Oleinik Type Estimates and Uniqueness for n x n Conservation Laws

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Abstract. Let $u_t + f(u)_x = 0$ be a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension. Relying on the existence of a semigroup of solutions, we first establish the uniqueness of entropy admissible weak solutions to the Cauchy problem, under a mild assumption on the local oscillation of u in a forward neighborhood of each point in the t-x plane. In turn, this yields the uniqueness of weak solutions which satisfy a decay estimate on positive waves of genuinely nonlinear families, thus extending a classical result proved by Oleinik in the scalar case.

1 - Introduction.

Consider a scalar conservation law in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

If f is strictly convex, say $f''(u) \geq \kappa > 0$ for every u, a well known estimate of Oleinik [20, 22] states that

$$u(t,y) - u(t,x) \leq \frac{y-x}{\kappa t} \tag{1.2}$$

for all $t > 0$, $x < y$ and every entropy-admissible solution of (1.1). Viceversa, if $u = u(t,x)$ is a weak solution satisfying (1.2), then u is entropy-admissible. In particular, given an initial condition

$$u(0,x) = \bar{u}(x), \tag{1.3}$$

the above decay estimate singles out a unique weak solution to the Cauchy problem, continuously depending on the initial data $\bar{u}$ in the $\mathbf{L}^1$ norm.

Aim of the present paper is to prove an analogous uniqueness theorem, valid for BV solutions of $n \times n$ hyperbolic systems. The following standard conditions [16, 22] will be assumed throughout:

(♣) The function f is smooth, defined for u in a neighborhood Ω of the origin. The system (1.1) is strictly hyperbolic. Each characteristic field is either linearly degenerate or genuinely nonlinear.

Under these assumptions, it was proved in [4, 6] that there exists a family of entropy weak solutions to (1.1) continuously depending on the initial data. More precisely, there exists a closed domain $\mathcal{D} \subset \mathbf{L}^1(I\!R;\ I\!R^n)$, constants $\eta_0, L > 0$, and a continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ with the properties:

(i) Every function $\bar{u} \in \mathbf{L}^1$ with Tot.Var.$(\bar{u}) \leq \eta_0$ lies in $\mathcal{D}$.

(ii) For all $\bar{u}, \bar{v} \in \mathcal{D}$, $t, s \geq 0$ one has $\big\|S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \leq L\big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}\big)$.

(iii) If $\bar{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t, \cdot) = S_t\bar{u}$ coincides with the solution of (1.1), (1.3) obtained by piecing together the standard self-similar solutions of the corresponding Riemann problems.

(iv) Each trajectory $t \mapsto u(t, \cdot) = S_t\bar{u}$ is a weak, entropy-admissible solution of the corresponding Cauchy problem (1.1), (1.3).

(v) Every weak solution obtained as limit of Glimm or front tracking approximations coincides with the corresponding trajectory of the semigroup.

An alternative, much shorter proof of this same result was recently given in [8]. The positively invariant domain $\mathcal{D}$ has the form

$$\mathcal{D} = cl\Big\{u \in \mathbf{L}^1(I\!R; I\!R^n); \quad u \text{ is piecewise constant}, \quad V(u) + C_0 \cdot Q(u) < \delta_0\Big\}, \tag{1.4}$$

for some constants $C_0, \delta_0 > 0$. Here $V(u)$ and $Q(u)$ denote the total strength of waves and the wave interaction potential of u, while cl denotes closure in $\mathbf{L}^1$. On a given domain $\mathcal{D}$, the semigroup S with the above properties is unique.

Following [1], we say that a map S with the properties (i)–(iii) is a *Standard Riemann Semigroup* (SRS). See [2] for a general survey. These results provide a new method for proving the uniqueness of the solution u to a given Cauchy problem (1.1), (1.3). Namely, it now suffices to show that u coincides with the corresponding semigroup trajectory:

$$u(t,\cdot) = S_t\bar{u} \qquad \text{for all } t \geq 0. \tag{1.5}$$

In turn, a convenient way to prove (1.5) is to use the error estimate [3]

$$\big\|u(T) - S_T u(0)\big\|_{\mathbf{L}^1} \leq L \cdot \int_0^T \left\{ \liminf_{h\to 0+} \frac{\big\|u(t+h) - S_h u(t)\big\|_{\mathbf{L}^1}}{h} \right\} dt, \tag{1.6}$$

valid for every Lipschitz continuous map $u : [0,T] \mapsto \mathcal{D}$. By showing that the integrand on the right hand side of (1.6) vanishes for almost every time t, one can thus establish (1.5). This approach was adopted in [7], proving the uniqueness of the entropy weak solution $u = u(t,x)$ which satisfies an additional regularity assumption. This additional condition, called *Tame Variation*, controls the total variation of u along space-like segments in the t-x plane.

In the first part of this paper we show that the Tame Variation can be replaced by a weaker assumption, restricting the oscillation of u on a forward neighborhood of each given point. Observing that a weak solution of (1.1) is defined up to a set of measure zero in the t-x plane, for sake of definitness we shall henceforth consider its right continuous version, so that

$$u(t,x) = \lim_{y\to x+} u(t,y) \qquad \text{for all } \ (t,x). \tag{1.7}$$

This is meaningful since each map $x \mapsto u(t,x)$ has bounded variation. By $\lambda_1(u) < \cdots < \lambda_n(u)$ we denote the characteristic speeds, i.e. the eigenvalues of the Jacobian matrix $Df(u)$. Following [16, p.555], we say that a shock is entropy admissible if the characteristics of the same genuinely nonlinear family impinge on the shock line from both sides.

For clarity, our main assumptions are listed below.

(A1) (Conservation Equations) The function $u = u(t,x)$ is a weak solution of the Cauchy problem (1.1), (1.3), taking values within the domain $\mathcal{D}$ of a Standard Riemann Semigroup S. More precisely, $u : [0,T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The initial condition (1.3) holds, together with

$$\iint \big(u\varphi_t + f(u)\varphi_x\big)\, dxdt = 0 \tag{1.8}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0,T[\times I\!R$.

(A2) (Entropy Condition) Let u have an approximate jump discontinuity at some point $(\tau,\xi) \in]0,T[\times I\!R$. More precisely, let there exist states $u^-, u^+ \in \Omega$ and a speed $\lambda \in I\!R$ such that, calling

$$U(t,x) \doteq \begin{cases} u^- & \text{if} \quad x < \xi + \lambda(t-\tau), \\ u^+ & \text{if} \quad x > \xi + \lambda(t-\tau), \end{cases} \tag{1.9}$$

there holds

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi-\rho}^{\xi+\rho} \Big|u(t,x) - U(t,x)\Big|\, dxdt = 0. \tag{1.10}$$

Then, for some $i \in \{1,\ldots,n\}$, one has the entropy inequality:

$$\lambda_i(u^-) \geq \lambda \geq \lambda_i(u^+). \tag{1.11}$$

(A3) (Tame Oscillation Condition) There exists constants K and λ^* such that, at every point (τ,ξ), one has

$$\limsup_{t\to\tau+,\ x\to\xi} \big|u(t,x) - u(\tau,\xi)\big| \leq K \cdot \big|u(\tau,\xi+) - u(\tau,\xi-)\big| \tag{1.12}$$

and

$$\lim_{\substack{t\to\tau+,\ x\to\xi\pm \\ |x-\xi|>\lambda^*(t-\tau)}} u(t,x) = u(\tau,\xi\pm). \tag{1.13}$$

By $u(\tau,\xi\pm)$ we denote here the right and left limits of the map $x \mapsto u(\tau,x)$ at the point ξ. They certainly exist because this map has bounded variation. Observe that the Tame Oscillation consists of local $\mathbf{L}^\infty$ type estimates. These are much weaker than the global BV type estimates required by the Tame Variation assumption in [7]. Yet, (A3) suffices to guarantee uniqueness of entropy weak solutions:

Theorem 1. *Let the basic assumptions* **(♣)** *hold, so that the system (1.1) generates a Standard Riemann Semigroup $S : \mathcal{D}\times[0,\infty[\mapsto \mathcal{D}$. Then, for every $\bar u \in \mathcal{D}$, $T > 0$, the Cauchy problem (1.1), (1.3) has a unique weak solution $u : [0,T] \mapsto \mathcal{D}$ satisfying the assumptions* (A1)–(A3). *Indeed, these conditions imply the identity (1.5)*

In the second part of this paper we show that, for genuinely nonlinear $n \times n$ systems, the Oleinik type estimates on the decay of positive waves imply both the Entropy Condition and the Tame Oscillation Condition stated above. This will provide an additional uniqueness theorem for weak solutions of the Cauchy problem (1.1), (1.3).

