L}^1([0,t])}, \qquad \forall t > 0;$

iii) each trajectory $t \to S_t(\bar{u}, \tilde{u})$ yields a solution (in the sense of Definition 1) to the initial-boundary value problem (1.1)-(1.3) which admits the explicit representation (2.8) given by Theorem 1.

Remark 2.4. The above properties indicate that the solution to (1.1)-(1.3) given by the explicit formula (2.8) should be regarded as the unique "good" weak entropy solution of the corresponding mixed problem. Moreover, by similar arguments to the ones in [16], such a solution has right and left limits in t and x at every point in the interior of Ω.

2.3 A Comparison Theorem

Let $(t, \bar{u}, \tilde{u}) \to S_t(\bar{u}, \tilde{u})$ be the map of Theorem 2 yielding the solution to the initial-boundary value problem (1.1)-(1.3). In [18] it is extended a pointwise comparison principle for the Cauchy problem ([10]) to the case of initial-boundary value problem:

$$\bar{u} \leq \bar{v}, \quad \tilde{u} \leq \tilde{v} \quad \text{a.e.} \quad t > 0 \qquad \Longrightarrow \qquad S_t(\bar{u}, \tilde{u}) \leq S_t(\bar{v}, \tilde{v}) \quad \text{a.e} \quad t, x > 0.$$

Here we shall estabilish a comparison property envolving the integral of the solution to (1.1)-(1.3) rather than the solution itself.

Theorem 3. *Let $\bar{u}, \bar{v} \in \mathbb{L}^1(\mathbb{R}^+)$, $f(\tilde{u}), f(\tilde{v}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$, with $\tilde{u}, \tilde{v} \geq b(0)$. Denote by $\Upsilon(\bar{u}, \tilde{u})$, $\Upsilon(\bar{v}, \tilde{v})$, respectively the trace of $f(S_t(\bar{u}, \tilde{u}))$, $f(S_t(\bar{v}, \tilde{v}))$ at $x = 0$. Then the following properties hold.*

i) If

$$\int_x^{+\infty} \bar{u}(\xi)\, d\xi \leq \int_x^{+\infty} \bar{v}(\xi)\, d\xi \qquad \forall\, x \geq 0 \tag{2.16}$$

$$\int_0^t \Upsilon(\bar{u}, \tilde{u})(s)\, ds \leq \int_0^t \Upsilon(\bar{v}, \tilde{v})(s)\, ds \qquad \forall\, t \geq 0, \tag{2.17}$$

then

$$\int_x^{+\infty} S_t(\bar{u}, \tilde{u})(\xi)\, d\xi \leq \int_x^{+\infty} S_t(\bar{v}, \tilde{v})(\xi)\, d\xi \qquad \forall\, t, x \geq 0. \tag{2.18}$$

ii) If $\big(f(\tilde{u}) - c\big)$, $\big(f(\tilde{v}) - c\big) \in \mathbb{L}^1(\mathbb{R}^+)$ for some constant c and

$$\int_0^x \bar{u}(\xi)\, d\xi \leq \int_0^x \bar{v}(\xi)\, d\xi \qquad \forall\, x \geq 0 \tag{2.19}$$

$$\int_t^{+\infty} \big(\Upsilon(\bar{u}, \tilde{u})(s) - c'\big)\, ds \leq \int_t^{+\infty} \big(\Upsilon(\bar{v}, \tilde{v})(s) - c'\big)\, ds \qquad \forall\, t \geq 0, \tag{2.20}$$

with

$$c' = \begin{cases} \max\{c, f(0)\} & \textit{if} \quad f^{-1}(c) > 0, \\ c & \textit{otherwise}, \end{cases}$$

then

$$\int_t^{+\infty} \big(f\left(S_s(\bar{u}, \tilde{u})(x)\right) - c'\big)\ ds \leq \int_t^{+\infty} \big(f\left(S_s(\bar{v}, \tilde{v})(x)\right) - c'\big)\ ds \qquad \forall\ t, x \geq 0. \tag{2.21}$$

Remark 2.5. Notice that if $\bar{u}, \tilde{u}$ are piecewise constant and $\big(f(\tilde{u}) - c\big) \in \mathbb{L}^1(\mathbb{R}^+)$, then

$$\lim_{t \to +\infty} \int_t^{+\infty} \left|S_t(\bar{u}, \tilde{u})(\xi) - f^{-1}(c')\right|\ d\xi = 0 \qquad \forall\ x > 0,$$

and $\big(\Upsilon(\bar{u}, \tilde{u}) - c'\big) \in \mathbb{L}^1(\mathbb{R}^+)$, with c' as above. It follows that using arguments similar to the ones in the proof of Theorem 2, we can extend by density this formula to the case where $\bar{u}, f(\tilde{u}) - c \in \mathbb{L}^1(\mathbb{R}^+)$, and hence from Lemma 3.2 in [9] we derive the conservation equality

$$\int_t^{+\infty} \big(f\left(S_s(\bar{u}, \tilde{u})(x)\right) - c\big)\ ds = \int_0^x \big(S_t(\bar{u}, \tilde{u})(\xi) - f^{-1}(c)\big)\ d\xi + \int_t^{+\infty} \big(\Upsilon(\bar{u}, \tilde{u})(s) - c\big)\ ds.$$

Remark 2.6. For solutions to the mixed problem with the same initial data we obtain the following comparison principle:

$$\begin{aligned} &\int_0^t \Upsilon(\bar{u}, \tilde{u})(s)\ ds \leq \int_0^t \Upsilon(\bar{u}, \tilde{v})(s)\ ds \qquad \forall\ t \geq 0 \\ &\qquad\qquad \Longrightarrow \quad \int_0^t f(S_s(\bar{u}, \tilde{u})(x))\ ds \leq \int_0^t f(S_s(\bar{u}, \tilde{v})(x))\ ds \qquad \forall\ t, x \geq 0. \end{aligned} \tag{2.22}$$

Indeed, it can be easily verified that if $\bar{u}, \tilde{u}$ are piecewise constant we have

$$\lim_{x \to +\infty} \int_0^t \left[f\left(S_s(\bar{u}, \tilde{u})\right)(x) - f(0)\right]\ ds = 0 \qquad \forall\ t > 0.$$

Then using arguments similar to the ones in the proof of Theorem 2, we can extend by density this formula to the case where $\bar{u} \in \mathbb{L}^1(\mathbb{R}^+)$, $f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$, and hence from Lemma 3.2 in [9] we derive the conservation equality

$$\int_0^t f(S_s(\bar{u}, \tilde{u})(x))\ ds = \int_x^{+\infty} S_t(\bar{u}, \tilde{u})(\xi)\ d\xi - \int_x^{+\infty} \bar{u}(\xi)\ d\xi + f(0)t \qquad \forall\ t, x \geq 0, \quad \forall\ (\bar{u}, \tilde{u}) \in \mathcal{D}. \tag{2.23}$$

Therefore (2.22) follows from property *i)* in the above Theorem.

