Nonclassical Shocks and the Cauchy problem for Nonconvex Conservation Laws

Debora Amadori

Dipartimento di Matematica, Università degli Studi di Milano
Via Saldini 50, Milano, 20133, Italy
E-mail: amadori@mat.unimi.it

Paolo Baiti

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy
E-mail: baiti@sissa.it

Philippe G. LeFloch

Centre de Mathématiques Appliquées
& Centre National de la Recherche Scientifique
UA 756, Ecole Polytechnique, 91128 Palaiseau Cedex, France
E-mail: lefloch@cmapx.polytechnique.fr

Benedetto Piccoli

S.I.S.S.A., Via Beirut 4, Trieste 34014, Italy
E-mail: piccoli@sissa.it

Abstract. The Riemann problem for a conservation law with a nonconvex (cubic) flux can be solved in a class of admissible nonclassical solutions that may violate the Oleinik entropy condition but satisfy a single entropy inequality and a kinetic relation. We use such a nonclassical Riemann solver in a front tracking algorithm, and prove that the approximate solutions remain bounded in the total variation norm. The nonclassical shocks induce an increase of the total variation and, therefore, the classical measure of total variation must be modified accordingly. We prove that the front tracking scheme converges strongly to a weak solution satisfying the entropy inequality.

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1 Introduction

This paper considers the Cauchy problem for a conservation law with nonconvex flux-function (taken to be cubic, for simplicity):

$$\partial_t u + \partial_x u^3 = 0, \qquad u(x,t) \in \boldsymbol{R}, \quad x \in \boldsymbol{R},\ t > 0, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \boldsymbol{R}, \tag{1.2}$$

where the initial data u_0 is a function with bounded total variation. It is well-known that singularities develop in finite time in initially smooth solutions to (1.1) and the class of continuous functions is too narrow to encompass all of the relevant solutions. In the broader class of (discontinuous) weak solutions understood in the sense of distributions, one attempts to recover the uniqueness of the solutions by adding a selection principle, called an entropy condition [11, 12].

The selection of weak solutions is customarily based on the vanishing viscosity method. The (classical) entropy solutions to (1.1), by definition, are limits of sequences of smooth solutions to the regularised equation

$$\partial_t u_\epsilon + \partial_x u_\epsilon^3 = \epsilon\, \partial_{xx} u_\epsilon, \tag{1.3}$$

when $\epsilon \to 0$. It is known that $u = \lim_{\epsilon \to 0} u_\epsilon$ satisfies the so-called Oleinik entropy condition [14] and, particularly, the Lax shock admissibility inequalities. The latter requires that the characteristics near the shock impinge on the discontinuity [11].

However in certain applications in continuum mechanics, dispersive phenomena (due to capillarity effects in fluid flows, for instance, or to the Hall effect in magnetohydrodynamics) may actually contribute – together with dissipative effects – to drive the propagating shock. In such situations, dispersion should be taken into account when defining entropy solutions to the hyperbolic equation (1.1). One of the simplest example where dissipation and dispersion are in balance is to select the weak solutions to (1.1)-(1.2) that can be realized as limits of solutions to

$$\partial_t u_\epsilon + \partial_x u_\epsilon^3 = \epsilon\, \partial_{xx} u_\epsilon + A\, \epsilon^2\, \partial_{xxx} u_\epsilon, \tag{1.4}$$

with $\epsilon \to 0$ but A kept fixed. When $A > 0$, the limiting solution generally does not coincide with the one obtained in (1.3) and depends upon A; see Jacobs-MacKinney-Shearer [9] and Hayes-LeFloch [6]. In this context, the limiting solution $u = \lim_{\epsilon \to 0} u_\epsilon$ given by (1.4) is referred to as the *admissible nonclassical solution*. Observe that it is necessary to provide the constant A in order to formulate a Cauchy problem for (1.1)-(1.2).

More generally, a framework to encompass all of the nonclassical solutions that are limits of a (continuous or discrete) approximation scheme like (1.4) was proposed by Hayes-LeFloch [6, 8] (and extended to systems in [7]).

The focus is on solutions that satisfy the conservation law (1.1) in the weak sense together with a single entropy inequality of the form

$$\partial_t U(u) + \partial_x F(u) \le 0, \tag{1.5}$$

where U is strictly convex and $F'(u) = 3\,U'(u)\,u^2$. A shock wave satisfying (1.1)-(1.5) is called a *nonclassical shock* if it does not satisfy the Oleinik entropy condition. Such a discontinuity is actually *undercompressive*, i.e. it travels with a speed either slower or faster than *both* characteristics on either sides of the shock. The main concern in [6] was the Riemann problem, i.e. the case when u_0 is piecewise constant. It was established that the Riemann problem has a unique solution provided an additional condition is imposed, called a *kinetic relation*, which is required for nonclassical shocks only and determines them uniquely. In contrast with Oleinik construction, the nonclassical Riemann solution may be non-monotone. Thus the total variation of the Riemann solution may be larger than the initial total variation. However, in the $\mathbf{L}^1$-norm, the Riemann solution depends continuously upon its end states.

The purpose of the present paper is to consider an arbitrary nonclassical Riemann solver, and to apply to the Cauchy problem a front tracking algorithm, developed by Baiti-Jenssen [1], Bressan [3], Dafermos [4], and Risebro [15]. With the classical Riemann solver, the analysis of convergence toward a classical, entropy solution of (1.1)-(1.2) is straightforward: the total variation of the approximate (and limiting) solutions decreases in time, and a uniform bound for the total variation follows. To handle nonclassical Riemann solutions, we are forced to introduce a modified measure of total variation which still is decreasing for approximate solutions, even in the presence of nonclassical shocks. This is done by carefully estimating the strength of wave fronts across interactions, distinguishing between classical and nonclassical waves, and modifying the usual total variation accordingly. See Sections 2 and 3.

In Sections 4 and 5, we prove that the front tracking scheme converges to a weak solution of (1.1) satisfying the entropy inequality (1.5). Further properties on the limiting solutions are given in [2] together with an extension to systems of equations.

2 Riemann Problem and Interaction Patterns

In this section we first introduce the nonclassical Riemann solver and then list all of the possible wave interaction configurations. Their graphical representation can be found at the end of the paper. The Oleinik entropy condition [14] selects a unique weak solution to the Riemann problem, depending continuously upon its initial states. The concept of nonclassical solution extends Oleinik's construction to encompass all limits of dispersive-diffusive schemes.

In the following, for definiteness, we choose the entropy-entropy flux pair

$$U(u) := \frac{u^2}{2}, \qquad F(u) := \frac{3u^4}{4}, \tag{2.1}$$

to serve in the entropy inequality (1.5). This choice is consistent with the approximation (1.4), as well as with (2.3) introduced below. It can be checked that, given a left state u_l, a shock connecting u_l to a right state u_r satisfies (1.5) with this choice of (U, F) iff $u_r \in \big[-|u_l|, |u_l|\big]$. In addition, taking for example u_l to be positive, the shock wave

(a) either satisfies the Oleinik condition if $u_r \in [-u_l/2, u_l[$;

(b) or violate this condition if $u_r \in [-u_l, -u_l/2)$.

In Case (b) we will say that the shock is *nonclassical*. In particular, when $u_r = -u_l/2$, the shock speed coincides with the characteristic speed at the right state, and a rarefaction wave can be attached to the shock: this is the limiting case between classical and nonclassical behavior.

In [6] it was proved that, in order to have uniqueness for the Riemann problem, for every left state u one has to single out a unique right state $\varphi(u)$ that can be connected to u with a nonclassical shock. The function $\varphi : \boldsymbol{R} \mapsto \boldsymbol{R}$ is called a *kinetic function*, which depends on the regularisation adopted for (1.1).

Therefore, in the present paper, we suppose that a Lipschitz continuous function φ is given. In view of Case (b) above, it is natural to assume that

$$-u < \varphi(u) \le -\frac{u}{2}, \quad \text{for all } u > 0, \qquad -\frac{u}{2} \le \varphi(u) < -u, \quad \text{for all } u < 0. \tag{H1}$$

It will be convenient to set

$$\alpha(u) = -u - \varphi(u). \tag{2.2}$$

Observe that (H1) is equivalent to

$$-\frac{u}{2} \le \alpha(u) < 0 \quad \text{for all } u > 0, \qquad 0 < \alpha(u) \le -\frac{u}{2} \quad \text{for all } u < 0.$$

Also, note that $\varphi(0) = \alpha(0) = 0$. We refer to Figure 2.1 for an illustration of φ and α.

