SINGULAR PERTURBATIONS OF FIRST-ORDER HYPERBOLIC SYSTEMS WITH STIFF SOURCE TERMS

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Abstract. This work develops a singular perturbation theory for initial-value problems of nonlinear first-order hyperbolic systems with stiff source terms in several space variables. It is observed that under reasonable assumptions, many equations of classical physics of that type admit a *structural stability condition*. This condition is equivalent to the well-known subcharacteristic condition for one-dimensional 2×2-systems and the well-known time-like condition for scalar second-order hyperbolic equations with a small positive parameter multiplying the highest derivatives. Under this stability condition, we construct a formal asymptotic expansion for the solution to the nonlinear problems. Furthermore, assuming some regularity of the solutions to the limiting inner problems and the reduced problems, we prove the existence of classical solutions in the uniform time interval where the reduced problem has a smooth solution and justify the validity of the formal expansion.

The stability condition seems to be a key to the problems of this kind and can be easily verified. Moreover, this presentation unifies and improves earlier works for some specific equations.

1. Introduction

This work deals with nonlinear first-order hyperbolic systems with stiff source terms in several space variables

$$
U_t + \sum_{j=1}^{d} A_j(U) U_{x_j} = \frac{Q(U)}{\epsilon}, \qquad U(x,0) = \bar{U}(x,\epsilon). \tag{1.1}
$$

Here U is the unknown n-vector function of $(x,t) \equiv (x_1, x_2, \cdots, x_d, t) \in \mathbf{R}^d \times [0, +\infty)$, $A_j = A_j(U) (j = 1, 2, \cdots, d)$ and $Q = Q(U)$ are the respective $n \times n$-matrix and n-vector smooth functions of $U \in G \subset \mathbf{R}^n$ (an open set called *state space*), $\bar{U}(x,\epsilon)$ is a given initial value function and ϵ is a small positive parameter. The subscripts (except j) denote the corresponding partial derivatives. For simplicity, we assume that $A_j (j = 1, 2, \cdots, d)$ and Q do not depend on x, t and ϵ; moreover, $\bar{U}(x,\epsilon)$ is periodic in x with period $(1, 1, \cdots, 1) \in \mathbf{R}^d$. For the general cases, the reader may consult my thesis [20].

The aim of this work is to investigate the limiting problem as ϵ goes to zero from the viewpoint of singular perturbations.

First-order hyperbolic systems with stiff source terms model a large number of different physical phenomena. In particalar, important examples occur in the kinetic theory [16],

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inviscid reactive flow [17] and [12], magnetohydrodynamics, inviscid gas dynamics with relaxation, traffic flow, river flow, certain chemical exchange processes, and so on. For the others we refer to chapters 3 and 10 of [19].

Next, we present our main results together with some historical comments. One of the main contributions of this work is the following observation that under reasonable assumptions, many equations of classical physics of the form (1.1) admit the following structure (also called *stability condition*):

(i). There is an invertible $n \times n$ matrix $P(U)$ and an invertible $r \times r (0 < r \leq n)$ matrix $\hat{S}(U)$, defined on the *equilibrium manifold* $\mathcal{E} := \{U \in G : Q(U) = 0\}$, such that

$$P(U)Q_U(U) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}(U) \end{pmatrix} P(U) \qquad \text{for} \quad U \in \mathcal{E};$$

(ii). as a hyperbolic system, (1.1) is symmetrizable, that is, there is a positive definite Hermitian matrix $A_0(U)$ such that

$$A_0(U)A_j(U) = A_j^*(U)A_0(U) \qquad \text{for} \quad U \in G \quad \text{and} \quad j = 1, 2, \cdots, d;$$

(iii). the hyperbolic part and the source term are coupled in the following sense

$$A_0(U)Q_U(U) + Q_U^*(U)A_0(U) \leq -P^*(U) \begin{pmatrix} 0 & 0 \\ 0 & I_r \end{pmatrix} P(U) \qquad \text{for} \quad U \in \mathcal{E}.$$

Here Q_U is the Jacobian of Q, the superscript "$*$" denotes the transpose operator acting on matrices and I_r is the r-order unit matrix. We know from [5] that (i) is just the usual assumption in the corresponding theory for ordinary differential equations, that is, $d = 0$.

