Continuous Dependence in the Large for Some Equations of Gas Dynamics

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Abstract. The main result of this paper is the construction of the *Standard Riemann Semigroup* for a class of 2×2 Conservation Laws without the requirement that the total variation of the initial data be *small*. More precisely, fix a positive M. For a class of equations of the form

$$u_t + \left[f(u)\right]_x = 0 \quad \text{with} \quad t \geq 0\,,\ x \in \mathbf{R}\,,\ u \in \mathbf{R}^2 \qquad (\star)$$

it is shown that there exists an $\mathbf{L^1}$–Lipschitz continuous semigroup S defined on functions of bounded variation, such that the trajectories of S are weak entropy solutions to $(\star)$. This class includes the classical and relativistic isentropic Euler equations and all the 2×2 Temple class systems.

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1. Introduction

We consider the Cauchy problem for a 2×2 system of strictly hyperbolic conservation laws:

$$\begin{cases} u_t + [f(u)]_x = 0 \\ u(0,x) = \bar{u}(x) \end{cases} \qquad \text{where} \qquad t \geq 0,\ x \in \mathbf{R},\ u = (u_1, u_2) \in \Omega \subseteq \mathbf{R}^2 . \tag{1.1}$$

Here, f is a smooth function $\mathbf{R}^2 \mapsto \mathbf{R}^2$ such that the eigenvalues of the Jacobian matrix $Df(u)$ are real and different for all u.

The existence of global weak solutions to (1.1) was first established in [G] under the assumption that $\mathrm{TV}(\bar{u})$ is sufficiently small, in the case of n equations. The problems of uniqueness and continuous dependence were solved more recently in a series of papers, see [BC1], [BCP], [BC2] and [BL]. Except for [BC2] and [BB], all the other results require the total variation of the initial data to be small. In [BC2], existence, uniqueness and continuous dependence were proved under the assumption that the initial data is a small perturbation of Riemann initial data with a single large jump. The construction in [BB] applies to $n \times n$ Temple class systems.

In the present paper we consider 2×2 systems and prove the continuous dependence of the solution from data $\bar{u}$ with $\mathrm{TV}(\bar{u})$ *finite*, not necessarily small. More precisely, we constructively define the *Standard Riemann Semigroup* (SRS), see [B2], [B3]. The result here obtained applies to the Euler equations of isentropic relativistic and classical gas dynamics. Moreover, the present construction applies also to the 2×2 Temple class systems, as considered in [BB].

The most interesting example is

The relativistic isentropic Euler equations, where

$$(\mathbf{R}) \qquad \begin{cases} u_1 = \rho \dfrac{1 + (\varsigma^2 v^2)/c^4}{1 - v^2/c^2} \\ u_2 = \rho v \dfrac{1 + (\varsigma^2 v^2)/c^4}{1 - v^2/c^2} \end{cases} \qquad f(u_1, u_2) = \left(\rho v \frac{1 + (\varsigma^2 v^2)/c^4}{1 - v^2/c^2}, \rho \frac{v^2 + \varsigma^2}{1 - v^2/c^2} \right)$$

here, ς denotes the sound speed, c speed of light, ρ and v are the classical density and coordinate gas speed. The state equation is $p(\rho) = \varsigma^2 \rho$, see [ST] (and the literature cited therein) for a discussion of these equations and for an existence result. Here, we only observe that in the limit $c \to \infty$, the system **(R)** reduces to:

The classical Euler isentropic model with $\gamma = 1$, where

$$(\mathbf{E}) \qquad \begin{cases} u_1 = \rho & \text{is the gas density,} \\ u_2 = \rho v & \text{is the momentum density} \end{cases} \qquad f(u_1, u_2) = \Big(\rho v, \rho (v^2 + \varsigma^2) \Big)$$

and ς is the sound speed. The state equation is given by the relation $p(\rho) = \varsigma^2 \rho$.

In the two cases above, the flow function f is defined on $\Omega =]0, +\infty[\times \mathbf{R}$ and is unbounded when $u_1 \to 0$.

For the sake of completeness, we recall that a 2×2 Temple class system is a strictly hyperbolic system of the form (1.1), with each characteristic field either linearly degenerate or genuinely non linear and with coinciding shock and rarefaction curves, see [BB] and the bibliography cited therein. We shall refer to this case by **(T)**.

The main result of this paper is the following:

Theorem 1.1. *Consider the Cauchy Problem (1.1) with f as defined in* **(R)**, **(E)** *or* **(T)**. *Fix two states $u_{-\infty}$, $u_{+\infty}$ in Ω and a positive M. Then there exists a domain $\mathcal{D} \subseteq \mathbf{BV}(\mathbf{R})$, a semigroup $S: [0, +\infty[\times \mathcal{D} \mapsto \mathcal{D}$ and a positive constant L such that*

(i) $\mathcal{D}$ contains the $\mathbf{L^1}$-closure of the set of those functions $u: \mathbf{R} \mapsto \Omega$ such that

$$\begin{cases} u - u_{-\infty} \in \mathbf{L^1}\left(]-\infty, 0]\right) \\ u - u_{+\infty} \in \mathbf{L^1}\left([0, +\infty[\right) \end{cases} \qquad \textit{and} \qquad \mathrm{TV}(u) \leq M. \tag{1.2}$$

(iii) S is $\mathbf{L^1}$–Lipschitz continuous with constant L, i.e.,

$$\left\| S_{t''} u'' - S_{t'} u' \right\|_{\mathbf{L^1}} \leq L \cdot \left(\left\| u'' - u' \right\|_{\mathbf{L^1}} + \left| t'' - t' \right| \right) .$$

(iv) If $\bar u$ is piecewise constant, then for t small, the map $(t,x) \mapsto (S_t)\, \bar u(x)$ coincides with the gluing of the Lax solutions of the Riemann problems at the jumps of $\bar u$.

Due to the results in [B2] and [BC2], the SRS above is unique. Furthermore, as can be proved by the same techniques used in the cited papers, the orbit $t \mapsto S_t \bar u$ of the SRS enjoys several other properties:

1. it is a weak entropy solution to (1.1);
2. it coincides with Glimm solution;
3. it is characterized as being a *viscosity solution* with interaction potential locally uniformly bounded;
4. it is unique in the class of solutions satisfying 3.

The rest of this paper is organized as follows. In section 2 we introduce the notation and outline the proof of Theorem 1.1. The technical proofs common to the two cases are in Section 3, while the estimates specific to **(R)**, **(E)** and **(T)** are left to Section 4.

2. Outline of the Proof

Let us first introduce the basic notation. Let $A(u) = Df(u)$ denote the Jacobian matrix of f at u. We assume that (1.1) is strictly hyperbolic with each characteristic family either genuinely non linear or linearly degenerate. Note that this holds for **(R)**, **(E)** and **(T)**. Define the averaged matrix

$$A(u,u') = \int_0^1 Df\left(\theta u + (1-\theta) u'\right)\, d\theta .$$

For $i = 1,2$, let $\lambda_i(u,u')$, $r_i(u,u')$ and $l_i(u,u')$ be the i–th eigenvalue and a right and left i–th eigenvectors of $A(u,u')$ respectively. We normalize these so that

$$l_i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

If $u' = u$ then $A(u) = A(u,u)$ and $\lambda_i(u)$, $r_i(u)$, $l_i(u)$ denote the eigenvalues and eigenvectors of $A(u)$. By strict hyperbolicity, we may assume that $\lambda_1(u) < \lambda_2(u)$ for all states u. Since we consider initial data with (possibly) large total variation, it may well happen that for some $u', u'' \in \Omega$ the opposite inequality $\lambda_1(u') > \lambda_2(u'')$ holds.