A special case of interest is when there exists a constant $\beta \in (0, \infty]$ such that

$$\alpha(u) = \varphi(u) = -\frac{u}{2} \qquad \text{for all } |u| \le \beta. \tag{H2}$$

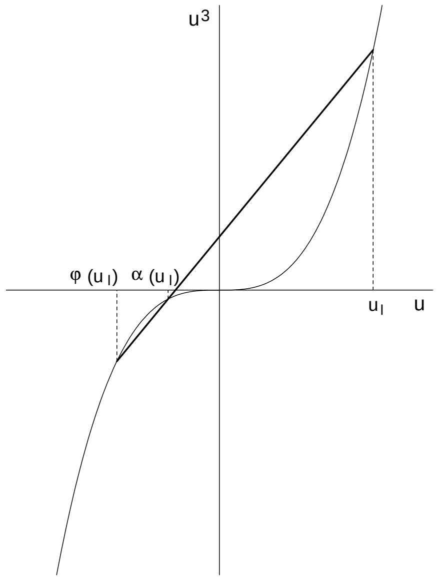

Figure 2.1

We will see that, under the assumption (H2), the solutions are always classical for values u in the interval $[-\beta, \beta]$. This hypothesis is compatible with the choice of the regularisation (1.4), but is not compatible with the following model having nonlinear diffusion and linear dispersion

$$\partial_t u_\varepsilon + \partial_x u_\varepsilon^3 = \varepsilon\, \partial_x \big(|\partial_x u_\varepsilon| \partial_x u_\varepsilon \big) + A\varepsilon\, \partial_{xxx} u_\varepsilon. \tag{2.3}$$

In this case there does not exist a "classical region" near the origin, or, equivalently, $\beta = 0$ (see [6] for details). Two other assumptions will be needed for some of our results:

$$\text{the function } \alpha \text{ is non-increasing,} \tag{H3}$$

$$\text{the function } \varphi \text{ is non-increasing.} \tag{H4}$$

Both (1.4) and (2.3) yield kinetic functions that are consistent with (H3)-(H4).

Based on a function φ satisfying (H1), Hayes-LeFloch [6] introduced a solution u to the Riemann problem (1.1)-(1.2) with

$$u_0(x) = \begin{cases} u_l, & \text{for } x < 0, \\ u_r, & \text{for } x > 0, \end{cases} \tag{2.4}$$

u_l and u_r being constant states. When $u_l > 0$, one has:

(i) If $u_r \geq u_l$, the function u is a rarefaction wave connecting u_l to u_r.

(ii) If $u_r \in \big[\alpha(u_l), u_l\big)$, u is a classical shock wave connecting u_l to u_r.

(iii) If $u_r \in \big(\varphi(u_l), \alpha(u_l)\big)$, u contains a (slower) nonclassical shock connecting u_l to $\varphi(u_l)$ followed by a (faster) classical shock connecting to u_r.

(iv) If $u_r \leq \varphi(u_l)$, the solution contains a nonclassical shock connecting u_l to $\varphi(u_l)$ followed by a rarefaction connecting to u_r.

The four cases are depicted in Figure 2.2.

The Riemann problem with left data $u_l < 0$ is solved in a completely similar fashion, using the value $\varphi(u_l) > 0$. Finally for $u_l = 0$, the Riemann problem is a single rarefaction wave, connecting monotonically u_l to u_r. The function u is called the *admissible nonclassical solution to the Riemann problem (based on the function φ).* Different choices for φ yield different solutions u. This is related to the fact that limits of solutions to equations like (1.4) do depend on the ratio A.

The Oleinik's solution to the Riemann problem is recovered by choosing φ to be the function in (H2) with $\beta = \infty$. Observe that the nonclassical solution coincides with the Oleinik's solution when u_l and u_r have the same sign. At the critical value $u_r = \alpha(u_l)$, one switches from a single wave pattern (Case ii) to a two-wave pattern (Case iii): the wave speeds in the Riemann solution change continuously, but the intermediate state $\varphi(u_l)$ does not approach the left or right states of the shock, u_l and u_r. This is a typical feature of the nonclassical solution, not encountered when the Oleinik entropy condition is enforced.

We have immediately:

Proposition 2.1 *When* (H1) *holds, the admissible nonclassical solution u to the Riemann problem, as defined above, depends continuously in the $\mathbf{L}^1$-norm upon its initial states and satisfies the maximum principle:*

$$\|u\|_{\mathbf{L}^\infty} \leq \max\big(|u_l|, |u_r|\big). \tag{2.5}$$

When (H2) *holds for some $\beta > 0$, the nonclassical Riemann solution coincides with Oleinik's solution when $|u_l| \leq \beta$.*

We stress that (2.5) holds for the flux-function $f(u) = u^3$ and the entropy $U(u) = u^2/2$ but need not hold true for more general flux-functions or entropies.

In the rest of this section we study the interactions of two wave fronts issued from nonclassical Riemann solutions. We show that certain types of interactions increase the total variation norm, which is the main source of difficulty.

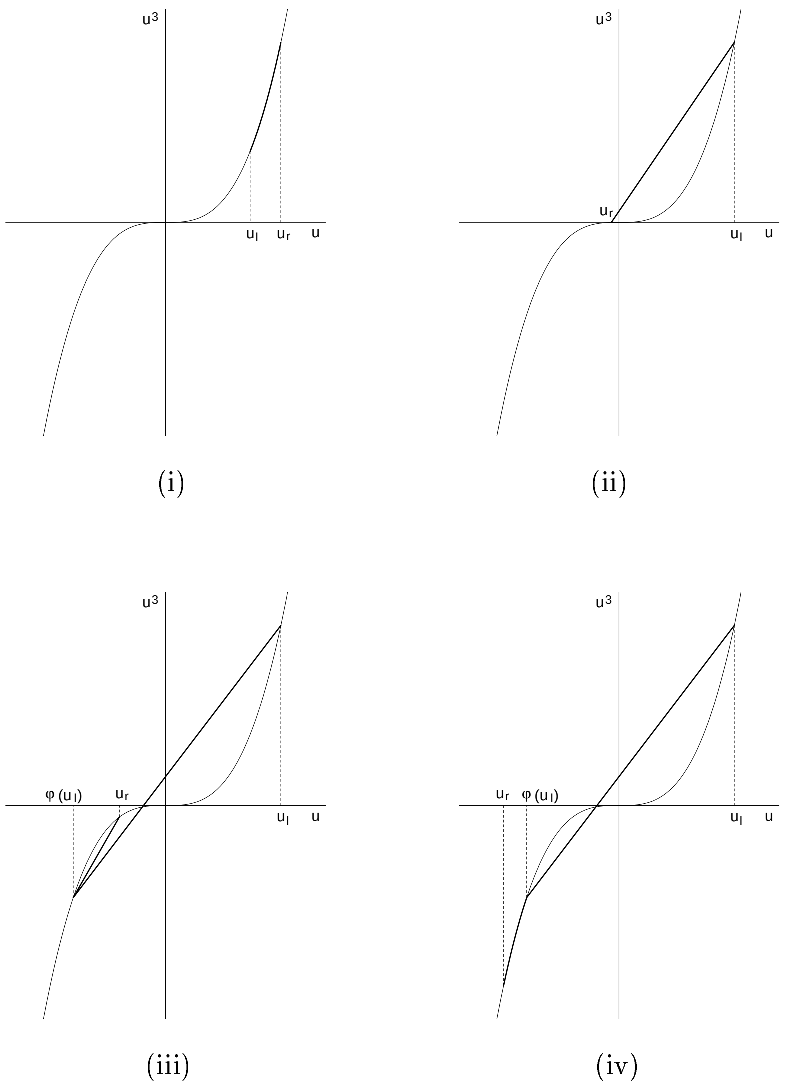

Figure 2.2

Assume that the three states u_l, u_m and u_r, are separated by two wave-fronts, each being generated by a (nonclassical) Riemann solution. The notation $(u_l : u_m)$ will be used for a wave front connecting u_l on the left to u_m on the right; by definition the two states are connected by a (single) admissible wave. Denote by s_1 and s_2 the speeds of the wave-fronts on the left and the right, respectively. For the two waves to meet one requires that $s_1 > s_2$.

Each front can be either a classical (C) or nonclassical shock (NC), or a rarefaction wave (R). Since we will need to consider piecewise constant approximations, in the following sections when constructing approximate solutions to the Cauchy problem, rarefaction waves are divided into n_ν propagating discontinuities (at the ν-stage of the construction). By convention, they travel at the (characteristic) speed of the right state. The speeds s of a (classical or nonclassical) shock, or s' of an approximate rarefaction between the states u_1 and u_2, are given by

$$s = s(u_1, u_2) = u_1^2 + u_1\, u_2 + u_2^2, \qquad s' = \lambda(u_2) = 3\, u_2^2, \tag{2.6}$$

respectively.

To classify the possible cases of interaction, we distinguish between the various types of incoming and outgoing wave-front, and discuss on the sign of u_m. We denote, for instance, by $\big[$(C, NC)-(NC, R)$\big]$ an interaction with incoming left classical shock and right nonclassical shock, and outgoing left nonclassical shock and right rarefaction.

In the trivial case $\varphi(u_l) = \alpha(u_l) = -u_l/2$, the nonclassical shock cannot be distinguished from the classical one. Thus we do not distinguish between the wave patterns (NC, R) and (C, R) in that case.

For definiteness we treat the case $u_l > 0$, the case $u_l < 0$ being completely analogous. We assume also $u_l \neq u_m$ and $u_m \neq u_r$. For each possible wave configuration, we determine the sign of the increment in the total variation norm $\Delta\mathbf{TV}$ (total variation after the interaction minus the one before).

The following discussion only requires (H1)-(H2) to hold.

