STABILITY OF CONSERVATION LAWS WITH DISCONTINUOUS COEFFICIENTS

RUNHILD AAE KLAUSEN AND NILS HENRIK RISEBRO

ABSTRACT. We prove L_1 contractivity of weak solutions to a conservation law with a flux function that may depend discontinuously on the space variable. Furthermore, we show that the L_1 difference between solutions to conservation laws with different flux functions, is bounded by the total variation with respect to the space variable, of the difference between the flux functions.

1. INTRODUCTION

In this paper we study scalar conservation laws of the form

$$u_t + f(a(x), u)_x = 0 \tag{1.1}$$

where $u = u(x,t)$ is the unknown function, and $(x,t) \in \mathbb{R} \times \mathbb{R}_0^+$. This equation expresses conservation of u, with a flux density given by $f(a,u)$. Independently of the smoothness of the initial data $u(x,0)$ and of the flux function $f(a(x),u)$, the solution will in general be discontinuous, therefore (1.1) is interpreted in the weak sense. We may also write (1.1) as the 2×2 system of conservation laws

$$\begin{aligned} a_t \qquad\quad &= 0 \\ u_t + f(a,u)_x &= 0 \\ a = a(x), \qquad u(x,0) &= u_0(x). \end{aligned} \tag{1.2}$$

Viewed in this fashion, (1.2) has two characteristic speeds, $\lambda_1 = 0$ and $\lambda_2 = {}^{\partial f}/_{\partial u}$. If $\lambda_2 = 0$ for some (a,u), then (1.2) is non-strictly hyperbolic, and is said to be resonant in sense of Temple and Isaacson [7, 8]. Resonant systems of conservation laws may exhibit unexpected behavior. Explicit examples show that the total variation of the solution u can increase without bound, see [16]. Furthermore, approximations based on the Glimm scheme may also have unbounded total variation as the discretization becomes small, see [19].

The conservation laws (1.1) or (1.2) occur in many models. Perhaps the simplest case is the so-called traffic flow model, see Whitham [23]. In this case (1.1) reads

$$\rho_t + (\rho v(k(x), \rho))_x = 0 \tag{1.3}$$

with $\rho = \rho(x,t)$ denoting the local density of cars at (x,t). The function v models the velocity and depends on road conditions through the function k. Typically $v(k,\rho) = k(1 - \rho/\rho_{\max})$, where k is the local speed limit.

Another occasion where such models arise is in two-phase flow in porous media. If s denotes the saturation of one of the phases, then mass conservation of this phase reads

$$s_t + \left(f(s)\left(1 - k(x)\kappa(s)\right)\right)_x = 0, \tag{1.4}$$

see [4, 5]. Here f is the so-called fractional flow function, and $k(x)$ models the gravitational pull multiplied by the absolute permeability of the medium. The function $\kappa(s)$ denotes the relative permeability of the relevant phase. Related to this model is a model of continuous sedimentation, see Diehl [3].

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Written as a system (1.2) is equivalent, see Wagner [22], to the polymer system

$$\begin{aligned} s_t + f(s,c)_x &= 0 \\ (sc)_t + (cf(s,c))_x &= 0 \end{aligned} \tag{1.5}$$

modeling the injection of water and polymer into a porous medium. Here the water saturation is denoted by the variable s and the concentration of polymer in the water is denoted by c. The equivalence of (1.5) and (1.2) is obtained through the Euler–Lagrange transformation.

Existence and uniqueness of solutions to scalar conservation laws of the type (1.1) were first obtained in the classical papers by Oleinik [17] and Kružkov [11]. In particular, under the assumption that a is twice differentiable, and $a''(x)$ bounded, Kružkov proved an L_1 contraction property of the solution operator. Recently, this result was generalized by Lin *et al.*, [14] who showed that it was sufficient to assume a' of bounded variation. Furthermore, Jenssen [9] showed this property also in the case where a is piecewise smooth and of bounded variation. More recently, Baiti and Jenssen [1] showed that the initial value problem for (1.2) was L_1 stable in both u and a, provided $\partial f/\partial u > 0$, i.e., the system is strictly hyperbolic.

Regarding the related polymer model (1.5), Tveito and Winther [20] showed the following L_1 stability estimate

$$\|s(\cdot,t)-\bar{s}(\cdot,t)\|_1 + \|c_x(\cdot,t)-\bar{c}_x(\cdot,t)\|_1 \leq M\left(\|s(\cdot,0)-\bar{s}(\cdot,0)\|_1 + \|c_x(\cdot,0)-\bar{c}_x(\cdot,0)\|_1\right) \tag{1.6}$$

for some constant M. Here (s,c) and $(\bar{s},\bar{c})$ are two solutions of (1.5), where $c_x(x,0)$ and $\bar{c}_x(x,0)$ are assumed to be of bounded variation. One key ingredient in obtaining this result is that in this case also $c_x(x,t)$ and $\bar{c}_x(x,t)$ have bounded variation.

The initial value problem for (1.1) or (1.2) is more difficult in the case where a is not continuous. For the two-phase problem existence of a solution was proved by Gimse and Risebro [5], and for the related sedimentation model by Diehl [3]. In [19], Temple showed existence of a weak solution to the polymer model (1.5), in the case of general $s(x,0)$ and $c(x,0)$ of bounded variation. For the model studied in the present paper, existence and uniqueness, in the class of solutions satisfying a wave entropy condition, was proved in [10].

Smoothing the discontinuous coefficient a in (1.1), one obtains a conservation law satisfying the hypothesis of Kružkov [11]. Hence the smoothed version of (1.1)

$$u_t + f(a_\varepsilon(x),u)_x = 0, \tag{1.7}$$

where a_ε is some smooth approximation to a, has unique solutions in the sense of [11]. It would be natural to expect that the solution of this equation tend to the solution of (1.1) as $a_\varepsilon \to a$. However, Lyons [16] showed that in general the limit of the corresponding solutions are dependent on how the smoothed coefficients a_ε tend to a.

