Numerical Passage from Systems of Conservation Laws to Hamilton-Jacobi Equations, and a Relaxation Scheme

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Dedicated to Ami Harten

Abstract

In this paper we study the numerical transition from a Hamilton-Jacobi (H-J) equation to its associated system of conservation laws in arbitrary space dimensions. We first study how, in a very generic setting, one can recover the viscosity solutions of the H-J equation using the numerical solutions to the system of conservation laws. We then introduce a simple, second order relaxation scheme to solve the underlying weakly hyperbolic system of conservation laws.

Key words. Hamilton-Jacobi equations, weakly hyperbolic system of conservation laws, shock capturing methods, numerical viscosity, relaxation schemes.

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1. Introduction

Hamilton-Jacobi equations arise in a variety of applications, such as optimal control, game theory, geometric optics and front propagation. In this paper we study several numerical issues associated with a general Hamilton-Jacobi equation in n space dimensions

$$\partial_t u + H(\nabla u) = 0\,. \tag{1.1}$$

Here ∇ is the gradient with respect to $\mathbf{x} = (x_1, \cdots, x_n)$, defined by $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$, unless otherwise specified. It is known that this equation has a close relation with a system of conservation laws. Let $\mathbf{p} = (p_1, \cdots, p_n)$ where

$$\mathbf{p} = \nabla u\,, \quad \text{or} \quad p_1 = \partial_{x_1} u\,, \quad \cdots, \quad p_n = \partial_{x_n} u\,, \tag{1.2}$$

then $\mathbf{p}$ formally solves the n-dimensional system of conservation laws

$$\partial_t \mathbf{p} + \nabla H(\mathbf{p}) = 0\,. \tag{1.3a}$$

Clearly if $\mathbf{p}$ is known then one may recover u from $\mathbf{p}$ by integrating the ordinary differential equation

$$\partial_t u + H(\mathbf{p}) = 0\,. \tag{1.3b}$$

We would like to point out that system (1.3) is only *weakly* hyperbolic in high dimension ($n \geq 2$), in that there is no complete set of eigenvectors. Thus it is not well-posed in the strong sense.

Understanding the relation between (1.1) and (1.3), both theoretically and numerically, has been a rich research subject. Theoretically, one gains a better understanding of the solution to (1.1) by knowing that of (1.3), the latter has been studied intensively in the past several decades. For example, it is known that the generic solutions to the system of conservation laws develop shocks (discontinuous solutions) in finite time even if the initial data are smooth [21]. The relation (1.2) suggests that the solution of u will develop a cusp where $\mathbf{p}$, the derivative of u, is discontinuous. The concept of weak entropy solutions, in the sense that they are limits of the solutions to the corresponding viscous regularization problems, was introduced to obtain a unique weak solution for both systems of conservation laws [21] and the Hamilton-Jacobi equations [18, 20,8]. The equivalence between the entropy weak solutions to (1.3) and the viscosity solutions to (1.1) has been known in the literatures, see [18,2,6,23]. For completeness and convenience for the readers, we present a simple elementary argument in Section 2, in which we will show that, in any space dimensions, the vanishing viscosity limit solution to (1.3) is equivalent to the viscosity solution of (1.1) for convex Hamiltonian H.

Numerous numerical methods have been developed to solve (1.1) (see [1,19, 5, 6, 26, 27, 28] and the references therein). The designs of most of these front capturing schemes are strongly guided by the idea of shock capturing methods for (1.3). For example, The idea of upwind scheme for (1.1), developed by Osher and Sethian [26], is based on the characteristic information on (1.3). Thus it is of considerable interest to understand the numerical transition from (1.3) to (1.1). Of course, if one is only interested in solving (1.1), one will just need to use the characteristic information of (1.3) as a guide to determine the correct "upwind" direction for a stable discretization of (1.1), rather than solving (1.3) directly. However, solving (1.3) itself is an interesting and challenging numerical problem, since it is only weakly hyperbolic. In addition, as our study here finds, the correct numerical transition from (1.3) to (1.1) involves an interesting numerical issue –the correct usage of numerical viscosity, in particular, in higher dimension– is of interest and importance to the shock capturing community for general systems of conservation laws in high dimensions.

In this article we will address the following two numerical issues. First, given a numerical solution for (1.3), obtained by a shock capturing method, how can one recover the viscosity solution of (1.1)? Second, what is an appropriate way to discretize the weakly hyperbolic system (1.3)?

Our numerical study, carried out in Sections 3 and 4, shows that in order to recover u by numerically integrating (1.3b), special attention needs to be taken to guarantee that (1.3b) is numerically consistent to (1.3a). This is to say that $\mathbf{p} = \nabla u$ needs to be preserved numerically. Such a numerical consistency is not automatically guaranteed, for example, if one integrates (1.3a) and (1.3b) both over the same spatial cell. This will lead to anomalous oscillations in u even if $\mathbf{p}$ looks perfectly normal. Using the analysis introduced by Jin and Liu for the effect of numerical viscosity for 1-D systems of conservation laws [13], we show why such a spurious phenomenon appears and give a generic way to fix this. This approach applies in any space dimensions to at least all first and second order shock capturing schemes. This study, as the 1-D results of [13], also contributes to a better understanding of the effect of numerical viscosity in higher space dimensions for more general system of conservation laws.

It is known that, for a weakly hyperbolic system, shock capturing methods may face the difficulty to achieve the desired order of accuracy [28, 10]. In Section 6 we introduce a second order relaxation scheme for this weakly hyperbolic system, which seems to produce quite satisfactorily results, as demonstrated by our numerical examples in Section 7.

Relaxation approximation to systems of conservation laws was first introduced by Jin and Xin in [16]. (See also [3,17,25] for recent studies). The idea there is to introduce a relaxation approximation to a system of conservation laws that can generate entropy solution in the zero relaxation limit. It also gives a class of relaxation schemes that are Riemann solver free. The new relaxation approximation introduced here for the Hamilton-Jacobi equation and its equivalent system of conservation laws, while maintaining the basic advantage of that constructed for general systems of conservation laws in [16], enjoys greater simplicity since here we only need to add one rate equation for any space dimensions, using the special structure of (1.3). The two reasons that such a second order scheme works well for a weakly hyperbolic system, as our understanding, are the following. First, the first order relaxation scheme, as the relaxation time approaches zero, is of Lax-Friedrichs type. It is known that the first order Lax-Friedrichs scheme and its second order, multidimensional extension produce quite good results for weakly hyperbolic systems [10, 11]. Second, the relaxation system (6.3) is (strongly) hyperbolic. The relaxation scheme is a discretization of this strongly well-posed system, rather than a weakly hyperbolic system. We believe both factors contribute to the good performance of the second order relaxation scheme for the weakly hyperbolic system (1.3), and further investigation along this line could be promising for more general weakly hyperbolic systems.

We finally make some concluding remarks in Section 8.

2. Remark on the Conservation Law Formulation of the H-J Equations

It is known [18, 22, 6, 2] that for a convex Hamiltonian, the question of entropy solutions in $W^{1,\infty}$ to the Cauchy problem for the H-J Equation

$$\begin{aligned} \partial_t u + H(\nabla u) &= 0\,, \\ u(\mathbf{x},0) &= u_0(\mathbf{x})\,, \end{aligned} \tag{2.1}$$

is equivalent to that of the entropy solutions in L^∞ to the Cauchy problem for the corresponding conservation law

$$\begin{aligned} \partial_t \mathbf{p} + \nabla H(\mathbf{p}) &= 0\,, \\ \mathbf{p}(\mathbf{x},0) = \mathbf{p}_0(\mathbf{x}) &= \nabla u_0(\mathbf{x})\,. \end{aligned} \tag{2.2}$$

For completeness of the presentation and the convenience for the readers, we will give an elementary argument for such an equivalence. For simplicity, we will assume that $u_0(\mathbf{x})$ is smooth, say $u_0 \in C^1$.

First, let $u \in W^{1,\infty}$ be the viscosity solution to (2.1). It is straightforward to show that $\mathbf{p} = \nabla u \in L^\infty$ is a weak solution to (2.2). Moreover, $\mathbf{p}$ is the limit of a viscous perturbation of (2.2), i.e., there exists a smooth solution $q^\epsilon(\mathbf{x}, t)$ to

$$\begin{aligned} \partial_t \mathbf{q}^\epsilon + \nabla H(\mathbf{q}^\epsilon) &= \epsilon \Delta \mathbf{q}^\epsilon \,, \\ \mathbf{q}^\epsilon(\mathbf{x}, 0) &= \nabla u_0(\mathbf{x}) \,, \end{aligned} \tag{2.3}$$

such that $\mathbf{q}^\epsilon$ converges to $\mathbf{p}$ in L^m for any $m < \infty$ and in the weak-* topology in L^∞.