2.1 The left incoming wave is a rarefaction.

Consider first the case when the front $(u_l : u_m)$ is a rarefaction, hence $u_m > u_l$. The front $(u_m : u_r)$ cannot be also a rarefaction, since it would have $u_r > u_m$ and the speed s_2 would be larger than s_1. Hence the front $(u_m : u_r)$ may be a classical shock, when $u_r \in \big[a(u_m), u_m\big)$, or a nonclassical shock. In the latter case we have $u_r = \varphi(u_m)$.

Various possibilities arise. If the right incoming wave is a shock, then

Case 1: $\big[$**(R, C)-(R)**$\big]$ when $u_r \in [u_l, u_m]$. A rarefaction wave is outgoing and, clearly, due to the cancellation of the rarefaction (u_l, u_m) by the shock, one gets $\Delta\mathbf{TV} < 0$. Observe that this case occurs only if the incoming rarefaction speed is larger than the incoming shock speed, which is the case because, by our convention, the rarefaction travels with the characteristic speed on its right side.

Case 2: $\big[$**(R, C)-(C)**$\big]$ when $u_r \in \big[\alpha(u_l), u_l\big)$. The incoming rarefaction cancels out with a part of the incoming shock, and again $\Delta\mathbf{TV} < 0$.

Case 3: $\big[$**(R, C)-(NC, C)**$\big]$ when $u_r \in \big(\varphi(u_l), \alpha(u_l)\big)$. If this happens, then it must be that $u_r \geq \alpha(u_m)$, and therefore $\alpha(u_m) < \alpha(u_l)$.

Two outgoing wave-fronts are produced here, a nonclassical shock connecting u_l to $\varphi(u_l)$, plus a classical one between $\varphi(u_l)$ and u_r. Here $\Delta\mathbf{TV} = 2\big(u_r - \varphi(u_l) - (u_m - u_l)\big)$.

Case 4: $\big[$**(R, C)-(NC, R)**$\big]$ when $u_r \in (\alpha(u_m), \varphi(u_l)]$: this, obviously, can happen only if $\alpha(u_m) < \varphi(u_l)$. The resulting wave pattern is a nonclassical shock plus a rarefaction between $\varphi(u_l)$ and u_r. The incoming rarefaction is canceled and $\Delta\mathbf{TV} < 0$.

If the right incoming wave is a nonclassical shock then $u_r = \varphi(u_m)$. This together with $u_l < u_m$ implies that $u_r \le \alpha(u_l)$, hence we have an outgoing NC shock followed by a classical shock or a rarefaction depending on the relative position of $\varphi(u_l)$ and $\varphi(u_m)$. More precisely, we have

Case 5: $\big[$**(R, NC)-(NC, C)**$\big]$ in the case $\varphi(u_l) < \varphi(u_m)$. Here $\Delta\mathbf{TV} = 2\big(\varphi(u_m) - u_m - \varphi(u_l) + u_l\big) > 0$.

Case 6: $\big[$**(R, NC)-(NC, R)**$\big]$ when $\varphi(u_m) \le \varphi(u_l)$. In this case $\Delta\mathbf{TV} < 0$.

The previous cases cover all possibilities; indeed, in all cases not considered, there cannot be a single wave connecting u_m and u_r.

2.2 The left wave is a classical shock.

If the left wave $(u_l : u_m)$ is a classical shock, then $u_m \in \big[\alpha(u_l), u_l\big)$. Assume first that $u_m \ge 0$. Then we have the following cases.

Case 7: $\big[$**(C, R)-(C)**$\big]$. The wave $(u_m : u_r)$ is a rarefaction, with $u_r \in (u_m, u_l)$. Observe that the incoming shock indeed travels faster than the rarefaction which, by convention, travels with the right characteristic speed. The resulting pattern is a shock connecting u_l and u_r. The rarefaction is canceled and $\Delta\mathbf{TV} < 0$.

Case 8: $\big[$**(C, C)-(C)**$\big]$ when $u_m > u_r \ge \max\big\{\alpha(u_l), \alpha(u_m)\big\}$. A shock is generated and $\Delta\mathbf{TV} = 0$.

Case 9: $\big[$**(C, C)-(NC, C)**$\big]$ when $u_r \in \big(\alpha(u_m), \alpha(u_l)\big)$. Necessarily we have $\alpha(u_m) < \alpha(u_l)$. Here the total variation increases: $\Delta\mathbf{TV} = 2\big(u_r - \varphi(u_l)\big) > 0$.

Case 10: $\big[$**(C, NC)-(C)**$\big]$. If $u_m > \beta$ and $u_r = \varphi(u_m)$ with $\varphi(u_m) > \alpha(u_l)$. Both the incoming and outgoing wave patterns are monotone functions and $\Delta\mathbf{TV} = 0$.

Case 11: $\big[$**(C, NC)-(NC, C)**$\big]$ if $u_m > \beta$ and $u_r = \varphi(u_m)$ with $\varphi(u_l) < \varphi(u_m) < \alpha(u_l)$. In this case the total variation increases: $\Delta\mathbf{TV} = 2\big(u_r - \varphi(u_l)\big) > 0$.

Case 12: $\big[$**(C, NC)-(NC, R)**$\big]$. If $u_m > \beta$ and $u_r = \varphi(u_m)$ with $\varphi(u_l) \geq \varphi(u_m)$. In this case both wave patterns are monotone functions and $\Delta\mathbf{TV} = 0$.

Consider now cases in which $u_m \leq 0$. Then u_r can assume values in the set $\big(-\infty, \alpha(u_m)\big) \cup \{\varphi(u_m)\}$. Observe that now $\alpha(u_m) > 0$.

Case 13: $\big[$**(C, NC)-(C)**$\big]$. This can happen only if $u_m < -\beta$ and $u_r = \varphi(u_m)$. The nonclassical shock has been canceled out and $\Delta\mathbf{TV} < 0$.

Case 14: $\big[$**(C, C)-(C)**$\big]$ when $u_m < u_r < \alpha(u_m)$. Then one incoming classical shock is completely canceled out and $\Delta\mathbf{TV} < 0$.

Case 15: $\big[$**(C, R)-(C)**$\big]$ when $\alpha(u_l) \leq u_r < u_m$. Then both incoming and outgoing wave patterns are monotone and $\Delta\mathbf{TV} = 0$.

Case 16: $\big[$**(C, R)-(NC, C)**$\big]$ when $\varphi(u_l) < u_r < \alpha(u_l)$. A nonclassical shock is generated and the total variation increases: $\Delta\mathbf{TV} = \big(u_r - \varphi(u_l)\big) > 0$.

Case 17: $\big[$**(C, R)-(NC, R)**$\big]$ when $\varphi(u_l) > u_r$. Then both wave patterns are monotone and $\Delta\mathbf{TV} = 0$.

2.3 The left wave is a nonclassical shock.

Suppose now that $(u_l : u_m)$ is a non-trivial nonclassical shock. Then $u_m = \varphi(u_l)$ and $\alpha(u_m)$ is positive. The second wave cannot be a rarefaction, with $u_r < u_m$, because the speed s_2 would be greater than s_1. The same would happen if $u_r \in \big(u_m, \alpha(u_l)\big)$. Hence it must be that $u_r \in \big(\alpha(u_l), \alpha(u_m)\big] \cup \big\{\varphi(u_m)\big\}$ and there are two possibilities.

Case 18: $\big[$**(NC, C)-(C)**$\big]$ when $\alpha(u_l) < u_r \leq \alpha(u_m)$. The nonclassical shock canceled out and a classical shock is produced: $\Delta\mathbf{TV} < 0$.

Case 19: $\big[$**(NC, NC)-(C)**$\big]$ when $u_r = \varphi(u_m)$. The two incoming nonclassical shocks canceled out and, also in this case, a shock wave is produced and $\Delta\mathbf{TV} < 0$.

Finally, the case $u_l = 0$ should be considered. This time there are only three types of interactions, but these can be included in Cases 1, 4, 6 if we admit the convention of having a non-classical shock with zero strength connecting the state 0 to itself. This completes the classification of all the possible interaction patterns.

It should be noticed that, under the assumptions (H3)-(H4), Cases 5, 9 and 12 can never occur, because of the monotonicity of either α or φ. But, even after eliminating these cases there are still Cases 3, 11 and 16 in which the total variation increases across the interaction.

One of the main problems here can be seen in Case 16 where the interaction of a shock and a (possibly very small) rarefaction, produces a (large) nonclassical shock plus a classical one, with a significant increase of the total variation. Therefore a nonclassical solution is not total variation diminishing and consequently the semigroup of solutions to the Cauchy problem, if it exists, can not be a contraction in the $\mathbf{L}^1$-norm, as is the case when the Oleinik entropy condition is enforced [10].

However, given a suitable sequence of approximate solutions u_ν of (1.1)-(1.2), in order to pass to the limit we need only to prove that the total variation of $u_\nu(t,\cdot)$ is uniformly bounded in t and ν. To reach this goal, we now construct a modified total variation functional that decreases along approximate solutions. More precisely, let $u : \boldsymbol{R} \mapsto \boldsymbol{R}$ be a piecewise constant function and let x_α, $\alpha = 1, \dots, N$, be the points of discontinuity of u. Define

$$\mathbf{V}(u) := \sum_{\alpha=1}^{N} \sigma\big(u(x_\alpha-), u(x_\alpha+)\big), \tag{2.7}$$

where $\sigma(u_l, u_r)$ measures the strength of the wave connecting u_l to u_r. If $\sigma(u_l, u_r)$ was equal to $|u_r - u_l|$, then $\mathbf{V}(u)$ would be precisely the total variation of u. However, to compensate the increase of the total variation we redefine the strength of a wave by weighting less all nonclassical shocks and classical shocks which change sign, i.e. for which $u_l\, u_r < 0$.