We shall assume that the flux function $f(a,u)$ satisfies the following

$$\left|\frac{\partial f}{\partial u}\right|, \left|\frac{\partial f}{\partial a}\right| \text{ and } \left|\frac{\partial^2 f}{\partial u^2}\right| \text{ are bounded,} \tag{1.8}$$

$$\frac{\partial^2 f}{\partial u^2} \leq 0 \quad (\geq 0), \tag{1.9}$$

$$\text{if } \frac{\partial f}{\partial u}(\bar{a},\bar{u}) = 0 \text{ for some } (\bar{a},\bar{u}), \text{ then } \frac{\partial^2 f}{\partial u^2}(\bar{a},\bar{u}) < 0 \quad (>0), \tag{1.10}$$

$$\text{there are constants } \beta < \gamma \text{ such that } f(a,\beta) = f(a,\gamma) = 0 \text{ for all } a. \tag{1.11}$$

In particular, (1.10) and (1.11) imply that for each a, there is a value u_T in the interval $[\beta,\gamma]$ such that $f'(u_T) = 0$. These assumptions also imply that the interval $[\beta,\gamma]$ is an invariant region for (1.1). In order to ensure existence of a solution we also assume that

$$\inf_a f(a,u_T(a)) \geq \alpha > 0 \quad (\leq -\alpha < 0), \tag{1.12}$$

for some constant γ.

Regarding the coefficient a we assume that

(1.13) $$a(x) \text{ is of bounded variation,}$$

(1.14) $$a(x) \text{ is discontinuous at finitely many points,}$$

(1.15) $$a'(x) \text{ is bounded whenever defined, and has one sided limits at points of discontinuity.}$$

The initial function $u(x,0) = u_0(x)$ is assumed to satisfy

(1.16) $$u_0(x) \in [\beta, \gamma] \quad \text{for all } x,$$

(1.17) $$u_0(x) \text{ is of bounded variation.}$$

Our main results are then the following:

Theorem 1.1. *Assume that f, a and u_0 satisfies (1.8)–(1.17). Let u and u_ε be the unique weak solutions to*

$$\begin{aligned} u_t + f(a(x), u)_x &= 0, \qquad & u(x,0) &= u_0(x) \\ u_{\varepsilon t} + f\left(a_\varepsilon(x), u_\varepsilon\right)_x &= 0, \qquad & u_\varepsilon(x,0) &= u_0(x), \end{aligned}$$

where $a_\varepsilon = a * \omega_\varepsilon$, with ω_ε being the standard mollifier defined in (2.22) and (2.23). Then

(1.18) $$\lim_{\varepsilon \to 0} \left\| u_\varepsilon(\cdot, t) - u(\cdot, t) \right\|_1 = 0.$$

This theorem is then shown to imply:

Theorem 1.2. *Assume that f satisfies (1.8)–(1.12), that a_1 and a_2 satisfies (1.13)–(1.15), and that $u_{1,0}$ and $u_{2,0}$ satisfy (1.16) and (1.17). Let u_1 and u_2 be the unique weak solutions of*

$$\begin{aligned} u_{1t} + f\left(a_1, u_1\right)_x &= 0, \qquad & u_1(x,0) &= u_{1,0}(x) \\ u_{2t} + f\left(a_2, u_2\right)_x &= 0, \qquad & u_2(x,0) &= u_{2,0}(x). \end{aligned}$$

Then

(1.19) $$\left\| u_1(\cdot,t) - u_2(\cdot,t) \right\|_1 \leq \left\| u_{1,0} - u_{2,0} \right\|_1 + t \left[\left\| \frac{\partial f}{\partial a} \right\|_\infty T.V.\left(a_1 - a_2\right) + K \left\| a_1 - a_2 \right\|_\infty \right],$$

where T.V.( ) denotes the total variation with respect to x, and the constant K depends on $u_{1,0}$, $u_{2,0}$ and f.

We remark that this theorem answers some of the questions posed in [21].

The rest of this paper is organized as follows: In section one we introduce the necessary preliminaries, and prove some estimates on the weak solution of (1.1). In section two we first show theorem 1.1, and then show stability of (1.1) with respect to a when a is smooth. In addition we show that the solution operator to (1.1) is monotone with respect to u_0 also for discontinuous a. Then the stability estimate for smooth a is used to prove theorem 1.2. In addition, for smooth a, we generalize Lucier's stability result [15], to equations of type (1.1).

2. Preliminaries

For simplicity we will study the initial value problem for the conservation law

(2.1) $$u_t + \left(k(x) f(u)\right)_x = 0, \qquad u(x,0) = u_0(x),$$

where the flux function $f(u)$ and the coefficient $k(x)$ satisfies the following hypothesis

$$|f'(u)|,\ |f(u)|,\ \text{and}\ |f''(u)|\ \text{are bounded,} \tag{2.2}$$

$$f''(u) \leq 0\ (\geq 0), \tag{2.3}$$

$$\text{If } f'(\bar{u}) = 0 \text{ for some } \bar{u}, \text{ then } f''(\bar{u}) < 0 \quad (> 0), \tag{2.4}$$

$$\text{there is a constant } \beta > 0 \text{ such that } f(0) = f(\beta) = 0, \tag{2.5}$$

$$k(x) \text{ is of bounded variation,} \tag{2.6}$$

$$k(x) \text{ has a finite number of discontinuities} \tag{2.7}$$

$$k' \text{ is bounded whenever defined, and has one sided limits at points of discontinuity,} \tag{2.8}$$

$$\text{there is a constant } \alpha > 0 \quad (< 0) \text{ such that } k(x) \geq \alpha \quad (\leq -\alpha) \text{ for all } x. \tag{2.9}$$

This is a restatement of equations (1.8)–(1.15) for the particular equation (2.1). Studying (2.1) with the assumptions (2.2)–(2.9) instead of (1.1) and (1.8)–(1.15), merely simplifies the calculations, and the results in this section can quite easily be modified to cover (1.1) and (1.8)–(1.15).