At this point, several comments are proper. The above stability condition is equivalent to the well-known subcharacteristic condition in [19] and [13] for one-dimensional 2×2-systems with $r = 1$ and the well-known time-like condition used in [7], [4] and [6] for scalar second-order hyperbolic equations with a small positive parameter multiplying the highest derivatives. See also [18]. In addition, it has been pointed out respectively in [17] and [1] that a simplified model for reacting flow and the one-dimensional Broadwell model of the Boltzmann equation, both consisting of three equations with $r = 1$, satisfy a *weaker stability condition*, which consists of (i), (ii) and

(iii)': the hyperbolic part and the source term are coupled in the following sense

$$A_0(U)Q_U(U) + Q_U^*(U)A_0(U) \leq 0 \qquad \text{for} \quad U \in \mathcal{E}.$$

More stability conditions for the system of the form (1.1) can be found in my thesis [20] and they provide the reader how the above stability condition was proposed.

After the completion of this work in 1992 the author got to know [2] (and [3]) where a notion of a strictly convex entropy is introduced for the system of the form (1.1) with the source term admitting a nontrivial constant *annihilator*. We note that there exist physical examples in magnetohydrodynamics whose source terms have not such an annihilator. On the other hand, the following has been pointed out in the proof of Theorem 2.2 in [2] that the existence of a convex entropy implies the above weaker stability condition. We will prove that the weaker one implies the stability condition if $r = 1$ or $A_0(U)Q_U(U)$ is Hermitian.

We also remark that Kreiss in [11] studied the system of the form (1.1) under the assumption that $A_j (j = 1, 2, \cdots, d)$ are Hermitian and Q_U is skew Hermitian. The latter implies that all eigenvalues of Q_U are pure imaginary, while in present case Q_U has always eigenvalues with non-zero real parts ($r > 0$).

In addition, the author would like to mention that this work was completed and reported in 1992 (see [20] and [21]). Afterwards, many papers on this kind of problems have shown up (see, e.g., [8] and [15]). In particular, it is worthwhile to point out that based on the subcharacteristic condition, the authors in [8] proposed a new approach to construct shock-capturing numerical schemes, without using Riemann solvers, for conservation laws. It seems that the above stabilty condition plays a key role in studying the problems of this kind. See [22] to know that the stability condition implies the stability of relaxation shock profiles for the simplified model system for reacting flow in [17] and [12].

Assume the structural stability condition and that the *limiting inner problem* (for fixed x)

$$\frac{d\tilde{I}}{d\tau} = Q(\tilde{U}) \qquad \text{with} \qquad \tilde{I}(x,0) = \bar{U}(x,0) \tag{1.2}$$

has a unique solution $\tilde{I}(x,\tau)$ defined on $[0,+\infty)$ whose limit $\tilde{I}(x,+\infty)$ exists. Then a formal asymptotic expansion

$$U_\epsilon = \sum_{k=0}^{m} \epsilon^k U_k(x,t) + \sum_{k=0}^{m} \epsilon^k I_k(x,t/\epsilon)$$

can be constructed. Here m is a prescribed positive integer; the leading term U_0 of the *outer expansion* solves the *reduced problem*

$$\begin{aligned} Q(U_0) =&0, \\ P^I(U_0)\Big(U_{0t} + \sum_{j=1}^{d} A_j(U_0)U_{0x_j}\Big) =&0, \\ U_0(x,0) =&\tilde{I}(x,+\infty), \end{aligned} \tag{1.3}$$

where $P^I(U_0)$ is the $(n-r)\times n$-matrix consisting of the first $(n-r)$ rows of $P(U_0)$; and the leading term I_0 of the *boundary-layer correction expansion* is defined as follows

$$I_0(x,t/\epsilon) = \tilde{I}(x,t/\epsilon) - U_0(x,0).$$

Furthermore, the formal asymptotic expansion U_ϵ satisfies

$$\begin{aligned} U_{\epsilon t} + \sum_{j=1}^{d} A_j(U_\epsilon)U_{\epsilon x_j} =&\frac{Q(U_\epsilon)}{\epsilon} + \epsilon^m F_1(x,t) + \epsilon^m O(\epsilon + e^{-\mu t/\epsilon}), \\ P^I(U_0)F_1 \equiv&0 \end{aligned} \tag{1.4}$$

with μ being a positive constant. The system of equations in (1.3) is called the *equilibrium system*.