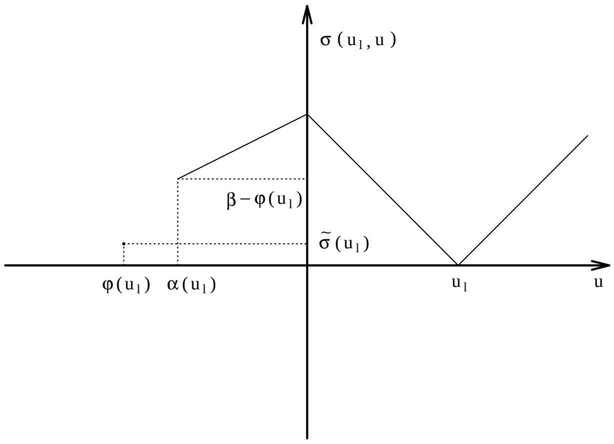

Figure 2.3

In the rest of this paper, we assume that (H1)-(H4) hold. We will show in Section 3 that a good choice for σ is

$$\sigma(u_l, u_r) := \begin{cases} |u_r - u_l|, & \text{if } u_l u_r \geq 0 \text{ or } |u_l| \leq \beta, \\ \theta(u_l) u_r + |u_l|, & \text{if } u_l u_r < 0 \text{ and } |u_r| \leq \big|\alpha(u_l)\big| \text{ and } |u_l| > \beta, \\ \beta + \big|\alpha(u_l)\big|, & \text{if } u_r = \varphi(u_l) \text{ and } |u_l| > \beta, \end{cases}$$

where

$$\theta(u_l) := \frac{\beta - |\alpha(u_l)|}{\alpha(u_l)} \in [-1, 1].$$

Observe that $\sigma(u_l, u_r)$ is not defined for all values of u_l and u_r, as this is not necessary in our analysis. We define also the strength of a nonclassical shock having u_l as left state by

$$\tilde{\sigma}(u_l) := \sigma\big(u_l, \varphi(u_l)\big).$$

The graph of the function $u \mapsto \sigma(u_l, u)$, in the case $u_l > \beta$ and $\theta(u_l) > 0$, is shown in Figure 2.3.

Now we want to compare the total variation of a piecewise function u with $\mathbf{V}(u)$. Clearly it is enough to consider the case when u contains only a single wave connecting the states u_l and u_r, say. First of all, since $\sigma(u_l, u_r) \leq |u_r - u_l|$, then one has

$$\mathbf{V}(u) \leq \mathbf{TV}(u). \tag{2.8}$$

Without loss of generality we can assume $u_l > 0$. First let the wave in u be classical. If $u_r \geq 0$ or $u_l \leq \beta$ then the two definitions coincide and we have $\mathbf{TV}(u) = \mathbf{V}(u)$. Now assume that $u_l > \beta$ and $0 > u_r \geq \alpha(u_l)$. Since $\theta(u_l) < 1$ and $\alpha(u_l) \geq -u_l/2$, it follows that

$$\frac{\mathbf{TV}(u)}{\mathbf{V}(u)} = \frac{u_l - u_r}{u_l + \theta(u_l)u_r} \leq \frac{u_l - u_r}{u_l + u_r} \leq \frac{u_l + u_l/2}{u_l - u_l/2} = 3. \tag{2.9}$$

Assume now that the states u_l and u_r are connected by a nonclassical wave. Then $u_l > \beta$ and $u_r = \varphi(u_l)$. In this case, by (2.2) we have

$$\frac{\mathbf{TV}(u)}{\mathbf{V}(u)} = \frac{u_l - \varphi(u_l)}{\beta - \alpha(u_l)} = \frac{2u_l + \alpha(u_l)}{\beta - \alpha(u_l)} \leq \frac{2u_l}{\beta + \beta/2} = \frac{4}{3\beta}u_l. \tag{2.10}$$

In conclusion, if u is a piecewise constant function containing only single waves then

$$\mathbf{V}(u) \leq \mathbf{TV}(u) \leq \max\left\{3, \frac{4}{3\beta}\|u\|_{\mathbf{L}^\infty}\right\}\mathbf{V}(u). \tag{2.11}$$

Therefore, in order to control the total variation of u, it is sufficient to control $\mathbf{V}(u)$ and the $\mathbf{L}^\infty$-norm of u. For the latter, we will use Proposition 2.1, while for the former we will need the estimate to be derived in the next section.

3 Interaction Estimates

In this section we prove that, under the hypotheses (H1)–(H4), the functional $\mathbf{V}$ (see (2.7) in Section 2) decreases across each interaction. In turn, this will imply that $\mathbf{V}$ decreases along our approximate solutions.

As before, we assume that two waves $(u_l : u_m)$ and $(u_m : u_r)$ are interacting. Consider the case $u_l > 0$, the case $u_l < 0$ being similar. Recall that $\alpha(u_l) = -u_l/2$ when $|u_l| \leq \beta$. In each case identified in Section 2, we compute the jump $\Delta\mathbf{V}$ of the functional $\mathbf{V}$.

CASE 1. This case is easy due to cancellations, and is omitted. CASE 2.

Recall that $u_m > u_l$, $u_r > \alpha(u_l)$ and $\alpha(u_m) < 0$. In particular one has $\alpha(u_m) - \alpha(u_l) \leq 0$ and $\varphi(u_m) - \varphi(u_l) \leq 0$. If $u_r > 0$ or $u_m < \beta$, it is trivial. Thus we assume $u_r < 0$ and $u_m \geq \beta$. We have, if $u_l > \beta$,

$$\begin{aligned}
\Delta\mathbf{V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r) \\
&= \big(u_l + \theta(u_l)u_r\big) - (u_m - u_l) - \big(u_m + \theta(u_m)u_r\big) \\
&= 2(u_l - u_m) + \beta u_r \left(\frac{1}{\alpha(u_l)} - \frac{1}{\alpha(u_m)}\right) \\
&\leq 2(u_l - u_m) + \beta\alpha(u_l) \left(\frac{1}{\alpha(u_l)} - \frac{1}{\alpha(u_m)}\right) \\
&= 2(u_l - u_m) + \beta \left(\frac{\alpha(u_m) - \alpha(u_l)}{\alpha(u_m)}\right) \\
&\leq 2(u_l - u_m) + \beta \left(\frac{u_l - u_m}{\alpha(u_m)}\right) = 2\frac{u_l - u_m}{\alpha(u_m)} \left(\alpha(u_m) + \frac{\beta}{2}\right) < 0,
\end{aligned}$$

while if $u_l < \beta$

$$\begin{aligned}
\Delta\mathbf{V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r) \\
&= (u_l - u_r) - (u_m - u_l) - \big(u_m + \theta(u_m)u_r\big) \\
&= 2(u_l - u_m) - \big(\theta(u_m) + 1\big)u_r \\
&\leq 2(u_l - u_m) - \big(\theta(u_m) + 1\big)\alpha(u_m) \\
&= -\big(u_m + 2\alpha(u_m)\big) - (\beta - u_l) - (u_m - u_l) < 0.
\end{aligned}$$

CASE 3. In this case $u_m > u_l > \beta$ and $\alpha(u_l) > u_r \geq \alpha(u_m)$. We have

$$\begin{aligned}
\Delta\mathbf{V} &= \tilde{\sigma}(u_l) + \sigma\big(\varphi(u_l), u_r\big) - \sigma(u_l, u_m) - \tilde{\sigma}(u_m) \\
&= \big(\beta - \alpha(u_l)\big) + \big(u_r - \varphi(u_l)\big) - (u_m - u_l) - \big(u_m + \theta(u_m)u_r\big) \\
&= 2(u_l - u_m) + \beta - \frac{\beta}{\alpha(u_m)}u_r \\
&\leq 2(u_l - u_m) + \beta \left(\frac{\alpha(u_m) - \alpha(u_l)}{\alpha(u_m)}\right),
\end{aligned}$$

and we conclude as in the Case 2. CASE 4. We have $u_m > u_l > \beta$ and $\varphi(u_l) > u_r \geq \alpha(u_m)$. Hence by (H2), $\alpha(u_l) > \alpha(u_m)$ and it follows that

$$\begin{aligned}\mathbf{\Delta V} &= \tilde{\sigma}(u_l) + \sigma\big(\varphi(u_l), u_r\big) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= \big(\beta - \alpha(u_l)\big) + \big(\varphi(u_l) - u_r\big) - (u_m - u_l) - \big(u_m + \theta(u_m)u_r\big)\\ &= \beta - 2\big(u_m + \alpha(u_l)\big) - \big(\theta(u_m) + 1\big)u_r\\ &\leq \beta - 2\big(u_m + \alpha(u_m)\big) - \big(\theta(u_m) + 1\big)\alpha(u_m)\\ &= -2\big(u_m + 2\alpha(u_m)\big) < 0.\end{aligned}$$

CASE 6. Since $u_m > u_l$ we have $\alpha(u_l) > \alpha(u_m)$ and

$$\begin{aligned}\mathbf{\Delta V} &= \tilde{\sigma}(u_l) + \sigma\big(\varphi(u_l), \varphi(u_m)\big) - \sigma(u_l, u_m) - \tilde{\sigma}(u_m)\\ &= \big(\beta - \alpha(u_l)\big) + \big(\alpha(u_m) + u_m - \alpha(u_l) - u_l\big) +\\ &\quad - (u_m - u_l) - \big(\beta - \alpha(u_m)\big)\\ &= 2\big(\alpha(u_m) - \alpha(u_l)\big) < 0.\end{aligned}$$

CASE 7. Trivial case.