By a solution of (2.1) we mean a weak solution in the usual weak sense, i.e.,

$$\iint_{0 \leq t \leq T} u\phi_t + k(x)f(u)\phi_x \,dxdt - \int_{\mathbb{R}} u(x,T)\phi(x,T)\,dx + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,dx = 0 \tag{2.10}$$

for all test functions $\phi \in C_0^1(\mathbb{R} \times \mathbb{R}^+)$.

In [10] the general initial value problem for (2.1) is studied using a singular mapping Ψ. For later use, we now give the details of this mapping. The hypothesis (2.2)-(2.5) imply that there is a unique u_T in the interval $\langle 0, \beta\rangle$ where f reaches its maximum so that $\|f\|_\infty = f(u_T)$. The mapping Ψ is defined as

$$\Psi(u,k) \stackrel{\text{def}}{=} k \operatorname{sign}(u - u_T)\, \frac{f(u_T) - f(u)}{f(u_T)}. \tag{2.11}$$

Note that Ψ is injective, and regular everywhere except on the line $u = u_T$. This mapping is ultimately borrowed from [19]. In passing, we note that the interval $[0,\beta]$ is an invariant region for (2.1), hence $\|u\|_\infty \leq \beta$ if $u_0(x) \in [0,\beta]$.

The initial data is assumed to be such that the total variation of $\Psi(u_0,k)$ is bounded. This will definitely be the case if both $T.V.(k)$ and $T.V.(u_0)$ are bounded. Then we have the following existence and uniqueness result:

Theorem 2.1. *Assume that f and k satisfy the hypothesis (2.2)–(2.8). If u_0 is such that $\Psi(u_0,k)(x)$ is of bounded variation, and $u_0(x)$ is in the interval $[0,\beta]$ for all x, then there exists one and only one weak solution to (2.1), which also satisfies the wave entropy condition*

$$\partial_x \left(k(x)f'(u)\right) \leq K\left(\frac{1}{t} + |k'|\right) \tag{2.12}$$

for some constant K, weakly in all intervals where k is continuous.

For a proof of this theorem, see [10]. Here we shall only mention that the key ingredient in obtaining the existence of a solution was showing compactness of a sequence of approximate solutions generated by a front tracking method. To show compactness, the total variation of the approximate solutions had to be bounded independently of the level of discretization. Since an explicit example, see Temple [19], or [16], shows that the total variation of u may increase without bound as the discretization level converges to zero, the bounds on the total variation are obtained $z = \Psi(u,k)$ rather than in u.

If $k(x)$ is differentiable, we can use the following definition of a weak solution, see [11],

$$\begin{aligned} \iint_{0 \leq t \leq T} & |u - c|\,\phi_t + \operatorname{sign}(u-c)\,k(x)\left(f(u) - f(c)\right)\phi_x - \operatorname{sign}(u-c)\,k'(x)\,f(c)\,\phi\,dxdt \\ & - \int_{\mathbb{R}} |u(x,T) - c|\,\phi(x,T)\,dx + \int_{\mathbb{R}} |u_0(x) - c|\,\phi(x,0)\,dx \geq 0, \end{aligned} \tag{2.13}$$

for all constants c and for all nonnegative test functions ϕ with compact support.

Later we shall be needing the following lemma.

Lemma 2.1. *Assume that $k(x)$ satisfies the hypothesis (2.6)–(2.9), and let u be a the weak solution to (2.1) satisfying the entropy condition (2.12), then we have*

$$T.V._x\left(f(u(\cdot,t))\right) \leq K\left(f,k,u_0\right), \tag{2.14}$$

where the constant K is given in (2.20). Furthermore, u is L_1 Lipschitz continuous in time, or more precisely

$$\|u(\cdot,t+\tau)-u(\cdot,t)\|_1 \leq \tau\Big[\|f\|_\infty T.V.(k) + K\|k\|_\infty\Big] \tag{2.15}$$

where K is the constant in (2.14).

Proof. We first show (2.14). This will be a consequence of the approximation procedure in [10], and for completeness of argument, we briefly recapitulate this method.

To construct approximate solutions, we first fix a small parameter $\delta > 0$ controlling the accuracy of the approximation. Assume that $\max[k(x)] < N\delta$ for some N, and set $k_i = i\delta$ for $i = 0,\ldots,N$, and let $z_{ij} = j\delta$ for $|j| \leq i$, finally let $u_{ij} = \Psi^{-1}(z_{ij},k_i)$. For each i we then define $f_{\delta,i}(u)$ to be the piecewise linear interpolant of $f(u)$ between the points u_{ij}, $-i \leq j \leq i$. Furthermore we approximate the initial function $v_{\delta 0}(x) \stackrel{\text{def}}{=} (z_0(x),k(x)) = \left(\Psi\left(u_0(x),k(x)\right),k(x)\right)$ using values from the grid $\{(z_{ij},k_i)\}$. By extending the front tracking construction in [2] or [6], we are then able to solve the initial value problem for the approximate equation

$$v_{\delta t} + F_\delta\left(v_\delta\right)_x = 0, \quad v_\delta(x,0) = v_{\delta 0}(x) \tag{2.16}$$

exactly, where the flux function F_δ is given by

$$F_\delta = \begin{pmatrix} f_{\delta,i} \\ 0 \end{pmatrix}.$$

The approximate u component is found by applying the inverse of Ψ, i.e., $u_\delta \stackrel{\text{def}}{=} \Psi^{-1}(v_\delta,k_\delta)$.

This exact solution is constructed by first solving the initial Riemann problems defined by $v_{\delta 0}$. This solution gives a series of discontinuities, called "waves", moving with constant speed. At some later time, two of these will collide, and at this point we solve the new Riemann problem defined by the states to the left and right of the collision. This solution will give new waves, and the process can be continued. In [10] this was shown to be a well defined algorithm, and the approximate solutions $\{u_\delta\}$ converged in L_1 to the weak entropy solution, in the sense of (2.12), as $\delta \to 0$.