Decay estimates were proved in [19] for approximate solutions constructed by the Glimm scheme, and in [5] for exact solutions obtained as limit of front tracking approximations. A careful statement of these results requires some notations.

Let $A(u) = Df(u)$ be the Jacobian matrix of f at u. Smooth solutions of (1.1) thus satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0\,. \tag{1.14}$$

Call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of $A(u)$. Moreover, choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, \ldots, n$, normalized so that

$$|r_i| \equiv 1\,, \qquad \langle l_i,\ r_j\rangle = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{1.15}$$

for every $i \in \{1, \ldots, n\}$ and all u in the domain of f. The assumption on the genuine nonlinearity of the i-th characteristic family can be written as

$$\nabla\lambda_i \cdot r_i(u) \doteq \lim_{h\to 0} \frac{\lambda_i\big(u + hr_i(u)\big) - \lambda_i(u)}{h} \geq \kappa_i > 0\,. \tag{1.16}$$

For a given state $u \in I\!\!R^n$ and $i = 1, \ldots, n$, we denote by

$$\sigma \mapsto S_i(\sigma)(u), \qquad \sigma \mapsto R_i(\sigma)(u)$$

respectively the i-shock and the i-rarefaction curve through u, parametrized by arc-length. Moreover, we consider the composite curve

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if} \quad \sigma \geq 0, \\ S_i(\sigma)(u) & \text{if} \quad \sigma < 0. \end{cases} \tag{1.17}$$

As in [5], the definition of the Glimm interaction functional can be extended to general BV functions. Let $u : I\!\!R \mapsto I\!\!R^n$ have bounded variation. Then the distributional derivative $\mu = Du$ is a vector measure. Let $x_1, x_2, \ldots$ be the points where u has a jump, say $\Delta u(x_\alpha) = u(x_\alpha+) - u(x_\alpha-)$. Call $\sigma^1_\alpha, \ldots, \sigma^n_\alpha$ the waves generated by the corresponding Riemann problem at x_α. Recalling (1.17), this means

$$u(x_\alpha+) = \Psi_n(\sigma^n_\alpha) \circ \cdots \circ \Psi_1(\sigma^1_\alpha)\big(u(x_\alpha-)\big)\,. \tag{1.18}$$

For $i = 1, \ldots, n$ we can now define μ^i as the signed measure such that, for every open interval J,

$$\mu^i(J) = \int_J l_i \cdot Du\,, \tag{1.19}$$

where

$$\begin{cases} l_i(x) = l_i\big(u(x)\big) & \text{if} \quad u \text{ is continuous at } x, \\ l_i(x_\alpha) \cdot \Delta u(x_\alpha) = \sigma_\alpha^i & \text{if} \quad u \text{ has a jump at } x_\alpha. \end{cases} \tag{1.20}$$

Observe that, for a point of jump as in (1.18), the definitions (1.19)-(1.20) simply mean $\mu^i\big(\{x_\alpha\}\big) = \sigma_\alpha^i$. In this case, (1.20) does not uniquely determine the value of $l_i(x_\alpha)$. However, since

$$\sigma_\alpha^i = l_i\big(u(x_\alpha)\big) \cdot \Delta u(x_\alpha) + \mathcal{O}(1) \cdot |\Delta u(x_\alpha)|^2 \,, \tag{1.21}$$

where $\mathcal{O}$ is the Landau order symbol, we can choose the vector l_i so that

$$\Big| l_i(x_\alpha) - l_i\big(u(x_\alpha)\big) \Big| = \mathcal{O}(1) \cdot \big|\Delta u(x_\alpha)\big| \,. \tag{1.22}$$

Call μ^{i+}, μ^{i-} the positive and negative parts of the signed measure μ^i, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad\qquad |\mu^i| = \mu^{i+} + \mu^{i-}. \tag{1.23}$$

The *total strength of waves* in u is defined as

$$V(u) \doteq \sum_{i=1}^n V_i(u), \qquad\qquad V_i(u) \doteq |\mu^i|(\mathbb{R}), \tag{1.24}$$

Let $\mathcal{N}$ be the set of those indices $i \in \{1, \ldots, n\}$ such that the i-th characteristic family is genuinely nonlinear. The *interaction potential* of waves in u is then defined as

$$Q(u) \doteq \sum_{1 \leq i < j \leq n} \big(|\mu^j| \times |\mu^i|\big)\big(\{(x,y); \quad x < y\}\big) + \sum_{i \in \mathcal{N}} \big(\mu^{i-} \times |\mu^i|\big)\big(\{(x,y); \quad x \neq y\}\big) \,. \tag{1.25}$$

The decay estimates in [5] can be stated as follows.

Proposition 1. *Let the i-th characteristic field be genuinely nonlinear. Then there exist constants $C_1, \kappa > 0$ such that, for every solution u with small total variation obtained as limit of wave-front tracking approximations, one has*

$$\mu_t^{i+}\big([a,b]\big) \leq \frac{b-a}{\kappa(t-s)} + C_1 \cdot \big[Q(u(s)) - Q(u(t))\big]. \tag{1.26}$$

for every interval $[a,b]$ and all $t > s \geq 0$.

Here and in the sequel, by μ_t^{i+} we denote the measure of positive i-waves in $u(t, \cdot)$. Intuitively, (1.26) says that these positive waves can be split in two parts:

- The "old" waves, generated before time s, that have decayed throughout the interval $[s,t]$ due to genuine nonlinearity. Their density is $\mathcal{O}(1) \cdot (t-s)^{-1}$.
- The "new" waves, generated after time s. Their density can be arbitrarily large, but their total strength is controlled by the decrease in the interaction potential.

Our second main result provides a converse to Proposition 1, showing that the uniqueness of solutions to the Cauchy problem can also be derived from the following.

(A4) (Decay Assumption) There exist a constant $\kappa > 0$ and a nonincreasing function Z such that

$$\mu_t^{i+}\big([a,b]\big) \leq \frac{b-a}{\kappa(t-s)} + \big[Z(s) - Z(t)\big]. \tag{1.27}$$

for every interval $[a,b]$ and all $i = 1,\ldots,n$, $t > s \geq 0$.

Theorem 2. *Let the system (1.1) be strictly hyperbolic, with each characteristic field genuinely nonlinear, so that it generates a Standard Riemann Semigroup S on a domain $\mathcal{D}$ as in (1.4), with $\delta_0 > 0$ sufficiently small. Then, for every $\bar{u} \in \mathcal{D}$, $T > 0$, the Cauchy problem (1.1), (1.3) has a unique solution $u : [0,T] \mapsto \mathcal{D}$ satisfying* **(A1)** *and* **(A4)** *for some constant κ and some nonincreasing function Z. Indeed, these conditions imply the identity (1.5).*

Several uniqueness results for entropy weak solutions to hyperbolic systems of conservation laws have appeared in the literature. For scalar conservation laws, the uniqueness and stability problem was completely solved by Kruzhkov [15]. In the case of systems, however, until recently all available theorems required additional regularity hypotheses on the solutions. In [21, 18], a uniqueness result is proved in the class of piecewise smooth functions, while in [10] it is shown that if a piecewise regular solution exists, then it is unique within a class of BV functions. The main result in [17] establishes the uniqueness of solutions under a restriction on the locations of centered rarefaction waves. For Temple class systems, stronger uniqueness theorems can be found in [9, 13]. Relying on the semigroup approach proposed in [1], the uniqueness theorem in [7] was the first one which could be applied within the same class of BV functions where an existence theorem is known. The present assumptions (A3) further weaken the regularity condition used in [7] and are clearly satisfied by the weak solutions obtained as limits of the Glimm scheme [12] or front tracking approximations [2]. As shown in [5], any limit of front tracking approximations satisfies the decay assumption (A4), provided that all characteristic fields of the system (1.1) are genuinely nonlinear.