Indeed, since u is the viscosity solution to (2.1) and H is convex, there exists a smooth solution $u^\epsilon(x, t)$ to

$$\begin{aligned} \partial_t u^\epsilon + H(\nabla u^\epsilon) &= \epsilon \Delta u^\epsilon \,, \\ u^\epsilon(\mathbf{x}, 0) &= u_0(\mathbf{x}) \,, \end{aligned} \tag{2.4}$$

such that u^ϵ is uniformly bounded in $W^{1,\infty}$, and $u^\epsilon \to u$ in $W^{1,m}$, see [18, 19, 22]. Set $\mathbf{p}^\epsilon = \nabla u^\epsilon$, then (2.4) implies that

$$\begin{aligned} \partial_t \mathbf{p}^\epsilon + \nabla H(\mathbf{p}^\epsilon) &= \epsilon \Delta \mathbf{p}^\epsilon \,, \\ \mathbf{p}^\epsilon(\mathbf{x}, 0) &= \nabla u_0(\mathbf{x}) \,. \end{aligned} \tag{2.5}$$

Denoting by $\mathbf{p}$ the weak-* limit of $\mathbf{p}^\epsilon$ in L^∞, multiplying (2.5) by a test function and integrating by parts, one can show easily that, as $\epsilon \to 0^+$, $\mathbf{p}$ is a weak solution to (2.2). Thus, by construction, $\mathbf{p}$ is the desired vanishing viscosity limit solution to (2.2).

Conversely, let $\mathbf{p}(\mathbf{x}, t) \in L^\infty$ be a weak solution to (2.2) obtained by the vanishing viscosity method, i.e., (2.5) has smooth solutions $\mathbf{p}^\epsilon$ which is uniformly bounded in L^∞ such that

$$\mathbf{p}^\epsilon(\mathbf{x}, t) \to \mathbf{p}(\mathbf{x}, t) \quad \text{in} \quad L^m \,, \tag{2.6}$$

and in weak-* topology in L^∞. Then there exists the unique viscosity solution $u(\mathbf{x}, t)$ to (2.1) satisfying

$$\nabla u(\mathbf{x}, t) = \mathbf{p}(\mathbf{x}, t) \qquad \text{a.e.}. \tag{2.7}$$

Indeed, let $\mathbf{p}^\epsilon(\mathbf{x}, t)$ be the solution to (2.5) satisfying the property (2.6). Let $u^\epsilon(\mathbf{x}, t)$ be the unique solution of the following problem

$$\begin{aligned} \partial_t u^\epsilon - \epsilon \Delta u^\epsilon &= -H(\mathbf{p}^\epsilon) \,, \\ u^\epsilon(\mathbf{x}, 0) &= u_0(\mathbf{x}) \,. \end{aligned} \tag{2.8}$$

Problem (2.8) has a global smooth solution $u^\epsilon(x, t)$. Moreover, by the maximal principle and the assumption (2.6), there exists a constant $C = C(T)$ independent of ϵ such that

$$|u^\epsilon| \le C(T) \qquad \text{for} \quad 0 < t \le T \,. \tag{2.9}$$

To estimate the derivative of u^ϵ, one may differentiate (2.8) with respect to $\mathbf{x}$ to get

$$\begin{aligned} \partial_t(\nabla u^\epsilon) - \epsilon\Delta(\nabla u^\epsilon) &= -\nabla H(\mathbf{p}^\epsilon)\,, \\ \nabla u^\epsilon(\mathbf{x},0) &= \nabla u_0(\mathbf{x})\,. \end{aligned} \tag{2.10}$$

It follows from (2.5) and (2.10) that

$$\begin{aligned} \partial_t(\nabla u^\epsilon - \mathbf{p}^\epsilon) - \epsilon\Delta(\nabla u^\epsilon - p^\epsilon) &= 0\,, \\ (\nabla u^\epsilon - \mathbf{p}^\epsilon)(\mathbf{x},0) &= 0\,. \end{aligned} \tag{2.11}$$

The uniqueness to the Cauchy problem of the heat equation (2.11) now yields

$$\nabla u^\epsilon(\mathbf{x},t) \equiv \mathbf{p}^\epsilon(\mathbf{x},t)\,. \tag{2.12}$$

This, together with (2.8), shows

$$\begin{aligned} \partial_t u^\epsilon + H(\nabla u^\epsilon) &= \epsilon\Delta u^\epsilon\,, \\ u^\epsilon(\mathbf{x},0) &= u_0(\mathbf{x})\,. \end{aligned} \tag{2.13}$$

Moreover, u^ϵ is uniformly bounded in $W^{1,\infty}$. Thus there is an $u \in W^{1,\infty}$ such that

$$\begin{aligned} u^\epsilon \to u &\qquad \text{uniformly}, \\ \nabla u^\epsilon \to \nabla u &\qquad \text{in} \quad L^m\,, \end{aligned} \tag{2.14}$$

where m is any positive integer less than infinity. It follows from (2.14), (2.12) and (2.6) that

$$\mathbf{p}(\mathbf{x},t) = \nabla u \qquad \text{a.e..} \tag{2.15}$$

Moreover, u is the unique viscosity solution to (2.1) by the classical theory, see [20, 21, 8, 9].

3. Numerical Transition: an 2D Example

Consider the 2D Hamilton-Jacobi equation

$$\partial_t u + H(\partial_x u, \partial_y u) = 0\,, \tag{3.1a}$$

where

$$H(p,q) = \sqrt{1 + p^2 + q^2} \tag{3.1b}$$

is a convex function. The corresponding system of conservation laws is

$$\partial_t p + \partial_x H(p,q) = 0\,, \tag{3.2a}$$

$$\partial_t q + \partial_y H(p,q) = 0\,, \tag{3.2b}$$

with

$$\partial_t u + H(p,q) = 0\,. \tag{3.2c}$$

Integrating over the grid cell $c_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, one can define the cell average for a generic variable v over c_{ij} by

$$v_{ij} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} v(x,y)\, dxdy\,. \tag{3.3}$$

Integrating (3.2) over c_{ij}, one gets

$$\partial_t p_{ij} + \frac{1}{\Delta x}(H_{i+1/2,j} - H_{i-1/2,j}) = 0\,, \tag{3.4a}$$

$$\partial_t q_{ij} + \frac{1}{\Delta y}(H_{i,j+1/2} - H_{i,j-1/2}) = 0\,, \tag{3.4b}$$

$$\partial_t u_{ij} + H_{ij}(p,q) = 0\,. \tag{3.4c}$$

To define $H_{ij}(p,q)$ in (3.4c), since

$$H_{ij}(p,q) = H(p_{ij}, q_{ij}) + O(\Delta x^2), \tag{3.5}$$

it is natural to approximate (3.4c) by

$$\partial_t u_{ij} + H(p_{ij}, q_{ij}) = 0\,. \tag{3.6}$$

We call (3.6) the "Scheme-I", to distinguish from "Scheme-II, III and IV" to be introduced in Section 4. The only difference among these schemes lie in the integration of (3.2c).

We use the second order relaxation scheme to be introduced in Section 6 to define the numerical flux $H_{i+1/2,j}$ and $H_{i,j+1/2}$ in (3.4a), and then update u by (3.6) using the classical second order Runge-Kutta method. The numerical phenomenon, as well as the analysis and the remedy, however, is method independent and applies to at least all first and second order shock capturing methods.

We use the following initial data [26]:

$$\begin{aligned} u(x,y,0) &= -.25[\cos(2\pi x) - 1][\cos(2\pi y) - 1] + 1\,, \\ p(x,y,0) &= u_x(x,y,0) = .5\pi \sin(2\pi x)[\cos(2\pi y) - 1]\,, \\ q(x,y,0) &= u_y(x,y,0) = .5\pi \sin(2\pi y)[\cos(2\pi x) - 1]\,, \end{aligned} \tag{3.7}$$