2.4 Properties of the Attainable Set for Scalar Conservation Laws with Integrable Boundary Control

Following [1] we turn now to study the mixed initial-boundary value problem

$$u_t + [f(u)]_x = 0, \tag{2.24}$$

$$u(0,x) = 0, \qquad t, x \geq 0, \tag{2.25}$$

$$u(t,0) = \tilde{u}(t), \tag{2.26}$$

from the point of view of control theory regarding the boundary data $\tilde{u}$ as a control. Here we extend the results given in [1] to the case where the admissible boundary controls are assumed to be integrable (possibly unbounded) functions. In this framework we will adopt the semigroup notation $S_t\tilde{u} \doteq S_t(0,\tilde{u})$ for the unique solution of (2.24)-(2.26) at time t. We shall be concerned with basic properties of the attainable sets for (2.24)-(2.26)

$$\mathcal{A}\,(T, \mathcal{U}) \doteq \{S_T\tilde{u} : \ \tilde{u} \in \mathcal{U}\}\,, \tag{2.27}$$

$$\mathcal{A}\,(\bar{x}, \mathcal{U}) \doteq \left\{S_{(\cdot)}\tilde{u}(\bar{x}) : \ \tilde{u} \in \mathcal{U}\right\}, \tag{2.28}$$

which consist of all profiles that can be attained at a fixed time $T > 0$ and at a fixed point $\bar{x} > 0$ by solutions of (1.1)-(1.2) with boundary data that varies inside a given class $\mathcal{U} \subseteq \mathbb{L}^1_{loc}$ of admissible boundary controls. In particular we give a characterization of

$$\mathcal{A}(T) \doteq \left\{S_T\tilde{u} : \ f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+), \ \tilde{u} \geq b(0)\right\}, \tag{2.29}$$

$$\mathcal{A}(\bar{x}) \doteq \left\{S_{(\cdot)}\tilde{u}(\bar{x}) : \ f(\tilde{u}) \in \mathbb{L}^1_{loc}(\mathbb{R}^+), \ \tilde{u} \geq b(0)\right\}, \tag{2.30}$$

and we establish the compactness of (2.27),(2.28) in connection with a special class of admissible boundary controls.
Throughout the following

$$D^- w(x) = \liminf_{h\to 0} \frac{w(x+h) - w(x)}{h}, \qquad D^+ w(x) = \limsup_{h\to 0} \frac{w(x+h) - w(x)}{h},$$

will denote respectively the lower and upper Dini derivatives of a function w at x.

Theorem 4. *In connection with problem (2.24)-(2.25), for any fixed $T > 0$, $\mathcal{A}(T)$ is the set of all integrable functions w which have right and left limits at each point and satisfy the following conditions*

$$w(x) \neq 0 \Longrightarrow f'(w(x)) \geq \frac{x}{T}, \tag{2.31}$$

$$w(x-) \neq 0 \quad \text{and} \quad w(y) = 0 \ \forall\, y > x \Longrightarrow f'(w(x-)) > \frac{x}{T}, \tag{2.32}$$

$$D^+ w(x) \leq \frac{f'(w(x))}{x f''(w(x))}, \tag{2.33}$$

for every $x > 0$.

Remark 2.7. By definition an element $\tilde{w} \in \mathcal{A}(T) \subseteq \mathbb{L}^1(\mathbb{R}^+)$ is an equivalence class of integrable functions. Hence the above characterization must be interpreted in the sense that $\tilde{w} \in \mathcal{A}(T)$ iff there exists a representative w in the class $\tilde{w}$ admitting right and left limits in any point and satisfying (2.31)-(2.33). In §4 we will prove that if a function $w \in \mathcal{L}^1(\mathbb{R}^+)$ has right and left limits in any point and satisfies (2.31) and (2.33) then it is uniformly bounded on subsets bounded away from the origin and there exists $\alpha > 0$ such that $w(x) = 0$ for all $x \geq \alpha$. Thus w has finite total increasing variation (and hence finite total variation as well) on such subsets.

Theorem 5. *In connection with problem (2.24)-(2.25), for any fixed $\bar{x} > 0$, $\mathcal{A}(\bar{x})$ is the set of all locally bounded functions ρ which satisfy the following conditions*

$$\rho(t) \neq 0 \Longrightarrow f'(\rho(t)) \geq \frac{\bar{x}}{t}, \tag{2.34}$$

$$\rho(\tau+) \neq 0 \quad \text{and} \quad \rho(t) = 0 \;\forall\, t < \tau \Longrightarrow f'(\rho(\tau+)) > \frac{\bar{x}}{\tau}, \tag{2.35}$$

$$D^-\rho(t) \geq \frac{f'(\rho(t))}{t f''(\rho(t))}, \tag{2.36}$$

for every $t > 0$.

Remark 2.8. The above characterization must be interpreted as for Theorem 3 in the sense that $\tilde{\rho} \in \mathcal{A}(\bar{x}) \subseteq \mathbb{L}^\infty_{loc}$ iff there exists a representative ρ in the class of $\tilde{\rho}$ satisfying (2.34)-(2.36). In particular (2.36) together with the local boundedness of ρ imply that such a map has finite total variation on compact subsets of $\mathbb{R}^+$ bounded away from the origin and hence (2.35) makes sense. Moreover (2.34) and the local boundedness of ρ imply that there exists $\beta > 0$ such that $\rho(t) = 0$ for any $0 \leq t \leq \beta$.

As in [1] in order to achieve the closure of the attainable sets for (2.24)-(2.25) we need to restrict the class of admissible boundary controls by means of a suitable multifunction G.

Theorem 6. *Let $G : \mathbb{R}^+ \hookrightarrow [b(0), +\infty)$ be a measurable multifunction with convex closed values satisfying*

$$|f(\tilde{u}(t))| \leq \eta(t) \qquad \text{a.e. } t > 0, \quad \forall\ \tilde{u} \text{ measurable selection of } G \tag{2.37}$$

for some $\eta \in \mathbb{L}^1_{loc}(\mathbb{R}+)$. Let $q_i : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, N$, be measurable maps convex w.r.t. the second variable, $g_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, \ldots, N$, measurable maps and let J be a possibly empty subset of $\mathbb{R}^+$. Denote

$$\begin{aligned} \mathcal{U} = \Big\{ \tilde{u} \in \mathbb{L}^\infty(\mathbb{R}^+) : \tilde{u}(t) \in G(t), \quad &\text{for a.e. } t, \\ &\int_0^t q_i\big(s, f(\tilde{u}(s))\big)\, ds \leq g_i(t) \quad \forall\ t \in J,\ \forall\ i = 1, \ldots, N \Big\}. \end{aligned} \tag{2.38}$$

Then $\mathcal{A}(T, \mathcal{U})$, $T > 0$, and $\mathcal{A}(\bar{x}, \mathcal{U})$, $\bar{x} > 0$, are compact subsets of $\mathbb{L}^1(\mathbb{R}^+)$ and $\mathbb{L}^1_{loc}(\mathbb{R}^+)$ respectively.

Remark 2.9. The convexity assumption on the multifunction G and on the functions q_i cannot be relaxed in order to ensure the closure of the attainable set, as shown in the Remarks 2.5 and 2.6 in [1].