The proof of Theorem 1.1 makes use of the technique introduced in [BC1] and developed in [BC2]; see also [AC] and [CC]. In this paper we therefore provide only the details pertaining to the setting under consideration. We refer to [BC1] and [BC2] for all the necessary details.

The first step is the introduction of global Riemann coordinates, which are the basic tool in the construction below. In these coordinates the rarefaction curves take the form

$$\phi_1^+(v,\sigma) = (v_1 + \sigma, v_2) , \qquad \phi_2^+(v,\sigma) = (v_1, v_2 + \sigma) .$$

Lemma 2.1. *In cases **(R)**, **(E)** and **(T)**, the i–shock curve through the point v can be parameterized w.r.t. the Riemann coordinates as*

$$\begin{aligned} \phi_1^-(v,\sigma) &= \left(v_1 + \sigma + \phi(\sigma), v_2 + \phi(\sigma)\right) , \\ \phi_2^-(v,\sigma) &= \left(v_1 + \phi(\sigma), v_2 + \sigma + \phi(\sigma)\right) . \end{aligned} \tag{2.1}$$

for a suitable smooth function $\phi\colon]-\infty, 0] \mapsto \mathbf{R}$ satisfying:

$$\phi(\sigma) \leq 0 \textit{ and } \phi'(\sigma) \geq 0, \textit{ for all } \sigma \in]-\infty, 0] . \tag{2.2}$$

Standard results, see [S], ensure that $\phi(\sigma) = o(\sigma^2)$ for $\sigma \to 0$. Lemma 2.1 implies also that in the Riemann coordinates plane, the shock and rarefaction curves are translation invariant. Furthermore, the 1–shock–rarefaction curve through (v_1^o, v_2^o) is symmetric to the 2–shock–rarefaction curve with respect to the axis $v_1 - v_2 = v_1^o - v_2^o$. Therefore, in a certain sense, systems satisfying Lemma 2.1, can be considered *stiff* in the phase plane.

In case **(T)**, Lemma 2.1 is proved setting simply $\phi = 0$. In cases **(R)** and **(E)**, the proof is sketched in Section 4.

Using the above parameterization of the shock–rarefaction curves, we now define the approximate solution to Riemann problems, a basic step in our construction. Choose any $\mathbf{C}^\infty$ function $\varphi\colon \mathbf{R} \mapsto \mathbf{R}$ such that

$$\begin{cases} \varphi(s) = 1 & \text{for all } s \leq -2, \\ \varphi'(s) \in [-2, 0] & \text{for all } s \in [-2, -1], \\ \varphi(s) = 0 & \text{for all } s \geq -1, \end{cases} \tag{2.3}$$

and, for a fixed $\varepsilon > 0$, define the interpolation between the i–shock and the i–rarefaction curve

$$\begin{aligned} \psi_1^\varepsilon(v,\sigma) &= \Big(v_1 + \sigma + \varphi(\sigma/\sqrt{\varepsilon})\phi(\sigma), v_2 + \varphi(\sigma/\sqrt{\varepsilon})\phi(\sigma)\Big), \\ \psi_2^\varepsilon(v,\sigma) &= \Big(v_1 + \varphi(\sigma/\sqrt{\varepsilon})\phi(\sigma), v_2 + \sigma + \varphi(\sigma/\sqrt{\varepsilon})\phi(\sigma)\Big). \end{aligned} \tag{2.4}$$

Given ε, a right and a left state u^l, u^r with Riemann coordinates $v^r = (v_1^r, v_2^r)$ and $v^l = (v_1^l, v_2^l)$, an approximate solution to the Riemann Problem

$$\begin{cases} u_t + \big(f(u)\big)_x = 0 \\ u(0,x) = \begin{cases} u^l & \text{if } x < 0 \\ u^r & \text{if } x > 0 \end{cases} \end{cases} \tag{2.5}$$

is constructed as follows. First, using the implicit function theorem, determine unique values σ_1 and σ_2 and a middle state v^m such that

$$v^r = \psi_2^\varepsilon\left(v^m, \sigma_2\right), \qquad v^m = \psi_1^\varepsilon\left(v^l, \sigma_1\right). \tag{2.6}$$

If $\sigma_1 \geq 0$, then the states v^l, v^m are connected by a rarefaction wave. We approximate this rarefaction wave by a series of discontinuities by introducing a fixed ε grid in the (v_1, v_2) plane. Let the integers h, k be such that $h\varepsilon \leq v_1^l < (h+1)\varepsilon$ and $k\varepsilon \leq v_1^m < (k+1)\varepsilon$. Introducing the states $\omega_1^j = (j\varepsilon, v_2^l)$ and $\widehat{\omega}_1^j = \left((j+1/2)\varepsilon, v_2^l\right)$ for $j = h, \ldots, k$, we construct the ε–approximate solution as an approximate rarefaction fan:

$$v^\varepsilon(t,x) = \begin{cases} v^l & \text{if } x < \lambda_1(\widehat{\omega}_1^h)\, t, \\ \omega_1^j & \text{if } \lambda_1\left(\widehat{\omega}_1^{j-1}\right) t < x < \lambda_1\left(\widehat{\omega}_1^j\right) t, \qquad j = h+1, \ldots, k, \\ v^m & \text{if } \lambda\left(\widehat{\omega}_1^k\right) t < x. \end{cases} \tag{2.7}$$

On the other hand, if $\sigma_1 < 0$, the states v^l and v^m are connected by a single shock:

$$v^\varepsilon(t,x) = \begin{cases} v^l & \text{if } x < \lambda_1^\varphi\left(v^l, \sigma_1\right) t, \\ v^m & \text{if } \lambda_1^\varphi\left(v^l, \sigma_1\right) t < x. \end{cases} \tag{2.8}$$

The shock speed λ_1^φ is defined as

$$\lambda_1^\varphi\left(v^l, \sigma_1\right) = \varphi\left(\sigma_1/\sqrt{\varepsilon}\right) \cdot \lambda_1^s\left(v^l, \sigma_1\right) + \left(1 - \varphi(\sigma_1/\sqrt{\varepsilon})\right) \cdot \lambda_1^r\left(v^l, \sigma_1\right),$$

with

$$\lambda_1^s(v^l,\sigma_1) = \lambda_1\left(v^l, \phi_1^-(v^l,\sigma_1)\right),$$
$$\lambda_1^r(v^l,\sigma_1) = \sum_j \frac{\mathrm{meas}\left([j\varepsilon,(j+1)\varepsilon] \cup [v_1^m, v_1^l]\right)}{|\sigma_1|}\,\lambda_1\left(\widehat{\omega}_1^j\right).$$

The construction of the ε–approximate solution for waves of the second family is entirely similar to the above construction, see [BC1] for details.