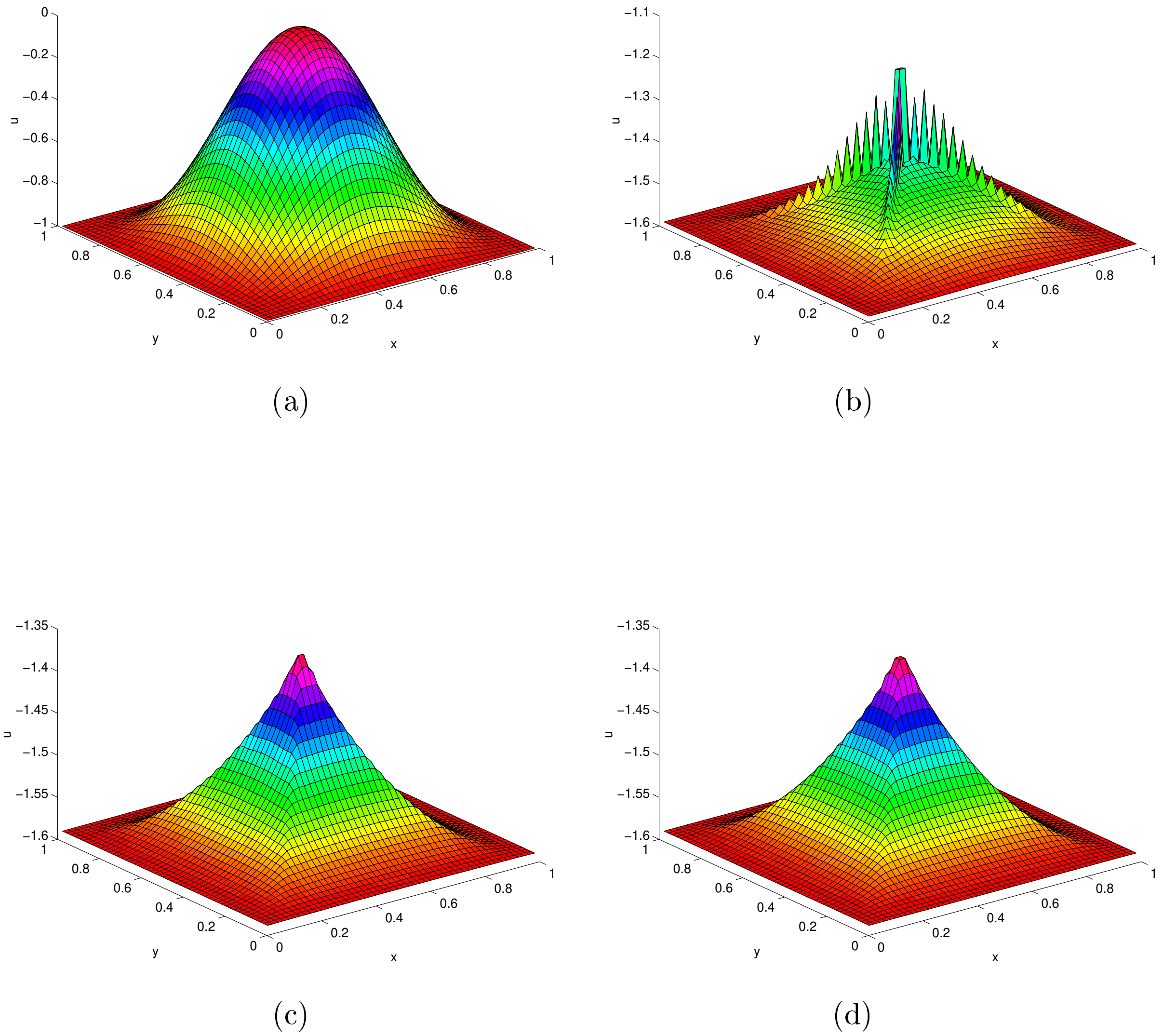


FIG.3.1. (a) $-u_{ij}$ at $t = 0$. (b) $-u_{ij}$ at $t = 0.6$ by Scheme-I. (c) $-u_{i+1/2,j}$ at $t = 0.6$ by Scheme-II. (d) $-u_{ij}$ at $t = 0.6$ by Scheme-IV.

which gives a flat surface in the boundary of the unit square with a global minimum at $(.5, .5)$. We use $\Delta t = 0.1$ and 50 grid points in each direction and periodic boundary conditions. The viscosity solution for u is a surface with its minimum focusing into a deep dent which opens up. The surface moves upward with unit speed, asymptotically approaching a flat sheet. For a better visualization we plot $-u$ in Fig.3.1. In Fig. 3.1a the initial profile is displayed. In Fig.3.1b we plot the numerical solution at $t = 0.6$ by Scheme-I. We also solve the same problem using Scheme-II and Scheme-IV to be introduced in section 4, which we define as good schemes. One can see

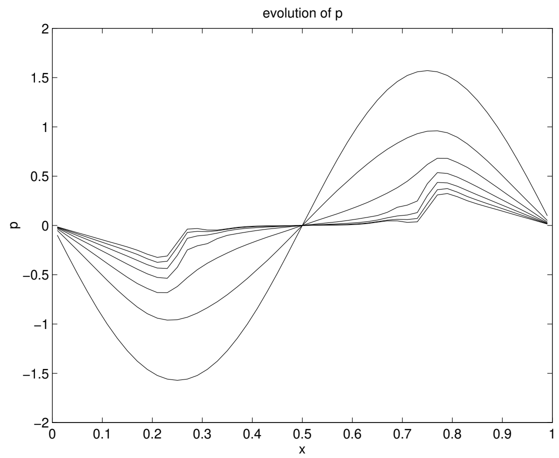
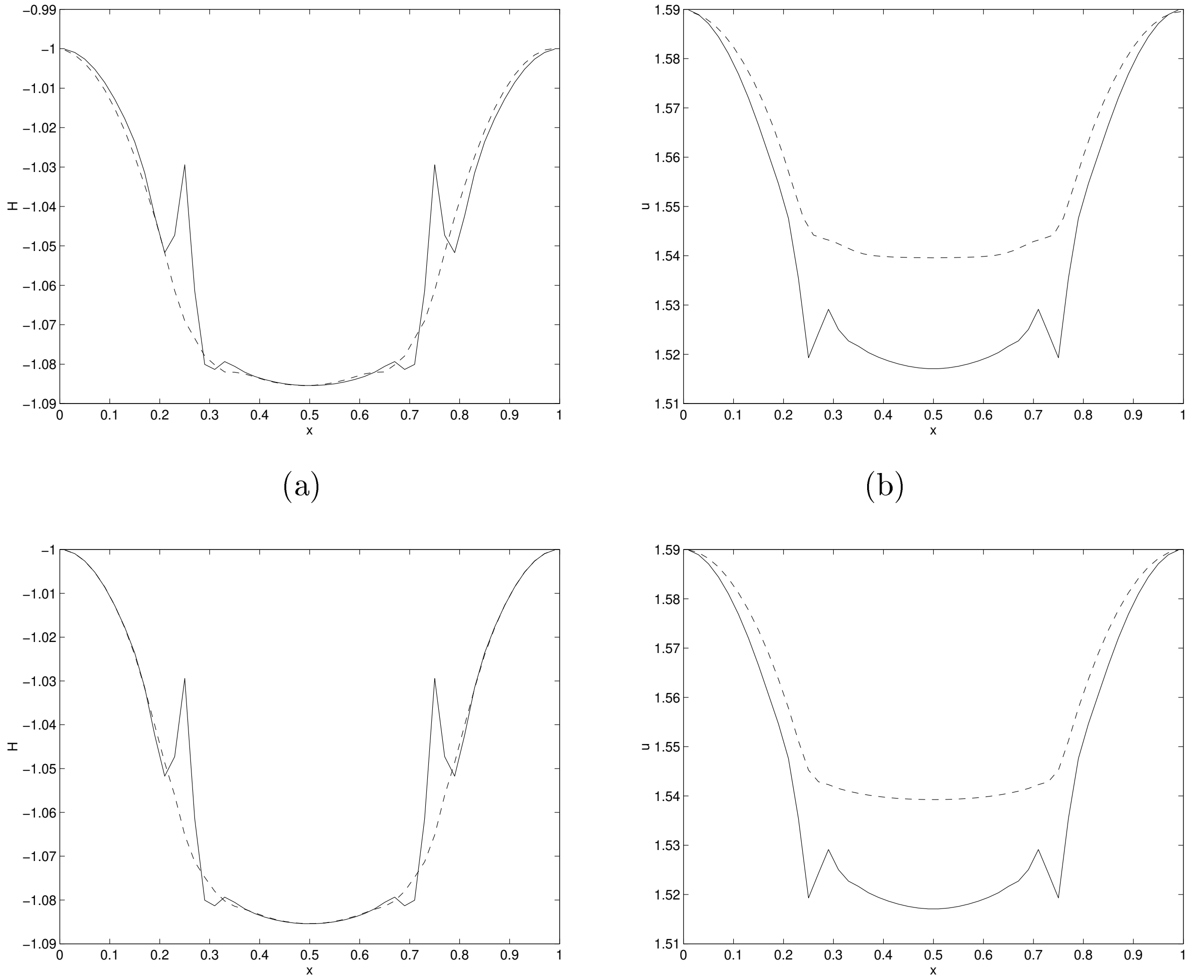


FIG.3.2. Numerical solution of p at $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ for fixed $y = 0.25$.

from these pictures that Scheme-I yields very different results from the other good ones. Besides the severe oscillations around the cusp, Scheme-I also gives unphysical u even in smooth region(notice the scaling difference in Fig.3.1a and Fig.3.1b). The Scheme-II and Scheme-IV both give the same viscosity solution obtained by Osher and Sethian [26] using a direct solver to (3.1).

In Fig.3.2 we depict a cross-section of p at $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6$ for fixed $y = 0.25$. Beginning with a smooth function, p soon develops a shock, and the evolution becomes slower at later times. In Fig.3.3a we depict $H_{ij} = H(p_{ij}, q_{ij})$ used by Scheme-I and $H_{i+1/2,j}$ used by Scheme-II and $\bar{\bar{H}}_{ij}$. The H used by Scheme-I has two spikes at the two discontinuities of p, while the H used in Scheme-II and IV are monotone profiles which differ from the first H only in the the spike regions. Such a difference in H, however, leads to a significant difference in u, as shown in Fig.3.3b and 3.3d.