Note that our construction of the formal asymptotic expansions needs fewer structural conditions than those of the earlier works (see [5] and the references cited therein) in the context of ordinbary differential equations, that is, $d = 0$ in (1.1).

We use an energy method to analyse the asymptotic expansions. Assuming the stability condition and some regularity of the solutions to the limiting inner problem and the reduced problem, we show the existence of classical solutions U^ϵ in the ϵ-independent time interval where the reduced problem has a smooth solution and the validity of the formal asymptotic expansion.

A direct corollary of the validity theorem is

$$U^\epsilon = \sum_{k=0}^{m} \epsilon^k U_k(x,t) + \sum_{k=0}^{m} \epsilon^k I_k(x, t/\epsilon) + O(\epsilon^{m+1/2}). \tag{1.5}$$

Note that the error estimate improves that of [1] for the one-dimensional Broadwell model of the Boltzmann equation, in that the convergence rate here is independent of the dimension d and we require only $m \geq 1$ for any $d \geq 0$.

In [2], the equilibrium system in conservation form is derived for the system in conservation form with the source term having a nontrivial constant annihilator. It is pointed out there that the equilibrium system is endowed with a naturally reduced entropy when the original system possesses a strictly convex entropy. Thus, one might expect the validity of asymptotic expansions for non-smooth solutions.

This paper is organized as follows. Section 2 is concerned with some relations between our stability condition and the weaker one, the strictly convex entropy condition in [2], and the subcharacteristic condition in [19] and [13]. In Section 3, we show that the stability condition is satisfied by some important physical examples. A formal asymptotic expansion is constructed in Section 4 and estimated in Section 5. In section 6 we justify the validity of the asymptotic expansion and prove the existence result.

2. On the Stability Condition

In this section we point out some relations between our stability condition and the weaker one, the strictly convex entropy condition in [2], and the subcharacteristic condition in [19] and [13].

We begin with the following elementary fact.

Lemma 2.1. *Let A_{12} and A_{22} be $(n-r)\times r$-, $r\times r$-matrices, respectively. Assume*

$$\begin{pmatrix} 0 & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \leq 0.$$

Then $A_{12} = 0$.

Proof. Assume $A_{12} \neq 0$. Then there exists an r-vector ξ_2 so that $A_{12}\xi_2 \neq 0$. Moreover, there exists an $(n-r)$-vector ξ_1 so that $\mathrm{Re}\xi_1^* A_{12}\xi_2 \neq 0$. Let μ be a real number and take

$$\xi = \begin{pmatrix} \mu\xi_1 \\ \xi_2 \end{pmatrix}.$$

By the assumption we have

$$\xi^* \begin{pmatrix} 0 & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix} \xi \leq 0,$$

that is, $2\mu\mathrm{Re}\xi_1^* A_{12}\xi_2 + \xi_2^* A_{22}\xi_2 \leq 0$ for all real μ. This is obviously false. ☐

Lemma 2.2. *Assume the system in (1.1) satisfies the weaker stability condition, that is, (i), (ii) and (iii)' in the previous section. Then $P^{-*}(U)A_0(U)P^{-1}(U)$ is a block-diagonal* *matrix (with the same partition as that in (i) and (iii)) and there is a positive semidefinite Hermitian $r \times r$-matrix $\Lambda(U)$ such that*

$$A_0(U)Q_U(U) + Q_U^*(U)A_0(U) = -P^*(U)\begin{pmatrix}0 & 0\\ 0 & \Lambda(U)\end{pmatrix}P(U).$$

Furthermore, if either $r = 1$ or $A_0(U)Q_U(U)$ is Hermitian, then $\Lambda(U)$ is positive definite.