Let now $\bar{u}$ be a piecewise constant initial condition. A piecewise constant ε–approximate solution to the Cauchy problem (1.1) is constructed as follows. At time $\tau_0 = 0$ solve the Riemann problems defined by the jumps in $\bar{u}$ using the above algorithm. This yields a piecewise constant approximate solution $(t,x) \mapsto u^\varepsilon(t,x)$ defined up to the time $\tau_1 > \tau_0$ where the first set of interactions takes place. The Riemann problems arising at time τ_1 are again approximately solved using the algorithm above. Then, u^ε can be defined up to the time τ_2 when the next interaction takes place, etc. ...

We shall consider piecewise constant functions $u = u(x)$ of the form

$$u = u_{-\infty}\cdot\chi_{]-\infty,\,x_1]} + \sum_{\alpha=1}^{n-1} u^\alpha\cdot\chi_{]x_\alpha,\,x_{\alpha+1}]} + u_{+\infty}\cdot\chi_{]x_n,\,+\infty[}\,. \tag{2.9}$$

In the present setting, the Riemann problem at x_α can always be uniquely, albeit approximately, solved by the above algorithm in terms of waves with sizes $\sigma_{1,\alpha},\sigma_{2,\alpha}$. Recalling (2.6), this means

$$v^\alpha = \psi_2^\varepsilon\left(\psi_1^\varepsilon(v^{\alpha-1},\sigma_{1,\alpha}),\sigma_{2,\alpha}\right) \qquad \text{for } \alpha = 0,\ldots,n\,. \tag{2.10}$$

where $v^\alpha = (v_1^\alpha, v_2^\alpha)$ are the Riemann coordinates of u^α (here $u^0 = u_{-\infty}$ and $u^n = u_{+\infty}$).

Let $\mathcal{A}$ denote the set of *all approaching waves*. With a slight abuse of notation, we say that a pair of waves $(\sigma_{i,\alpha},\sigma_{j,\beta})$ is approaching if either, $\alpha < \beta$ and $j < i$, or if $j = i$, $\sigma_{i,\alpha} < 0$ or $\sigma_{i,\beta} < 0$, see [S]. On the set of all functions of the form (2.9)–(2.10), we introduce the functionals

$$V^\varepsilon(u) = \sum_\alpha\sum_i \left(1-\eta\,\mathrm{sign}\,\sigma_{i,\alpha}\right)\left|\sigma_{i,\alpha}\right|, \quad Q^\varepsilon(u) = \sum_{\sigma_{i,\alpha},\sigma_{j,\beta}\in\mathcal{A}} \left|\sigma_{i,\alpha}\sigma_{j,\beta}\right|, \quad \Upsilon^\varepsilon = K\cdot V^\varepsilon + Q^\varepsilon \tag{2.11}$$

where $\eta \in\,]0,1[$ and $K > 0$ are constants depending only on M, f and $u_{\pm\infty}$, whose value will be defined below. The dependence of Υ^ε on ε is motivated by the dependence on ε of the wave sizes in the ε–solution to a Riemann problem, see (2.4)–(2.10). Clearly

$$\frac{1}{C}\cdot\mathrm{TV}(u) \le \Upsilon^\varepsilon(u) \le C\cdot\mathrm{TV}(u) \tag{2.12}$$

for any piecewise constant function u in $\mathcal{D}$, where C is a suitable constant independent of ε. To simplify the notation, in the remainder of this paper, C denotes a generic *"large"* constant dependent only on f, M and $u_{\pm\infty}$. The actual value of C is unimportant for the results obtained here.

The functional Υ^ε changes value only when waves interact, and in Section 3 we prove that

$$t \mapsto \Upsilon^\varepsilon\left(u^\varepsilon(t)\right) \quad \text{is strictly decreasing at interaction times.} \tag{2.13}$$

By means of Υ^ε it is possible to define the domain

$$\mathcal{D}^\varepsilon = \left\{u \text{ as in (2.9)–(2.10)} : \Upsilon^\varepsilon(u) \le \bar{M}\right\} \tag{2.14}$$

Note that $\mathcal{D}^\varepsilon$ depends on ε, $u_{\pm\infty}$ and on $\bar{M}$. The latter is chosen so that all piecewise constant functions u of the form (2.9)–(2.10) with

$$K\cdot\mathrm{TV}(u) + \left(\mathrm{TV}(u)\right)^2 \le M \tag{2.15}$$

are also elements of $\mathcal{D}^\varepsilon$ for all ε, which is possible due to (2.12).

Proposition 2.2. *Fix $M > 0$ and f as in* **(R)**, **(E)** *or* **(T)**. *Then there exists a constant $\eta > 0$, independent of ε, such that, for any $\bar{u} \in \mathcal{D}^\varepsilon$, the wave–front tracking algorithm constructs an approximate weak solution $u^\varepsilon : [0, +\infty[\times \mathbf{R} \mapsto \Omega$ of (1.1), with the following properties:*

(i) $u^\varepsilon(t, \cdot) \in \mathcal{D}^\varepsilon$ for all $t \geq 0$;
(ii) the function $t \mapsto \Upsilon^\varepsilon\left(u^\varepsilon(t, \cdot)\right)$ is non–increasing;
(iii) Any strip of the form $[0, T] \times \mathbf{R}$ contains finitely many interaction points of u^ε;
(iv) $\mathrm{TV}\left(u^\varepsilon(t, \cdot)\right)$ is uniformly bounded.

Lemmas 3.1 and 3.2 in Section 3 provide the details for the proof of *(ii)*, the other claims being proved as in [BC1].

To denote this globally defined, ε–approximate solution, we use the semigroup notation

$$u^\varepsilon(t, \cdot) = S_t^\varepsilon \bar{u}\,. \tag{2.16}$$

The proof works then towards an estimate independent of ε of the Lipschitz constant for the semigroup S^ε in the $\mathbf{L}^1$ norm. The basic technique is to shift the locations x_α of the jumps in the initial condition $\bar{u}$ at constant rates ξ_α, and estimate the rates at which the jumps in the corresponding solution $u^\varepsilon(t, \cdot)$ are shifted, for any fixed $t > 0$.

Definition 2.3. *Let $]a, b[$ be an open interval. An elementary path in $\mathcal{D}^\varepsilon$ is a map $\gamma :]a, b[\mapsto \mathcal{D}^\varepsilon$ of the form*

$$\gamma(\theta) = u_{-\infty} \cdot \chi_{]-\infty, x_0^\theta[} + \sum_{\alpha=1}^{n-1} u^\alpha \cdot \chi_{]x_{\alpha-1}^\theta, x_\alpha^\theta[} + u_{+\infty} \cdot \chi_{]x_n^\theta, +\infty[}\,, \qquad x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha \theta$$

with $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a, b[$ and $\alpha = 0, \ldots, n$.

Definition 2.4. *A continuous map $\gamma : [a, b] \mapsto \mathbf{L^1}_{\mathrm{loc}}$ is a pseudopolygonal if there exist countably many disjoint open intervals $J_h \subset [a, b]$ such that:*

(i) The restriction of γ to each J_h is an elementary path.
(ii) The set $[a, b] \setminus \bigcup_{h \geq 1} J_h$ is countable.