In next section we show why the spike in H appears by a traveling wave analysis on a viscous regularization to (3.4a,b), and how one can recover a monotone H. In section 4 we introduce algorithms for (3.4c) (the Scheme-II, III and IV), which use the monotone H to solve for u. We show that using such H's we obtain the viscosity solution of the H-J equation. This study follows into the framework of [13], where the effect of numerical viscosity for slowly moving shock in 1-D is examined and similar numerical phenomenon resolved.



(a) (b)

(c) (d)

FIG.3.3. (a) H_{ij} at $t = 0.6$ by solid line and $H_{i+1/2,j}$ at $t = 0.6$ by the dashed line. (b) The solid line is u at $t = 0.6$ by Scheme-I, and the dashed line is u at $t = 0.6$ by Scheme-II. (c) H_{ij} at $t = 0.6$ by solid line and $\bar{\bar{H}}_{i,j}$ at $t = 0.6$ by the dashed line. (d) The solid line is u at $t = 0.6$ by Scheme-I, and the dashed line is u at $t = 0.6$ by Scheme-IV.

4. The Effect of Numerical Viscosity

4.1. An Illustrative Example

The simplest way to understand the numerical phenomenon reported here is to

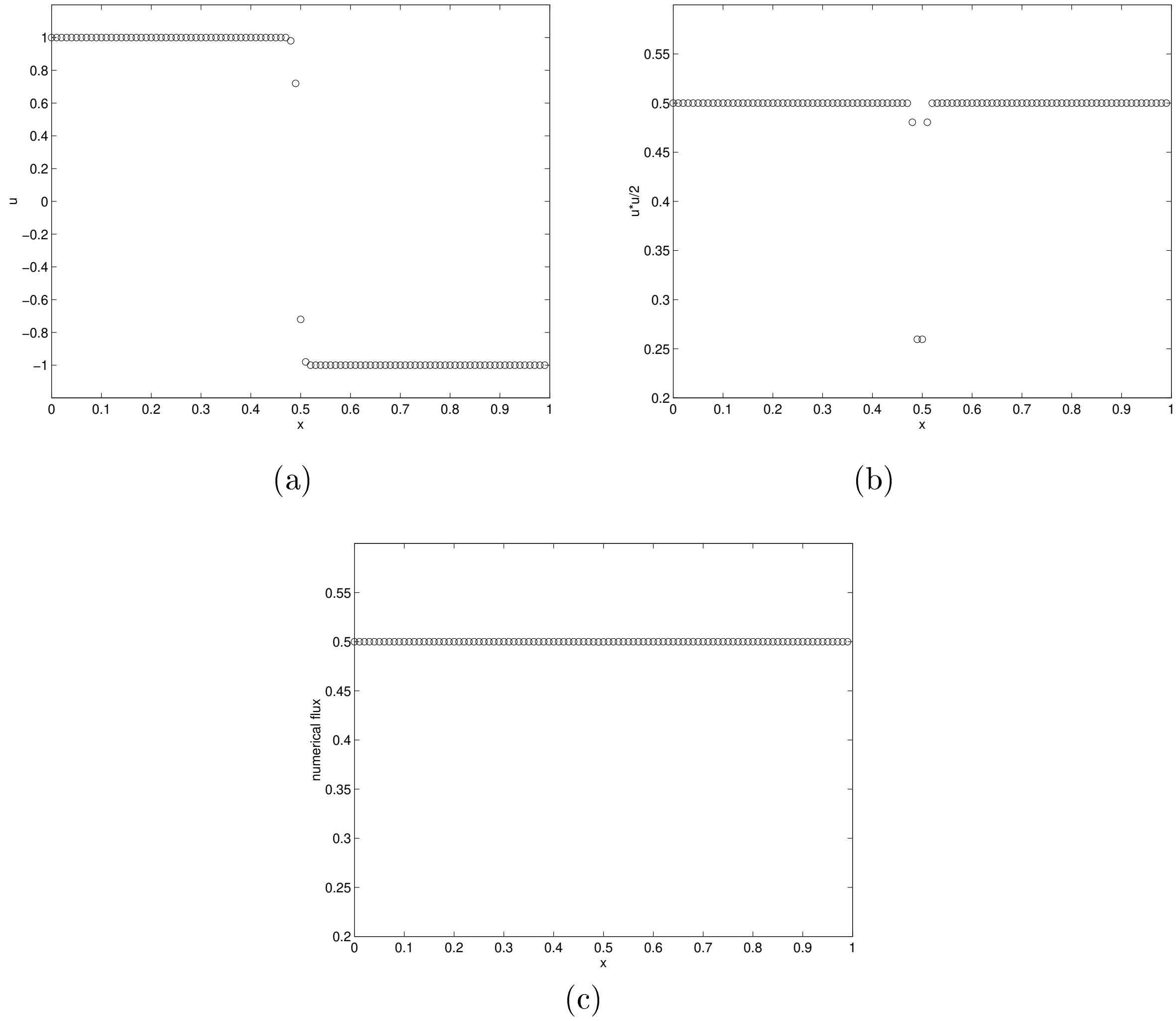


FIG.4.1. The numerical solution to (3.0) at $t = 0.1$. (a) u; (b) $u^2/2$; (c) numerical flux.

consider the 1-D inviscid Burger's equation

$$\partial_t u + \partial_x \frac{u^2}{2} = 0\,, \tag{4.1a}$$

$$u(0,x) = 1 \quad \text{if} \quad x \le 0.5\,; \quad u(0,x) = -1 \quad \text{if} \quad 0.5 < x \le 1\ . \tag{4.1b}$$

The exact solution to this problem is a stationary shock sitting in the origin, so the solution at later time is always the same as the initial one. This exact solution to $u^2/2$ is $u^2 \equiv 0.5$ for all x. We also use the second order relaxation scheme [16] to solve this problem numerically, and display the numerical solutions of u and $u^2/2$ at $t = 0.1$ in Fig.4.1. While u_i is a monotone profile across the discontinuity, $u_i^2/2$ exhibits a spike at the shock layer of u, which significantly differs from the analytic solution

$u^2/2 \equiv 0.5$. We also plot the numerical flux $(U^2/2)_{i+1/2}$, which agrees perfectly with the analytic solution without the spike.

This simple example shows that one should be very careful when using the information in the unphysical shock layer to obtain other physical quantities (such as $u^2/2$ here). When $u^2/2$ is needed it is clearly better to use the numerical flux $(u^2/2)_{i+1/2}$ rather than the cell average value $(u^2/2)_i$.

For nonlinear systems the spike appears for physical quantities, such as the momentum in the Euler equations, when the shock is stationary or move slowly. For general systems of conservation laws such a phenomenon was studied by Jin and Liu [13], while earlier similar approaches had been taken for specific systems of conservation laws [4,12,29]. Here we extend these studies to the 2-D problem, and show that the numerical viscosity causes the spike, which can be eliminated using a change of variable. Moreover this analysis also leads to several working schemes that give the correct viscosity solution to the H-J equation. Scheme-II.

4.2. The Existence of the Spike

In order to understand the behavior of the solution obtained by the shock capturing methods, we study the following viscous system of conservation laws for p and q:

$$\partial_t p + \partial_x H = \epsilon\, \partial_{xx} p\,, \tag{4.2a}$$

$$\partial_t q + \partial_y H = \epsilon\, \partial_{yy} q\,. \tag{4.2b}$$

This is the modified equation for a linear, first order scheme such as the Lax-Friedrichs scheme. For nonlinear schemes the numerical viscosity is nonlinear, but similar analysis can also be carried out to study similar numerical phenomenon.

To study the behavior for small viscosity ϵ, we look for planar traveling wave solution to (4.2). First we look for the planar traveling waves moving in x-direction of the form

$$p(x,y,t) = \tilde{p}(\xi)\,, \quad q(x,y,t) = \tilde{q}(\xi) \tag{4.3}$$

where $\xi = \frac{x-st}{\epsilon}$, s is the speed of wave (a constant), with asymptotic boundary values at infinities as

$$\tilde{p}(\pm\infty) = p_\pm\,, \quad \tilde{q}(\pm\infty) = q_\pm\,. \tag{4.4}$$

Substituting (4.3) in (4.2) one obtains

$$-s\tilde{p}_\xi + H_\xi(\tilde{p}(\xi),\tilde{q}(\xi)) = \tilde{p}_{\xi\xi}\,, \tag{4.5a}$$

$$-s\tilde{q}_\xi = 0\,. \tag{4.5b}$$

Thus the problem for $\tilde{q}(\xi)$ becomes

$$-s\tilde{q}_\xi = 0\,, \qquad \tilde{q}(\pm\infty) = q_\pm\,. \tag{4.6}$$

Clearly for any non-stationary wave, $s \neq 0$, the only solution to (4.6) is

$$\tilde{q}(\xi) \equiv q_+ = q_- = q_\pm\,. \tag{4.7}$$

Then the problem for $\tilde{p}(\xi)$ becomes

$$-s\tilde{p}_\xi + \partial_\xi H(\tilde{p}(\xi), q_\pm) = \partial_{\xi\xi}\tilde{p}(\xi)\,, \tag{4.8a}$$

$$\lim_{\xi\to\pm\infty} \tilde{p}(\xi) = p_\pm\,. \tag{4.8b}$$

This becomes the 1-D scalar equation, so the existence and monotonicity of the solution is guaranteed for shock wave [21].