2 - Preliminary results.

In this section we collect some technical lemmas, for later use.

Lemma 1. *Let $u : [0,T] \times I\!\!R \mapsto I\!\!R^n$ be a function which satisfies the Conservation Equations* (A1), *say with Tot.Var.$u(t,\cdot) \leq M$ for all $t \in [0,T]$. Then the map $t \mapsto u(t,\cdot)$ is Lipschitz continuous,* *i.e.*

$$\big\|u(\tau,\cdot)-u(\tau',\cdot)\big\|_{\mathbf{L}^1}\leq L'|\tau-\tau'| \tag{2.1}$$

for some constant L' and all $\tau,\tau'\in[0,T]$.

Proof. Fix any $\tau>\tau'>0$ and construct a smooth approximation to the characteristic function of the interval $[\tau',\tau]$. For this purpose, take a smooth nondecreasing function $\alpha:I\!R\to[0,\ 1]$, such that

$$\alpha(x)=\begin{cases}0 & \text{if}\quad x\leq -1,\\ 1 & \text{if}\quad x\geq 1,\end{cases}$$

and define $\alpha_h(x)\doteq\alpha(x/h)$. As $h\to 0+$, α_h thus approaches the Heaviside function. Consider any smooth function $\psi=\psi(x)$ with compact support, and define

$$\varphi_h(t,x)=[\alpha_h(t-\tau')-\alpha_h(t-\tau)]\psi(x).$$

By the assumption on u, using φ_h in (1.8) and letting $h\to 0$, thanks to the $\mathbf{L}^1$-continuity of the function $t\mapsto u(t,\cdot)$ we obtain

$$\int\psi(x)[u(\tau,x)-u(\tau',x)]dx+\int_{\tau'}^{\tau}\int\psi_x(x)f(u)dxdt=0.$$

It thus follows

$$\begin{aligned}\big\|u(\tau,\cdot)-u(\tau',\cdot)\big\|_{\mathbf{L}^1}&=\sup_{\psi\in\mathcal{C}^1_c,\quad|\psi|\leq 1}\int\psi(x)[u(\tau,x)-u(\tau',x)]dx\\ &\leq\int_{\tau'}^{\tau}\text{Tot.Var.}\big\{f(u(t,\cdot))\big\}dt\\ &\leq M\cdot Lip(f)\cdot|\tau-\tau'|,\end{aligned}$$

where by $Lip(f)$ we denote the Lipschitz constant of the function f on the domain $|u|\leq M$.

Lemma 2. *Let $w:]a,\ b[\mapsto I\!R^n$ be an integrable function such that, for some measure μ, one has*

$$\left|\int_{\zeta_1}^{\zeta_2}w(x)\,dx\right|\leq\mu\big([\zeta_1,\ \zeta_2]\big),\qquad \textit{whenever } a<\zeta_1<\zeta_2<b. \tag{2.2}$$

Then

$$\int_a^b\big|w(x)\big|\,dx\leq\mu\big(]a,\ b[\big). \tag{2.3}$$

Proof. Observe that, in (2.2), one can replace the closed interval $[\zeta_1,\zeta_2]$ with an open one. Namely,

$$\left|\int_{\zeta_1}^{\zeta_2}w(x)\,dx\right|=\lim_{\varepsilon\to 0}\left|\int_{\zeta_1+\varepsilon}^{\zeta_2-\varepsilon}w(x)\,dx\right|\leq\mu\big(]\zeta_1,\ \zeta_2[\big) \tag{2.4}$$

Next, fix any $\varepsilon > 0$. Then there exists a piecewise constant function v such that

$$\int_a^b \big|w(x) - v(x)\big| dx \le \varepsilon.$$

Calling $a = x_0 < \cdots < x_N = b$ the points of discontinuity of v, we compute

$$\begin{aligned}
\int_a^b \big|w(x)\big| dx &\le \int_a^b \big|w(x) - v(x)\big| dx + \int_a^b \big|v(x)\big| dx \\
&\le \varepsilon + \sum_j \int_{x_{j-1}}^{x_j} \big|v(x)\big| dx \\
&= \varepsilon + \sum_j \left| \int_{x_{j-1}}^{x_j} v(x) dx \right| \\
&\le \varepsilon + \sum_j \left| \int_{x_{j-1}}^{x_j} w(x) dx \right| + \sum_j \left| \int_{x_{j-1}}^{x_j} \big(v(x) - w(x)\big) dx \right| \\
&\le \varepsilon + \sum_j \mu\big(]x_{j-1},\ x_j[\big) + \int_a^b \big|v(x) - w(x)\big| dx \\
&\le 2\varepsilon + \mu\big(]a,\ b[\big).
\end{aligned}$$

Since ε was arbitrary, this proves the lemma.

Lemma 3. *Let $u : [0, T] \mapsto \mathcal{D}$ be Lipschitz continuous. At a given point (τ, ξ), let the conditions (1.9)-(1.10) hold, for some $u^-, u^+ \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$. Then, for each $\lambda^* > 0$ one has*

$$\lim_{\rho \to 0+} \sup_{|h| \le \rho} \frac{1}{\rho} \int_{\xi - \lambda^* \rho}^{\xi + \lambda^* \rho} \big|u(\tau + h,\ x) - U(\tau + h,\ x)\big|\ dx = 0. \tag{2.5}$$

Proof. Assume, on the contrary, that there exists $\delta > 0$ and sequences $\rho_\nu \to 0$, $|h_\nu| < \rho_\nu$, such that

$$\int_{\xi - \lambda^* \rho_\nu}^{\xi + \lambda^* \rho_\nu} \big|u(\tau + h,\ x) - U(\tau + h,\ x)\big|\ dx > \delta \rho_\nu \tag{2.6}$$

for every $\nu \ge 1$. To fix the ideas, assume $h_\nu \ge 0$ for all ν, the other cases being entirely similar. By the Lipschitz continuity proved in (2.1), we have

$$\int_a^b \Big\{ \big|u(t, x) - U(t, x)\big| - \big|u(t', x) - U(t', x)\big| \Big\} dx \le L^* |t - t'| \tag{2.7}$$

for some constant L^*, for every $t, t' \ge 0$ and every interval $[a, b]$. It is not restrictive to assume $L^* > 1/\delta$. The assumption (1.10) implies

$$\lim_{\rho \to 0+} \frac{1}{\rho^2} \int_{\tau - \rho}^{\tau + \rho} \int_{\xi - \lambda^* \rho}^{\xi + \lambda^* \rho} \Big|u(t, x) - U(t, x)\Big|\ dx dt = 0. \tag{2.8}$$

However, if (2.6) holds, by (2.7) it follows

$$
\begin{aligned}
\int_{\tau-\rho_\nu}^{\tau+\rho_\nu} \int_{\xi-\lambda^*\rho_\nu}^{\xi+\lambda^*\rho_\nu} & \Big| u(t,x) - U(t,x) \Big| \, dxdt \\
&\geq \int_{\tau+h_\nu-(\delta/L^*)\rho_\nu}^{\tau+h_\nu} \int_{\xi-\lambda^*\rho_\nu}^{\xi+\lambda^*\rho_\nu} \Big\{ \big| u(\tau+h_\nu,\ x) - U(\tau+h_\nu,\ x) \big| - L^*|\tau+h_\nu-t| \Big\} \, dxdt \\
&\geq \int_{\tau+h_\nu-(\delta/L^*)\rho_\nu}^{\tau+h_\nu} \delta\rho_\nu - L^*(\tau+h_\nu-t) \; dt \\
&= \delta(\delta/L^*)\rho_\nu^2
\end{aligned}
\tag{2.9}
$$

for all ν, in contradiction with (2.8). This proves the lemma.