3. The Explicit Formula, the Contraction Property and the Comparison Principle

3.1 Proof of Theorem 1

In this paragraph we extend the explicit formula stated in [16] for the problem (1.1)-(1.3) to the case $\bar{u}$, $f(\tilde{u}) \in \mathbb{L}^1$. To this aim we first observe that the classical Lax formula for the Cauchy problem ([14]) can be easily extended to $\mathbb{L}^1$ initial data. Furthermore the following result holds:

Proposition 1. *Let $(t,x) \mapsto u(t,x)$ be the solution of (1.1)-(1.2) with $\bar{u} \in \mathbb{L}^1$. Then $f(u(\cdot, x\pm)) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$ for any $x > 0$.*

Proof. We will prove that $f(u(\cdot, x+)) \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$, the other case being entirely similar. Hence we shall denote $u(t,x) = u(t,x+)$. Since u is obtained by the Lax formula, it follows that $u(\cdot, x) \in \mathbb{L}^\infty_{loc}((0,+\infty))$ for any $x > 0$. Thus it remains to show that $f(u(\cdot,x)) \in \mathbb{L}^1([0,T])$, for some $T > 0$. Indeed, being f superlinear $f(u(\cdot,x))^- = \max\{-f(u(\cdot,x)), 0\}$ is bounded. Thus it suffices to show that $f(u(\cdot,x))^+ = \max\{f(u(\cdot,x)), 0\} \in \mathbb{L}^1([0,T])$. Observe that for any $s > 0$

$$\int_s^T f(u(t,x))\ dt = \int_s^T f(u(t,x))^+\ dt - \int_s^T f(u(t,x))^-\ dt$$

so that $f(u(\cdot,x))^+ \in \mathbb{L}^1([0,T])$ iff

$$\lim_{s \to 0^+} \int_s^T f(u(t,x))\ dt$$

exists and is finite.

Choose $T > 0$ in order that $u(T,x) \neq b(0)$: if such T does not exist, then the conclusion follows easily. By considering the maximal backward characteristic $t \mapsto \theta(t) = x + (t-T) f'(u(T,x+))$ departing from (T,x) and using Lemma 3.2 in [9], for any $s > 0$ we get

$$\int_s^T f(u(t,x+))\ dt - (T-s)\left[f(u(T,x+)) - f'(u(T,x+))\, u(T,x+)\right] = -\int_x^{\theta(s)} u(s,\xi)\ d\xi. \tag{3.1}$$

Hence

$$\lim_{s \to 0^+} \int_s^T f(u(t,x+))\ dt = -T\ g\left[f'(u(T,x+))\right] - \int_x^{\theta(0)} \bar{u}(\xi)\ d\xi \tag{3.2}$$

which concludes the proof. □

The proof of Theorem 1 follows from the next Proposition by using the same arguments as in §2.4.2 in [16].

Proposition 2. *For any positive number t, let $y = y(t)$ be a point which minimizes the function $\Psi_{\bar{u}}(t, \cdot)$ defined at (2.5). Then there exists a unique positive absolutely continuous function $m = m(t)$, $t > 0$ such that $m(0+) = 0$ and (2.6) holds. Moreover the function $\Upsilon = \Upsilon(t)$ defined by (2.7) belongs to $\mathbb{L}^1_{loc}([0, +\infty))$ and satisfies (2.10).*

Proof. We first show that the function $t \mapsto f(b(-y(t)/t))$ belongs to $\mathbb{L}^1_{loc}([0, +\infty))$. Fix $T > 0$ and set

$$\bar{x} \doteq \begin{cases} 0 & \text{if} \quad f'(0) \leq 0 \\ -Tf'(0) & \text{if} \quad f'(0) > 0, \end{cases} \tag{3.3}$$

$$\bar{u}'(x) \doteq \begin{cases} 0 & \text{if} \quad x \leq \bar{x} \\ b(x) & \text{if} \quad \bar{x} < x \leq 0 \\ \bar{u}(x) & \text{if} \quad x > 0. \end{cases} \tag{3.4}$$

We claim that, for all $t \leq T$, $x > 0$, if y_{min} solves

$$\min_{y \geq 0} \int_0^y \bar{u}(\xi)\ d\xi + tg\left(\frac{x-y}{t}\right) \tag{3.5}$$

then it solves

$$\min_{y \in \mathbb{R}} \int_0^y \bar{u}'(\xi)\ d\xi + tg\left(\frac{x-y}{t}\right) \tag{3.6}$$

as well. Indeed in the case $\bar{x} = 0$ this follows from observing that $g(x/t) < g((x-y)/t)$ for all $y < 0$. When $\bar{x} < 0$ observe that

$$\int_0^y \bar{u}'(\xi)\ d\xi + tg\left(\frac{x-y}{t}\right) = g(y) + f(b(0)) + tg\left(\frac{x-y}{t}\right) \geq tg\left(\frac{x}{t}\right), \qquad \forall\ \bar{x} \leq y < 0,$$

since $y \mapsto g(y) + f(b(0)) + tg((x-y)/t)$ is decreasing on $(-\infty, x/(t+1))$, while

$$g(\bar{x}) + f(b(0)) + tg\left(\frac{x-y}{t}\right) \geq g(\bar{x}) + f(b(0)) + tg\left(\frac{x-\bar{x}}{t}\right) \geq tg\left(\frac{x}{t}\right), \qquad \forall\ y \leq \bar{x},$$

since $y \mapsto g(\bar{x}) + f(b(0)) + tg((x-y)/t)$ is decreasing on $(-\infty, x - tf'(0))$ and $x - tf'(0) \geq \bar{x}$, which yields the claim. Therefore, if $u = u(t, x)$ denotes the solution of problem (1.1)-(1.2) with initial data $\bar{u}'$ obtained with Lax formula ([14]), for any $t \leq T$ we have $f(b(-y(t)/t)) = f(u(t, 0))$, which is locally integrable by Proposition 1.

Next we show that $m = m(t)$ is a positive absolutely continuous function satysfying (2.6) iff it is the maximal forward solution in the sense of Filippov ([11]) of the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t} m(t) = F(t, m(t)) \\ m(0) = 0, \end{cases} \tag{3.7}$$

$$F(t, m) \doteq \begin{cases} 0 & \text{if} \quad m < 0 \\ f(\tilde{u}(t)) - f\left(b\left(-\dfrac{y(t)}{t}\right)\right) & \text{if} \quad m \geq 0. \end{cases} \tag{3.8}$$

Indeed, assume that m is a positive absolutely continuous function solving (2.6). Let t be a point of differentiability for m and suppose that (2.6) holds in t. If $m(t) > 0$, then clearly

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = f\left(\tilde{u}(t)\right) - f\left(b\left(-\frac{y(t)}{t}\right)\right). \tag{3.9}$$

Otherwise, assume that $m(t) = 0$ for any t belonging to a set U with positive measure. We claim that

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) = \max\left\{0, f\left(\tilde{u}(t)\right) - f\left(b\left(-\frac{y(t)}{t}\right)\right)\right\}. \tag{3.10}$$

for a.e. $t \in U$. Since for any $\varepsilon > 0$ we can choose a set $E \subseteq \mathbb{R}^+$ such that $\mathrm{d}m/\mathrm{d}t|_E$ and $[f(\tilde{u}) - f(b(-y/t))]\,|_E$ are continuous and $meas(E^c) < \varepsilon$, it suffices to prove (3.10) for $t \in E$. Observe that E contains at most contably many isolated points. Let $t \in E$ be not isolated and assume $m(t) = 0$ and $f\left(\tilde{u}(t)\right) - f\left(b(-y(t)/t)\right) < 0$. Being m positive $\mathrm{d}m/\mathrm{d}t\ (t) \geq 0$. We show that $\mathrm{d}m/\mathrm{d}t\ (t) = 0$. By contradiction suppose $\mathrm{d}m/\mathrm{d}t\ (t) > 0$. Then there exists $(t_\nu)_{\nu \in \mathbb{N}} \subset E$ converging to t such that $m(t_\nu) > 0$, $\mathrm{d}m/\mathrm{d}t\ (t_\nu) > 0$, $f\left(\tilde{u}(t_\nu)\right) - f\left(b(-y(t_\nu)/t_\nu)\right) < 0$ which contradicts (3.9). In the case $f\left(\tilde{u}(t)\right) - f\left(b(-y(t)/t)\right) \geq 0$, using

$$\frac{\mathrm{d}}{\mathrm{d}t} m(t) \geq f\left(\tilde{u}(t)\right) - f\left(b\left(-\frac{y(t)}{t}\right)\right)$$

the conclusion follows in a similar way.
Assume now that m is the maximal forward solution of (3.7). If $m(t) > 0$, then $m(t)$ satisfies (3.9). Otherwise, being m the maximal forward solution, $m(t) = 0$ and (3.10) holds. Hence it is a positive solution of (2.10). This concludes the proof of the part of the Proposition concerning the existence and uniqueness of m, while the properties of Υ follow easily from (2.7) and (3.9)-(3.10).
□