We let $\mathrm{Var}_{z,k} v_\delta$ denote the total x variation in the variables $(z,k) = (\Psi(u,k),k)$ of v_δ, i.e.,

$$\mathrm{Var}_{z,k} v = T.V._x(z) + T.V._x(k).$$

The key element used in the convergence analysis in [10] was that

$$\mathrm{Var}_{z,k} v_\delta \leq 5\mathrm{Var}_{z,k} v_{\delta 0}. \tag{2.17}$$

Recall that for each δ, $v_\delta(x,t)$ is a piecewise constant function in x, whose discontinuities, or waves, are of two types. So-called z-waves, across which k is constant, or k-waves, across which the product $kf(u)$ is constant. To estimate the total variation of $f(u_\delta)$, we first estimate the variation in f over the z-waves. If Δy denotes the jump in the quantity y over a wave, then by the definition of the mapping Ψ, we have that

$$|\Delta f_\delta| \leq \frac{\|f\|_\infty}{k}|\Delta z| \tag{2.18}$$

for all z waves, with equality holding only if both z_l and z_r are on the same side of u_T. For the variation of f over k waves, we have that the product kf is constant over the wave, therefore

$$\Delta f_\delta = f_l - f_r = \frac{k_r - k_l}{k_l k_r} kf, \tag{2.19}$$

where we label the states to the left and right of a wave by subscript "l" and "r" respectively. Together, (2.18) and (2.19) and (2.17) imply that

$$
\begin{aligned}
T.V.\left(f\left(u_\delta\right)\right) &\le \frac{\|f\|_\infty}{\alpha}\mathrm{Var}_z\left(v_\delta\right) + \frac{\|k\|_\infty \|f\|_\infty}{\alpha^2}\mathrm{Var}_k\left(v_\delta\right) \\
&\le K\left(f,k,u_0\right) \stackrel{\text{def}}{=} \frac{5\left[\|k\|_\infty \vee 1\right]\|f\|_\infty}{\left[\alpha \wedge \alpha^2\right]}\mathrm{Var}_{z,k} v_0,
\end{aligned}
\tag{2.20}
$$

where $\left[a \vee b\right]$ denotes the maximum of a and b, and $\left[a \wedge b\right]$ denotes the minimum of a and b. Once the bound on $T.V.(f)$ is established, the second part of the lemma follows by standard arguments, see e.g. [13]. □

If k is differentiable, and k' is of bounded variation, we have the stronger result:

Theorem 2.2. *Assume that k is differentiable and that $T.V.(k')$ is bounded. Let u be a weak entropy solution to* (2.1)*, then the following holds:*

$$
T.V.\left(u(\cdot,t)\right) \le T.V.\left(u_0\right) + K_1 t,
\tag{2.21}
$$

where the constant K_1 depends on $T.V.(k)$, $T.V.(k')$, $Var_{z,k} v_{\delta 0}$, $\|f\|_\infty$, $\|f\|_{Lip}$ and $\|f''\|_\infty$.

For a proof of this theorem, see [14].

At this point it is convenient to introduce the standard mollifier ω_ε defined by

$$
\omega_\varepsilon(x) = \frac{1}{\varepsilon}\omega\left(\frac{x}{\varepsilon}\right),
\tag{2.22}
$$

where ω is a smooth nonnegative function with support in $[-1,1]$, such that

$$
\int \omega(x)\,dx = 1, \qquad \omega(-x) = \omega(x), \qquad \omega'(x) \ge 0 \ \text{ for } x \le 0.
$$

Furthermore, we denote the convolution product

$$
\int_{\mathbb{R}} \omega_\varepsilon(x-y)h(y)\,dy
\tag{2.23}
$$

by $\omega_\varepsilon * h(x)$ for any suitable function h.

3. Stability analysis

If one smoothes the coefficient k, for instance by a convolution with ω_ε, then the resulting equation is covered by classical theory [11, 17]. First we show that the solutions to the regularized equations converge (in L_1) to the solution of the original equation.

Let k be a function satisfying (2.6)–(2.9), and let k_ε be the "smoothed coefficient" defined by $k_\varepsilon(x) = \omega_\varepsilon * k(x)$. We define u_ε to be the unique weak entropy solution to the "smoothed equation"

$$
u_{\varepsilon t} + \left(k_\varepsilon(x) f\left(u_\varepsilon\right)\right)_x = 0, \qquad u_\varepsilon(x,0) = u_0(x).
\tag{3.1}
$$

Note that u_ε will satisfy both entropy conditions (2.12) and (2.13). Regarding u_ε we have the following theorem:

Theorem 3.1. *Let u be the weak solution of* (2.1) *which also satisfies* (2.12)*, and let u_ε be the weak solution of* (3.1) *which also satisfies* (2.13)*. For any sequence $\{\varepsilon_j\}$ such that $\varepsilon_j \to 0$ as $j \to \infty$, we have that $u_{\varepsilon_j} \to u$ in L_1 as $j \to \infty$.*

Proof. Let $v_{\varepsilon_j} = \left(u_{\varepsilon_j}, k_{\varepsilon_j}\right)$ as in [10], and let $z = \Psi(u,k)$. The key observation needed in the proof of this theorem is then that

$$
\mathrm{Var}_{z,k} v_{\varepsilon_j}(x,t) \le 5\mathrm{Var}_{z,k} v_{\varepsilon_j}(x,0),
\tag{3.2}
$$

which follows by letting $\delta \to 0$ in [10, (1.18)]. Since $k_{\varepsilon_j} = k * \omega_{\varepsilon_j}$

$$
\mathrm{Var}_{z,k} v_{\varepsilon_j}(x,0) \le \mathrm{Var}_{z,k} v(x,0).
\tag{3.3}
$$