Corollary 1. *Under the same assumptions of Lemma 3, the states u^-, u^+ in (1.9)-(1.10) satisfy*

$$
u^- = u(\tau, \xi-), \qquad\qquad u^+ = u(\tau, \xi+). \tag{2.10}
$$

Indeed, taking $h = 0$ and letting $\rho \to 0+$ in (2.5) one obtains

$$
\begin{aligned}
\big| u(\tau,\xi+) - u^+ \big| &\leq \limsup_{\rho\to 0+} \frac{1}{\rho} \left(\int_\xi^{\xi+\rho} \big| u(\tau,\xi+) - u(\tau,x) \big| \, dx \; + \; \int_\xi^{\xi+\rho} \big| u(\tau,x) - u^+ \big| \, dx \right) \\
&= 0
\end{aligned}
$$

The other identity is proved similarly.

Lemma 4. *Let u be a weak solution of (1.1). Let the conditions (1.9)-(1.10) hold at some point (τ, ξ). Then the states u^-, u^+ and the speed λ satisfy the Rankine-Hugoniot equations*

$$
\lambda(u^+ - u^-) = f(u^+) - f(u^-). \tag{2.11}
$$

Together with the entropy condition in (A2), Lemma 4 shows that the states u^-, u^+ are connected either by an admissible shock or by a contact discontinuity, propagating with speed λ. To prove the lemma, define the function

$$
V(s,y) \doteq \begin{cases} u^- & \text{if} \quad y < \lambda s, \\ u^+ & \text{if} \quad y > \lambda s. \end{cases}
$$

By (1.10) it follows

$$
\lim_{\rho\to 0+} \iint_B \big| u(\tau+\rho s,\ \xi+\rho y) - V(s,y) \big| \, dyds = 0
$$

for every bounded set $B \subset I\!\!R^2$. Moreover, (1.8) implies

$$
\iint u(\tau+\rho s,\ \xi+\rho y)\phi_s(s,y) + f\big(u(\tau+\rho s,\ \xi+\rho y)\big)\phi_y(s,y) \; dyds = 0
$$

for all $\phi \in \mathcal{C}_c^1$ and all $\rho > 0$. The two above relations imply

$$\iint V\phi_s + f(V)\phi_y \; dyds = 0,$$

showing that the function V itself is a solution of (1.1). Therefore the Rankine-Hugoniot equations (2.11) must hold.

We conclude this section by recalling two local integral estimates that characterize the trajectories of a Standard Riemann Semigroup.

Two types of local approximate solutions for (1.1) will be considered. One is derived from the self-similar solution of a Riemann problem, the other is obtained by "freezing" the coefficients of the corresponding quasilinear hyperbolic system in a neighborhood of a given point.

Let $w : I\!R \mapsto I\!R^n$ be any BV function and fix any point $\xi \in I\!R$ where w has a jump. Call $\omega = \omega(t,x)$ the unique self-similar entropy solution of the Riemann problem

$$\omega_t + f(\omega)_x = 0, \qquad \omega(0,x) = \begin{cases} w(\xi-) & \text{if} \quad x < 0, \\ w(\xi+) & \text{if} \quad x > 0. \end{cases} \tag{2.12}$$

Let $\hat\lambda$ be an upper bound for the absolute values of all wave speeds. For $t \geq 0$, define

$$U^\sharp(t,x) \doteq \begin{cases} \omega(t,\ x-\xi) & \text{if} \quad |x-\xi| \leq \hat\lambda t, \\ w(x) & \text{if} \quad |x-\xi| > \hat\lambda t. \end{cases} \tag{2.13}$$

Observe that the function $t \mapsto U^\sharp(t,\cdot)$ is Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance, and approaches w as $t \to 0+$.

Next, call $\widetilde{A} \doteq Df\big(w(\xi)\big)$ the Jacobian matrix of f computed at the point $w(\xi)$. For $t \geq 0$, define $U^\flat(t,x)$ as the solution of the linear hyperbolic Cauchy problem with constant coefficients

$$U_t^\flat + \widetilde{A} U_x^\flat = 0, \qquad U^\flat(0) = w. \tag{2.14}$$

Lemma 5. *Let S be a Standard Riemann Semigroup generated by the system (1.1), with domain $\mathcal{D}$ as in (1.4). Then, there exists a constant C_1 such that the following holds. For every function $w \in \mathcal{D}$, every $\xi \in I\!R$ and $h, \rho > 0$, with the above definitions one has*

$$\frac{1}{h}\int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda} \Big|\big(S_h w\big)(x) - U^\sharp(h,x)\Big| \, dx \;\leq\; C_1 \cdot \text{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi[\, \cup\,]\xi,\ \xi+\rho[\big), \tag{2.15}$$

$$\frac{1}{h}\int_{\xi-\rho+h\hat\lambda}^{\xi+\rho-h\hat\lambda} \Big|\big(S_h w\big)(x) - U^\flat(h,x)\Big| \, dx \leq C_1 \cdot \Big(\text{Tot.Var.}\big(w;\ ]\xi-\rho,\ \xi+\rho[\big)\Big)^2. \tag{2.16}$$

For a proof, see [1, p.217].

3 - Proof of Theorem 1.

As before, let $\hat\lambda$ be an upper bound for all wave speeds. Let u satisfy the hypotheses of Theorem 1. For every R, it suffices to show that $u(T,\cdot)=S_T\bar u$, restricted to the interval

$$J_T \doteq [-R+\hat\lambda T,\ R-\hat\lambda T], \tag{3.1}$$

for all $T>0$. In turn, this can be deduced from the error estimate

$$\big\|u(T)-S_T\bar u\big\|_{\mathbf{L}^1(J_T)} \le L\cdot\int_0^T \left\{\liminf_{h\to0+}\frac{\big\|u(\tau+h)-S_h u(\tau)\big\|_{\mathbf{L}^1(J_{\tau+h})}}{h}\right\}\,d\tau, \tag{3.2}$$

if we prove that the integrand on the right hand side of (3.2) vanishes almost everywhere. Recalling Lemma 1 and the assumption $u(t,\cdot)\in\mathcal{D}$, we conclude that $u=u(t,x)$ is a BV function on the strip $[0,T]\times I\!R$, in the sense that the distributional derivatives $D_t u, D_x u$ are Radon measures [11]. By a well known structure theorem [11, 23], there exists a set $\widetilde{\mathcal N}\subset\,]0,T[\,\times I\!R$ of 1-dimensional Hausdorff measure zero such that, at every point $(\tau,\xi)\notin\widetilde{\mathcal N}$, u either is approximately continuous or has an approximate jump discontinuity. We claim that, at every such point of discontinuity, the jump cannot have a horizontal tangent. In other words, one can never have (1.10) in the case

$$U(t,x)=\begin{cases}u^- & \text{if}\quad t<\tau,\\ u^+ & \text{if}\quad t>\tau,\end{cases}\qquad u^+\neq u^-. \tag{3.3}$$

Indeed, for every positive constant c (arbitrarily large), from (3.3), (1.10) and (2.1) we deduce

$$\begin{aligned}
c|u^+-u^-| &= \lim_{\rho\to0+}\frac{1}{\rho^2}\int_\tau^{\tau+\rho}\int_\xi^{\xi+c\rho}|u^+-u^-|\,dxdt\\
&\le \limsup_{\rho\to0+}\frac{1}{\rho^2}\int_\tau^{\tau+\rho}\int_\xi^{\xi+c\rho}\big|u(t,x)-u(2\tau-t,x)\big|\,dxdt\\
&\le \lim_{\rho\to0+}\frac{1}{\rho^2}\int_\tau^{\tau+\rho}L'\cdot2(t-\tau)\,dt\\
&= L'.
\end{aligned}$$

Hence $|u^+-u^-|\le L'/c$ for every c, against the second assumption in (3.3).