Note that (i) always holds if the rank of $Q_U(U)$ for $U \in \mathcal{E}$ is equal to the number of its non-zero eigenvalues. This lemma provides a method to check whether a given system satisfies the stability condition.

Proof. Set $A = P^{-*}(U)A_0(U)P^{-1}(U)$. Since PQ_UP^{-1} is block-diagonal, the inequality in (iii)' implies that

$$0 \geq P^{-*}(U)[A_0(U)Q_U(U) + Q_U^*(U)A_0(U)]P^{-1}(U) = \begin{pmatrix}0 & A_{12}\hat{S}\\ \hat{S}^*A_{12}^* & A_{22}\hat{S} + \hat{S}^*A_{22}\end{pmatrix}. \tag{2.1}$$

By Lemma 2.1 we see $A_{12}\hat{S} = 0$. Since $\hat{S}$ is invertible, $A_{12} = 0$ and therefore A is block-diagonal. Moreover, (2.1) shows that $A_{22}\hat{S} + \hat{S}^*A_{22}$ is negative semidefinite. Taking $\Lambda = -A_{22}\hat{S} - \hat{S}^*A_{22}$, we prove the first part.

For the other part, it suffices to show that Λ is invertible. In fact, if $A_0(U)Q_U(U)$ is Hermitian, then so is $A_{22}\hat{S}$. Therefore $A_{22}\hat{S} + \hat{S}^*A_{22} = 2A_{22}\hat{S}$ is invertible, since A_{22} is positive definite and $\hat{S}$ is invertible. In addition, it is obvious that $A_{22}\hat{S} + \hat{S}^*A_{22}$ is invertible in case $r = 1$. □

Remark 2.1. Indeed, there exist models for which $A_{22}\hat{S} + \hat{S}^*A_{22}$ is not invertible. To see this, let us consider the system in (1.1) with

$$d = 2,\ n = 2,\ A_1 = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix},\ A_2 = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix},\ \text{and } S = \begin{pmatrix}0 & \sqrt{2}\\ -\sqrt{2} & -2\end{pmatrix}.$$

Since A_1 is diagonal with distinct diagonal entries and A_2 is symmetric, the symmetrizer A_0 must be a scaler matrix. Without loss of generality, we take A_0 to be the unit matrix. On the other hand, two eigenvalues of S are -1 + i and -1 - i, thus S is stable and $r = 2$. However,

$$A_0S + S^*A_0 = \begin{pmatrix}0 & 0\\ 0 & -4\end{pmatrix}$$

is not invertible.

Next, we point out the relation between our stability condition and the *entropy condition* in [2]. In that paper, the authors considered the system in (1.1) with

$$CQ(U) \equiv 0 \tag{2.2}$$

for some constant $(n - r) \times n$ full-rank matrix C (annihilator). The system satisfies the entropy condition if there is a strictly convex smooth function $\Phi(U)$ such that

(1). $\Phi_{UU}(U)A_j(U)$ is symmetric for each j and $U \in G$;
(2). $\Phi_U(U)Q(U) \leq 0$ for $U \in G$;
(3). for any $U \in G$ the following three relations are equivalent

$$Q(U) = 0, \quad \Phi_U(U)Q(U) = 0, \quad \Phi_U(U) = v^*C \quad \text{for some } (n - r)\text{-vector } v.$$

In addition, it is assumed in some way in [2] that the rank r of $Q_U(U)$ is equal to the number of its non-zero eigenvalues.

Lemma 2.3. *Assume the rank of $Q_U(U)$ for $U \in \mathcal{E}$ is equal to the number of its non-zero eigenvalues. Then the entropy condition implies the weak stability condition.*

Proof. The assumption of the lemma immediately implies (i) with some $P(U)$. Let $\Phi(U)$ be the strictly convex entropy. Then $A_0(U) =: \Phi_{UU}(U)$ is symmetric positive definite and therefore (ii) is just (1).