Integrating (4.8a) over ξ, one gets

$$-s\tilde{p} + H = \tilde{p}_\xi + C\,, \tag{4.9}$$

where $C = -sp_- + H(p_-, q_-)$ is the integration constant. Clearly (4.9) shows that

$$H = s\tilde{p} + \tilde{p}_\xi + C\,. \tag{4.10}$$

Since $\tilde{p}$ is a monotone profile from p_- to p_+, $\tilde{p}_\xi$ has the spike structure. (4.9) indicates that H is a superposition of a monotone profile $s\tilde{p}$ and a spike $\tilde{p}_\xi$. When $|s\tilde{p}| << |\tilde{p}_\xi|$ (this is particularly the case when the shock speed is small compared to other characteristic speeds), the spike dominates and the solution of H becomes a spike in the ξ direction.

In the case that $s = 0$, (4.5b) has no constraint on $\tilde{q}(\xi)$. (4.5a) becomes

$$\partial_\xi H(\tilde{p}, \tilde{q}) = \tilde{p}_{\xi\xi}\,,$$

which implies

$$\partial_\xi \tilde{p} = H(\tilde{p}(\xi), \tilde{q}(\xi)) - H(p_-, q_-) \tag{4.11}$$

Then the solution is not unique. Thus, we have to be careful with approximation to (4.2) since it may pick up the wrong solution !! However, a natural requirement on the solution of (4.2) is that $p_y = q_x$, which implies $\partial_\xi \tilde{q}(\xi) = 0$, so even in the stationary case one should have $\tilde{q}(\xi) \equiv q_+ = q_-$.

Remark: If instead of (4.2), one considers the fully regularized problem

$$\partial_t p + \partial_x H = \epsilon\, \Delta p\,, \tag{4.12a}$$

$$\partial_t q + \partial_y H = \epsilon\, \Delta q\,, \tag{4.12b}$$

where $\Delta = \partial_{xx} + \partial_{yy}$ is the standard Laplacian. Then by repeating our previous analysis, one sees that a planary traveling wave solution of the form (4.3) satisfies

$$-s\partial_\xi \tilde{q}(\xi) = \partial_{\xi\xi}\tilde{q} \qquad \lim_{\xi\to\pm\infty} \tilde{q}(\xi) = q_\pm\,. \tag{4.13}$$

(4.13) has a smooth solution if and only if $\tilde{q}(\xi) \equiv q_\pm$ no matter $s = 0$ or not. The equation for $\tilde{p}(\xi)$ is now

$$\begin{aligned} -s\partial_\xi \tilde{p}(\xi) + \partial_\xi H(\tilde{p}(\xi), q_\pm) &= \partial_{\xi\xi}\tilde{p}\,, \\ \lim_{\xi\to\infty} \tilde{p}(\xi) &= p_\pm\,, \end{aligned} \tag{4.14}$$

which again has a spiky profile for H since integrating (4.14) over ξ gives (4.9).

Since most shock capturing methods use dimensional splitting, (4.2) is more typical than (4.12).

4.3. Recovering the Monotone H

As done in [13] for 1-D system of conservation law, here we can also recover a monotone H. Note from (4.10),

$$\bar{H} \equiv H - \tilde{p}_\xi = s\tilde{p} + C \tag{4.15}$$

is a monotone profile. The equivalence of (4.15) for (4.2a) is the change of variable

$$\bar{H} = H - \epsilon\partial_x p\,. \tag{4.16}$$

Similarly one can also recover a monotone H from (4.2b) using

$$\bar{H} = H - \epsilon\partial_y q\,. \tag{4.17}$$

5. The Working Schemes

The traveling wave analysis in section 4 not only explains the source of the spike, which is solely due to the presence of the numerical viscosity, but also shows how to remove the spike: *an analog of the transformation (4.16) or (4.17) should be performed numerically.*

The numerical flux for a general shock capturing method can be written as

$$H_{i+1/2,j} = \tfrac{1}{2}(H_{ij} + H_{i+1,j}) - P_{i+1/2,j}\Delta_{i+1/2}p\,, \tag{5.1a}$$

$$H_{i,j+1/2} = \tfrac{1}{2}(H_{ij} + H_{i,j+1}) - Q_{i,j+1/2}\Delta_{j+1/2}q\,, \tag{5.1b}$$

where $\Delta_{j+1/2}\cdot = \cdot_{j+1} - \cdot_j$, and P, Q are the numerical viscosity matrices, which are method specific. We leave P, Q unspecified since this study is general for *all* shock capturing methods. Applying (5.1) into (3.4a,b) one gets

$$\partial_t p_{ij} + \frac{1}{2\Delta x}(H_{i+1,j} - H_{i-1,j}) = \frac{1}{\Delta x}(P_{i+1/2,j}\Delta_{i+1/2}p - P_{i-1/2,j}\Delta_{i-1/2}p)\,, \tag{5.2a}$$

$$\partial_t q_{ij} + \frac{1}{2\Delta y}(H_{i,j+1} - H_{i,j-1}) = \frac{1}{\Delta y}(Q_{i,j+1/2}\Delta_{j+1/2}q - Q_{i,j-1/2}\Delta_{j-1/2}q)\,. \tag{5.2b}$$

The continuous limit of (5.2) is

$$\partial_t p + \partial_x H(p,q) = \partial_x(P\partial_x p)\,, \tag{5.3a}$$

$$\partial_t q + \partial_y H(p,q) = \partial_y(Q\partial_y q)\,. \tag{5.3b}$$

A analogous transformation of (4.16) for (5.3) is then

$$\bar{H} = H - P\partial_x p\,. \tag{5.4}$$

The discrete analog of (5.4) for (5.2a) is

$$\bar{H}_{ij} = \frac{1}{2}(H_{ij} + H_{i+1,j}) - P_{i+1/2,j}\Delta_{i+1/2}p\,. \tag{5.5}$$

This is a consistent, second order discretization of (5.4). By comparing (5.5) with (5.1a) one sees that

$$\bar{H}_{ij} = H_{i+1/2,j}\,. \tag{5.6}$$

Thus the discrete transformation (5.5) simply transfers the cell average value or the value at cell center H_{ij} into the numerical flux in x-direction $H_{i+1/2,j}$!

This defines our recovering of a monotone H. While the cell average H_{ij} has a spike, the numerical flux $H_{i+1/2,j}$ does not have the spike (this is also justified numerically in Fig3.3a and 3.3c)! This is the quantity to be used to solve for u in (5.2c), instead of (5.6). Thus the Scheme-II for u to replace (5.6) should be

$$\partial_t u_{i+1/2,j} + H_{i+1/2,j} = 0\,. \tag{5.7}$$

Clearly this is also consistent with (3.4a). Since $p = u_x$, (3.4a) is just

$$\partial_t \partial_x u_{ij} + \frac{1}{\Delta x}(H_{i+1/2,j} - H_{i-1/2,j}) = 0\,. \tag{5.8}$$

If one thinks of $\partial_x u_{ij} = \frac{1}{\Delta x}(u_{i+1/2,j} - u_{i-1/2,j})$ then (5.7) is clearly consistent to (5.8).

A natural question is whether such an u is the viscosity solution to the H-J equation. Using (5.5) and (5.6) in (5.7) one sees that

$$\partial_t u_{i+1/2,j} + \frac{1}{2}(H_{ij} + H_{i+1,j}) = P_{i+1/2,j}\Delta_{i+1/2}p\,. = 0\,. \tag{5.9}$$

This is consistent to

$$\partial_t u + H = \Delta x P \partial_x p = \Delta x P \partial_{xx} u\,. \tag{5.10}$$

Thus it selects the viscosity solution with viscosity in x-direction.