CASE 8. We have $u_l > u_m > 0$ and $u_m > u_r > \alpha(u_m)$. If $u_r > 0$ or $u_l < \beta$ the case is trivial. Otherwise $0 > u_r > \alpha(u_m)$, and if $u_m > \beta$ we have

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= \big(u_l + \theta(u_l)u_r\big) - (u_l - u_m) - \big(u_m + \theta(u_m)u_r\big)\\ &= \big(\theta(u_l) - \theta(u_r)\big)u_r = \beta u_r\left(\frac{1}{\alpha(u_l)} - \frac{1}{\alpha(u_m)}\right) < 0,\end{aligned}$$

while if $u_m < \beta$ we have

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= \big(u_l + \theta(u_l)u_r\big) - (u_l - u_m) - (u_m - u_r)\\ &= \big(\theta(u_l) + 1\big)u_r < 0.\end{aligned}$$

CASE 10. Since $u_l > u_m > \beta$ and $\alpha(u_l) < \varphi(u_m)$, we get

$$\begin{aligned}\mathbf{\Delta V} &= \sigma\big(u_l, \varphi(u_m)\big) - \sigma(u_l, u_m) - \tilde{\sigma}(u_m)\\ &= \big(u_l + \theta(u_l)\varphi(u_m)\big) - (u_l - u_m) - \big(\beta - \alpha(u_m)\big)\\ &= -\beta + \big(\theta(u_l) - 1\big)\varphi(u_m) = \beta\left(\frac{\varphi(u_m)}{\alpha(u_l)} - 1\right) < 0.\end{aligned}$$

CASE 11. We get easily

$$\begin{aligned}\mathbf{\Delta V} &= \tilde\sigma(u_l) + \sigma\big(\varphi(u_l), \varphi(u_m)\big) - \sigma(u_l, u_m) - \tilde\sigma(u_m)\\ &= \big(\beta - \alpha(u_l)\big) + \big(\alpha(u_l) + u_l - \alpha(u_m) - u_m\big) +\\ &\qquad - (u_l - u_m) - \big(\beta - \alpha(u_m)\big) = 0.\end{aligned}$$

Case 13. Here $u_m < -\beta$ and necessarily $u_l > \beta$. Moreover $\alpha(u_l) < 0$, hence

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \tilde\sigma(u_m)\\ &= \big(u_l - \varphi(u_m)\big) - \big(u_l + \theta(u_l)u_m\big) - \big(\beta + \alpha(u_m)\big)\\ &= -\beta\left(1 + \frac{u_m}{\alpha(u_l)}\right) < 0.\end{aligned}$$

Case 14. This is possible when $u_l > u_m$, $\alpha(u_l) < u_m < 0$ and $u_m < u_r < \alpha(u_m)$. If $u_l < \beta$, this case is trivial. So assume now $u_l \geq \beta$. If $u_r < 0$ then

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= \big(u_l + \theta(u_l)u_r\big) - \big(u_l + \theta(u_l)u_m\big) - (u_r - u_m)\\ &= \frac{\beta}{\alpha(u_l)}(u_r - u_m) < 0.\end{aligned}$$

If $u_r > 0$ and $u_m > -\beta$, then

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= (u_l - u_r) - \big(u_l + \theta(u_l)u_m\big) - (u_r - u_m)\\ &= -\frac{\beta}{\alpha(u_l)}u_m - 2u_r < 0.\end{aligned}$$

Finally, if $u_m < -\beta$, then

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= (u_l - u_r) - \left(u_l + \Big(\frac{\beta}{\alpha(u_l)} + 1\Big)u_m\right) - \left(\Big(\frac{\beta}{\alpha(u_m)} - 1\Big)u_r - u_m\right)\\ &= -\beta\left(\frac{u_m}{\alpha(u_l)} + \frac{u_r}{\alpha(u_m)}\right) < 0.\end{aligned}$$

Case 15. We have $0 > u_m > u_r$. If $u_l < \beta$, the case is trivial. Otherwise, since $\theta(u_l) \geq -1$, we have

$$\begin{aligned}\mathbf{\Delta V} &= \sigma(u_l, u_r) - \sigma(u_l, u_m) - \sigma(u_m, u_r)\\ &= \big(u_l + \theta(u_l)u_r\big) - \big(u_l + \theta(u_l)u_m\big) - (u_m - u_r)\\ &= \big(\theta(u_l) + 1\big)(u_r - u_m) \leq 0.\end{aligned}$$

CASE 16. In this case $0 \geq u_m > \alpha(u_l) > u_r$. It follows that

$$\begin{aligned}\Delta\mathbf{V} &= \tilde{\sigma}(u_l) + \sigma\big(\varphi(u_l), u_r\big) - \sigma(u_l, u_m) - \sigma(u_m, u_r) \\ &= \big(\beta - \alpha(u_l)\big) + \big(u_r + u_l + \alpha(u_l)\big) - \big(u_l + \theta(u_l)u_m\big) - (u_m - u_r) \\ &= \beta + 2u_r - \big(\theta(u_l) + 1\big)u_m \leq \beta + 2\alpha(u_l) - \left(\frac{\beta}{\alpha(u_l)} + 2\right)\alpha(u_l) = 0.\end{aligned}$$

CASE 17. Here $0 \geq u_m > \alpha(u_l)$, hence

$$\begin{aligned}\Delta\mathbf{V} &= \tilde{\sigma}(u_l) + \sigma\big(\varphi(u_l), u_r\big) - \sigma(u_l, u_m) - \sigma(u_m, u_r) \\ &= \big(\beta - \alpha(u_l)\big) + \big(-u_l - \alpha(u_l) - u_r\big) - \big(u_l + \theta(u_l)u_m\big) - (u_m - u_r) \\ &= \beta - 2u_l - 2\alpha(u_l) - \big(\theta(u_l) + 1\big)u_m \\ &\leq \beta - 2u_l - 2\alpha(u_l) - \big(\beta + 2\alpha(u_l)\big) \\ &= -2\big(u_l + 2\alpha(u_l)\big) < 0.\end{aligned}$$

CASE 18. It can happen when $u_l > \beta$, $u_m = \varphi(u_l)$ and $\alpha(u_m) > u_r > \alpha(u_l)$.

Many subcases. If $u_r < 0$ then

$$\begin{aligned}\Delta\mathbf{V} &= \sigma(u_l, u_r) - \tilde{\sigma}(u_l) - \sigma\big(\varphi(u_l), u_r\big) \\ &= \big(u_l + \theta(u_l)u_r\big) - \big(\beta - \alpha(u_l)\big) - \big(u_r - \varphi(u_l)\big) \\ &= \big(u_l + \theta(u_l)u_r\big) - \big(\beta - \alpha(u_l)\big) - \big(u_r + u_l + \alpha(u_l)\big) \\ &= \beta\left(\frac{u_r}{\alpha(u_l)} - 1\right) < 0.\end{aligned}$$

Let now $u_r > 0$. If $\varphi(u_l) > -\beta$ then

$$\begin{aligned}\Delta\mathbf{V} &= \sigma(u_l, u_r) - \tilde{\sigma}(u_l) - \sigma\big(\varphi(u_l), u_r\big) \\ &= (u_l - u_r) - \big(\beta - \alpha(u_l)\big) - \big(u_r + u_l + \alpha(u_l)\big) \\ &= -\beta - 2u_r < 0.\end{aligned}$$

If, instead, $\varphi(u_l) < -\beta$ then

$$\begin{aligned}\Delta\mathbf{V} &= \sigma(u_l, u_r) - \tilde{\sigma}(u_l) - \sigma\big(\varphi(u_l), u_r\big) \\ &= (u_l - u_r) - \big(\beta - \alpha(u_l)\big) - \left(\Big(\frac{\beta}{\alpha(\varphi(u_l))} - 1\Big)u_r - \varphi(u_l)\right) \\ &= -\beta\left(1 + \frac{u_r}{\alpha(\varphi(u_l))}\right) < 0.\end{aligned}$$

CASE 19. Here $u_m = \varphi(u_l)$ and $u_r = \varphi(u_m)$, hence

$$
\begin{aligned}
\Delta \mathbf{V} &= \sigma\big(u_l, \varphi(u_m)\big) - \tilde{\sigma}(u_l) - \tilde{\sigma}(u_m) \\
&= \big(u_l + \alpha(u_m) + u_m\big) - \big(\beta - \alpha(u_l)\big) - \big(\beta + \alpha(u_m)\big) \\
&= \big(u_l + \alpha(u_m) - u_l - \alpha(u_l)\big) - \big(\beta - \alpha(u_l)\big) - \big(\beta + \alpha(u_m)\big) = -2\beta < 0.
\end{aligned}
$$

Recall that, under the assumptions (H1)–(H4), Cases 5, 9 and 12 never occur. Hence all of the possible interactions have been considered, and we have proved that $\mathbf{V}$ decreases across each of them.