3.2 Proof of Theorem 2

Consider the domain

$$\widehat{\mathcal{D}} \doteq \left\{ (\bar{u}, \tilde{u}) \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^\infty(\mathbb{R}^+) :\ \tilde{u}(t) \geq b(0) \quad \text{a.e.} \quad t \right\}. \tag{3.11}$$

Then for every $(\bar{u}, \tilde{u}) \in \widehat{\mathcal{D}}$ let $\widehat{S}_t(\bar{u}, \tilde{u})$ be the value at time t of the solution to (1.1)-(1.3) which, by Theorem 4 in [1], is unique, admits the representation (2.8) of Theorem 1 and satisfies the $\mathbb{L}^1$-contraction property ii). Since

$$\widehat{\mathcal{D}}' \doteq \left\{ (\bar{u}, f(\tilde{u})) \in \mathbb{L}^\infty(\mathbb{R}^+) \cap \mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^\infty(\mathbb{R}^+) :\ \tilde{u}(t) \geq b(0) \quad \text{a.e.} \quad t \right\}$$

is a dense subset of

$$\mathcal{D}' \doteq \left\{ (\bar{u}, f(\tilde{u})) \in \mathbb{L}^1(\mathbb{R}^+) \times \mathbb{L}^1_{loc}(\mathbb{R}^+) :\ \tilde{u}(t) \geq b(0) \quad \text{a.e.} \quad t \right\},$$

the flow $\widehat{S} : \mathbb{R}^+ \times \widehat{\mathcal{D}} \to \mathbb{L}^1(\mathbb{R}^+)$ can be uniquely extended by continuity to a map $S : \mathbb{R}^+ \times \mathcal{D} \to \mathbb{L}^1(\mathbb{R}^+)$ satisfying ii) as well. Thus the proof will be completed if we show that $t \to S_t(\bar{u}, \tilde{u})$ admits the representation (2.8) of Theorem 1 for every $(\bar{u}, \tilde{u}) \in \mathcal{D}$.
Let $(\bar{u}_\nu, \tilde{u}_\nu)_{\nu\in\mathbb{N}} \subseteq \widehat{\mathcal{D}}$ be a sequence such that

$$\bar{u}_\nu \to \bar{u} \qquad \text{in} \qquad \mathbb{L}^1(\mathbb{R}^+) \tag{3.12}$$

$$f(\tilde{u}_\nu) \to f(\tilde{u}) \qquad \text{in} \qquad \mathbb{L}^1_{loc}(\mathbb{R}^+). \tag{3.13}$$

Then for every fixed $t > 0$, one has $S_t(\bar{u}_\nu, \tilde{u}_\nu)(x) = b\left((x - y_\nu(t,x))/t\right)$ for a.e. $x > 0$, $y_\nu(t,x)$ denoting the unique miminum point for the function $y \mapsto \Psi_{\Upsilon_\nu}(t,x,y)$ defined by (2.9) in connection with the trace Υ_ν at $x = 0$ of $f\big(S_{(\cdot)}(\bar{u}_\nu, \tilde{u}_\nu)\big)$. We will show now that $f\left[b\left(-y_\nu(t)/t\right)\right]$ converges in $\mathbb{L}^1_{loc}(\mathbb{R}^+)$ to $f\left[b\left(-y(t)/t\right)\right]$, where $y_\nu(t)$, $y(t)$ realize the minimum of the functions $\Psi_{\bar{u}_\nu}(t,\cdot)$ and $\Psi_{\bar{u}}(t,\cdot)$ respectively. To this purpose consider the Cauchy propblems with initial data $\bar{u}'_\nu$ and $\bar{u}'$ defined in connection with $\bar{u}_\nu$ and $\bar{u}$ as in (3.4) and let $u_\nu(\cdot,\cdot)$ and $u(\cdot,\cdot)$ denote the corresponding solutions. Since $u_\nu(T,0)$ converges to $u(T,0)$ for a.e. $T > 0$, and hence $\theta_\nu(0) = -Tf'(u_\nu(T,0+))$ converges to $\theta(0) = -Tf'(u(T,0+))$ for a.e. T, by the same arguments of the proof of Proposition 2 and using (3.2) we have

$$\begin{aligned}
\lim_{\nu\to+\infty} \int_0^T f\left(b\left(-\frac{y_\nu(t)}{t}\right)\right)\, dt &= \lim_{\nu\to+\infty} \int_0^T f\left(u_\nu(t,0)\right)\, dt \\
&= \lim_{\nu\to+\infty} \left\{-Tg\left[f'\left(u_\nu(T,0)\right)\right] - \int_0^{\theta_\nu(0)} \bar{u}_\nu(\xi)\, d\xi\right\} \\
&= -Tg\left[f'\left(u(T,0)\right)\right] - \int_0^{\theta(0)} \bar{u}(\xi)\, d\xi \\
&= \int_0^T f\left(u(t,0)\right)\, dt \\
&= \int_0^T f\left(b\left(-\frac{y(t)}{t}\right)\right)\, dt
\end{aligned} \tag{3.14}$$

for a.e. $T > 0$. Observe now that the sequence $\left(\Psi_{\bar{u}_\nu}(t,\cdot)\right)_{\nu\in\mathbb{N}}$ converges uniformly to $\Psi_{\bar{u}}(t,\cdot)$ and hence, for a.e. $t > 0$, the corresponding minimum points $y_\nu(t)$, being unique (see [16]), converge to $y(t)$. It follows that, being $f(u)^- = \max\{-f(u), 0\}$ uniformly bounded, $f\left[b(-y_\nu(\cdot)/(\cdot))\right]^-$ converges in $\mathbb{L}^1_{loc}$ to $f\left[b(-y(\cdot)/(\cdot))\right]^-$. Hence denoting $f(u)^+ = \max\{f(u), 0\}$ and using (3.14),

$$\lim_{\nu\to\infty} \int_0^T f\left(b\left(-\frac{y_\nu(t)}{t}\right)\right)^+\, dt = \int_0^T f\left(b\left(-\frac{y(t)}{t}\right)\right)^+\, dt,$$