Due to (2.13), exactly as in [BC1] one can prove:

Proposition 2.5. *Let γ_0 be a pseudopolygonal in $\mathcal{D}^\varepsilon$. Then, for all $\tau > 0$, the path $\gamma_\tau = S_\tau^\varepsilon \circ \gamma_0$ is also a pseudopolygonal.*

For a fixed $\varepsilon > 0$, the *weighted length* of the elementary path $\gamma :]a, b[\mapsto \mathcal{D}^\varepsilon$ is

$$\|\gamma\| = (b - a) \cdot \Upsilon_\xi^\varepsilon(u)$$

where the functional Υ_ξ^ε will be defined below. In the above definition, $\|\gamma\|$ does not depend on the particular choice of θ such that $\gamma(\theta) = u$, since the map $\theta \mapsto \Upsilon_\xi^\varepsilon\left(\gamma(\theta)\right)$ is constant along elementary paths.

Definition 2.6. *The weighted length of a pseudopolygonal is the sum of the weighted lengths of its elementary paths. For any two piecewise constant functions $u, w \in \mathcal{D}^\varepsilon$, their weighted distance is*

$$d_\varepsilon(u, w) = \inf\left\{\|\gamma\| \text{ such that } \gamma : [0, 1] \mapsto \mathcal{D}^\varepsilon \text{ is a pseudopolygonal joining } u \text{ with } w\right\}\,.$$

Let $u \in \mathcal{D}^\varepsilon$ be as in (2.9). We define

$$S_{i,\alpha}^\varepsilon = 2\left(\sum_{\beta=1}^{n}\sum_{j=1}^{2} [\![\, \sigma_{j,\beta} \,]\!]_-\right) - [\![\, \sigma_{i,\alpha} \,]\!]_- \qquad R_\alpha^\varepsilon = \sum_{\beta=1}^{\alpha-1} \left|\sigma_{2,\beta}\right| + \sum_{\beta=\alpha+1}^{n} \left|\sigma_{1,\beta}\right| \tag{2.17}$$

$$\Upsilon_\xi^\varepsilon = \sum_{\alpha=1}^{n}\sum_{i=1}^{2} \left|\sigma_{i,\alpha}\xi_\alpha\right| e^{H_1 S_{i,\alpha}^\varepsilon + H_2 R_\alpha^\varepsilon + H_3 V^\varepsilon} \tag{2.18}$$

where $[\![s]\!]_- = (|s| - s)/2$ denotes the negative part of s. A precise value for the constants H_1, H_2, H_3 will be determined later. The definition of Υ^ε_ξ is chosen so that the following propositions hold:

Proposition 2.7. *Let M and f be as in Theorem 1. Then there exists positive constants H_1, H_2 and H_3, independent of ε in (2.18), such that the following holds: if γ_0 is a pseudopolygonal, then the weighted length $\|\gamma_\tau\|$ of the pseudopolygonal $\gamma_\tau = S^\varepsilon_\tau \circ \gamma_0$ is a non–increasing function of time, i.e.,*

$$\text{the map} \quad t \mapsto \Upsilon^\varepsilon_\xi\left(S^\varepsilon_t \circ \gamma\right) \quad \text{is non–increasing.} \tag{2.19}$$

The proof of (2.19) is in Section 3, while the proofs of the remaining lemmas are entirely similar to the proofs of the corresponding lemmas in [BC1].

Proposition 2.8. *Any two functions u, u' in $\mathcal{D}^\varepsilon$ can be joined by a pseudopolygonal entirely contained in $\mathcal{D}^\varepsilon$. Moreover, the weighted length of this pseudopolygonal is uniformly equivalent to the usual $\mathbf{L^1}$–distance, i.e.,*

$$\frac{1}{C} \cdot \|\gamma\|_{\mathbf{L^1}} \le (b-a) \cdot \Upsilon^\varepsilon_\xi(\gamma) \le C \cdot \|\gamma\|_{\mathbf{L^1}} \,. \tag{2.20}$$

The proof of (2.20) is an immediate consequence of (2.17)–(2.18) via the bound provided by (2.15).

Proposition 2.9. *Let M and f be as in Theorem 1. Then the semigroup $S^\varepsilon : [0, +\infty[\times \mathcal{D}^\varepsilon \mapsto \mathcal{D}^\varepsilon$ defined by (2.16) is uniformly Lipschitz continuous w.r.t. the $\mathbf{L^1}$ distance, with a constant independent of ε.*

As in [BC1], to complete the proof of Theorem 1.1 we consider a sequence of semigroups S^{ε_n} with $\lim_{n\to+\infty} \varepsilon_n = 0$, and construct the limit semigroup as $S = \lim_{n\to+\infty} S^{\varepsilon_n}$. More precisely, for $\bar{u} \in \mathcal{D}$ and $t \ge 0$, we define

$$S_t \bar{u} = \lim_{n\to+\infty} S^{\varepsilon_n}_t \bar{u}_n$$

where $\bar{u}_n \in \mathcal{D}^{\varepsilon_n}$ is any sequence converging to $\bar{u}$ in $\mathbf{L^1}$. The conclusion of the proof is exactly as in [BC1].

3. Technical Proofs

The core of the wave front tracking algorithm lies in the control of the interactions. Aiming at continuous dependence, the usual slight modification of wave speeds to avoid multiple interactions cannot be adopted. On the other hand, we treat below in detail the case of simple interactions, leaving to the inductive procedures developed in [BC1] and [BC2] the case of many waves interacting simultaneously.

Throughout this section we refer to the notation displayed in Figures 1 and 2. The shift and the propagation speed of a wave will be labeled with the same super/subscripts of the wave sizes.

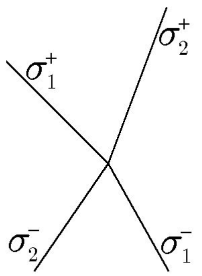

Figure 1.

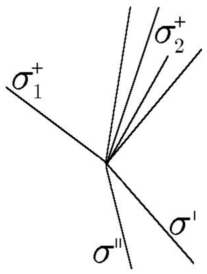

Figure 2.

Above, σ_2^+ stands for the *total* size of the 2–waves exiting the interaction. If $\sigma_2^+ > 0$, the single rarefaction "wavelets" are denoted by $\sigma^+_{2,\alpha}$.

The next few lemmas deal with the basic interaction estimates in the case of shock–rarefaction curves of the form (2.4). Recall the function ϕ as defined in (2.2) and set $\phi(\sigma) = 0$ for positive σ. Let

$$\Psi^\varepsilon : \begin{array}{ccc} \mathbf{R}^2 & \mapsto & \mathbf{R}^2 \\ \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} & \mapsto & \begin{pmatrix} \sigma_1 + \phi_\varepsilon(\sigma_1) + \phi_\varepsilon(\sigma_2) \\ \sigma_2 + \phi_\varepsilon(\sigma_1) + \phi_\varepsilon(\sigma_2) \end{pmatrix} \end{array} \quad \text{with} \quad \phi_\varepsilon(\sigma) = \varphi(\sigma/\sqrt{\varepsilon}) \cdot \phi(\sigma) \,. \tag{3.1}$$

The Riemann problem (2.5) is approximately solved by i–waves of (total) sizes σ_i if and only if

$$v^r = v^l + \Psi^\varepsilon(\sigma_1, \sigma_2) \tag{3.2}$$

Observe that Ψ^ε is injective, due to (2.2) and (2.3).

Below, ε will be kept fixed and, hence, omitted. With a slight abuse of notation, denote by $\mathcal{A}(\sigma)$ the set of waves approaching σ, according to the usual definition of approaching waves. Below, ΔV, ΔQ and $\Delta\Upsilon$ denote the variation in the value of V, Q and Υ at an interaction time.