Hence we can use Helly's theorem and standard arguments, notably using (2.15), as in [18], to show that there is a subsequence of $\{\varepsilon_j\}$, for convenience also denoted $\{\varepsilon_j\}$, and a function $\bar{u}(x,t)$, such that $u_{\varepsilon_j} \to \bar{u}$ uniformly in L_1^{loc} as $\varepsilon_j \to 0$. If φ is a test function, let now

$$\begin{aligned} W_\varphi(\bar{u}) &= \iint_{t\geq 0} \bar{u}\varphi_t + kf(\bar{u})\,\varphi_x\,dxdt + \int u_0(x)\varphi(x,0)\,dx \\ &= \iint_{t\geq 0} \left(u_{\varepsilon_j} - \bar{u}\right)\varphi_t + \left(k_\varepsilon f\left(u_{\varepsilon_j}\right) - kf(\bar{u})\right)\,dxdt \end{aligned}$$

since u_{ε_j} is a weak solution to (3.1). Consequently, for some constant K,

$$|W_\varphi(\bar{u})| \leq K\left(\left\|\bar{u} - u_{\varepsilon_j}\right\|_1 + \left\|k_{\varepsilon_j} - k\right\|_1\right),$$

and $|W_\varphi(\bar{u})|$ can be made arbitrarily small by making ε_j small. Hence $\bar{u}$ is a weak solution to (2.1).

Furthermore we have that u_{ε_j} satisfies (2.12), and

$$\partial_x\left(k_{\varepsilon_j} f'(u_\varepsilon)\right) \leq K\left(\frac{1}{t} + \left|k'_{\varepsilon_j}\right|\right)$$

weakly. Let the discontinuities of $k(x)$ be located at $\{x_j\}_1^n$. Fix a positive small number $\delta < \min_i(x_{i+1} - x_i)$. If x is in the interval $\langle x_j + \delta, x_{j+1} - \delta\rangle$, then $k_{\varepsilon_j}(x) \to k(x)$ and $k'_{\varepsilon_j}(x) \to k'(x)$ pointwise. Hence

$$\partial_x\left(kf'(\bar{u})\right) \leq K\left(\frac{1}{t} + |k'|\right)$$

weakly in all intervals $\langle x_j + \delta, x_{j+1} - \delta\rangle$ for any $\delta > 0$. Thus $\bar{u}$ is the unique entropy solution, in the sense of (2.12), to (2.1). Since we can repeat this argument for any subsequence of the original sequence $\{\varepsilon_j\}$, we conclude that the whole sequence $\left\{u_{\varepsilon_j}\right\}$ converges to the unique solution of (2.1). ☐

This convergence result has the following immediate corollary

Corollary 3.1. *Let k and u be as in the previous theorem, and let v be a solution to* (2.1)*, but with initial data $v_0(x)$. If*

$$u_0(x) \leq v_0(x)$$

almost everywhere, then

$$u(x,t) \leq v(x,t)$$

almost everywhere.

Proof. In [11] Kružkov showed that

$$\int \Phi\left(u_\varepsilon(x,t) - v_\varepsilon(x,t)\right)\,dx \leq \int \Phi\left(u_0(x) - v_0(x)\right)\,dx$$

for $\Phi(s) = s + |s|$, and where u_ε and v_ε are the weak solutions to the smoothed equation (3.1) with initial data u_0 and v_0 respectively. We now use the Lebesgue bounded convergence theorem and theorem 3.1 to conclude that

$$\int \Phi\left(u(x,t) - v(x,t)\right)\,dx \leq \int \Phi\left(u_0(x) - v_0(x)\right)\,dx.$$

This proves the corollary. ☐

Now that we have shown convergence with respect to regularizations of the coefficients, we are in a position to analyze stability with respect to variations in the initial data and in the coefficients. First we will do this when the coefficients are smooth, and then we shall use theorem 3.1 to obtain estimates in the general case.

In the following let $k(x)$ and $l(x)$ be two functions satisfying (2.6), (2.8) and (2.9), and in addition we assume that k and l are at least twice differentiable, with bounded second derivatives.

Let u and v be the weak entropy solutions to

$$u_t + (kf(u))_x = 0, \qquad u(x,0) = u_0(x), \tag{3.4}$$

$$v_t + (lf(v))_x = 0, \qquad v(x,0) = v_0(x) \tag{3.5}$$

respectively. If we start with the Kružkov form (2.13) for u, set $c = v(y,s)$, choose $\varphi = \omega_{\varepsilon_1}(x-y)\omega_{\varepsilon_0}(t-s)$, and finally integrate with respect to y and s we obtain

(3.6)

$$\begin{aligned} D_{\varepsilon_0\varepsilon_1}(u,v) \stackrel{\text{def}}{=} \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} &|u(x,t) - v(y,s)|\, \varphi_t \\ &+ \operatorname{sign}(u-v)\, k(x)\, (f(u(x,t)) - f(v(y,s)))\, \varphi_x - k'(x) f(v((y,s)))\, \varphi \, dxdt\, dyds \\ - \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} &|u(x,T) - v(y,s)|\, \varphi((x,T))\, dx\, dyds \\ + \int_0^T \int_{\mathbb{R}} \int_{\mathbb{R}} &|u_0(x) - v(y,s)|\, \varphi(x,0)\, dx\, dyds \geq 0. \end{aligned}$$

Adding and subtracting $k(y)f(v(y,s))$ in the second term under the quadruple integral in (3.6), we can write $D_{\varepsilon_0\varepsilon_1}(u,v)$ as

$$D_{\varepsilon_0\varepsilon_1}(u,v) = \iiiint I_1 + I_2 + I_3 \, dxdt\, dyds - R^T_{\varepsilon_0\varepsilon_1}(u,v) + R^0_{\varepsilon_0\varepsilon_1}(u,v),$$

where the I terms are defined as

(3.7)