which implies the convergence in $\mathbb{L}^1_{loc}$ of $f\left[b(-y_\nu(t)/t)\right]$ to $f\left[b(-y(t)/t)\right]$. By (2.10) and using Dunford-Pettis Theorem, this implies that the sequence $(\Upsilon_\nu)_{\nu\in\mathbb{N}}$ is weakly compact in $\mathbb{L}^1_{loc}$. Thus there exists a subsequence still denoted $(\Upsilon_\nu)_{\nu\in\mathbb{N}}$ which converges weakly in $\mathbb{L}^1_{loc}$ to some function $\Upsilon \in \mathbb{L}^1_{loc}(\mathbb{R}^+)$. Therefore for every $x > 0$ the sequence of maps $\left(\Psi_{\Upsilon_\nu}(t,x,\cdot)\right)_{\nu\in\mathbb{N}}$ converges uniformly to $\Psi_\Upsilon(t,x,\cdot)$. This implies that for a.e. $x>0$ the corresponding minimum points $y_\nu(t,x)$, being unique (see [16]), converge to the minimum point $y(t,x)$ of $\Psi_\Upsilon(t,x,\cdot)$ and hence $\big(b\left((x-y_\nu(t,x))/t\right)\big)_{\nu\in\mathbb{N}}$ converges to $b\left((x-y(t,x))/t\right)$ for a.e. $x>0$ proving that $S_t(\bar u,\tilde u)$ satisfies (2.8). To conclude the proof we only need to show that if we set

$$m(t) \doteq \int_0^t \left[\Upsilon(s) - f\Big(b\Big(-\frac{y(s)}{s}\Big)\Big)\right]\, ds, \qquad t\geq 0, \tag{3.15}$$

then $m=m(t)$ satisfies the variational inequality (2.6). Let m_ν, $\nu\in\mathbb{N}$, be the absolutely continuous maps satisfying $m_\nu(0+)=0$, and

$$\int_0^t \left\{\frac{\mathrm{d}}{\mathrm{d}s} m_\nu(s) - f(\tilde u_\nu(s)) + f\left(b\left(-\frac{y_\nu(s)}{s}\right)\right)\right\}\{\varphi(s)-m_\nu(s)\}\ ds \geq 0, \tag{3.16}$$

for all $t\geq 0$ and for any $\varphi\geq 0$. Since $\Upsilon_\nu \rightharpoonup \Upsilon$ and $f(b(-y_\nu(\cdot)/\cdot)) \to f(b(-y(\cdot)/\cdot))$ in $\mathbb{L}^1_{loc}(\mathbb{R}^+)$, and since by (2.7) $d/dm_\nu(t) = \Upsilon_\nu(t) - f(b(-y_\nu(t)/t))$, it follows that the sequence $(m_\nu)_{\nu\in\mathbb{N}}$ converges to m uniformly on compact subsets of $\mathbb{R}^+$. Hence, letting $\nu\to\infty$ in (3.16) and due to the arbitrary choice of $\varphi\geq 0$, we find that m satisfies (2.6). $\square$

3.3 Proof of Theorem 3

We first establish property *i)*. We claim that

$$\mathcal{U}(t,x) \doteq -\int_x^{+\infty} S_t(\bar u,\tilde u)(\xi)\ d\xi, \qquad t,x\geq 0, \tag{3.17}$$

is the viscosity solution of the mixed problem for a Hamilton-Jacobi equation

$$\mathcal{U}_t + f(\mathcal{U}_x) = f(0), \tag{3.18}$$

$$\mathcal{U}(0,x) = \overline{\mathcal{U}}(x) \doteq -\int_x^{+\infty} \bar u(\xi)\ d\xi, \qquad\qquad t,\ x\geq 0, \tag{3.19}$$

$$\mathcal{U}(t,0) = \widetilde{\mathcal{U}}(t) \doteq -\int_0^{+\infty} \bar u(\xi)\ d\xi - \int_0^t \Upsilon(\bar u,\tilde u)(s)\ ds + tf(0). \tag{3.20}$$

This suffices to conclude since then the thesis of Theorem 3 is a consequence of the well-known comparison property for solutions of the mixed problem for Hamilton-Jacobi equations (see [8], Theorem V.2 and [7]).
To prove the claim observe that, if $\overline{\mathcal{V}}$, $\widetilde{\mathcal{V}}$, are uniformly continuous functions with $\overline{\mathcal{V}}(0)=\widetilde{\mathcal{V}}(0)$, the viscosity solution $\mathcal{V}=\mathcal{V}(t,x)$ of

$$\mathcal{V}_t + f(\mathcal{V}_x) = f(0), \tag{3.21}$$

$$\mathcal{V}(0,x) = \overline{\mathcal{V}}(x), \qquad\qquad t,\ x\geq 0, \tag{3.22}$$

$$\mathcal{V}(t,0) = \widetilde{\mathcal{V}}(t), \tag{3.23}$$

has the explicit representation in the interior of the domain (see [6], [3]):

$$\mathcal{V}(t,x) = \min_{y\in\mathbb{R}} H_{\overline{\mathcal{V}},\widetilde{\mathcal{V}}}(t,x;y), \qquad t,x>0, \tag{3.24}$$

where

$$H_{\overline{\mathcal{V}},\widetilde{\mathcal{V}}}(t,x;y) = \begin{cases} \overline{\mathcal{V}}(y) + t\Big(g\Big(\dfrac{x-y}{t}\Big) + f(0)\Big) & \text{if} \quad y \geq 0 \\ \widetilde{\mathcal{V}}(\tau) + (t-\tau)\,\Big(g\Big(\dfrac{x}{t-\tau}\Big) + f(0)\Big) & \text{if} \quad y < 0, \end{cases} \tag{3.25}$$

with g denoting the Legendre transform of f and τ satisfying

$$\frac{x-y}{t} = \frac{x}{t-\tau}, \qquad y \leq 0.$$

On the other hand, using the explicit representation of solutions to problem (1.1)-(1.3) given by Theorem 2, denoting with $\Psi_{\Upsilon(\bar{u},\tilde{u})}$ the map defined by (2.9) and with $y(t,x)$ a point of minimum value for $\Psi_{\Upsilon(\bar{u},\tilde{u})}(t,x,\cdot)$, we find that for a.e. $t,x>0$

$$\begin{aligned} \min_{y\in\mathbb{R}} H_{\overline{\mathcal{U}},\widetilde{\mathcal{U}}}(t,x;y) &= -\int_0^{+\infty} \bar{u}(\xi)\ d\xi + tf(0) + \min_{y\in\mathbb{R}} \Psi_{\Upsilon(\bar{u},\tilde{u})}(t,x;y) \\ &= -\int_0^{+\infty} \bar{u}(\xi)\ d\xi + tf(0) + \Psi_{\Upsilon(\bar{u},\tilde{u})}(t,x;y(t,x)). \end{aligned} \tag{3.26}$$

If $y(t,x) \geq 0$, we get

$$\begin{aligned} \min_{y\in\mathbb{R}} H_{\overline{\mathcal{U}},\widetilde{\mathcal{U}}}(t,x;y) &= -\int_{y(t,x)}^{+\infty} \bar{u}(\xi)\ d\xi + t\Big(g\Big(\frac{x-y(t,x)}{t}\Big) + f(0)\Big) \\ &= -\int_{x-tf'(u(t,x))}^{+\infty} \bar{u}(\xi)\ d\xi + t\Big(g\Big(f'(u(t,x)\Big) + f(0)\Big). \end{aligned} \tag{3.27}$$