To see the inequality in (iii)', we observe from (2) that $\Phi_U(U)Q(U)$ takes maximum values at $U \in \mathcal{E}$. Thus, at such U, the Hessian matrix is non-positive, that is,

$$\begin{aligned} 0 \geq\ & (\Phi_U Q)_{UU}(U) \\ =\ & \Phi_{UUU}(U)Q(U) + \Phi_{UU}(U)Q_U(U) + Q_U^*(U)\Phi_{UU}(U) + \Phi_U(U)Q_{UU}(U) \\ =\ & A_0(U)Q_U(U) + Q_U^*(U)A_0(U) + v^*CQ_{UU}(U) \\ =\ & A_0(U)Q_U(U) + Q_U^*(U)A_0(U) + v^*(CQ)_{UU}(U) \\ =\ & A_0(U)Q_U(U) + Q_U^*(U)A_0(U). \end{aligned}$$

Here we have used that $Q(U) = 0$, $\Phi_U(U) = v^*C$ due to (3) and the relation in (2.2). □

Now we point out an equivalent version of the stability condition for multi-dimensional 2×2-systems with $r = 1$

$$\begin{aligned} u_t + \sum_{j=1}^{d} f_j(u,v)_x &= 0, \\ v_t + \sum_{j=1}^{d} g_j(u,v)_x &= \frac{q(u,v)}{\epsilon}. \end{aligned} \tag{2.3}$$

Here u, v are both scalars and it is assumed that $q(u,v) = 0$ is equivalent to $v = h(u)$ with a given function h. Moreover, $q_v(u,v) \neq 0$ for all (u,v) under consideration.

The relevant result can be stated as

Lemma 2.4. *The system in (2.3) admits the stability condition if and only if $q_v < 0$ and there exist two positive functions $\kappa_1(u,v)$ and $\kappa_2(u,v)$ such that for all j,*

$$\kappa_1 q_v f_{jv} = \kappa_2(q_v g_{ju} + q_u f_{ju} - q_u g_{jv} - q_u^2 q_v^{-1} f_{jv}). \tag{2.4}$$

Proof. Assume $q_v < 0$ and the existence of positive functions κ_1, κ_2 satisfying the relation in (2.4). Set $\alpha(u,v) = \dfrac{q_u(u,v)}{q_v(u,v)}$ and define

$$P_0(u,v) = \begin{pmatrix} 1 & 0 \\ \alpha(u,v) & 1 \end{pmatrix} \quad \text{and} \quad \Lambda_0 = P_0^* \begin{pmatrix} \kappa_1(u,v) & 0 \\ 0 & \kappa_2(u,v) \end{pmatrix} P_0.$$

Then

$$P_0 \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & q_v \end{pmatrix} P_0 \quad \text{and} \quad \Lambda_0 \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix} = \left\{ \Lambda_0 \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix} \right\}^*.$$

Moreover,

$$\Lambda_0 \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ q_u & q_v \end{pmatrix}^* \Lambda_0 = 2\kappa_2 q_v P_0^* \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P_0 \leq 0.$$

Thus, the stability condition is satisfied.

Conversely, assume the stability condition is satisfied with some P and A_0. In view of Lemma 2.2, there exist $\kappa_1, \kappa_2 > 0$ such that

$$P^{-*}A_0P^{-1} = \text{diag}(\kappa_1, \kappa_2).$$

Here P may be assumed to be P_0. In fact, since Q_U has distinct eigenvalues 0 and $q_v(\neq 0)$, the general form of P satisfying (i) of the stability condition is $P = DP_0$ with D being an arbitrary invertible diagonal matrix. It follows from (iii) in the stability condition that $\kappa_2 q_v < 0$ and thereby $q_v < 0$. On the other hand, since A_0A_j are all symmetric with

$$A_j = \begin{pmatrix} f_{ju} & f_{jv} \\ g_{ju} & g_{jv} \end{pmatrix},$$

so are $P^{-*}A_0P^{-1}PA_jP^{-1} = \text{diag}(\kappa_1, \kappa_2)PA_jP^{-1}$. By writing out the explicit expression of PA_jP^{-1}, we see immediately that the relation in (2.4) is satisfied by κ_1 and κ_2. Hence, the proof is complete. $\square$