Taking the projection of $\widetilde{\mathcal N}$ on the t-axis, we conclude that there exists a set $\mathcal N\subset[0,T]$ of measure zero, containing the endpoints 0 and T, such that, at every point $(\tau,\xi)\in[0,T]\times I\!R$ with $\tau\notin\mathcal N$, the following property holds.

(P) Either u is approximately continuous at (τ,ξ), or it has an approximate jump discontinuity, in the sense that (1.9)-(1.10) hold for some states u^-,u^+ and some $\lambda\in I\!R$. In this second case, Corollary 1 implies $u^- = u(\tau, \xi-)$, $u^+ = u(\tau, \xi+)$. Moreover, by Lemma 4 the Rankine-Hugoniot equations (2.11) hold. Hence, by (A2) the entropy condition (1.11) holds as well.

Theorem 1 will be proved by establishing the basic relation

$$\liminf_{h\to 0+} \frac{1}{h} \int_{-R+\hat{\lambda}(\tau+h)}^{R-\hat{\lambda}(\tau+h)} \left|u(\tau+h, x) - \big(S_h u(\tau)\big)(x)\right| dx = 0 \qquad \text{for all } \ \tau \notin \mathcal{N}. \tag{3.4}$$

Let $\tau \in [0, T] \setminus \mathcal{N}$ and $\varepsilon > 0$ be given. For notational convenience, call μ the measure of total variation of $u(\tau, \cdot)$, so that

$$\mu\big(]a,\ b[\big) \doteq \text{Tot.Var.}\big\{u(\tau);\ ]a,\ b[\,\big\}. \tag{3.5}$$

Let $\xi'_1 < \ldots < \xi'_N$ be the points in $[-R,\ R]$ such that

$$\mu\big(\{\xi'_\alpha\}\big) \geq \varepsilon.$$

The boundedness of μ on any bounded interval implies that only finitely many such points exist. Since $\tau \notin \mathcal{N}$, for each $\alpha = 1, \ldots, N$ by the property (P) the two states $u^\pm_\alpha \doteq u(\tau, \xi'_\alpha\pm)$ are connected by an entropy-admissible shock, travelling with some speed λ^α. Moreover, introducing the functions

$$U_\alpha(t, x) = \begin{cases} u^-_\alpha & \text{if} \quad x < \xi'_\alpha + \lambda^\alpha(t-\tau), \\ u^+_\alpha & \text{if} \quad x > \xi'_\alpha + \lambda^\alpha(t-\tau), \end{cases} \tag{3.6}$$

one has

$$\lim_{\rho\to 0+} \frac{1}{\rho^2} \int_{\tau-\rho}^{\tau+\rho} \int_{\xi'_\alpha-\rho}^{\xi'_\alpha+\rho} \left|u(t, x) - U_\alpha(t, x)\right|\, dxdt = 0 \qquad (\alpha = 1, \ldots, N). \tag{3.7}$$

For any fixed $\lambda^* > 0$ (arbitrarily large), Lemma 3 implies

$$\lim_{h\to 0+} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \left|u(\tau+h, x) - U_\alpha(\tau+h, x)\right|\, dx = 0. \tag{3.8}$$

By (3.8) we can thus assume

$$\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \Big|u(\tau+h, x) - U_\alpha(\tau+h, x)\Big|\, dx \leq \varepsilon \tag{3.9}$$

for $h > 0$ sufficiently small. By (2.15) in Lemma 5 it follows that the semigroup trajectory satisfies an entirely similar estimate:

$$\begin{aligned} &\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha-\lambda^* h}^{\xi'_\alpha+\lambda^* h} \Big|\big(S_h u(\tau)\big)(x) - U_\alpha(\tau+h, x)\Big|\, dx \\ &\qquad \leq C_1 \cdot \sum_{\alpha=1}^{N} \text{Tot.Var.}\Big\{u(\tau);\ ]\xi'_\alpha - 2\lambda^* h,\ \xi'_\alpha[\, \cup\,]\xi'_\alpha,\ \xi'_\alpha + 2\lambda^* h[\,\Big\} \\ &\qquad \leq \varepsilon, \end{aligned} \tag{3.10}$$

for all $h > 0$ sufficiently small. Indeed, it is not restrictive to assume that λ^* is larger than all wave speeds. Combining (3.9) and (3.10) we deduce

$$\sum_{\alpha=1}^{N} \frac{1}{h} \int_{\xi'_\alpha - \lambda^* h}^{\xi'_\alpha + \lambda^* h} \Big| u(\tau + h, x) - \big(S_h u(\tau)\big)(x) \Big| \, dx \ \leq \ 2\,\varepsilon \qquad \text{for all } h \in \,]0, \ h^*], \tag{3.11}$$

with $h^* > 0$ sufficiently small.

Next, we consider all the remaining points, where $u(\tau, \cdot)$ is either continuous or has a jump of strength $< \varepsilon$. Choose $\rho > 0$ such that

$$2\rho < \min_{\alpha = 2, \cdots, N} |\xi'_\alpha - \xi'_{\alpha-1}|,$$

$$\mu\big(]a, \ b[\big) \leq \varepsilon \qquad \text{whenever} \ \ b - a \leq 2\,\rho, \quad]a, b[\, \cap \, \{\xi'_1, \ldots \xi'_N\} = \emptyset. \tag{3.12}$$

Such a ρ exists, because the total variation of $u(\tau)$ is bounded and $\mu\big(\{x\}\big) < \varepsilon$ whenever $x \notin \{\xi'_1, \ldots \xi'_N\}$.

We can now select a finitely many points $\xi_1 < \ldots < \xi_M$, such that the open intervals $I_\beta \doteq \,]\xi_\beta - \rho, \ \xi_\beta + \rho[$ satisfy:

(i) $\bigcup_{\beta=1}^{M} I_\beta \ \supseteq \ [-R, \ R] \setminus \{\xi'_1, \ldots, \xi'_N\}$.

(ii) Every point x is contained in at most two distinct intervals I_β.

Observe that the closed intervals

$$I'_{\alpha,h} \doteq [\xi'_\alpha - h\lambda^*, \ \xi'_\alpha + h\lambda^*]$$

together with the open intervals

$$I_{\beta,h} \doteq \,]\xi_\beta - \rho + h\lambda^*, \ \xi_\beta + \rho - h\lambda^*[$$

still cover $[-R, \ R]$, for $h > 0$ sufficiently small.