If $y(t,x) < 0$, then

$$\begin{aligned} \min_{y\in\mathbb{R}} H_{\overline{\mathcal{U}},\widetilde{\mathcal{U}}}(t,x;y) &= -\int_0^{+\infty} \bar{u}(\xi)\ d\xi - \int_0^{t-t(x/x-y(t,x))} \Upsilon(\bar{u},\tilde{u})(s)\ ds + t\Big(g\Big(\frac{x-y(t,x)}{t}\Big) + f(0)\Big) \\ &= -\int_0^{+\infty} \bar{u}(\xi)\ d\xi - \int_0^{t-t/f'(u(t,x))} \Upsilon(\bar{u},\tilde{u})(s)\ ds + t\Big(g\Big(f'(u(t,x)\Big) + f(0)\Big). \end{aligned} \tag{3.28}$$

Since Lemma 3.2 in [9] together with (2.23) and (3.2) implies

$$\int_{x-tf'(u(t,x))}^{+\infty} \bar{u}(\xi)\ d\xi - \int_x^{+\infty} S_t(\bar{u},\tilde{u})(\xi)\ d\xi = t\Big(g\Big(f'(u(t,x)\Big) + f(0)\Big)$$

$$\int_0^{+\infty} \bar{u}(\xi)\ d\xi + \int_0^{t-t/f'(u(t,x))} \Upsilon(\bar{u},\tilde{u})(s)\ ds - \int_x^{+\infty} S_t(\bar{u},\tilde{u})(\xi)\ d\xi = t\Big(g\Big(f'(u(t,x)\Big) + f(0)\Big),$$

from (3.27, (3.28) it follows that the map $\mathcal{U}$ defined by (3.17) satisfies

$$\mathcal{U}(t,x) = \min_{y\in\mathbb{R}} H_{\overline{\mathcal{U}},\widetilde{\mathcal{U}}}(t,x;y),$$

for a.e. $t,x>0$. Hence $\mathcal{U}=\mathcal{U}(t,x)$ is the viscosity solution of problem (3.18)-(3.20), proving the claim.

Property *ii)* is derived in a similar way by showing that

$$\mathcal{U}' = \int_0^x S_t(\bar{u},\tilde{u})(\xi)\ d\xi + \int_t^{+\infty} \left(\Upsilon(\bar{u},\tilde{u})(s) - c'\right)\ ds \tag{3.29}$$

is the viscosity solution of the mixed problem for the Hamilton-Jacobi equation

$$\mathcal{U}'_t + f\left(\mathcal{U}'_x\right) = c' \tag{3.30}$$

$$\mathcal{U}'(0,x) = \int_0^x \bar{u}(\xi)\ d\xi + \int_0^{+\infty} \left(\Upsilon(\bar{u},\tilde{u})(s) - c'\right)\ ds, \qquad t,\ x\geq 0, \tag{3.31}$$

$$\mathcal{U}(t,0) = \int_t^{+\infty} \left(\Upsilon(\bar{u},\tilde{u})(s) - c'\right)\ ds. \tag{3.32}$$

Therefore, the conservation equality of Remark 2.5 together with the comparison principle for viscosity solutions of the mixed problem (3.30)-(3.32) yields property *ii)*.

4. Properties of the Attainable Set

4.1 Proof of Theorems 4 and 5

Proof of Theorem 4. Since by Remark 2.4 any solution to (1.1)-(1.3) admits right and left limits in x, we can assume $w\in\mathcal{A}(T)$ to be right continuous. We claim that from the explicit representation (2.8) it follows that w is bounded on compact subsets of $\mathbb{R}^+$ bounded away from the origin. Indeed, assume by contradiction that there exists a sequence $(x_\nu)_{\nu\in\mathbb{N}}\subseteq[M,N]$, $M>0$, such that $\lim\limits_{\nu\to+\infty} w(x_\nu) = +\infty$. Let $y_\nu \doteq y(T,x_\nu)$ be the sequence of minimum points of $\Psi_\Upsilon(T,x_\nu,\cdot)$. Then by (2.8) $\lim\limits_{\nu\to+\infty} y_\nu = -\infty$. Thus $y_\nu = x - tx/(t-\tau_\nu)$ for ν sufficiently large and $\lim\limits_{\nu\to+\infty}\tau_\nu = t$. Since g is superlinear, it follows that

$$\lim_{\nu\to+\infty}\min_y \Psi_\Upsilon(T,x_\nu,y) = +\infty, \tag{4.1}$$

which gives a contradiction being $\Psi_\Upsilon(T,x,0)$, and hence $\min\limits_y \Psi(T,x,y)$, uniformly bounded on $[M,N]$. The case $\lim\limits_{\nu\to+\infty} w(x_\nu) = -\infty$ cannot happen neither. In fact in such a case $\lim\limits_{\nu\to+\infty} y_\nu = +\infty$, and (4.1) holds again.

Now properties (2.31)-(2.33) are proved with the same arguments of Theorem 1 in [1].

Assume now that $w \in \mathbb{L}^1(\mathbb{R}^+)$ satisfies (2.31)-(2.33). By Remark 2.7 we can suppose that w is right continuous and bounded on subsets of $\mathbb{R}^+$ bounded away from the origin. Then using again the same arguments in Theorem 1 in [1], we can define two functions $u = u(t,x)$ and $\tilde{u} = \tilde{u}(t) \geq b(0)$ such that

1) u is a weak entropy solution of (1.1) in the interior of Ω and by construction it is bounded on $[0,t] \times \mathbb{R}^+$, for any $t < T$ and on $[0,T] \times [R, +\infty)$ for any $R > 0$;

2) for any $R > 0$
$$\lim_{t \to T^-} \int_R^{+\infty} |u(t,x) - w(x)| \; dx = 0;$$

3) there exist
$$\lim_{x \to 0^+} f(u(t,x)) \doteq \Upsilon(t)$$
$$\lim_{x \to 0^+} \operatorname{sgn} \; f'(u(t,x)) \doteq \mu(t)$$
for a.e. $t \in (0,T)$ and (2.4) holds.

We claim that $f(\tilde{u}) \in \mathbb{L}^1(0,T)$. Being f superlinear, it suffices to show that
$$\int_0^T f(\tilde{u}(s)) \; ds < +\infty. \tag{4.2}$$
Choose a sequence $x_\nu \to 0^+$. Since u is a solution to (1.1) on $[0,T] \times [x_\nu, +\infty)$ for any x_ν and using 2)-3) we find
$$\begin{aligned} \int_0^T f(\tilde{u}(s)) \; ds &\leq \int_0^T \Upsilon(s) \; ds \\ &\leq \liminf_{\nu \to +\infty} \int_0^T f(u(s, x_\nu)) \; ds \\ &= \liminf_{\nu \to +\infty} \int_{x_\nu}^{+\infty} w(\xi) \; d\xi + T f(0) \end{aligned}$$
which implies (4.2). Since (2.2) holds for any $t < T$, using 3) we get $u(t,x) = S_t \tilde{u}(x)$ for any $t < T$ and $x > 0$. Being $t \mapsto S_t \tilde{u}$ continuous as a map from $[0,T]$ into $\mathbb{L}^1(\mathbb{R}^+)$, from 2) it follows $S_T \tilde{u}(x) = w(x)$. □