Another working scheme (Scheme-III) is to use the numerical flux in y-direction, $H_{i,j+1/2}$. This will give a way to compute $u_{i,j+1/2}$ as

$$\partial_t u_{i,j+1/2} + H_{i,j+1/2} = 0\,, \tag{5.11}$$

which selects the viscosity solution u as the limit of

$$\partial_t u + H = \Delta y Q \partial_{yy} u\,. \tag{5.12}$$

Another reasonable way to define a monotone H is to take the following average H,

$$\bar{\bar{H}}_{ij} = \tfrac{1}{4}(H_{i+1/2,j} + H_{i-1/2,j} + H_{i,j+1/2} + H_{i,j-1/2}) \tag{5.13}$$

and solve for u_{ij} from

$$\partial_t u_{ij} + \bar{\bar{H}}_{ij} = 0 \tag{5.14}$$

(5.14) is called Scheme-IV. Since (5.10) takes numerical fluxes from both x and y directions, it is consistent to the viscosity solution defined by

$$\partial_t u + H = \Delta x P \partial_{xx} u + \Delta y Q \partial_{yy} u\,. \tag{5.15}$$

In section 3 we showed the numerical results obtained using Scheme-II, and Scheme-IV. Scheme-III gives pretty much the same results as the Scheme-II. They all give the correct (and good) viscosity solution. It should be remarked that such an approach is dimension independent and applies to problems in any space dimensions.

Remark: For (1.3a) and (1.3b) to be consistent, $\mathbf{p}$ needs to remain as the gradient of u. Numerically, this property is preserved if the discretization of (3.2c) is consistent to those of (3.2a,b). The Scheme-II, (5.7), is indeed its consistency with (3.4a). Similarly, Scheme-II, (5.11), is consistent to (3.4b). This formal argument provides a different perspective on why these schemes should work.

6. A Relaxation Approximation to the H-J Equation

6.1. A Relaxation Approximation to the **p***-formulation*

In this section we introduce a relaxation approximation to the n-dimensional Hamilton-Jacobi equation

$$\partial_t u + H(\nabla u) = 0\,. \tag{6.1}$$

This possibly provides a way to define the weak solution to (6.1). At least in 1-D one can show that our relaxation approximation defines an entropy solution to (6.1) [24,32].

Let $p_i = \partial_{x_i} u (i = 1, 2, \cdots, n)$, and $\mathbf{p} = (p_1, p_2, \cdots, p_n)$, then $\mathbf{p}$ formally satisfies the following systems of conservation laws

$$\partial_t \mathbf{p} + \nabla H(\mathbf{p}) = 0\,. \tag{6.2}$$

We introduce the following hyperbolic system with relaxation

$$\partial_t \mathbf{p} + \nabla w = 0\,, \tag{6.3a}$$

$$\partial_t w + a\nabla \cdot \mathbf{p} = -\frac{1}{\epsilon}(w - H(\mathbf{p}))\,, \tag{6.3b}$$

$$\partial_t u + w = 0\,, \tag{6.3c}$$

where a is a positive constant to be chosen later such that this system is stable. Note that in multi-D this relaxation approximation differs from the original one we proposed in [16] for general systems of conservation laws. Here, to regularize an n-dimensional system of n-equations we just add one additional rate equation (6.3b), due to the special structure of the Hamilton-Jacobi equation, while in our earlier formulation [16] we need to add n more equations. In 1-D it exactly recovers our original relaxation system for conservation laws.

Comparing with the original system of conservation laws (6.2), the relaxation system (6.3) has several advantages, such as finite speed of propagation, linear convection, and greater regularity. Furthermore, one checks easily that (6.3) is a symmetrizable system in 1-D. Moreover, the relaxation system is (strongly) hyperbolic, while (6.2) is only weakly hyperbolic. This property possibly provides a good framework to design shock capturing schemes for weakly hyperbolic system.

To derive a dissipative condition for (6.3), we may apply the Chapman-Enskog expansion to (6.3). By (6.3b),

$$w = H(\mathbf{p}) - \epsilon(\partial_t w + a\nabla\cdot\mathbf{p})\,. \tag{6.4}$$

Also, (6.3c) gives

$$\partial_t \nabla u + \nabla w = 0\,. \tag{6.5}$$

Combining (6.5) with (6.3a) gives

$$\partial_t(\nabla u - \mathbf{p}) = 0\,. \tag{6.6}$$

We take the same initial data so that there is no initial layer,

$$\mathbf{p}(\mathbf{x},0) = p_0(\mathbf{x}) \equiv \nabla u_0(\mathbf{x})\,. \tag{6.7}$$

With (6.7), this implies

$$\nabla u(\mathbf{x},t) = \mathbf{p}(\mathbf{x},t) \qquad \text{for} \quad t \geq 0\,. \tag{6.8}$$

Using (6.4),

$$w(\mathbf{x},t) = H(\nabla u) - \epsilon(\partial_t w + a\Delta u)\,. \tag{6.9}$$

To compute $\partial_t w$, from (6.4),

$$\partial_t w = (\nabla_{\mathbf{p}} H(\mathbf{p}))^T \partial_t \mathbf{p} - \epsilon(\partial_{tt} w + a\nabla\cdot\mathbf{p}_t)\,. \tag{6.10}$$

Using (6.3a),

$$\partial_t w = -(\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\nabla w) - \epsilon(\partial_{tt} w + a\nabla\cdot\mathbf{p}_t)\,. \tag{6.11}$$

Again using (6.4), we get

$$\begin{aligned} \nabla w =& \nabla H(\mathbf{p}) - \epsilon(\partial_t \nabla w + a\nabla(\nabla\cdot\mathbf{p})) \\ =& \nabla\mathbf{p}\cdot\nabla_{\mathbf{p}} H - \epsilon(\partial_t \nabla w + a\nabla(\nabla\cdot\mathbf{p}))\,, \end{aligned} \tag{6.12}$$

where

$$\nabla\mathbf{p} = \begin{pmatrix} \partial_{x_1} p_1 & \partial_{x_1} p_2 & \cdots & \partial_{x_1} p_n \\ \partial_{x_2} p_1 & \partial_{x_2} p_2 & \cdots & \partial_{x_2} p_n \\ \cdots & \cdots & \cdots & \cdots \\ \partial_{x_n} p_1 & \partial_{x_n} p_2 & \cdots & \partial_{x_n} p_n \end{pmatrix}, \qquad \nabla_{\mathbf{p}} H(\mathbf{p}) = \begin{pmatrix} \partial_{p_1} H \\ \partial_{p_2} H \\ \vdots \\ \partial_{p_n} H \end{pmatrix}. \tag{6.13}$$

Applying (6.12) in (6.11) gives

$$\begin{aligned}\partial_t w = &- (\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\nabla \mathbf{p})(\nabla_{\mathbf{p}} H(\mathbf{p})) + \epsilon (\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\partial_t \nabla w + a \nabla(\nabla \cdot \mathbf{p})) \\ &- \epsilon(\partial_{tt} w + a \nabla \cdot \mathbf{p}_t)\,.\end{aligned} \tag{6.14}$$

Finally, using (6.3c), (6.9) and (6.14), we have

$$\partial_t u + H(\nabla u) = \epsilon (a \Delta u - (\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\nabla \mathbf{p})(\nabla_{\mathbf{p}} H(\mathbf{p}))) + O(\epsilon^2)\,. \tag{6.15}$$

Thus the dissipative condition should be that

$$a \Delta u - (\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\nabla \mathbf{p}) \nabla_{\mathbf{p}} H(\mathbf{p}) \qquad \text{is an elliptic operator.} \tag{6.16}$$

Note

$$\begin{aligned}&(\nabla_{\mathbf{p}} H(\mathbf{p}))^T (\nabla \mathbf{p})(\nabla_{\mathbf{p}} H(\mathbf{p}))) \\ =&(\partial_{p_1} H, \cdots, \partial_{p_n} H) \begin{pmatrix} \partial^2_{x_1 x_1} u & \partial^2_{x_1 x_2} u & \cdots & \partial^2_{x_1 x_n} u \\ \partial^2_{x_2 x_1} u & \partial^2_{x_2 x_2} u & \cdots & \partial^2_{x_2 x_n} u \\ \cdots & \cdots & \cdots & \cdots \\ \partial^2_{x_n x_1} u & \partial^2_{x_n x_2} u & \cdots & \partial^2_{x_n x_n} u \end{pmatrix} \begin{pmatrix} \partial_{p_1} H \\ \partial_{p_2} H \\ \cdots \\ \partial_{p_n} H \end{pmatrix} \\ =& \sum_{ij} (\partial_{p_i} H \partial_{p_j} H) \partial^2_{x_i x_j} u\,.\end{aligned} \tag{6.17}$$

Therefore, the uniform ellipticity requires that

$$a \delta_{ij} - \partial_{p_i} H \partial_{p_j} H > 0\,. \tag{6.18}$$

That is,

$$a I - (\nabla_{\mathbf{p}} H)(\nabla_{\mathbf{p}} H)^T > 0\,. \tag{6.19}$$

For any $\xi \in R^n$,

$$\begin{aligned}&\xi^t (a I - (\nabla_{\mathbf{p}} H)(\nabla_{\mathbf{p}} H)^T) \xi \\ =& a |\xi|^2 - (\xi_1, \cdots, \xi_n) \begin{pmatrix} \partial_{p_1} H \\ \partial_{p_2} H \\ \cdots \\ \partial_{p_n} H \end{pmatrix} (\partial_{p_1} H, \cdots, \partial_{p_n} H) \begin{pmatrix} \xi_1 \\ \xi_2 \\ \cdots \\ \xi_n \end{pmatrix} \\ =& (a - |\nabla_{\mathbf{p}} H|^2) |\xi|^2\,.\end{aligned} \tag{6.20}$$

Thus the sufficient condition should be

$$a > |\nabla_{\mathbf{p}} H|^2\,. \tag{6.21}$$

This condition is analogous to the *subcharacteristic condition* for the relaxation approximation for conservation laws [3, 16].