We claim that, for every β, there exists $h_\beta > 0$ such that

$$\big| u(\tau + h, x) - u(\tau, \xi_\beta) \big| \leq (K+2)\varepsilon \qquad \text{for all } x \in I_{\beta,h}, \quad h \in [0, h_\beta]. \tag{3.13}$$

To see this, we apply (1.13) with ξ replaced by $\xi_\beta - \rho$. By (3.12), there exists $\delta^* > 0$ such that

$$\begin{aligned} &\big| u(\tau + h, \ x) - u(\tau, \xi_\beta) \big| \\ &\qquad \leq \Big| u(\tau + h, \ x) - u\big(\tau, \ (\xi_\beta - \rho) + \big) \Big| + \Big| u\big(\tau, \ (\xi_\beta - \rho) + \big) - u(\tau, \xi_\beta) \Big| \\ &\qquad \leq \varepsilon + \varepsilon \end{aligned} \tag{3.14}$$

for all $h > 0$ and $x \in I_{\beta,h}$ with $x < \xi_\beta - \rho + \delta^*$. Similarly,

$$\big|u(\tau + h,\ x) - u(\tau, \xi_\beta)\big| \leq 2\varepsilon \tag{3.15}$$

for all $h > 0$ and $x \in I_{\beta,h}$ with $x > \xi_\beta + \rho - \delta^*$. For each point (τ, x) with $|x - \xi_\beta| \leq \rho - \delta^*$ we can now apply (1.12) and deduce

$$\big|u(\tau + h,\ y) - u(\tau, x)\big| \leq K\big|u(\tau, x+) - u(\tau, x-)\big| + \varepsilon \tag{3.16}$$

for $h > 0$ and $|y - x|$ sufficiently small. By (3.16) and (3.12), for some $\delta_x > 0$ there holds

$$\begin{aligned} \big|u(\tau + h,\ y) - u(\tau, \xi_\beta)\big| &\leq \big|u(\tau + h,\ y) - u(\tau, x)\big| + \big|u(\tau, x) - u(\tau, \xi_\beta)\big| \\ &\leq K\varepsilon + 2\varepsilon, \end{aligned} \tag{3.17}$$

for $h \in [0, \delta_x]$, $|y - x| < \delta_x$. Covering the compact interval $[\xi_\beta - \rho + \delta^*,\ \xi_\beta + \rho - \delta^*]$ with finitely many subintervals $[x_\ell - \delta_\ell,\ x_\ell + \delta_\ell]$, $\ell = 1, \ldots, N$, and choosing $h_\beta \doteq \min\{\delta^*, \delta_1, \ldots, \delta_N\}$, we deduce (3.13).

Next, for each β, define $U^\flat$ to be the solution of the linear hyperbolic problem with constant coefficients

$$U^\flat_t + \widetilde{A} U^\flat_x = 0, \qquad U^\flat(\tau) = u(\tau), \tag{3.18}$$

where $\widetilde{A} \doteq Df\big(u(\tau, \xi_\beta)\big)$. Call $\tilde{\lambda}_1, \ldots \tilde{\lambda}_n$ the eigenvalues of $\widetilde{A}$ and let $\tilde{l}_i$, $\tilde{r}_i$ be the corresponding left and right eigenvectors, normalized as in (1.15). For every $i = 1, \ldots, n$ and every choice of $\zeta_1, \zeta_2 \in I_{\beta,h}$, we now estimate the quantity

$$E_i \doteq \int_{\zeta_1}^{\zeta_2} \big\langle \tilde{l}_i,\ u(\tau + h, x) - U^\flat(\tau + h, x)\big\rangle\, dx.$$

Observing that (3.18) implies

$$\big\langle \tilde{l}_i,\ U^\flat(\tau + h, x)\big\rangle = \big\langle \tilde{l}_i,\ U^\flat(\tau, x - h\tilde{\lambda}_i)\big\rangle = \big\langle \tilde{l}_i,\ u(\tau, x - h\tilde{\lambda}_i)\big\rangle,$$

and integrating (1.1) over the domain

$$D_i \doteq \Big\{(t, x);\ \ t \in [\tau,\ \tau + h],\ \ \zeta_1 + (t - \tau - h)\tilde{\lambda}_i \leq x \leq \zeta_2 + (t - \tau - h)\tilde{\lambda}_i\Big\},$$

we obtain

$$\begin{aligned} E_i = &\int_\tau^{\tau+h} \Big\langle \tilde{l}_i,\ \big(f(u) - \tilde{\lambda}_i u\big)\big(t,\ \zeta_1 + (t - \tau - h)\tilde{\lambda}_i\big)\Big\rangle\, dt \\ &- \int_\tau^{\tau+h} \Big\langle \tilde{l}_i,\ \big(f(u) - \tilde{\lambda}_i u\big)\big(t,\ \zeta_2 + (t - \tau - h)\tilde{\lambda}_i\big)\Big\rangle\, dt. \end{aligned} \tag{3.19}$$

To estimate the quantity in (3.19), consider the states

$$u' \doteq u\big(t,\ \zeta_1 + (t-\tau-h)\tilde\lambda_i\big), \qquad u'' \doteq u\big(t,\ \zeta_2 + (t-\tau-h)\tilde\lambda_i\big), \qquad \tilde u \doteq u(\tau, \xi_\beta).$$

We then have

$$\Big\langle \tilde l_i,\ f(u'') - f(u') - \tilde\lambda_i(u''-u')\Big\rangle = \Big\langle \tilde l_i,\ Df(\tilde u)\cdot(u''-u') - \tilde\lambda_i(u''-u')\Big\rangle + \Big\langle \tilde l_i,\ A^*(u''-u')\Big\rangle,$$

where the matrix A^* is defined by

$$A^* \doteq \int_0^1 \Big[Df\big(su'' + (1-s)u'\big) - Df(\tilde u)\Big]\ ds.$$

Therefore,

$$\left|\Big\langle \tilde l_i,\ f(u'') - f(u') - \tilde\lambda_i(u''-u')\Big\rangle\right| \ \le\ C\cdot|u''-u'|\cdot\big(|u''-\tilde u| + |u'-\tilde u|\big), \tag{3.20}$$

for some constant C depending only on the second order derivatives of the function f. By (3.13),

$$|u''-\tilde u| + |u'-\tilde u| \le 2(K+2)\varepsilon. \tag{3.21}$$

Using (3.20)-(3.21) in (3.19) we deduce that, for some constant C_2,

$$\left|\int_{\zeta_1}^{\zeta_2} \big\langle \tilde l_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\big\rangle\ dx\right| \le C_2\ \varepsilon \int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t);\ J^i_{\zeta_1,\zeta_2}(t)\big\}dt, \tag{3.22}$$

where $J^i_{\zeta_1,\zeta_2}(t) \doteq\ ]\zeta_1 + (t-\tau-h)\tilde\lambda_i,\ \zeta_2 + (t-\tau-h)\tilde\lambda_i[$. Define the positive Radon measure $\mu_{i,h}$ by setting

$$\mu_{i,h}\big(]\zeta_1,\ \zeta_2[\big) \doteq \frac{1}{h}\int_\tau^{\tau+h} \text{Tot.Var.}\big\{u(t);\ J^i_{\zeta_1,\zeta_2}(t)\big\}dt,$$

for every open interval $]\zeta_1,\ \zeta_2[$. Thanks to (3.22), we can apply Lemma 2 and obtain

$$\begin{aligned}
&\int_{\xi_\beta-\rho+\lambda^*h}^{\xi_\beta+\rho-\lambda^*h} \left|\Big\langle \tilde l_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\Big\rangle\right|\ dx\\
&\quad \le C_2\varepsilon h\cdot \mu_{i,h}\big(]\xi_\beta-\rho+\lambda^*h,\ \xi_\beta+\rho-\lambda^*h[\big)\\
&\quad = C_2\varepsilon\cdot\int_\tau^{\tau+h} \text{Tot.Var.}\Big\{u(t);\ ]\xi_\beta-\rho+\lambda^*h+(t-\tau-h)\tilde\lambda_i,\ \xi_\beta+\rho-\lambda^*h+(t-\tau-h)\tilde\lambda_i[\Big\}\ dt\\
&\quad \le C_2\varepsilon\cdot\int_0^h \text{Tot.Var.}\big\{u(\tau+h');\ I_{\beta,h'}\big\}\ dh'.
\end{aligned}$$

We are now ready to estimate

$$
\begin{aligned}
\sum_{\beta=1}^{M} \int_{\xi_\beta-\rho+\lambda^* h}^{\xi_\beta+\rho-\lambda^* h} & \left|u(\tau+h,x) - U^\flat(\tau+h,x)\right| \, dx \\
&= \mathcal{O}(1)\cdot \sum_{\beta=1}^{M}\sum_{i=1}^{n} \int_{\xi_\beta-\rho+\lambda^* h}^{\xi_\beta+\rho-\lambda^* h} \left|\left\langle \tilde l_i,\ u(\tau+h,x) - U^\flat(\tau+h,x)\right\rangle\right| \, dx \\
&= \mathcal{O}(1)\cdot\varepsilon \sum_{\beta=1}^{M} \int_{\tau}^{\tau+h} \text{Tot.Var.}\big\{u(t);\ ]\xi_\beta-\rho+\lambda^* t,\ \xi_\beta+\rho-\lambda^* t[\big\}dt \\
&= \mathcal{O}(1)\cdot\varepsilon \int_{\tau}^{\tau+h} \text{Tot.Var.}\big\{u(t);\ I\!R\big\}dt \\
&\le C_3\ \varepsilon\ h,
\end{aligned}
\tag{3.23}
$$

for some constant C_3 and all $h > 0$ sufficiently small.