Lemma 3.1. *Consider an interaction as in Figure 2. Then there exists a constant C and positive weights η and K in definition (2.11) such that*

$$\begin{gathered}\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| \le C \cdot \left|\sigma'\sigma''\right| \cdot \left(\left|\sigma'\right| + \left|\sigma''\right|\right) \\ \Delta\Upsilon < -\left|\sigma'\sigma''\right|\end{gathered} \tag{3.3}$$

Proof. By the estimates in Lemma 3 of [BC1], there exists a positive δ such that if $\left|\sigma'\right| + \left|\sigma''\right| < \delta$, then the estimate (3.3) holds with a suitable positive C_1. Note that, due to the translation invariance of the shock–rarefaction curves (2.1), δ and C_1 do not depend on the state variables (v_1, v_2). Assume now that $\left|\sigma'\right| + \left|\sigma''\right| > \delta$. Consider the $\mathbf{C^2}$ function $F(\sigma', \sigma'') = \left(\sigma_1^+(\sigma', \sigma'') - (\sigma' + \sigma''), \sigma_2^+(\sigma', \sigma'')\right)$. From $F(\sigma', 0) = F(0, \sigma'') = (0, 0)$ it follows that $\left|F(x, y)\right| < C \cdot |xy|$ for all (x, y) in some fixed compact, see also [Y]. Hence

$$\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right| \le C \cdot \left|\sigma'\sigma''\right| \le C_2 \cdot \left|\sigma'\sigma''\right| \cdot \left(\left|\sigma'\right| + \left|\sigma''\right|\right)$$

and (3.3) follows by choosing $\max\{C_1, C_2\}$ as the constant in (3.3).

Note that from $\Psi(\sigma_1^+, \sigma_2^+) = \Psi(\sigma', 0) + \Psi(\sigma'', 0)$ it follows that

$$\sigma_2^+ = \sigma_1^+ - \sigma' - \sigma''. \tag{3.4}$$

To prove the second estimate, the first step is to choose the weights in (2.11). Choose first η sufficiently small (see Steps 2. and 3. below) and then a suitably large K (as in Steps 1. and 3. below), both weights depend only on f, $u_{\pm\infty}$ and $\bar{M}$. Various cases are in order:

1. $\sigma' < 0$ and $\sigma'' < 0$.

Simply compute

$$\begin{aligned}
\Delta V &= -2\eta\left|\sigma_2^+\right| \\
\Delta Q &\le -\left|\sigma'\sigma''\right| + \left(\left|\sigma_1^+ - (\sigma' + \sigma'')\right| + \left|\sigma_2^+\right|\right)\left(\sum_{\sigma_\alpha \in \mathcal{A}(\sigma_1^+)} |\sigma_\alpha|\right) \le -\left|\sigma'\sigma''\right| + C \cdot \left|\sigma_2^+\right| \\
\Delta\Upsilon &\le -\left|\sigma'\sigma''\right| + (C - 2K\eta)\left|\sigma_2^+\right| < -\left|\sigma'\sigma''\right|
\end{aligned} \tag{3.5}$$

provided $K > C/(2\eta)$. Above, we used the equalities $\mathcal{A}(\sigma_1^+) = \mathcal{A}(\sigma') \setminus \{\sigma''\} = \mathcal{A}(\sigma'') \setminus \{\sigma'\}$.

2. $\sigma' < 0$, $\sigma'' > 0$ and $\sigma_1^+ < 0$.

Using (3.4), (2.2), (2.3) and the assumptions on the signs of the waves, it follows that

$$\sigma' + \phi_\varepsilon(\sigma') < \sigma' + \sigma'' + \phi_\varepsilon(\sigma') = \sigma_1^+ + \phi_\varepsilon(\sigma_1^+) + \phi_\varepsilon(\sigma_2^+) < \sigma_1^+ + \phi_\varepsilon(\sigma_1^+)$$

which implies $\sigma' < \sigma_1^+$ and $\sigma_2^+ > -\sigma''$ by (3.4). Analogously

$$\sigma_2^+ > \sigma_2^+ + \phi_\varepsilon(\sigma_2^+) = \phi_\varepsilon(\sigma') - \phi_\varepsilon(\sigma_1^+) \ge C \cdot \left(\sigma' - \sigma_1^+\right), \text{ whence } \left|\sigma_2^+\right| \le C \cdot \left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) \tag{3.6}$$

where $C \geq \sup_{[-M,0]} \left|\phi'_\varepsilon\right|$ (note that C is independent of ε), so that by (2.11)

$$\Delta V = 2\left(\left|\sigma_1^+\right| - \left|\sigma'\right| + \eta\left|\sigma_2^+\right|\right) \leq 2(1 - C\eta)\left(\left|\sigma_1^+\right| - \left|\sigma'\right|\right) \leq \left|\sigma_1^+\right| - \left|\sigma'\right| \tag{3.7}$$

provided $\eta < 1/C$. Moreover

$$\begin{aligned}
\Delta Q &\leq -\left|\sigma'\sigma''\right| + \left(\left|\sigma_1^+\right| - \left|\sigma'\right|\right)\left(\sum_{\sigma_\alpha \in \mathcal{A}(\sigma_1^+)} \left|\sigma_\alpha\right|\right) + \left|\sigma_2^+\right|\left(\sum_{\sigma_\alpha \in \mathcal{A}(\sigma_2^+)} \left|\sigma_\alpha\right|\right) \\
&\leq -\left|\sigma'\sigma''\right| + C \cdot \left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) \\
\Delta\Upsilon &\leq -\left|\sigma'\sigma''\right| + (K - C)\left(\left|\sigma'\right| - \left|\sigma_1^+\right|\right) < -\left|\sigma'\sigma''\right|
\end{aligned}$$

provided $K > C$.

3. $\sigma' < 0$, $\sigma'' > 0$ and $\sigma_1^+ > 0$.

In this case $\sigma'' \leq \varepsilon$ and, due to (3.1), $\sigma_2^+ = 0$ for ε sufficiently small (but independently from the size of the interacting waves). Hence $\sigma_1^+ \leq \sigma''$. Compute now

$$\begin{aligned}
\Delta V &= -2\left|\sigma'\right| < -\left|\sigma'\right| \\
\Delta Q &= -\left|\sigma'\sigma''\right| + \left(\left|\sigma_1^+\right| - \left|\sigma''\right|\right)\left(\sum_{\sigma_\alpha \in \mathcal{A}(\sigma_1^+)} \left|\sigma_\alpha\right|\right) \leq -\left|\sigma'\sigma''\right| \\
\Delta\Upsilon &\leq -\left|\sigma'\sigma''\right| - K\left|\sigma'\right| < -\left|\sigma'\sigma''\right|
\end{aligned}$$

completing the proof. ✠

Lemma 3.2. *Consider an interaction as in Figure 1. Then, with the same weights chosen in Lemma 3.1*

$$\sigma_1^+ = \sigma_1^- \quad \textit{and} \quad \sigma_2^+ = \sigma_2^-, \quad \textit{moreover} \quad \Delta\Upsilon = -\left|\sigma_1^-\sigma_2^-\right|$$

Proof. The equalities follow from the injectivity of Ψ: $\Psi(\sigma_1^+, \sigma_2^+) = \Psi(\sigma_1^-, 0) + \Psi(0, \sigma_2^-) = \Psi(\sigma_1^-, \sigma_2^-)$. The estimate on $\Delta\Upsilon$ is immediate from (2.11), once the previous equalities are established. ✠

The next lemma collects the bounds on the variation of propagation speeds at interactions that will be of use below. The proofs can be found in [BC1], Section 7, Lemmas 15 to 20. These proofs rely solely on the regularity of f, namely $\mathbf{C^4}$, over a suitable compact set. Therefore we do not repeat them here.