4 Front Tracking Approximations

In this section we construct a sequence of piecewise constant approximate solutions to (1.1)-(1.2). For a fixed initial data $u_0 \in \mathbf{BV}$, let u_ν be a sequence of piecewise constant functions such that $u_\nu \to u_0$ in $\mathbf{L}^1$ and $\mathbf{TV}(u_\nu) \to \mathbf{TV}(u_0)$ as $\nu \to \infty$. Let δ_ν be a sequence of positive numbers converging to zero. For each ν, the approximate solution u_ν is constructed as follows. Solve approximatively the Riemann problem at each discontinuity point of u_ν. This is obtained by approximating the solution given by the nonclassical Riemann solver: every shock or non classical shock travels with the correct shock speed, while the rarefactions are approximated by a rarefaction fan. More precisely, every rarefaction wave connecting the states u_l and u_r, say, with $|u_l - u_r| > \delta_\nu$ is approximated by a finite number of small jumps traveling with speed equal to the right characteristic speed and with strength less than or equal to δ_ν. We patch together the local solutions to obtain a function $u_\nu(t,x)$. At a new interaction, we solve again the Riemann problem, etc. By slightly changing the speeds of the waves we can assume that only two waves interact at each interaction time. The function u_ν can be defined for all times if we prove that the total number of waves at every fixed time t, say $I_\nu(t)$, and the total number of interactions are both bounded.

First of all, note that at most two (shock, rarefaction) waves may leave a point of interaction. Thus, across every interaction,

$$\Delta I_\nu := I_\nu(t+) - I_\nu(t-) \le 0$$

as long as all possible outgoing rarefactions are not split. Since by construction only the rarefactions with strength greater than δ_ν are split, and since the strength of the rarefactions at time $t = 0+$ is less than δ_ν, then it will be sufficient to prove that their strength does not increase across each interaction. Let us now consider all the possible interactions when a rarefaction wave is produced. By the analysis in Section 2, this happens in Cases 1, 4, 6 and 17. In all of these cases, there is also an incoming rarefaction, connecting the states u_l and u_r, say. Call σ the rarefaction strength.

Case 1. Trivial case: $\Delta\sigma < 0$.

Case 4. The variation of the strength across the interaction is given by

$$\begin{aligned}\Delta\sigma &= \big(\varphi(u_l) - u_r\big) - (u_m - u_l) = -u_m - u_r - \alpha(u_l)\\ &\le 2\alpha(u_m) - u_r - \alpha(u_l) = \big(\alpha(u_m) - \alpha(u_l)\big) + \big(\alpha(u_m) - u_r\big) < 0.\end{aligned}$$

Case 6. In this case the variation is given by

$$\Delta\sigma = \big(\varphi(u_l) - \varphi(u_m)\big) - (u_m - u_l) = \alpha(u_m) - \alpha(u_l) < 0.$$

Case 17. Now the variation is given by

$$\Delta\sigma = \big(\varphi(u_l) - u_r\big) - (u_m - u_r) < 0.$$

So we have proved that the strength of a rarefaction does not increase across an interaction. Hence no new rarefaction can be generated at positive times and the total number $I_\nu(t)$ is bounded by $I_\nu(0+)$ for every time $t > 0$ and integer ν.

For a conservation law with convex flux-function, the bound on the total number of waves is sufficient to deduce a bound on the total number of interactions too. Indeed two interacting waves either join together or cancel, and therefore can interact only once. This fact is no longer true in our case due to non-convexity of the flux and the presence of nonclassical shocks. For example, consider a nonclassical shock and a classical shock interacting as in Case 11. After the interaction, a classical and a nonclassical fronts are produced. At a later time, the outgoing non-classical shock may interact with another shock coming from the left (again Case 11), producing a new classical shock and a non-classical one with greater speed that turns back and is again interacting with the first shock considered. So, it is not clear at this stage of the proof that the total number of interactions is finite. To prove this fact, we now construct a functional that decreases by a fixed positive quantity at each interaction. Note that $\mathbf{V}(u_\nu(t))$ does not satisfy this property, due to Cases 8, 10, 11 and 17.

Fix ν and let $I_\nu = I_\nu(0+) \ge I_\nu(t)$. We define

$$\begin{aligned}\mathbf{H}_\nu(t) := (I_\nu + 1)\,\#\{\text{waves in } u_\nu(t)\} \;+ \sum_{x_\alpha(t)\in C,R} \#\{\text{waves in } u_\nu(t)\} \;+\\ + \sum_{x_\alpha(t)\in NC} \#\{\text{waves in } u_\nu(t) \text{ to the left of } x_\alpha(t)\}\;, \qquad (4.1)\end{aligned}$$

where C, NC, R denote the sets of all classical, nonclassical shocks and rarefaction, respectively, at time t. By the analysis above, $\mathbf{H}_\nu(0+)$ is bounded by $(I_\nu + 1)I_\nu + I_\nu^2$. Moreover, by definition, $\mathbf{H}_\nu$ decreases by at least 1 across each interaction with only one outgoing wave, i.e. in Cases 1, 2, 7, 8, 10, 13, 14, 15, 18, 19: namely the first term in (4.1) decreases by at least $I_\nu + 1$

while the sum of the last two terms increases by at most I_ν. In all of the remaining cases a left outgoing nonclassical shock is produced. Hence, even though the number of waves is constant across these interactions and so the first term remains constant, the second part of the functional decreases by at least 1. Therefore $\Delta\mathbf{H}_\nu \le -1$ across each interaction. Since initially $\mathbf{H}_\nu$ is bounded and positive, this implies that the total number of interactions is finite. So $u_\nu(t,x)$ can be defined for all $t>0$ and $x \in \boldsymbol{R}$.

5 Convergence

We conclude by checking that there exists a subsequence of $\{u_\nu(t,x)\}$ that converges in $\mathbf{L}^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R})$ to a weak solution satisfying the entropy inequality.

First of all, a uniform control for $\|u_\nu\|_{\mathbf{L}^\infty}$ follows from Proposition 2.1, namely:

Lemma 5.1 (Maximum principle) *For every $t>0$ and $\nu \ge 1$, there holds*

$$\|u_\nu(t)\|_{\mathbf{L}^\infty} \le \|u_\nu(0)\|_{\mathbf{L}^\infty}. \tag{5.1}$$

Moreover $\|u_\nu(0)\|_{\mathbf{L}^\infty}$ is bounded by a constant C, since u_ν converges to u_0 in $\mathbf{L}^1$ and $\mathbf{BV}$. Combining the maximum principle with the analysis in Sections 3 and 4, $\mathbf{V}(u_\nu)$ and therefore the total variation of $u_\nu(t,\cdot)$ is uniformly bounded in $t>0$ and ν. Indeed

$$\begin{aligned}
\mathbf{TV}\big(u_\nu(t)\big) &\le \max\left\{3, \frac{4}{3\beta}\|u_\nu(t)\|_{\mathbf{L}^\infty}\right\}\mathbf{V}\big(u_\nu(t)\big)\\
&\le \max\left\{3, \frac{4}{3\beta}\|u_\nu(0)\|_{\mathbf{L}^\infty}\right\}\mathbf{V}\big(u_\nu(0+)\big)\\
&\le \max\left\{3, \frac{4C}{3\beta}\right\}\mathbf{TV}\big(u_\nu(0+)\big).
\end{aligned} \tag{5.2}$$

Hence, by Helly's Theorem there exists a subsequence of u_ν that converges in $\mathbf{L}^1_{loc}(\boldsymbol{R}_+ \times \boldsymbol{R})$ to a limiting function u. As in [1, 3], one can check that u is actually a weak solution of (1.1)-(1.2), i.e. for every function ϕ with compact support in $[0,\infty) \times \boldsymbol{R}$ it satisfies

$$\int_0^\infty \int_{-\infty}^\infty \Big[u(t,x)\partial_t\phi(t,x) + u(t,x)^3\partial_x\phi(t,x)\Big]\,dxdt + \int_{-\infty}^\infty u_0(x)\phi(0,x)\,dx = 0,$$

and that satisfies also the single entropy inequality (1.5), i.e. for every positive function ϕ with compact support in $\boldsymbol{R} \times \boldsymbol{R}$ it satisfies

$$\int_0^\infty \int_{-\infty}^\infty \Big[\frac{u(t,x)^2}{2}\partial_t\phi(t,x) + \frac{3u(t,x)^4}{4}\partial_x\phi(t,x)\Big]\,dxdt \ge 0.$$

Finally we summarize the main result obtained in this paper:

Theorem 5.2 *Consider the cubic conservation law* (1.1) *together with a nonclassical Riemann solver characterized by the function* $\varphi : \boldsymbol{R} \to \boldsymbol{R}$. *Suppose that* φ *satisfies the assumptions* (H1)–(H4) *listed in Section* 2.