On the other hand, applying (2.16) with $w = u(\tau)$, by (3.12) we deduce

$$
\begin{aligned}
\frac{1}{h}\int_{\xi_\beta-\rho+\lambda^* h}^{\xi_\beta+\rho-\lambda^* h} \left|\big(S_h u(\tau)\big)(x) - U^\flat(\tau+h,x)\right| \, dx &\le C_1 \Big(\text{Tot.Var.}\big\{u(\tau);\ ]\xi_\beta-\rho,\ \xi_\beta+\rho[\,\big\}\Big)^2 \\
&\le C_1\varepsilon\cdot \text{Tot.Var.}\big\{u(\tau);\ ]\xi_\beta-\rho,\ \xi_\beta+\rho[\,\big\}.
\end{aligned}
\tag{3.24}
$$

Indeed, it is not restrictive to assume λ^* larger than all wave speeds. The two inequalities (3.23) and (3.24) together yield

$$
\sum_{\beta=1}^{M} \frac{1}{h}\int_{\xi_\beta-\rho+\lambda^* h}^{\xi_\beta+\rho-\lambda^* h} \left|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\right| dx \le \Big(C_3 + 2C_1\cdot \text{Tot.Var.}\big\{u(\tau); I\!R\big\}\Big)\varepsilon. \tag{3.25}
$$

Since the intervals $I'_{\alpha,h}$, $I_{\beta,h}$ cover $[-R,R]$, from (3.9) and (3.25) we finally obtain

$$
\frac{1}{h}\int_{-R}^{R} \left|u(\tau+h,x) - \big(S_h u(\tau)\big)(x)\right| dx \le \Big(1 + C_3 + 2C_1\ \text{Tot.Var.}\big\{u(\tau); I\!R\big\}\Big)\varepsilon, \tag{3.26}
$$

for every $h > 0$ sufficiently small. Since $\varepsilon > 0$ was arbitrary, this establishes (3.4). Using (3.4) in (3.2), Theorem 1 is proved.

4 - Proof of Theorem 2.

We first recall a result of real analysis [14, p.320]. Let $Z : [0,T] \mapsto I\!R$ be a nonincreasing function. Then, for every $M > 0$ there holds

$$
\text{meas}\left\{\tau\in[0,T]\,;\quad \left|\frac{Z(t)-Z(\tau)}{t-\tau}\right| > M \quad \text{for some } t\neq\tau\right\} \le \frac{2\big[Z(0)-Z(T)\big]}{M}\,. \tag{4.1}
$$

From (4.1) we deduce the existence of a set $\mathcal{N}$ of measure zero such that, for each $\tau \in [0,T] \setminus \mathcal{N}$, there exists a constant $M(\tau)$ satisfying

$$\big|Z(t) - Z(\tau)\big| \leq M(\tau)\big|t - \tau\big| \qquad \text{for all } t \in [0,T]. \tag{4.2}$$

Recalling the structure theorem for BV functions [11, 23], we can also assume that, for all $\tau \notin \mathcal{N}$ and all $\xi \in I\!\!R$, either u is approximately continuous or it has an approximate jump discontinuity at the point (τ, ξ).

From now on, we fix $\tau \notin \mathcal{N}$ and call $M = M(\tau)$ the corresponding constant in (4.2). We first prove that, at time τ, the Decay Assumption (A4) implies the Entropy Condition (A2). Let ξ be a point of jump for the function $x \mapsto u(\tau, x)$, so that (1.9)-(1.10) and (2.11) hold. In particular, we have

$$u(\tau, \xi+) = \Psi_i(\sigma)\big(u(\tau, \xi-)\big)$$

for some $\sigma \in I\!\!R$ and some $i \in \{1, \ldots, n\}$. Applying (1.27) with $a = b = \xi$, we find

$$\sigma = \mu^i_\tau\big(\{\xi\}\big) \leq \big[Z(t) - Z(\tau)\big] \leq M(\tau - t) \tag{4.3}$$

for all $0 < t < \tau$. Letting $t \to \tau-$, we conclude $\sigma \leq 0$. Hence the entropy condition (1.11) holds.

It now remains to prove that the Tame Oscillation Condition (A3) also holds. Assuming that the constant $\delta_0 > 0$ in (1.4) is sufficiently small, we can choose a unit vector $\mathbf{e}$ and a constant $c_0 > 0$ such that

$$\frac{d}{d\sigma}\big\langle \mathbf{e},\ \Psi_i(\sigma)(u)\big\rangle \geq c_0 \tag{4.4}$$

for $|u| < \delta_0$, $|\sigma| < \delta_0$. Recalling that the maps $x \mapsto u(t,x)$ were assumed right continuous, from (4.4) it follows

$$\big\langle \mathbf{e},\ u(t,b) - u(t,a)\big\rangle \leq \sum_{i=1}^n \mu^{i+}_t\big(]a,b]\big) - c_0 \cdot \sum_{i=1}^n \mu^{i-}_t\big(]a,b]\big), \tag{4.5}$$

for all $a < b$ and $t \geq 0$. We claim that (1.12) holds with $K = 2/c_0$. Indeed, assume by contradiction that there exists a point ξ and sequences $h_\nu \to 0+$ and $x_\nu \to \xi$ such that

$$\big|u(\tau + h_\nu, x_\nu) - u(\tau, \xi)\big| > \frac{2}{c_0} \cdot \big|u(\tau, \xi+) - u(\tau, \xi-)\big|. \tag{4.6}$$

By Lemma 3, we can select sequences $a_\nu \to \xi-$ and $b_\nu \to \xi+$, with $a_\nu < x_\nu < b_\nu$, such that (by possibly taking a subsequence and relabelling)

$$u^-_\nu \doteq u(\tau + h_\nu, a_\nu) \to u(\tau, \xi-), \qquad u^+_\nu \doteq u(\tau + h_\nu, b_\nu) \to u(\tau, \xi+).$$

We can also assume that

$$\tilde{u}_\nu \doteq u(\tau + h_\nu, x_\nu) \to \tilde{u}.$$

for some $\tilde u$. Set $\delta_\nu \doteq b_\nu - a_\nu$. Applying the decay estimate (1.27) to the interval $[\tau - \sqrt{\delta_\nu},\ \tau + h_\nu]$ we obtain

$$\begin{aligned}\mu^{i+}_{\tau+h_\nu}\big([a_\nu, b_\nu]\big) &\leq \frac{\delta_\nu}{\kappa(\sqrt{\delta_\nu} + h_\nu)} + \big[Z(\tau - \sqrt{\delta_\nu}) - Z(\tau + h_\nu)\big] \\ &\leq \frac{\sqrt{\delta_\nu}}{\kappa} + M(\sqrt{\delta_\nu} + h_\nu).\end{aligned}\tag{4.7}$$