Lemma 3.3. *There exists a constant $C > 0$ such that for all u*

$$\lambda_2(u) - \lambda_1(u) > \frac{1}{C}. \tag{3.8}$$

In the case of the interaction in Figure 1:

$$\left|\lambda_1^+ - \lambda_1^-\right| < C \cdot \left|\sigma_2^-\right| \quad \textit{and} \quad \left|\lambda_2^+ - \lambda_2^-\right| \leq C \cdot \left|\sigma_1^-\right|. \tag{3.9}$$

In the case of the interaction in Figure 2:

$$\left|\lambda' - \lambda''\right| > C\left(\left|\sigma''\right| + \left|\sigma'\right|\right) \tag{3.10}$$

and if $\max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\} > \sqrt{\varepsilon}/2$, *then*

$$\left|\frac{(\sigma'+\sigma'')\lambda_1^+ - \sigma'\lambda' - \sigma''\lambda''}{\lambda''-\lambda'}\right| \leq C \cdot \left|\sigma'\sigma''\right|. \tag{3.11}$$

The basic interaction estimates on shifting interactions are the following.

Lemma 3.4. *Consider an interaction as in Figure 1. Then*

$$\sum_{i=1}^{2}\left|\sigma_i^+\xi_i^+\right| - \left|\sigma_1^-\xi_1^-\right| < C\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^+\right|\right)$$

while in the case of Figure 2

$$\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right| + \sum_\alpha \left|\sigma_{2,\alpha}^+\xi_{2,\alpha}^+\right| < C \cdot \left|\sigma'\sigma''\right| \cdot \left(\left|\xi'\right| + \left|\xi''\right|\right).$$

For a proof of the above estimates, see [BC1], Section 7, Lemmas 21 and 22.

We are now ready to provide the main proof.

Proof of Proposition 2.7. An elementary estimate that will be used throughout our calculations is the following:

$$a, b \in \mathbf{R}, \quad a < b \Rightarrow e^a - e^b < -|b-a|e^a \tag{3.12}$$

1. Choice of the constants H_1, H_2 and H_3.

First choose H_1 such that (3.17), and (3.20) hold. Then, choose H_2 satisfying (3.18), (3.21) and (3.24). Finally, choose H_3 such that (3.14), (3.17), (3.21), (3.19), and (3.20) hold.

It is useful to separate the waves taking part in the interaction from those on the left or on the right of the interaction point:

$$\Delta\Upsilon_\xi = \Delta\Upsilon_\xi^{\text{left}} + \Delta\Upsilon_\xi^{\text{int}} + \Delta\Upsilon_\xi^{\text{right}} \tag{3.13}$$

In the various steps below it is proved that each of the summands in the right hand side of this equation is nonpositive.

2. Interaction between shocks of the same family.

Concerning the waves on the left of the interaction point, by Lemma 3.1, definition (2.17)–(2.18) and (3.4), it follows that

$$\Delta S_{i,\alpha} = -2\left|\sigma_2^+\right|, \quad \Delta R_\alpha = -\left|\sigma_2^+\right|, \qquad \text{and} \qquad \Delta(\sigma_{i,\alpha}\xi_{i,\alpha}) = 0\,.$$

As a consequence

$$\Delta\Upsilon_\xi^{\text{left}} \leq 0\,.$$

Consider now the last summand in (3.13). Applying Lemma 3.1 and definition (2.17)–(2.18) to the waves to the right of the interaction point we get

$$\Delta S_{i,\alpha} = -2\left|\sigma_2^+\right|, \quad \Delta R_\alpha = \left|\sigma_2^+\right|, \quad \text{and} \quad \Delta(\sigma_{i,\alpha}\xi_{i,\alpha}) = 0.$$

Hence by (3.5), the condition

$$2\eta H_3 + 2H_1 > H_2 \tag{3.14}$$

ensures that $\Delta\left(H_1 S_{i,\alpha} + H_2 R_\alpha + H_3 V\right) \leq 0$ which in turn implies $\Delta\Upsilon_\xi^{\text{right}} \leq 0$.

Consider the waves entering or exiting the interaction point. By means of Lemma 3.1 and definition (2.17)–(2.18)

$$\begin{array}{lll} S_1^+ - S' = -\left|\sigma''\right| - \left|\sigma_2^+\right| & S_1^+ - S'' = -\left|\sigma'\right| - \left|\sigma_2^+\right| & S_{2,\alpha}^+ - S_1' \leq \left|\sigma_2^+\right| \\ R_1^+ = R' & R_1^+ = R'' - \left|\sigma'\right| & R_{2,\alpha}^+ - R' \leq \left|\sigma_2^+\right| \end{array} \tag{3.15}$$

thus, using also Lemma 3.4,

$$\begin{aligned} \Delta\Upsilon_\xi^{\text{int.}} &= \left|\sigma_1^+\xi_1^+\right| e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} + \sum_\alpha \left|\sigma_{2,\alpha}^+ \xi_{2,\alpha}^+\right| e^{H_1 S_{2,\alpha}^+ + H_2 R_{2,\alpha}^+ + H_3 V^+} \\ &\quad - \left|\sigma'\xi'\right| e^{H_1 S' + H_2 R' + H_3 V^-} - \left|\sigma''\xi''\right| e^{H_1 S'' + H_2 R'' + H_3 V^-} \\ &= \left(\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad + \left|\sigma_1^+\xi_1^+\right| e^{H_1 S_1^+ + H_2 R'} \left(e^{H_3 V^+} - e^{H_3 V^-}\right) \\ &\quad + \left|\sigma'\xi'\right| \left(e^{H_1 S_1^+} - e^{H_1 S'}\right) e^{H_2 R' + H_3 V^-} \\ &\quad + \left|\sigma''\xi''\right| \left(e^{H_1 S_1^+ + H_2 R'} - e^{H_1 S'' + H_2 R''}\right) e^{H_3 V^-} \\ &\quad + \sum_\alpha \left|\sigma_{2,\alpha}^+ \xi_{2,\alpha}^+\right| e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad + \sum_\alpha \left|\sigma_{2,\alpha}^+ \xi_{2,\alpha}^+\right| \left(e^{H_1 S_{2,\alpha}^+ + H_2 R_{2,\alpha}^+ + H_3 V^+} - e^{H_1 S_1^+ + H_2 R' + H_3 V^-}\right) \\ &\leq C\left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad - H_1 \left|\sigma'\xi'\right| \left(\left|\sigma''\right| + \left|\sigma_2^+\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad - H_1 \left|\sigma''\xi''\right| \left(\left|\sigma'\right| + \left|\sigma_2^+\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad + C \cdot \left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad + \sum_\alpha \left|\sigma_{2,\alpha}^+ \xi_{2,\alpha}^+\right| (H_1 + H_2 - \eta H_3) \left|\sigma_2^+\right| e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\leq (C - H_1) \left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\quad + C \cdot \sum_\alpha \left|\sigma_{2,\alpha}^+ \xi_{2,\alpha}^+\right| (H_1 + H_2 - \eta H_3) e^{H_1 S_1^+ + H_2 R' + H_3 V^-} \\ &\leq 0 \end{aligned} \tag{3.16}$$

Provided

$$H_1 > C \quad \text{and} \quad H_3 > (H_1 + H_2)/\eta\,. \tag{3.17}$$

3. Interaction between a 1–shock σ' and a 1–rarefaction σ'' resulting in a 1–shock. Note that by the "stiffness" of the system, this case is entirely analogous to the interaction between a 1–rarefaction σ' and a 1–shock σ''.