Then, given an initial data with bounded total variation, the front tracking algorithm introduced in Section 4 *generates a sequence of approximate solutions* $u_\nu = u_\nu(t, x)$ *which satisfy the maximum principle* (5.1) *and have uniformly bounded total variation. As* $\nu \to \infty$, *the scheme converges to a weak solution to the conservation law* (1.1) *satisfying the entropy inequality* (1.5).

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References

[1] P. Baiti and H. K. Jenssen, On the front tracking algorithm, *J. Math. Anal. Appl.*, to appear.

[2] P. Baiti, P. G. LeFloch and B. Piccoli, in preparation.

[3] A. Bressan, Global solutions to systems of conservation laws by wave-front tracking, *J. Math. Anal. Appl.* **170** (1992), 414–432.

[4] C. Dafermos, Polygonal approximations of solutions of the initial value problem for a conservation law, *J. Math. Anal. Appl.* **38** (1972), 33–41.

[5] J. Glimm, Solutions in the large for nonlinear hyperbolic systems of equations, *Comm. Pure. Appl. Math.* **18** (1965), 697–715.

[6] B. T. Hayes and P. G. LeFloch, Non-classical shock waves and kinetic relations: scalar conservation laws, *Arch. Rational Mech. Anal.*, to appear.

[7] B. T. Hayes and P. G. LeFloch, Non-classical shock waves and kinetic relations: systems of conservation laws, preprint CMAP, Ecole Polytechnique, France, 1997.

[8] B. T. Hayes and P. G. LeFloch, Non-classical shock waves and kinetic relations: difference schemes, *SIAM J. Numer. Anal.*, to appear.

[9] D. Jacobs, W. R. McKinney and M. Shearer, Traveling wave solutions of the modified Korteweg-deVries Burgers equation, *J. Differential Equations* **116** (1995), 448–467.

[10] S. N. Kružkov, First order quasilinear equations in several independent variables, *Math. USSR Sbornik* **10** No. 2 (1970), 217–243.

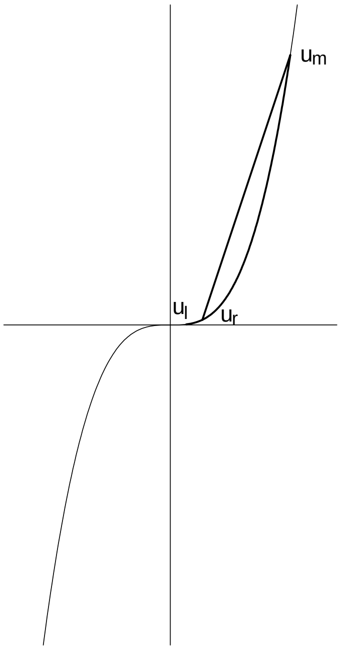
[11] P. D. Lax, Hyperbolic Systems of Conservation Laws II, *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[12] P. D. Lax, Shock waves and entropy, in "Contributions to Nonlinear Functional Analysis", E. A. Zarantonello ed., Academic Press, New York, 1971, pp. 603–634.

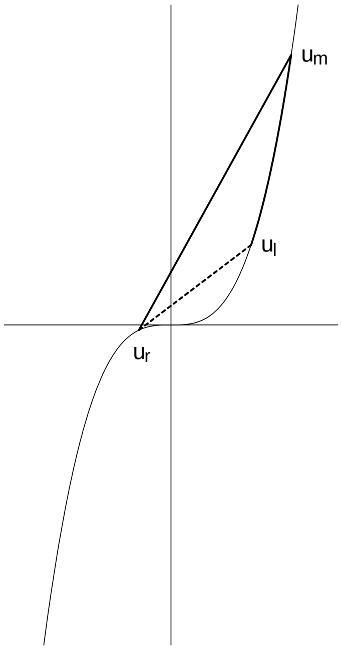
[13] T. P. Liu, Admissible solutions of hyperbolic conservation laws, *Amer. Math. Soc. Memoir* **240**, 1981.

[14] O. Oleinik, Discontinuous solutions of nonlinear differential equations, *Usp. Mat. Nauk.* **12** (1957), 3–73; English transl. in *Amer. Math. Soc. Transl.* Ser. 2 **26**, 95–172.

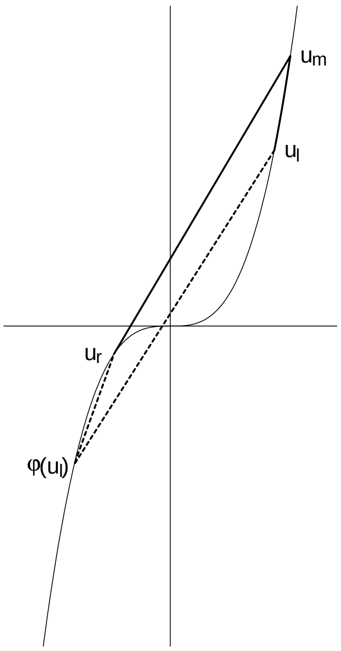
[15] N. H. Risebro, A front-tracking alternative to the random choice method, *Proc. Amer. Math. Soc.* **117**, no. 4 (1993), 1125–1139.



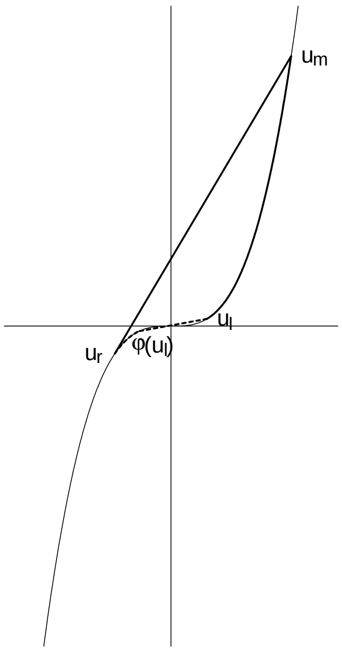
Case 1: (R, C)-(R)



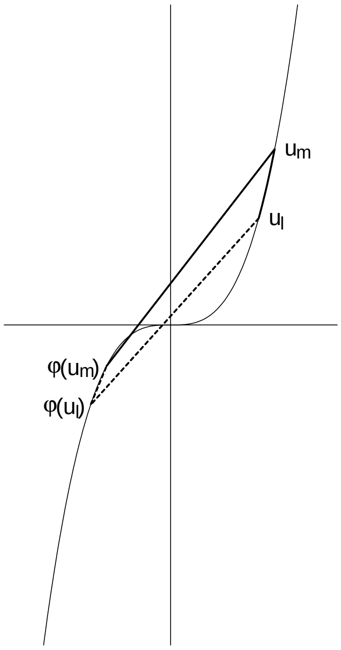
Case 2: (R, C)-(C)



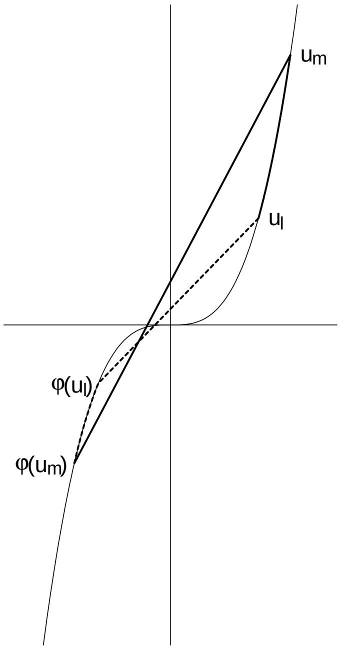
Case 3: (R, C)-(NC, C)



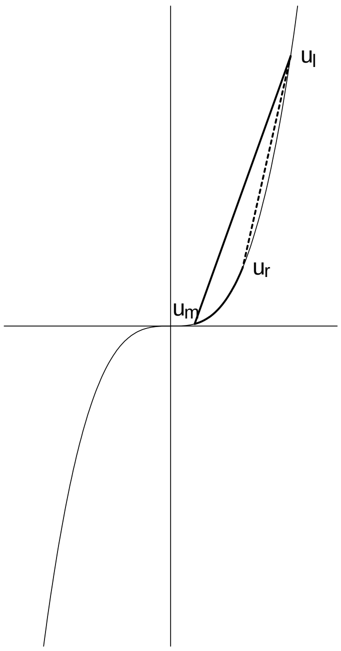
Case 4: (R, C)-(NC, R)



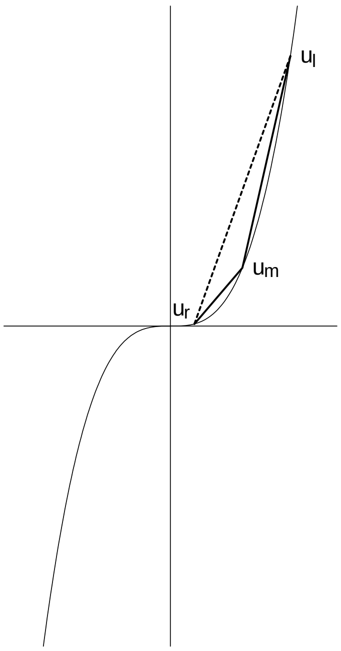
Case 5: (R, NC)-(NC, C)