$$\begin{aligned} I_1 &\stackrel{\text{def}}{=} |u(x,t) - v(y,s)|\, \omega_{\varepsilon_1}(x-y)\omega'_{\varepsilon_0}(t-s), \\ I_2 &\stackrel{\text{def}}{=} \operatorname{sign}(u(x,t) - v(y,s))\, [k(x)f(u(x,t)) - k(y)f(v(y,s))]\, \omega'_{\varepsilon_1}(x-y)\omega_{\varepsilon_0}(t-s), \\ I_3 &\stackrel{\text{def}}{=} \operatorname{sign}(u(x,t) - v(y,s))\, f(v(y,s)) \Big[(k(y) - k(x))\, \omega'_{\varepsilon_1}(x-y) - k'(x)\omega_{\varepsilon_1}(x-y) \Big] \omega_{\varepsilon_0}(t-s) \\ &= \operatorname{sign}(u(x,t) - v(y,s))\, f(v(y,s)) \left[\frac{\partial}{\partial x}\Big((k(y) - k(x))\, \omega_{\varepsilon_1}(x-y) \Big) \right] \omega_{\varepsilon_0}(t-s), \end{aligned}$$

and the boundary terms as

$$\begin{aligned} R^T_{\varepsilon_0\varepsilon_1}(u,v) &\stackrel{\text{def}}{=} \iiint |u(x,T) - v(y,s)|\, \omega_{\varepsilon_0}(T-s)\omega_{\varepsilon_1}(x-y)\, dx\, dyds, \\ R^0_{\varepsilon_0\varepsilon_1}(u,v) &\stackrel{\text{def}}{=} \iiint |u_0(x) - v(y,s)|\, \omega_{\varepsilon_0}(s)\omega_{\varepsilon_1}(x-y)\, dx\, dyds. \end{aligned}$$

Let Ω be defined as

$$\Omega \stackrel{\text{def}}{=} \iiiint I_1 + I_2 + I_3 \, dxdt\, dyds + R^T_{\varepsilon_0\varepsilon_1}(v,u) - R^0_{\varepsilon_0\varepsilon_1}(v,u).$$

Thus (3.6) can be written

$$0 \leq D_{\varepsilon_0\varepsilon_1}(u,v) = \Omega - R^T_{\varepsilon_0\varepsilon_1}(v,u) + R^0_{\varepsilon_0\varepsilon_1}(v,u) - R^T_{\varepsilon_0\varepsilon_1}(u,v) + R^0_{\varepsilon_0\varepsilon_1}(u,v),$$

or

$$R^T_{\varepsilon_0\varepsilon_1}(v,u) + R^T_{\varepsilon_0\varepsilon_1}(u,v) \leq R^0_{\varepsilon_0\varepsilon_1}(v,u) + R^0_{\varepsilon_0\varepsilon_1}(u,v) + \Omega.$$

This is the same as

(3.8)

$$\begin{aligned} \int_0^T \omega_{\varepsilon_0}(t-T) \iint \omega_{\varepsilon_1}(x-y) \Big(|u(x,T) - v(y,t)| + |v(y,T) - u(x,t)| \Big)\, dx\, dy\, dt \\ \leq \int_0^T \omega_{\varepsilon_0}(t) \iint \omega_{\varepsilon_1}(x-y) \Big(|u_0(x) - v(y,t)| + |v_0(y) - u(x,t)| \Big)\, dx\, dy\, dt + \Omega. \end{aligned}$$

By using the triangle inequality we find that

(3.9)

$$\begin{aligned}\iint \omega_{\varepsilon_1}(x-y)\Big(|u_0(x)-v(y,t)|+|v_0(y)-u(x,t)|\Big)\,dx\,dy \\ \leq 2\,\|u_0-v_0\|_1+\|u(\cdot,t)-u_0\|_1+\|v(\cdot,t)-v_0\|_1 \\ +\iint \omega_{\varepsilon_1}(x-y)\Big(\,|u_0(x)-u_0(y)|+|v_0(x)-v_0(y)|\,\Big)\,dx\,dy.\end{aligned}$$

Similarly by the inverse triangle inequality

(3.10)

$$\begin{aligned}\iint \omega_{\varepsilon_1}(x-y)\Big(|u(x,T)-v(y,t)|+|v(y,T)-u(x,t)|\Big)\,dxdy \\ \geq 2\,\|u(\cdot,T)-v(\cdot,T)\|_1-\|u(\cdot,t)-u(\cdot,T)\|_1-\|v(\cdot,t)-v(\cdot,T)\|_1 \\ -\iint \omega_{\varepsilon_1}(x-y)\Big(\,|u(x,T)-u(y,T)|+|v(x,T)-v(y,T)|\,\Big)\,dx\,dy.\end{aligned}$$

Using (3.9), (3.10), (2.15) and letting ε_0 and ε_1 to zero in (3.8), we find that

$$\|u(\cdot,T)-v(\cdot,T)\|_1 \leq \|u_0-v_0\|_1+\lim_{\varepsilon_0,\varepsilon_1\to 0}\Omega. \tag{3.11}$$

So what remains is to investigate the limit of Ω, that is

$$\lim_{\varepsilon_0,\varepsilon_1\to 0}\left(\iiiint I_1+I_2+I_3\,dxdt\,dyds+R^T_{\varepsilon_0\varepsilon_1}(v,u)-R^0_{\varepsilon_0\varepsilon_1}(v,u)\right). \tag{3.12}$$

Since v is an entropy solution to (3.5), we can proceed with v as we have done with u, to find an inequality of the type (3.6) with the roles of u and v reversed, and k substituted by l. This equation will take the form

$$\iiiint \tilde{I}_1+\tilde{I}_2+\tilde{I}_3\,dxdt\,dyds-R^T_{\varepsilon_0\varepsilon_1}(v,u)+R^0_{\varepsilon_0\varepsilon_1}(v,u)\geq 0. \tag{3.13}$$

In this equation the we use the test function

$$\varphi=\omega_{\varepsilon_0}(s-t)\omega_{\varepsilon_1}(y-x),$$

hence $I_1=-\tilde{I}_1$. Using the last equality in (3.7), and a partial integration, which is allowed since $u(x,t)$ is of bounded variation by theorem 2.2, we find that

$$\lim_{\varepsilon_0,\varepsilon_1\to 0}\iiiint I_3\,dxdt\,dyds=\lim_{\varepsilon_0,\varepsilon_1\to 0}\iiiint \tilde{I}_3\,dxdt\,dyds=0.$$