Proof of Theorem 5. It suffices to show that if $\rho \in \mathcal{A}(\bar{x})$, then ρ is locally bounded at the origin. The other properties follow from arguments similar to the ones used in the proof of Theorem 3. Assume $\rho(t) = S_t \tilde{u}(\bar{x})$ and by contradiction let $\tau_\nu \downarrow 0^+$ be a sequence such that $\rho(\tau_\nu) \to +\infty$ (the case $\rho(\tau_\nu) \to -\infty$ cannot happen since (2.31)). By (2.31), the maximal backward characteristic through $(\tau_\nu, \bar{x})$, $\theta_\nu : t \mapsto \bar{x} + (t - \tau_\nu) f'(\rho(\tau_\nu))$, must reach the positive t-axis. Fix $0 < x \leq \bar{x}$ and let $(t_n)_{n \in \mathbb{N}} \subset \mathbb{R}^+$ be a sequence such that
$$\lim_{n \to +\infty} t_n = 0 \tag{4.3}$$
$$\begin{cases} t_n > 0 & \text{if} \quad f'(0) \leq 0 \\ 0 < t_n < \dfrac{x}{f'(0)} & \text{if} \quad f'(0) > 0. \end{cases} \tag{4.4}$$

Choose $N \in \mathbb{N}$ such that if $\nu > N$ and $\theta_\nu(s_\nu) = x$, then

$$s_\nu < \frac{x}{f'(0)}$$

and let $t_n < s_\nu$. Since minimal and maximal backward characteristics cannot intersect and by (4.4) we get $S_{t_n}\tilde{u}(x) \neq 0$. Hence, by (2.31) and (4.3)

$$f'(S_{t_n}\tilde{u}(x)) \geq \frac{x}{t_n} \to +\infty.$$

This yields a contradiction, since $S_t\tilde{u} \to 0$ in $\mathbb{L}^1(\mathbb{R}^+)$. □

4.2 Proof of Remark 2.7

Here we shall prove that if $w \in \mathcal{L}^1(\mathbb{R}^+)$ has right and left limits at each point, is right continuous and satisfies (2.31) and (2.33), then it is bounded on subsets bounded away from the origin. This and (2.31) imply that there exists $\alpha > 0$ such that $w(x) = 0$ whenever $x > \alpha$.
For each $x > 0$ consider the line

$$\theta_x : t \mapsto x + f'(w(x))(t - T). \tag{4.5}$$

Using the same arguments of the proof of Theorem 1 in [1], from (2.33) we can desume that such lines do not intersect in the interior of Ω. Being $w \in \mathcal{L}^1(\mathbb{R}^+)$,

$$\liminf_{x\to+\infty} w(x) = 0.$$

Hence there exists $R > 0$ such that $f'(w(R)) < R/T$. By (2.31) $w(R) = 0$. This implies that $w(x) = 0$ for any $x > R$, since the lines $(\theta_x)_{x\in\mathbb{R}^+}$ do not intersect each other in the interior of Ω. Assume now by contradiction that there exist a sequence $(x_\nu)_{\nu\in\mathbb{N}}$ such that $\lim_{\nu\to+\infty} w(x_\nu) = +\infty$ (the case $w(x_\nu) \to -\infty$ being ruled out by (2.31)). By the previous arguments we can suppose that $(x_\nu)_{\nu\in\mathbb{N}}$ converges to some $\bar{x} > 0$. For any $0 < x < \bar{x}$, let $N \in \mathbb{N}$ be sufficiently large so that $0 < x < x_\nu$ for any $\nu > N$. Then there exists $s \in (0, T)$ such that $\theta_x(s) = 0$. Moreover

$$s \geq T - \frac{x_\nu}{f'(w(x_\nu))} \doteq \tau_\nu \to T^-$$

and hence

$$f'(w(x)) \geq \frac{x}{T - \tau_\nu} \to +\infty,$$

which yields a contradiction. □

4.3 Proof of Theorem 6

Let $(\tilde{u}_\nu)_{\nu\in\mathbb{N}} \subseteq \mathcal{U}$. Then by (2.37) from Dunford-Pettis Theorem it follows that the sequence $(f(\tilde{u}_\nu))_{\nu\in\mathbb{N}} \subseteq \mathcal{U}$ is weakly compact in $\mathbb{L}^1(0,T)$. Hence it can be assumed to converge w-$\mathbb{L}^1$ to some function Φ. Furthermore, if $(\Upsilon_\nu)_{\nu\in\mathbb{N}} \subset \mathbb{L}^1(0,T)$ is the sequence of the traces in the sense of (2.2) of $f(S_{(\cdot)}\tilde{u}_\nu)$ at the origin, then by (2.10) it is weakly compact in $\mathbb{L}^1(0,T)$ as well, and it can be assumed to converge w-$\mathbb{L}^1$ to some function Υ. Since $f(\tilde{u}_\nu(t)) \in G(t)$ and by (2.4) $f(\tilde{u}_\nu(t)) \le \Upsilon_\nu(t)$ for a.e. t, being f convex and G convex closed valued it follows that $\Phi(t) \in f(G(t))$ and $\Phi(t) \le \Upsilon(t)$ for a.e. t. Hence there exists a measurable selection $\tilde{u}$ from G such that

$$\Phi(t) = f(\tilde{u}(t)), \qquad f(\tilde{u}(t)) \in G(t), \qquad f(\tilde{u}(t)) \le \Upsilon(t), \qquad \text{for a.e. } t > 0.$$

Since for any $t \in J$ the functionals $y \mapsto \int_0^t q_i(s, y(s))\ ds$, $i = 1, \ldots, N$, are sequencially lower semicontinuous w.r.t. weak convergence on $\mathbb{L}^1$ (see Theorem 3 in [12]), it follows that $\tilde{u} \in \mathcal{U}$. We claim that $S_T\tilde{u}_\nu \to S_T\tilde{u}$ in $\mathbb{L}^1(\mathbb{R}^+)$. First of all observe that from similar arguments to the ones in the proof of Theorem 2, it follows that for any $t \in [0,T]$ $S_t\tilde{u}_\nu$ converges a.e. to a function $u(t,\cdot)$ that admits the representation (2.8) in the interior of Ω. Moreover from the proof of Theorem 3 in [1], it follows that Υ satisfies (2.4). Hence $u(t,\cdot) = S_t\tilde{u}$ for any $t \in [0,T]$. Let $\xi > 0$ be such that $S_T\tilde{u}(x) = 0$ for all $x > \xi$ and $S_T\tilde{u}_\nu(\xi+) \to S_T\tilde{u}(\xi+)$. There exists $N \in \mathbb{N}$ such that if $\nu > N$ then $S_T\tilde{u}_\nu(\xi+) < b(\xi/T)$. By (2.31) and since maximal backward characteristics do not intersect it follows that if $\nu > N$ then $S_T\tilde{u}_\nu(x) = 0$ for all $x \ge \xi$. Furthermore, observe that by (2.31) $S_T\tilde{u}_\nu \ge \min\{b(0), 0\}$. Therefore to prove that $S_T\tilde{u}_\nu \to S_T\tilde{u}$ in $\mathbb{L}^1(\mathbb{R}^+)$ it suffices to show that

$$\lim_{\nu\to+\infty} \int_0^{+\infty} S_T\tilde{u}_\nu(x)\ dx = \int_0^{+\infty} S_T\tilde{u}(x)\ dx.$$

By using (2.2), Lemma 3.2 in [9] and the weak convergence in $\mathbb{L}^1$ of Υ_ν to Υ, we get

$$\begin{aligned}\lim_{\nu\to+\infty} \int_0^{+\infty} S_T\tilde{u}_\nu(x)\ dx &= \lim_{\nu\to+\infty} \int_0^{T} \Upsilon_\nu(t)\ dt + Tf(0)\\ &= \int_0^{T} \Upsilon(t)\ dt + Tf(0)\\ &= \int_0^{+\infty} S_T\tilde{u}(x)\ dx,\end{aligned}$$

which concludes the proof.