Therefore,

$$\lim_{\nu\to\infty} \mu^{i+}_{\tau+h_\nu}\big([a_\nu, b_\nu]\big) = 0.\tag{4.8}$$

Using (4.5) and (4.8), and recalling that all functions $x \mapsto u(t,x)$ are right continuous, we now estimate

$$\begin{aligned}\big|u(\tau,\xi+) - u(\tau,\xi-)\big| &\geq \Big|\big\langle \mathbf{e},\ u(\tau,\xi+) - u(\tau,\xi-)\big\rangle\Big| \\ &= \lim_{\nu\to\infty} \Big|\langle \mathbf{e},\ u^+_\nu - u^-_\nu\rangle\Big| \\ &\geq c_0 \sum_{i=1}^n \liminf_{\nu\to\infty} \mu^{i-}_{\tau+h_\nu}\big(]a_\nu,\ b_\nu]\big).\end{aligned}\tag{4.9}$$

Moreover,

$$\begin{aligned}\big|u(\tau,\xi+) - \tilde u\big| &\leq \lim_{\nu\to\infty} \big|u^+_\nu - \tilde u_\nu\big| \\ &\leq \sum_{i=1}^n \liminf_{\nu\to\infty} \mu^{i-}_{\tau+h_\nu}\big(]x_\nu,\ b_\nu]\big).\end{aligned}\tag{4.10}$$

Together, (4.9) and (4.10) yield

$$\big|u(\tau,\xi+) - \tilde u\big| \leq \frac{1}{c_0}\big|u(\tau,\xi+) - u(\tau,\xi-)\big|,\tag{4.11}$$

which proves (1.12), since $u(\tau,\xi) = u(\tau,\xi+)$ by the convention (1.7).

Finally, we show that the decay assumption (1.27) also implies (1.13), where λ^* is any upper bound for all characteristic speeds. If the map $x \mapsto u(\tau,x)$ is continuous at $x = \xi$, then (1.13) is an easy consequence of (1.12). Now consider the other case, where u has an approximate jump discontinuity at (τ,ξ), so that (1.9)-(1.10) hold, for some u^-, u^+, λ satisfying (2.11). Assume by contradiction that exist sequences $h_\nu \to 0+$ and $x_\nu \to \xi+$, with $x_\nu - \xi > \lambda^* h_\nu$, such that

$$\lim_{\nu\to\infty} u(\tau + h_\nu,\ x_\nu) = \tilde u \neq u(\tau,\xi+).\tag{4.12}$$

Since $\lambda < \lambda^*$, by (2.5) there exist sequences $a_\nu, b_\nu \to \xi+$ with $\xi + \lambda h_\nu < a_\nu < x_\nu < b_\nu$ such that

$$u^-_\nu \doteq u(\tau + h_\nu,\ a_\nu) \to u(\tau,\xi+), \qquad u^+_\nu \doteq u(\tau + h_\nu,\ b_\nu) \to u(\tau,\xi+).$$

Repeating the same arguments in (4.9)-(4.11), we conclude

$$\big|\tilde u - u(\tau,\xi+)\big| \leq \frac{1}{c_0}\big|u(\tau,\xi+) - u(\tau,\xi+)\big| = 0.$$

This establishes the limit (1.13) in the sector where $x > \xi + \lambda^*(t-\tau)$. The case $x < \xi - \lambda^*(t-\tau)$ is entirely similar.

Thanks to the above analysis, for all $\tau \notin \mathcal{N}$ and $\xi \in I\!R$, the assumptions (A2)-(A3) hold. Therefore, we can repeat the arguments in Section 3 and establish (3.4). By (3.2), this implies $u(T) = S_T \bar u$ for all $T > 0$, proving Theorem 2.

5 - Concluding remarks.

We conclude with a few observations on the role of the regularity assumption (A3) toward the uniqueness of solutions of the Cauchy problem (1.1), (1.3). In our proof, this assumption is used at one single step, namely in the estimate (3.22) of the distance between a weak solution u and the solution $U^\flat$ of an approximating linear problem. While deriving this particular estimate, we are not relying on any entropy condition. Therefore we need an assumption which rules out the appearance of large oscillations immediately after time τ. For general $n \times n$ systems, it is not clear whether (A3) could actually be deduced from (A1)-(A2), as in the scalar case.

In [7], a stronger regularity assumption was used, namely

(A5) (Tame Variation Condition) There exists a constant C such that, for every horizontal segment Γ in the t-x plane and every space-like segment Γ' in the domain of dependency of Γ, one has

$$\text{Tot.Var.}\{u\,;\ \Gamma'\} \leq C \cdot \text{Tot.Var.}\{u\,;\ \Gamma\}. \tag{5.1}$$

It is not difficult to show that (A5) implies (A3). Indeed, fix any point (τ, ξ) and consider sequences $t_\nu \to \tau+$, $x_\nu \to \xi$. Let $\hat\lambda$ be an upper bound for all wave speeds and define

$$\rho_\nu \doteq |x_\nu - \xi| + \hat\lambda(t_\nu - \tau).$$

Clearly, $\rho_\nu \to 0$. Consider the horizontal segment Γ_ν with endpoints $(\tau,\ \xi - \rho_\nu)$, $(\tau,\ \xi + \rho_\nu)$, and the space-like segment Γ'_ν, with endpoints (t_ν, x_ν), $(\tau,\ \xi + \rho_\nu)$. Applying (5.1) we obtain

$$\begin{aligned}
&\limsup_{\nu\to\infty} \big|u(t_\nu, x_\nu) - u(\tau,\xi)\big| \\
&\qquad \leq \limsup_{\nu\to\infty} \Big\{ \big|u(t_\nu, x_\nu) - u(\tau, \xi+\rho_\nu)\big| + \big|u(\tau,\xi+\rho_\nu) - u(\tau,\xi)\big| \Big\} \\
&\qquad \leq (C+1)\cdot \limsup_{\nu\to\infty} \text{Tot.Var.}\big\{u(\tau)\,;\ \Gamma_\nu\big\} \\
&\qquad \leq (C+1)\cdot \big|u(\tau,\xi+) - u(\tau,\xi-)\big|.
\end{aligned}$$

This yields (1.12) with $K = C+1$.

To prove (1.13) with $\lambda^* = \hat{\lambda}$, assume $x_\nu \to \xi$, $t_\nu \to \tau+$, with $x_\nu > \xi + \hat{\lambda}(t_\nu - \tau)$. Calling Γ_ν the segment with endpoints $\big(\tau,\ x_\nu \pm \hat{\lambda}(t_\nu - \tau)\big)$ and Γ'_ν the segment with endpoints (t_ν, x_ν), $\big(\tau,\ x_\nu - \hat{\lambda}(t_\nu - \tau)\big)$, an application of (5.1) yields

$$\begin{aligned}&\limsup_{\nu\to\infty}\big|u(t_\nu, x_\nu) - u(\tau, \xi+)\big|\\ &\qquad \le \limsup_{\nu\to\infty} C \cdot \text{Tot.Var.}\big\{u(\tau)\,;\ \Gamma_\nu\big\} + \limsup_{\nu\to\infty} \big|u(\tau,\ x_\nu - \hat{\lambda}(t_\nu - \tau)) - u(\tau, \xi+)\big|\\ &\qquad = 0.\end{aligned}$$

It is worth observing that Theorem 1 remains valid under assumptions somewhat weaker than (A3). Namely, in (1.13) one can allow the positive number λ^* to depend arbitrarily on the point (τ, ξ). Moreover, instead of (1.12), one can assume that for almost every τ there exists a continuous function $K_\tau = K_\tau(\zeta)$ with $K_\tau(0) = 0$ such that

$$\limsup_{t\to\tau+,\ x\to\xi} \big|u(t,x) - u(\tau,\xi)\big| \le K_\tau\Big(\big|u(\tau,\xi+) - u(\tau,\xi-)\big|\Big).$$

This extension requires only minor modifications of the original proof.

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