Therefore, if we add (3.12) and (3.13) what remains is

(3.14)

$$\begin{aligned}&\lim_{\varepsilon_0,\varepsilon_1\to 0}\Omega \\ &\leq \lim_{\varepsilon_0,\varepsilon_1\to 0}\Bigg(\iiiint \operatorname{sign}(u-v)\,[k(x)f(u(x,t))-k(y)f(v(y,s))]\,\omega_{\varepsilon_0}(t-s)\omega'_{\varepsilon_1}(x-y) \\ &\quad-\iiiint \operatorname{sign}(u-v)\,[l(x)f(u(x,t))-l(y)f(v(y,s))]\,\omega_{\varepsilon_0}(t-s)\omega'_{\varepsilon_1}(x-y)\,dxdt\,dyds\Bigg) \\ &=\lim_{\varepsilon_1\to 0}\iiint \operatorname{sign}(u-v)\,\Big[(k(x)-l(x))\,f(u(x,t))-(k(y)-l(y))\,f(v(y,t))\Big]\omega'_{\varepsilon_1}(x-y)\,dx\,dy\,dt.\end{aligned}$$

Let $I(x,y)$ be defined as

$$I(x,y)=\operatorname{sign}(u-v)\,J(x,y), \tag{3.15}$$

where J is given by

$$J(x,y)=\Big[(k(x)-l(x))\,f(u(x,t))-(k(y)-l(y))\,f(v(y,t))\Big], \tag{3.16}$$

We now let $\sigma_\gamma(\,\cdot\,)$ be a smooth approximation to sign$(\,\cdot\,)$ such that $\sigma_\gamma' = 2\omega_\gamma$. Now we can choose $\gamma = \gamma(\varepsilon_1)$, $\lim_{\varepsilon_1\to 0}\gamma(\varepsilon_1) = 0$, so small that

$$\left|\iint \left(\operatorname{sign}(u-v) - \sigma_\gamma(u-v)\right) J(x,y)\omega_{\varepsilon_1}'(x-y)\,dxdy\right| < \varepsilon_1, \tag{3.17}$$

see [24]. We then proceed to estimate

$$\begin{aligned}\left|\iint \sigma_\gamma(u-v)\, J(x,y)\omega_{\varepsilon_1}'(x-y)\,dxdy\right| &\le \left|\iint \frac{\partial}{\partial x}\sigma_\gamma(u-v)\, J(x,y)\omega_{\varepsilon_1}(x-y)\,dxdy\right| \\ &\quad + \left|\iint \sigma_\gamma(u-v)\,\frac{\partial}{\partial x}J(x,y)\omega_{\varepsilon_1}(x-y)\,dxdy\right| \\ &= \left|\iint 2\omega_\gamma(u-v)\left[\frac{\partial u}{\partial x}\right] J(x,y)\omega_{\varepsilon_1}(x-y)\,dxdy\right| \\ &\quad + \left|\iint \sigma_\gamma(u-v)\,\frac{\partial}{\partial x}J(x,y)\omega_{\varepsilon_1}(x-y)\,dxdy\right|.\end{aligned} \tag{3.18}$$

Using (3.17), the fact that $J = 0$ when $u = v$ and $x = y$, and that u is of bounded variation, we see that the integral in (3.18) will tend to zero as $\varepsilon_1 \to 0$. Consequently

$$\begin{aligned}\lim_{\varepsilon_1\to 0}\left|\iint I(x,y)\omega_{\varepsilon_1}'(x-y)\,dxdy\right| &= \lim_{\varepsilon_1\to 0}\left|\iint \sigma_\gamma(u-v)\, J(x,y)\omega_{\varepsilon_1}'(x-y)\,dxdy\right| \\ &\le \lim_{\varepsilon_1\to 0}\iint \left|\frac{\partial}{\partial x}J(x,y)\right|\omega_{\varepsilon_1}(x-y)\,dxdy \\ &\le T.V.(k-l)\,\|f\|_\infty + \|k-l\|_\infty\, T.V.(f(u)) \\ &\le T.V.(k-l)\,\|f\|_\infty + \|k-l\|_\infty\, K(f,k,u_0),\end{aligned} \tag{3.19}$$

where $K(f,k,u_0)$ is the constant in lemma 1.2, (2.20). Similarly, if we start with a partial integration in the y direction, we find that

$$\lim_{\varepsilon_1\to 0}\left|\iint I(x,y)\omega_{\varepsilon_1}'(x-y)\,dxdy\right| \le \|f\|_\infty\, T.V.(k-l) + \|k-l\|_\infty\, K(f,l,v_0). \tag{3.20}$$

Thus

$$\lim_{\varepsilon_0,\varepsilon_1\to 0}\Omega \le T\big[\|f\|_\infty\, T.V.(k-l) + \big[K(f,k,u_0)\wedge K(f,l,v_0)\big]\,\|k-l\|_\infty\big]. \tag{3.21}$$

Summing up, we have proved the following stability result:

Theorem 3.2. *Let $k(x)$ and $l(x)$ be twice differentiable functions of bounded variation, with bounded derivatives, and let u and v be entropy solutions in the sense of* (2.13) *of*

$$\begin{aligned} u_t + (kf(u))_x &= 0, \qquad u(x,0) = u_0(x),\\ v_t + (lf(v))_x &= 0, \qquad v(x,0) = v_0(x).\end{aligned}$$

Then

$$\|u(\cdot,t) - v(\cdot,t)\|_1 \le \|u_0 - v_0\|_1 + t\big[\|f\|_\infty\, T.V.(k-l) + \big[K(f,k,u_0)\wedge K(f,l,v_0)\big]\,\|k-l\|_\infty\big]. \tag{3.22}$$

Remark. Since this theorem is independent of theorem 3.1, it might be tempting to try to use it to prove the convergence of the sequence $\{u_\varepsilon\}$ defined by (3.1). The problem is that $T.V.(k_\varepsilon - k_\gamma)$ does not become small as ε and γ tend to zero.