5. An Application

In this section we derive a characterization of the optimal boundary control for a optimization problem of traffic flow which was introduced in [1]. By treating a flow of traffic on a stretch of highway as a continuum one finds (see [19]) that the density of cars $u = u(t,x)$ satisfies the conservation law

$$u_t + [f(u)]_x = 0 \tag{5.1}$$

where the flux function f depends only on the density of cars by

$$f(u) = uv(u) \tag{5.2}$$

with $v(u)$ representing the velocity of cars usually taken $v(u) = a_1 \ln(a_2/u)$ for suitable constants a_1 and a_2. We are interested in the problem of minimizing the mean time spent by cars travelling through the stretch of highway between an entry at a point $x = 0$ and an exit at a point $x = \bar{x}$ by controlling the density of cars entering the highway $\tilde{u} = \tilde{u}(t)$. We suppose that no cars are on the stretch of highway $[0, \bar{x}]$ at the initial time $t = 0$ and that the function $g = g(t)$ which represents the flow of cars arriving at the entry $x = 0$ per unit time is continuous with compact support. This lead us to consider the minimization problem

$$\min_{\tilde{u} \in \mathcal{U}} \int_0^\tau t \ f\left(S_t \tilde{u}(\bar{x})\right) \ dt, \tag{5.3}$$

where τ is a time after which no car is assumed to be on the highway, $S_t \tilde{u}$ denotes the solution to the initial-boundary value problem (5.1),(2.25),(2.26), and the admissible set of boundary controls $\mathcal{U}$ consists of all $\mathbb{L}^\infty$ functions $\tilde{u}$ satisfying the following conditions.

i) The net flux of cars entering the stretch of highway must be equal to the total number of cars arriving at the entry:

$$\int_0^\tau f\left(\tilde{u}(s)\right) \ ds = \int_0^\tau g(s) \ ds. \tag{5.4}$$

ii) At any time $0 < t \leq \tau$ the total number of cars which have entered the highway untill that moment must be less or equal to the total number of cars arrived at the entry in the same period of time:

$$\int_0^t f\left(\tilde{u}(s)\right) \ ds \leq \int_0^t g(s) \ ds. \tag{5.5}$$

iii) The maximum number of cars entering the highway must be less or equal to the maximum density of cars u_m allowed on the highway:

$$\tilde{u}(t) \in [0, u_m]. \tag{5.6}$$

Moreover, since it is not restrictive to consider boundary data with characteristics entering the domain, one can suppose that $\tilde{u} \in [0, b(0)] \subseteq [0, u_m]$, for a.e. $t > 0$, and hence the solution to problem (5.1),(2.25),(2.26), assume always the boundary data at the boundary, i.e.

$$f(S_t \tilde{u})(0) = \Upsilon(\tilde{u})(t), \qquad \text{for a.e.} \qquad t > 0, \tag{5.7}$$

for any $\tilde{u} \in \mathcal{U}$. Notice that

$$\int_0^\tau t\ f\left(S_t\tilde{u}(\bar{x})\right)\ dt = \tau \int_0^\tau f\left(S_t\tilde{u}(\bar{x})\right)\ dt - \int_0^\tau \int_0^t f\left(S_s\tilde{u}(\bar{x})\right)\ ds\ dt$$

and hence by (5.4) the optimization problem (5.3) is equivalent to

$$\max_{\tilde{u}\in\mathcal{U}} \int_0^\tau \int_0^t f\left(S_s\tilde{u}(\bar{x})\right)\ ds\ dt. \tag{5.8}$$

Therefore if we find an admissible boundary data $\hat{u} = \hat{u}(t)$ which satisfies

$$\int_0^t f\left(S_s\hat{u}(\bar{x})\right)\ ds \geq \int_0^t f\left(S_s\tilde{u}(\bar{x})\right)\ ds, \qquad \forall t > 0, \qquad \forall \tilde{u} \in \mathcal{U}, \tag{5.9}$$

such a boundary control will be a optimal solution of our problem. But from the comparison principle given by Theorem 3 and by (5.7) it follows that in order to satisfy (5.9) it is sufficient to check that

$$\int_0^t f(\hat{u}(s))\ ds \geq \int_0^t f(\tilde{u}(s))\ ds, \qquad \forall t > 0, \qquad \forall \tilde{u} \in \mathcal{U}. \tag{5.10}$$

Then, recalling the constraints on the admissible boundary data given by (5.4)-(5.6), it is clear how to construct the optimal solution $\hat{u} = \hat{u}(t)$ of problem (5.3) (see fig. 1):

$$\hat{u}(t) = \begin{cases} f^{-1}(u_m) & \text{if} \quad \left(g(t) \leq f(u_m) \quad \text{and} \quad \int_0^t f(\hat{u}(s))\ ds < \int_0^t g(s)\ ds\right) \quad \text{or} \quad g(t) > f(u_m) \\ f^{-1}(g(t)) & \text{if} \quad \left(g(t) \leq f(u_m) \quad \text{and} \quad \int_0^t f(\hat{u}(s))\ ds = \int_0^t g(s)\ ds\right) \end{cases}$$

where f^{-1} denotes the inverse map of the restriction of f to the interval $[0, f(b(0))]$.

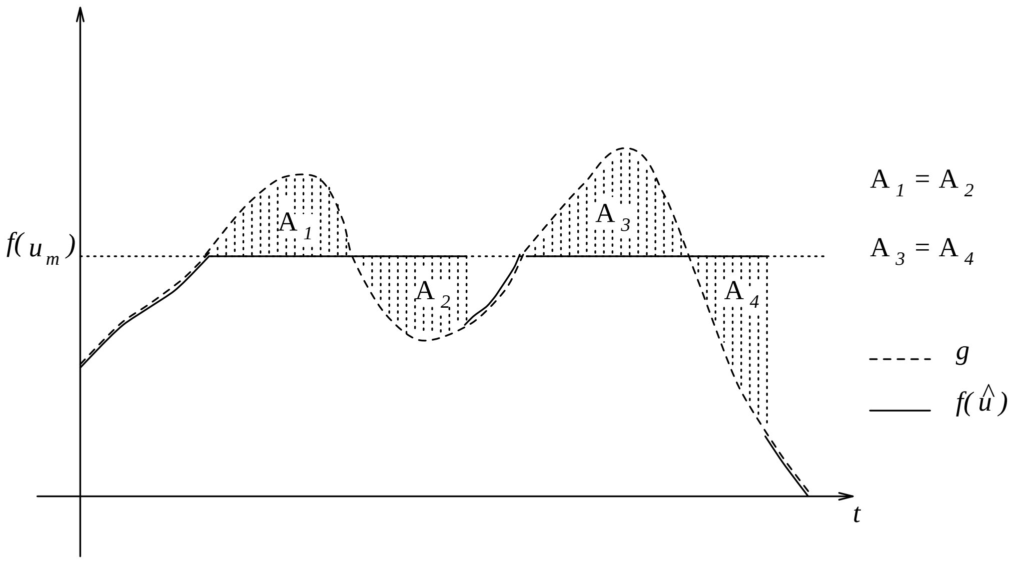

Figure 1

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