Remark: The equivalent relaxation approximation for the H-J equation (1.1) is clearly

$$\begin{aligned} \partial_t u + w &= 0\,, \\ \partial_t w + a\Delta u &= -\frac{1}{\epsilon}(w - H(\nabla u))\,. \end{aligned} \tag{6.22}$$

This is obtained after applying $\mathbf{p} = \nabla u$ in (6.3b). Upon eliminating w this becomes

$$\partial_t u + H(\nabla u) = -\epsilon(\partial_{tt} u - a\Delta u)\,, \tag{6.23}$$

which shows that (6.22) is a perturbation of the H-J equation by a linear wave operator.

6.2. A Relaxation Scheme

As was done in [16], we can also derive a class of relaxation schemes to the Hamilton-Jacobi equation based on a discretization of the relaxation approximation (6.3). For example, for an 2D problem,

$$\partial_t p_1 + \partial_x w = 0\,, \tag{6.24a}$$

$$\partial_t p_2 + \partial_y w = 0\,, \tag{6.24b}$$

$$\partial_t w + a\partial_x p_1 + a\partial_y p_2 = -\tfrac{1}{\epsilon}(w - H(p_1, p_2))\,, \tag{6.24c}$$

$$\partial_t u + w = 0\,. \tag{6.24d}$$

Since (6.24a-c) is a semi-linear system of hyperbolic equations, one can apply the upwind scheme dimension-by-dimension easily. Integrating (6.24a-c) over c_{ij}, we have

$$\partial_t p^1_{ij} + \frac{1}{\Delta x}(w_{i+1/2,j} - w_{i-1/2,j}) = 0\,, \tag{6.25a}$$

$$\partial_t p^2_{ij} + \frac{1}{\Delta y}(w_{i,j+1/2} - w_{i,j-1/2}) = 0\,, \tag{6.25b}$$

$$\partial_t w_{ij} + a\frac{1}{\Delta x}(p^1_{i+1/2,j} - p^1_{i-1/2,j}) + a\frac{1}{\Delta y}(p^2_{i,j+1/2} - p^2_{i,j-1/2}) = -\tfrac{1}{\epsilon}(w_{ij} - H(p^1_{ij}, p^2_{ij}))\,. \tag{6.25c}$$

For notation convenience we use $p^1 = p_1, p^2 = p_2$. To define the numerical fluxes we use the second order upwind scheme (with slope limiter [31]) dimension-by-dimension. In x-direction the Riemann invariants are $p^1 \pm \frac{1}{\sqrt{a}} w$, which move with characteristic speeds $\pm\sqrt{a}$, thus the second order upwind discretization to $p^1 \pm \frac{1}{\sqrt{a}} w$ is

$$\begin{aligned} (p^1 + \tfrac{1}{\sqrt{a}}\, w)_{i+1/2,j} &= (p^1 + \tfrac{1}{\sqrt{a}}\, w)_{ij} + \tfrac{1}{2} h^x_j \sigma^{x,+}_{ij}\,, \\ (p^1 - \tfrac{1}{\sqrt{a}}\, w)_{i+1/2,j} &= (p^1 - \tfrac{1}{\sqrt{a}}\, w)_{i+1,j} - \tfrac{1}{2} h^x_{j+1} \sigma^{x,-}_{i+1,j}\,, \end{aligned} \tag{6.26}$$

where $\sigma_{ij}^{\pm}$ are the limited slopes of $p^1 \pm \frac{1}{\sqrt{a}}w$ defined in (6.29), and h_i^x is the mesh size in x direction. This gives

$$
\begin{aligned}
p^1_{i+1/2,j} &= \tfrac{1}{2}(p^1_{ij} + p^1_{i+1,j}) - \frac{1}{2\sqrt{a}}\,(w_{i+1,j} - w_{ij}) \\
&\quad + \frac{1}{4}\,(h_i^x \sigma_{ij}^{x,+} - h_{i+1}^x \sigma_{i+1,j}^{x,-})\,, \\
w_{i+1/2,j} &= \tfrac{1}{2}(w_{ij} + w_{i+1,j}) - \frac{\sqrt{a}}{2}\,(p^1_{i+1,j} - p^1_{ij}) \\
&\quad + \tfrac{\sqrt{a}}{4}\,(h_i^x \sigma_{ij}^{x,+} + h_{i+1}^x \sigma_{i+1,j}^{x,-})\,.
\end{aligned}
\tag{6.27}
$$

Similarly in the y-direction we have the following numerical fluxes:

$$
\begin{aligned}
p^2_{i+1/2,j} &= \tfrac{1}{2}(p^2_{ij} + p^2_{i,j+1}) - \frac{1}{2\sqrt{a}}\,(w_{i,j+1} - w_{ij}) \\
&\quad + \frac{1}{4}\,(h_j^y \sigma_{ij}^{y,+} - h_{j+1}^y \sigma_{i,j+1}^{y,-})\,, \\
w_{i,j+1/2} &= \tfrac{1}{2}(w_{ij} + w_{i,j+1}) - \frac{\sqrt{a}}{2}\,(p^2_{i,j+1} - p^2_{ij}) \\
&\quad + \tfrac{\sqrt{a}}{4}\,(h_j^y \sigma_{ij}^{y,+} + h_{j+1}^y \sigma_{i,j+1}^{y,-})\,,
\end{aligned}
\tag{6.28}
$$

where h_j^y is the mesh size in y-direction. The limited slopes are defined as

$$
\begin{aligned}
\sigma_{ij}^{x,\pm} &= \frac{1}{h_i^x}\left(p^1_{i+1,j} \pm \frac{1}{\sqrt{a}}\,w_{i+1,j} - p^1_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}\right)\,\phi(\theta_{ij}^{x,\pm})\,, \\
\theta_{ij}^{x,\pm} &= \frac{p^1_{ij} \pm \frac{1}{\sqrt{a}}\,w_{ij} - p^1_{i-1,j} \mp \frac{1}{\sqrt{a}}\,w_{i-1,j}}{p^1_{i+1,j} \pm \frac{1}{\sqrt{a}}\,w_{i+1,j} - p^1_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}}\,; \\
\sigma_{ij}^{y,\pm} &= \frac{1}{h_j^y}\left(p^2_{i,j+1} \pm \frac{1}{\sqrt{a}}\,w_{i,j+1} - p^2_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}\right)\,\phi(\theta_{ij}^{y,\pm})\,, \\
\theta_{ij}^{y,\pm} &= \frac{p^2_{ij} \pm \frac{1}{\sqrt{a}}\,w_{ij} - p^2_{i,j-1} \mp \frac{1}{\sqrt{a}}\,w_{i,j-1}}{p^2_{i,j+1} \pm \frac{1}{\sqrt{a}}\,w_{i,j+1} - p^2_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}}\,,
\end{aligned}
\tag{6.29}
$$

with [30]

$$
\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}\,. \tag{6.30}
$$

For the time discretization, denoting (6.25a-c) by

$$
\partial_t U + A(U) = B(U) \tag{6.31}
$$

where $A(U)$ stands for the convection terms, and $B(U)$ for the relaxation term. Then one can just use a second order (for $\epsilon << \Delta t$) Runge-Kutta type splitting [16]:

$$U^* = U^n + B(U^*) \;; \tag{6.32a}$$
$$U^{(1)} = U^* - kA(U^*) \;, \tag{6.32b}$$
$$U^{**} = U^{(1)} + B(U^{**}) \;; \tag{6.32c}$$
$$U^{(2)} = U^{**} - kA(U^{**}) \;, \tag{6.32d}$$
$$U^{n+1} = \tfrac{1}{2}\left(U^n + U^{(2)}\right) \;, \tag{6.32e}$$

where k is the time step. Although we use implicit relaxation term, since the relaxation term is not fully-ranked, in (6.32a) and (6.32c) one can always solve for p and q first, using which to update $H(p,q)$, then solve for the unknown w which is linear in both equations. Thus (6.32) can be implemented explicitly.