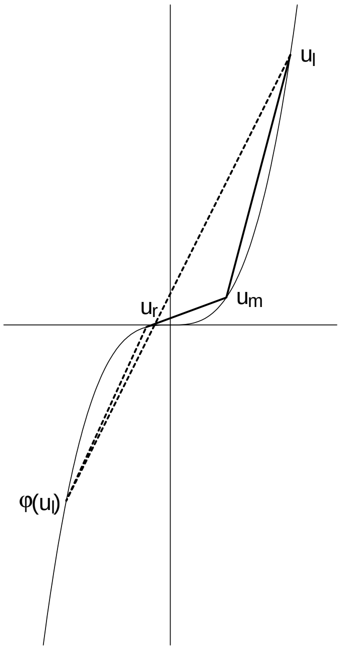
Case 6: (R, NC)-(NC, R)



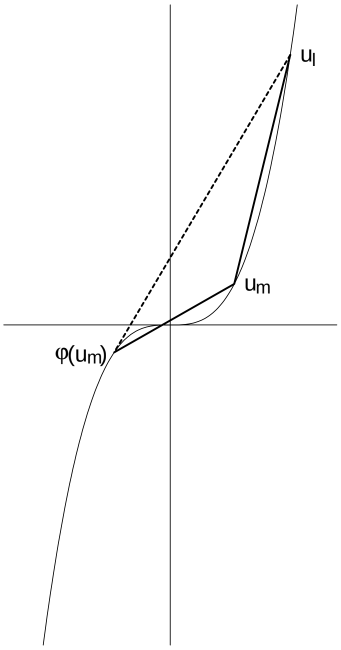
Case 7: (C, R)-(C)



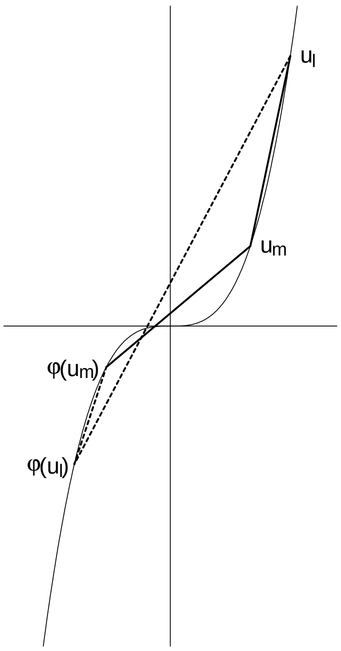
Case 8: (C, C)-(C)



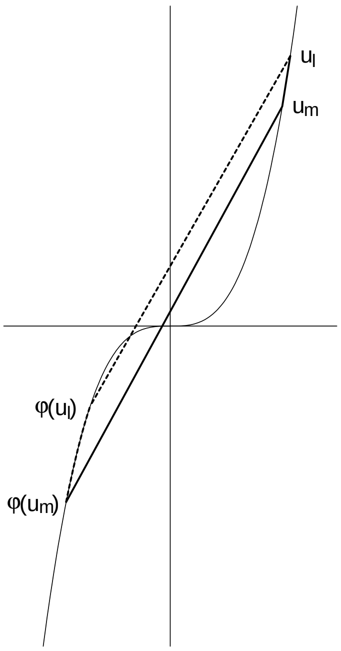
Case 9: (C, C)-(NC, C)



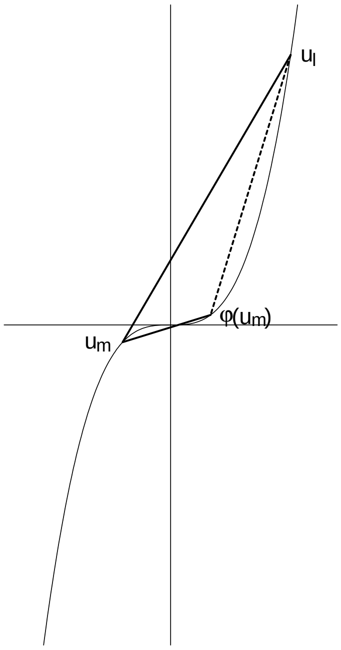
Case 10: (C, NC)-(C)



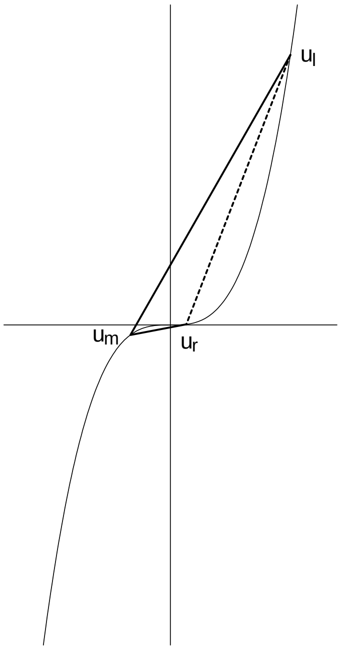
Case 11: (C, NC)-(NC, C)



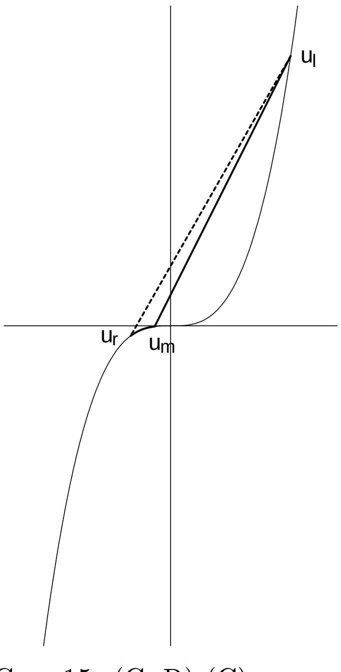
Case 12: (C, NC)-(NC, R)



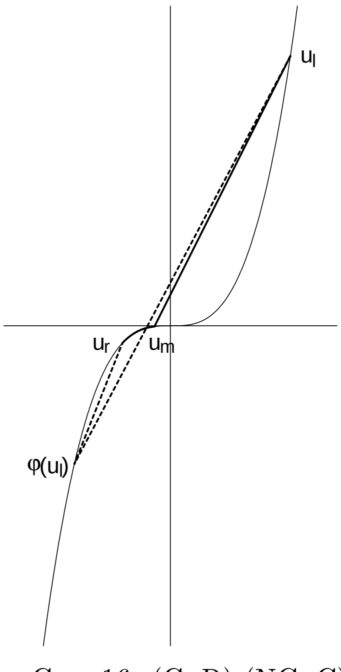
Case 13: (C, NC)-(C)



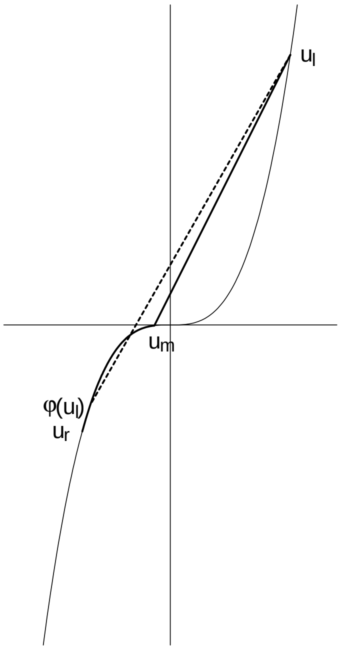
Case 14: (C, C)-(C)



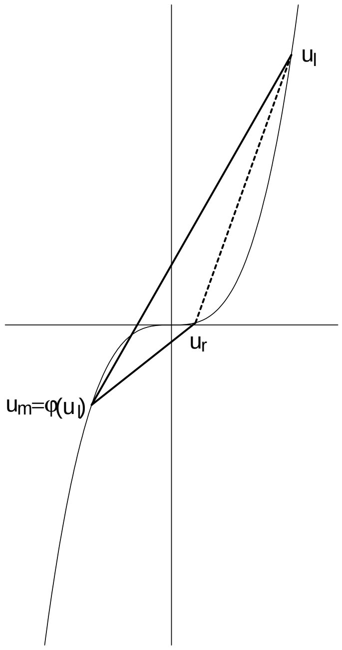
Case 15: (C, R)-(C)



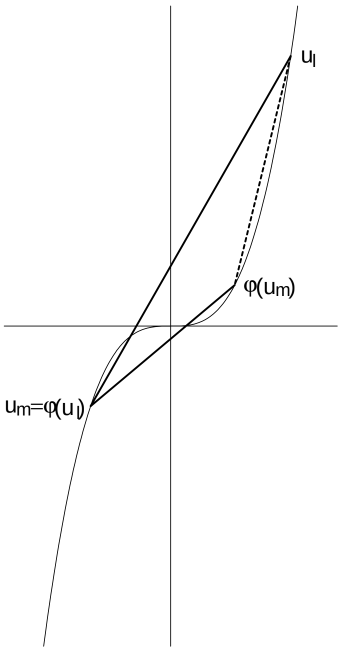
Case 16: (C, R)-(NC, C)



Case 17: (C, R)-(NC, R)



Case 18: (NC, C)-(C)



Case 19: (NC, NC)-(C)