If the coefficients k and l are differentiable, then we can use Oleinik's results [17], or the more recent result by Lin *et al.*, i.e., theorem 2.2, and infer that both u and v are of bounded variation. The estimates for the total variation obtained in [14] do however increase with $T.V.(k')$, and will therefore not suffice to generalize (3.22) to discontinuous k and l. Never the less, we can use the same method and prove the following slight generalization of Lucier's stability result for scalar conservation laws [15].

Theorem 3.3. *Let k and l be as in 3.2, assume also that k' and l' are of bounded variation. Let f and g satisfy the hypothesis (2.2)–(2.5), and assume that u and v are weak solutions in the sense of (2.13) of*

$$\begin{aligned} u_t + (k(x)f(u))_x &= 0, \qquad u(x,0) = u_0(x) \\ v_t + (l(x)g(v))_x &= 0, \qquad v(x,0) = v_0(x). \end{aligned}$$

Then[1]

(3.23)

$$\begin{aligned} \|u(\cdot,t) - v(\cdot,t)\|_1 \le \|u_0 - v_0\|_1 \\ + t\Bigg[\Big(\|f\|_\infty T.V.(k-l) + K(t,k)\left(\|f\|_{Lip} + \|k\|_\infty \|f-g\|_{Lip}\right) + T.V.(k)\|f-g\|_\infty\Big) \\ \wedge \Big(\|g\|_\infty T.V.(k-l) + K(t,l)\left(\|g\|_{Lip} + \|l\|_\infty \|f-g\|_{Lip}\right) + T.V.(l)\|f-g\|_\infty\Big)\Bigg], \end{aligned}$$

where $K(t,k)$ is defined as

$$K(t,k) \stackrel{def}{=} T.V.(u_0) + K_1 t$$

and K_1 is the constant defined in lemma 1.3, (2.21). The "constant" $K(t,l)$ is similarly defined.

Proof. We can proceed as in the proof of theorem 3.2, but instead of (3.15) we must estimate the total variation of

$$\tilde{I}(x,y) = \operatorname{sign}(u-v)\left[(k(x)f(u(x,t)) - l(x)g(u(x,t))) - (k(y)f(v(y,t)) - l(y)g(v(y,t)))\right]. \tag{3.24}$$

To estimate the variation, we can write the bracketed term as

$$(k(x) - l(x))\, f(u) + l(x)\,(f(u) - g(u)) - (k(y) - l(y))\, f(v) - l(y)\,(f(v) - g(v)) \tag{3.25}$$

or as

$$(k(x) - l(x))\, g(u) + k(x)\,(f(u) - g(u)) - (k(y) - l(y))\, g(v) - k(y)\,(f(v) - g(v))\,. \tag{3.26}$$

Thus

$$\begin{aligned} &\iint \tilde{I}\omega'_{\varepsilon_1}(x-y)\,dxdy \\ &\le \Bigg[\Big(\|f\|_\infty T.V.(k-l) + \|k-l\|_\infty T.V.(u)\|f\|_{Lip} + \|k\|_\infty \|f-g\|_{Lip} T.V.(u) + T.V.(k)\|f-g\|_\infty\Big) \\ &\quad \wedge \Big(\|g\|_\infty T.V.(k-l) + \|k-l\|_\infty T.V.(v)\|g\|_{Lip} + \|l\|_\infty \|f-g\|_{Lip} T.V.(v) + T.V.(l)\|f-g\|_\infty\Big)\Bigg]. \end{aligned}$$

Using theorem 2.2 to estimate $T.V.(u)$ and $T.V.(v)$ concludes the proof of the theorem. ☐

Our main result is a corollary of theorem 3.1 and theorem 3.2.

Corollary 3.2. *Assume that the flux function f and the coefficients k and l satisfy the hypothesis (2.2)–(2.9). Let u and v be weak entropy solutions in the sense of (2.12) to*

$$\begin{aligned} u_t + (k(x)f(u))_x &= 0, \qquad u(x,0) = u_0(x) \\ v_t + (l(x)f(v))_x &= 0, \qquad v(x,0) = v_0(x). \end{aligned}$$

Then

(3.27)

$$\|u(\cdot,t) - v(\cdot,t)\|_1 \le \|u_0 - v_0\|_1 + t\big[\|f\|_\infty T.V.(k-l) + \left[K(f,k,u_0) \wedge K(f,l,v_0)\right]\|k-l\|_\infty\big]$$

where $K(f,\cdot,\cdot)$ is defined by (2.20).

---

[1] $\|h\|_{Lip} \stackrel{\text{def}}{=} \sup_{u\neq v} \frac{h(u)-h(v)}{u-v}$

Proof. Let k_ε and l_ε be regularizations of k and l defined by convolutions with ω_ε, and let u_ε and v_ε be solutions of the regularized equations. Then, by theorem 3.2

$$\|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t)\|_1 \le \|u_0 - v_0\|_1 + t\left[\|f\|_\infty T.V.(k_\varepsilon - l_\varepsilon) + \left[K(f,k_\varepsilon,u_0) \wedge K(f,l_\varepsilon,v_0)\right]\|k_\varepsilon - l_\varepsilon\|_\infty\right]$$

By theorem 3.1, u_ε converges in L_1 to u and v_ε to v. The total variation of the difference $k_\varepsilon - l_\varepsilon$ will also converge to $T.V.(k-l)$, and finally (2.20) shows that $K(f,k_\varepsilon,u_0)$ and $K(f,l_\varepsilon,v_0)$ will converge to $K(f,k,u_0)$ and $K(f,l,v_0)$ respectively. This concludes the proof of the lemma. □

The proofs of theorem 1.1 and 1.2 carries over, *mutis mutandi*, from the proofs of theorem 3.1 and corollary 3.2, via theorem 3.2.

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(Klausen), Norsk Hydro ASA, Research Centre, 5200 Bergen
E-mail address: `Runhild.Aae.Klausen@nho.hydro.com`

(Risebro), Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, N–0316 Oslo, Norway
E-mail address: `nilshr@math.uio.no`