In this case, σ_2^+ is a 2–shock. Consider the waves on the left of the interaction point. Due to (3.4), (3.6) and (3.7)

$$\Delta S_{i,\alpha} \leq 2\left|\sigma_2^+\right|, \quad \Delta R_\alpha \leq -\left|\sigma_2^+\right|, \quad \Delta(\sigma_{i,\alpha}\xi_{i,\alpha}) = 0, \quad \Delta V \leq -\frac{1}{C}\left|\sigma_2^+\right|.$$

These inequalities imply

$$\Delta\left(H_1 S_{i,\alpha} + H_2 R_\alpha + H_3 V\right) \leq 0$$

provided

$$H_3 + CH_2 > 2CH_1 \tag{3.18}$$

Passing to waves on the right of the interaction point, we similarly get

$$\Delta S_{i,\alpha} \leq 2\left|\sigma_2^+\right|, \quad \Delta R_\alpha = \left|\sigma_2^+\right|, \quad \Delta(\sigma_{i,\alpha}\xi_{i,\alpha}) = 0.$$

while the other weights remain unchanged. Thus by Lemma 3.1

$$\Delta\left(H_1 S_{i,\alpha} + H_2 R_\alpha + H_3 V\right) \leq \left((2H_1 + H_2) - \frac{1}{C}H_3\right)\left(\left|\sigma_1^+\right| - |\sigma'|\right) \leq 0$$

provided

$$H_3 > C \cdot (2H_1 + H_2)\,. \tag{3.19}$$

Concerning the interacting waves, compute first

$$\begin{aligned}
S_1^+ - S' &= \left|\sigma_1^+\right| - |\sigma'| + 2\left|\sigma_2^+\right| \leq C \cdot \left(|\sigma'| - \left|\sigma_1^+\right|\right) - |\sigma''| & \qquad R_1^+ &= R' \\
S_1^+ - S'' &= \left|\sigma_1^+\right| - 2|\sigma'| + 2\left|\sigma_2^+\right| \leq C \cdot \left(|\sigma'| - \left|\sigma_1^+\right|\right) - |\sigma'| & \qquad R_1^+ &\leq R'' \\
S_2^+ - S_1^+ &= \left|\sigma_1^+\right| - \left|\sigma_2^+\right| & \qquad R_2^+ &= R_1^+
\end{aligned}$$

hence by (3.7)

$$\begin{aligned}
\Delta\Upsilon_\xi^{\mathrm{int.}} &= \left(\left|\sigma_1^+\xi_1^+\right| - |\sigma'\xi'| - |\sigma''\xi''|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\quad + |\sigma'\xi'|\left(e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} - e^{H_1 S' + H_2 R' + H_3 V^-}\right) \\
&\quad + |\sigma''\xi''|\left(e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} - e^{H_1 S'' + H_2 R'' + H_3 V^-}\right) \\
&\quad + \left|\sigma_2^+\xi_2^+\right| e^{H_1 S_2^+ + H_2 R_2^+ + H_3 V^+} \\
&\leq C|\sigma'\sigma''|\left(|\xi'| + |\xi''|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\quad + |\sigma'\xi'|\left(CH_1\left(|\sigma'| - \left|\sigma_1^+\right|\right) - H_1|\sigma''| - H_3\left(|\sigma'| - \left|\sigma_1^+\right|\right)\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\quad + |\sigma''\xi''|\left(CH_1\left(|\sigma'| - \left|\sigma_1^+\right|\right) - H_1|\sigma'| - H_3\left(|\sigma'| - \left|\sigma_1^+\right|\right)\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\quad + C|\sigma'\sigma''|\left(|\xi'| + |\xi''|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\leq (C - H_1)\,|\sigma'\sigma''|\left(|\xi'| + |\xi''|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\quad + (CH_1 - H_3)\left(|\sigma'| - \left|\sigma_1^+\right|\right)\left(|\sigma'\xi'| + |\sigma''\xi''|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\
&\leq 0
\end{aligned}$$

provided

$$H_1 > C \quad \text{and} \quad H_3 > CH_1\,. \tag{3.20}$$

are

will

4. Interaction between a 1–shock σ' and a 1–rarefaction σ'' resulting in a 1–rarefaction. Note that in this case, σ_2^+ will be zero. Hence

$$\Delta V = -2\left|\sigma'\right|,$$

and for waves not taking part in the interaction

$$\Delta S_{i,\alpha} = -2\left|\sigma'\right|.$$

For waves to the left of the interaction we have that

$$\Delta R_\alpha = -\left|\sigma'\right|,$$

while to the right

$$\Delta R_\alpha = 0.$$

Hence $\Delta\left(\Upsilon_\xi^{\text{left}} + \Upsilon_\xi^{\text{right}}\right) < 0$.

Regarding the interacting waves, we compute

$$\begin{aligned} S_1^+ &= S' - \left|\sigma'\right| = S'' - 2\left|\sigma'\right| \\ R_1^+ &= R'' = R' - \left|\sigma'\right| \end{aligned}$$

Consequently

$$\begin{aligned} \Delta\Upsilon_\xi^{\text{int.}} &= \left(\left|\sigma_1^+\xi_1^+\right| - \left|\sigma'\xi'\right| - \left|\sigma''\xi''\right|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\ &\quad + \left|\sigma'\xi'\right| \left(e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} - e^{H_1 S' + H_2 R' + H_3 V^-}\right) \\ &\quad + \left|\sigma''\xi''\right| \left(e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} - e^{H_1 S'' + H_2 R'' + H_3 V^-}\right) \\ &\leq C\left|\sigma'\sigma''\right| \left(\left|\xi'\right| + \left|\xi''\right|\right) \\ &\quad + \left|\sigma'\xi'\right| \left(-\left(H_1 + 2H_3\right)\left|\sigma'\right| - H_2\left|\sigma''\right|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\ &\quad + \left|\sigma''\xi''\right| \left(-2\left(H_1 + H_3\right)\left|\sigma'\right|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\ &\leq \left|\sigma'\sigma''\right| \left(\left(C - H_2\right)\left|\xi'\right| + \left(C - 2\left(H_1 + H_3\right)\right)\left|\xi''\right|\right) e^{H_1 S_1^+ + H_2 R_1^+ + H_3 V^+} \\ &\leq 0 \end{aligned}$$

provided

$$H_2 > C, \quad \text{and} \quad 2\left(H_1 + H_3\right) > C. \tag{3.21}$$

5. Interactions between waves of different families.

We use the same notation as in Figure 1. Simple computations lead to

$$\xi_1^+ - \xi_1^- = \frac{\lambda_1^+ - \lambda_1^-}{\lambda_2^- - \lambda_1^-}\left(\xi_2^- - \xi_1^-\right) \qquad \text{and} \qquad \xi_2^+ - \xi_2^- = \frac{\lambda_2^+ - \lambda_2^-}{\lambda_2^- - \lambda_1^-}\left(\xi_2^- - \xi_1^-\right) \tag{3.22}$$