While for (6.24d), we use the discretization discussed in section 5 for H and then use the classical second order Runge-Kutta method for time marching. Note for the relaxation scheme we define

$$H_{i+1/2,j} = w_{i+1/2,j}\,, \quad H_{i,j+1/2} = w_{i,j+1/2}\,. \tag{6.33}$$

It is known that, for a weakly hyperbolic system, shock capturing methods in general may have very poor accuracy [28, 10]. However, as demonstrated in next section, our second order relaxation schemes seem to produce quite satisfactorily results. The two reasons, in our understanding, are the following. First, the first order relaxation scheme, as the relaxation time approaches zero, is of Lax-Friedrichs type. It is known that the first order Lax-Friedrichs scheme and its second order, multidimensional sequel produce quite good results for weakly hyperbolic systems [10,11]. Second, the relaxation system (6.3) is (strongly) hyperbolic. The relaxation scheme is a discretization of this strongly well-posed system, rather than a weakly hyperbolic system. We believe both factors contribute to the good performance of the second order relaxation scheme for the weakly hyperbolic system (1.3).

7. Numerical Examples

In this section we present several numerical examples using the relaxation schemes. As in [27], we solve the 2D H-J equation

$$\begin{aligned} &\partial_t u + H(\partial_x u, \partial_y u) = 0\,, \\ &u(x,y,0) = -\cos\pi\left(\frac{x+y}{2}\right)\,, \end{aligned} \tag{7.1}$$

in domain $-2 \le x, y \le 2$, with a convex H (corresponding to the Burger's equation in conservation laws)

$$H(u,v) = \frac{1}{2}(u+v+1)^2 \tag{7.2a}$$

and a nonconvex H

$$H(u,v) = \cos(u+v+1)\,. \tag{7.2b}$$

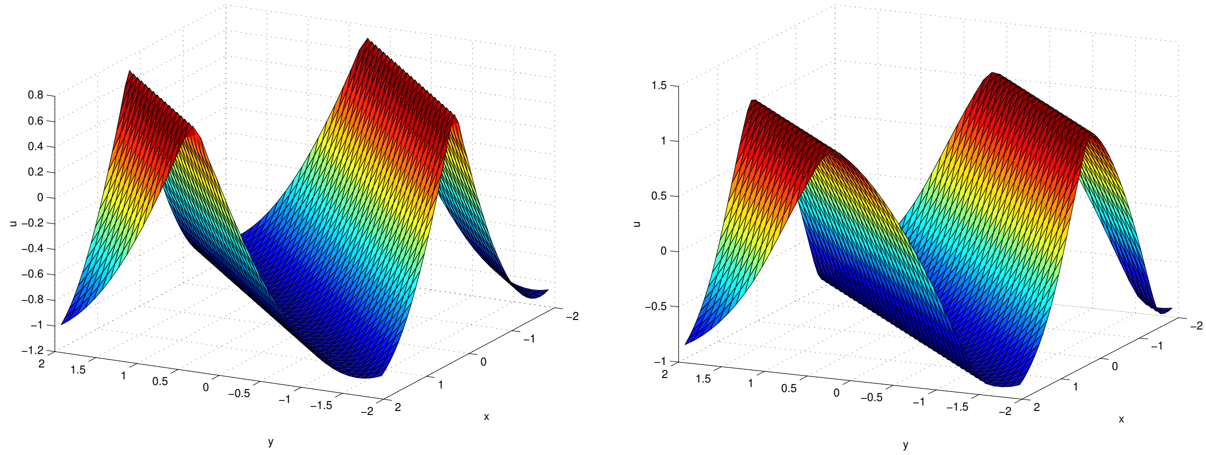
We use 40 grid points in both x and y directions, and compute to $t = 1.5/\pi^2$. The results are depicted in Fig.7.1. We also did a convergence study in both l_1 and l_∞ norms at $t = 0.5/\pi^2$, and the results are given in Table 7.1. Due to the large gradient near the local extrema the decay is not quite second order. Similar behavior exists also for the second order Osher-Sethian method [25] and the ENO schemes in [27]. Notice that the decay rates for the second order relaxation scheme and the second order Osher-Sethian scheme are quite comparable, even if the relaxation scheme is based on a weakly hyperbolic system. In the nonconvex case the solution is smoother, and we do observe a second order convergence in both l_1 and l_∞ norms, as shown in Table 7.2.

	Relaxation		Osher-Sethian	
No. Pts	L_1	L_∞	L_1	L_∞
10	0.26811	0.08256	0.21623	0.05743
20	0.05610	0.01916	0.05563	0.02832
40	0.01402	0.00718	0.01573	0.00708
80	0.00344	0.00205	0.00404	0.00189

Table 1. Numerical Errors of the Convex Case (7.2a).

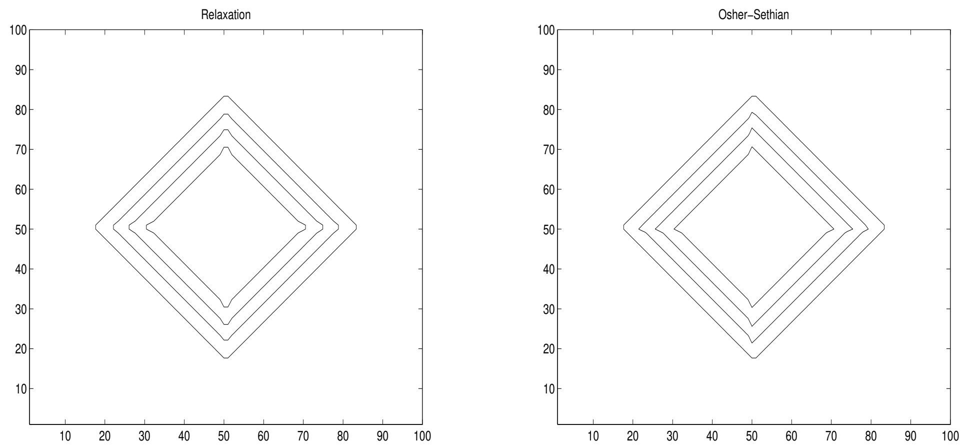
	Relaxation	
No. Pts	L_1	L_∞
10	0.18653	0.02560
20	0.04975	0.00788
40	0.01383	0.00197
80	0.00363	0.00050

Table 2. Numerical Errors of the Nonconvex Case (7.2b).



(a) (b)

FIG. 7.1. The numerical solutions by the second order relaxation scheme to (7.1) at $t = 1.5/\pi^2$ with: (a) convex H in (7.2a); (b) nonconvex H in (7.2b).



(a) (b)

FIG. 7.2. A shrinking diamond. (a) the relaxation scheme, (b) Osher-Sethian scheme. Output time $t = 0, 0.1, 0.2, 0.3$.

In the second example, we take $H(u, v) = \sqrt{u^2 + v^2}$ and the initial condition $u(x, y) = |x| + |y| - 1$. The zero level set of u is a square, centered in the origin, of side length $\sqrt{2}$. We define $p = \text{sign}x$ and $q = \text{sign}y$. This Hamilton-Jacobi equation describes a solution u whose zero level set (the diamond) moves in the inward normal direction with speed one [26]. The four corners of u corresponding to the shocks in p and q. We use 100 grid points between -1.5 and 1.5 in both x and y direction. The results given by the second order relaxation schemes and the second order Osher-

Sethian scheme [26] are depicted in Figure 7.2. It shows that both schemes give comparable results, although the solution of the Osher-Sethian scheme is slightly sharper.

8. Conclusion

In this article we have studied the numerical transition from a system of conservation laws and a Hamilton-Jacobi equation. We investigate how to numerically bridge the equivalence of the viscosity solution to the system of conservation laws and that of the corresponding Hamilton-Jacobi equation. Our study shows that the numerical solution to the system of conservation laws generated by a shock capturing method should be used carefully in order to convert to the viscosity solution of the Hamilton-Jacobi equation. In order to achieve this we analyze the structure of the two dimensional shock profile, and find a way to correctly produce the desired viscosity solution. This gives a natural way, in the discrete setting, on how to maintain the system of conservation laws as the gradient of the Hamilton-Jacobi equation. Such a study has an impact even beyond the scope of the Hamilton-Jacobi equation, since a good understanding of the effect of numerical viscosity is a central issue toward a good understanding of the higher dimensional shock capturing methods.

We then constructed a relaxation approximation to the system of conservation laws and the corresponding Hamilton-Jacobi equation. This new relaxation approximation, while sharing the spirit of the original one we proposed in [16] for general system of conservation laws, enjoys a greater simplicity thanks to the special structure of the Hamilton-Jacobi equation. Numerical results show that it is adequate to use it to solve a weakly hyperbolic system. This issue deserves further investigation for more general weakly hyperbolic systems.

Recently, it was observed in [14] that, when the subcharacteristic condition (6.21) is at the degenerated case (the equality holds), the $O(\epsilon)$ term in the Chapman-Enskog expansion of the relaxation approximation (6.3) becomes the mean-curvature term for a suitable choice of the Hamiltonian. Thus this relaxation system yields asymptotically the level set equation of Osher-Sethian [26] for front propagating in the normal direction with speed $V = 1 - \epsilon\kappa$, where κ is the mean curvature. More detailed analysis, as well as its impact on numerical approximation to such a motion, is carried out in [15].

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