By Lemma 3.2, the following estimates hold:

$$\begin{aligned} \sigma_1^+ &= \sigma_1^-, \quad S_1^+ = S_1^-, \quad R_1^+ = R_1^- - \left|\sigma_2^-\right|, \\ \sigma_2^+ &= \sigma_2^-, \quad S_2^+ = S_2^-, \quad R_2^+ = R_2^- - \left|\sigma_1^-\right|, \end{aligned} \quad R_1^+ = R_2^+, \quad \Delta V = 0. \tag{3.23}$$

Using (3.22), (3.23), we obtain

$$\Delta\Upsilon_\xi = \sum_{i=1}^{2}\left(\left|\sigma_i^+\xi_i^+\right|e^{H_1S_i^++H_2R_i^++H_3V^+} - \left|\sigma_i^-\xi_i^-\right|e^{H_1S_i^-+H_2R_i^-+H_3V^-}\right)$$
$$= \left(\left|\sigma_1^-\right|e^{H_1S_1^-}\left(\left|\xi_1^+\right| - \left|\xi_1^-\right|e^{H_2|\sigma_2|}\right)e^{H_2R_1^+} + \left|\sigma_2^-\right|e^{H_1S_2^-}\left(\left|\xi_2^+\right| - \left|\xi_2^-\right|e^{H_2|\sigma_1|}\right)e^{H_2R_2^+}\right)e^{H_3V^-}$$

Consider the first summand in the parentheses above:

$$\begin{aligned}\left|\xi_1^+\right| - \left|\xi_1^-\right|e^{H_2|\sigma_2|} &\le \left|\xi_1^+ - \xi_1^-\right| + \left|\xi_1^-\right|\left(1 - e^{H_2|\sigma_2|}\right)\\ &\le C\cdot\left|\sigma_2^-\right|\left(\left|\xi_1^+\right| + \left|\xi_2^-\right|\right) - H_2\left|\sigma_2^-\right|\left|\xi_2^-\right|,\end{aligned}$$

the second summand can be treated similarly. Hence

$$\Delta\Upsilon_\xi \le \left(C\cdot\left(e^{H_1S_1^-} + e^{H_1S_2^-}\right) - H_2\right)\left|\sigma_1^-\sigma_2^-\right|\left(\left|\xi_1^-\right| + \left|\xi_2^-\right|\right)e^{H_2R_1^++H_3V^+} \le 0$$

provided

$$H_2 > Ce^{CH_1\bar{M}} \tag{3.24}$$

The proof of Proposition 2.7 is now completed. ✠

4. The Relativistic Isentropic Model

Consider system **(R)** and introduce

$$\bar{\varsigma} = \frac{\varsigma}{1+(\varsigma/c)^2}\,. \tag{4.1}$$

Note that $\lim_{c\to\infty}\bar{\varsigma} = \varsigma$. A global set of Riemann coordinates (v_1, v_2) exists and is related to the conserved quantities $(\rho, \rho v)$ by

$$\begin{cases} v_1 = c\,\mathrm{arctanh}\left(\dfrac{v}{c}\right) - \bar{\varsigma}\ln\left(\dfrac{\rho}{\rho_*}\right)\\ v_2 = c\,\mathrm{arctanh}\left(\dfrac{v}{c}\right) + \bar{\varsigma}\ln\left(\dfrac{\rho}{\rho_*}\right)\end{cases} \qquad \begin{cases}\rho = \rho_*\exp\dfrac{v_2 - v_1}{2\bar{\varsigma}}\\ v = c\tanh\dfrac{v_1+v_2}{2c}\end{cases}$$

where ρ_* is a given density introduced to have dimensionally correct expressions. By direct computations, it is shown that the shock–rarefaction curves satisfy (2.1) with the parameterization

$$\phi(\sigma) = -\frac{1}{2}\sigma + c\,\mathrm{arcsinh}\left(2\frac{\bar{\varsigma}}{c}\sinh\frac{\sigma}{4\bar{\varsigma}}\right), \qquad \sigma \le 0$$

proving Lemma 2.1. For the sake of completeness, below we provide the expressions for the characteristic speeds and the shock speeds:

$$\lambda_1(v_1,v_2) = c\tanh\left(\frac{v_1+v_2}{2c} - \mathrm{arctanh}\,\frac{\varsigma}{c}\right) \qquad \lambda_2(v_1,v_2) = c\tanh\left(\frac{v_1+v_2}{2c} + \mathrm{arctanh}\,\frac{\varsigma}{c}\right)$$
$$\lambda_1^s(v_1,v_2,\sigma) = c\tanh\left(\frac{v_1+v_2}{2c} - \mathrm{arctanh}\,\frac{\varsigma}{c}\sqrt{\frac{(\varsigma/c)^2 + e^{\sigma/2\bar{\varsigma}}}{1+(\varsigma/c)^2e^{\sigma/2\bar{\varsigma}}}}\right)$$
$$\lambda_2^s(v_1,v_2,\sigma) = c\tanh\left(\frac{v_1+v_2}{2c} + \mathrm{arctanh}\,\frac{\varsigma}{c}\sqrt{\frac{(\varsigma/c)^2 + e^{\sigma/2\bar{\varsigma}}}{1+(\varsigma/c)^2e^{\sigma/2\bar{\varsigma}}}}\right).$$

Passing the various expressions above to the limit $c \to \infty$ we obtain the analogous expressions for case **(E)**. A global set of Riemann coordinates (v_1, v_2) is related to $(\rho, \rho v)$ by

$$\begin{cases} v_1 = v - \varsigma \ln \dfrac{\rho}{\rho_*} \\ v_2 = v + \varsigma \ln \dfrac{\rho}{\rho_*} \end{cases} \qquad \begin{cases} \rho = \rho_* \exp \dfrac{v_2 - v_1}{2\varsigma} \\ v = \dfrac{v_1 + v_2}{2} \end{cases}$$

Shock–rarefaction curves are parameterized as in (2.1) provided

$$\phi(\sigma) = -\frac{1}{2}\sigma + 2\varsigma \sinh \frac{\sigma}{4\varsigma}, \qquad \sigma \le 0$$

proving Lemma 2.1. The characteristic speeds and the shock speeds are, respectively

$$\begin{aligned} \lambda_1(v_1, v_2) &= \frac{v_1 + v_2}{2} - \varsigma & \lambda_2(v_1, v_2) &= \frac{v_1 + v_2}{2} + \varsigma \\ \lambda_1^s(v_1, v_2, \sigma) &= \frac{v_1 + v_2}{2} - \varsigma e^{-\sigma/(4\varsigma)} & \lambda_2^s(v_1, v_2, \sigma) &= \frac{v_1 + v_2}{2} + \varsigma e^{-\sigma/(4\varsigma)} . \end{aligned}$$

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