We conclude this section by showing the equivalence of our stability condition and the subcharacteristic condition in [19] and [13] for one-dimensional 2×2-systems with $r = 1$. Indeed, the subcharacteristic condition is that at $(u, v) = (u, h(u))$, $q_v < 0$ and

$$\det \begin{pmatrix} f_u - \lambda^* & f_v \\ g_u & g_v - \lambda^* \end{pmatrix} = (f_u - \lambda^*)(g_v - \lambda^*) - f_v g_u < 0. \tag{2.5}$$

Here $\lambda^* = f_u(u, h(u)) - \alpha(u, h(u)) f_v(u, h(u))$ is the characteristics of the equilibrium system. If the subcharacteristic condition holds, then $\kappa_1(u, v) \equiv f_v(g_u + \alpha f_u - \alpha g_v - \alpha^2 f_v) > 0$ at $(u, v) = (u, h(u))$. Taking $\kappa_2(u, v) = f_v^2$, we see that the relation in (2.4) is satisfied by κ_1 and κ_2 so chosen. Since the sign of $\kappa_1(u, v), \kappa_2(u, v)$ and $-q_v(u, v)$ do not change for (u, v) close to $(u, h(u))$, our stability condition is verified. Conversely, we only need to notice that the relation in (2.4) implied that in (2.5).

3. Examples and Applications

This section is devoted to some important physical examples satisfying our stability condition. We will often use Lemma 2.2.

3.1. Equations in the Kinetic Theory of Gases

Discrete velocity models of the Boltzmann equation in the kinetic theory of gases have the form

$$U_t + \sum_{j=1}^{d} A_j U_{x_j} = \frac{Q(U)}{\epsilon}. \tag{3.1}$$

Here U is an n-vector with components being the density functions; $A_j (j = 1, 2, \cdots, d)$ are (constant) diagonal matrices; $Q(U)$, known as the collision operator, is an n-vector with quadratic forms of U as components; ϵ is proportional to the mean free path of particles under consideration. For details we refer to [16].

On physical grounds, the solution components of (3.1) are non-negative. As in [1] which treats the one-dimensional Broadwell model, we assume that the solution components are positive. Thus, the state space G is taken to be

$$G = \Big\{ U \in \mathbf{R}^d : u_k > 0 \quad \text{for} \quad k = 1, 2, \cdots, n \Big\},$$

where u_k denotes the k-th component of U.

For lots of models widely studied, such as Carleman models, Broadwell models, etc., $Q_U(U)$ can be decomposed as

$$Q_U(U) = A(U)A_0(U), \tag{3.2}$$

where $A(U)$ is a negative semidefinite Hermitian matrix and $A_0(U)$ is a diagonal matrix.

Obviously, $A_0(U)$ symmetrizes the A_j's. Furthermore, we can prove that if $A_0(U)$ is positive definite, then the stability condition is satisfied. Indeed, we have

Lemma 3.1. *Let (3.2) hold with A, A_0 being negative semidefinite Hermitian and positive definite, respectively. Then there is an invertible matrix P such that*

$$PQ_U = \Lambda P \qquad \text{and} \qquad A_0 Q_U = P^* \Lambda P,$$

where Λ is a diagonal matrix with non-positive entries.

Proof. Since $A_0^{1/2} A A_0^{1/2}$ is Hermitian, there is a unitary matrix H such that $A_0^{1/2} A A_0^{1/2} = H^* \Lambda H$ with Λ being a diagonal matrix. Furthermore, Λ is negative semidefinite since so is A and A_0 is positive definite. Take $P = HA_0^{1/2}$. Then

$$\begin{aligned} PQ_U = &\ HA_0^{1/2} A A_0^{1/2} A_0^{1/2} = HH^* \Lambda H A_0^{1/2} = \Lambda P, \\ A_0 Q_U = &\ A_0^{1/2} A_0^{1/2} A A_0^{1/2} A_0^{1/2} = A_0^{1/2} H^* \Lambda H A_0^{1/2} = P^* \Lambda P. \end{aligned}$$

This completes the proof. □

Here are several examples. We denote by Q_k the k-the component of $Q(U)$.

The one-dimensional Carleman model: $n = 2$ and $Q(U) = \big(u_2^2 - u_1^2, u_1^2 - u_2^2\big)^*$. A direct calculation shows

$$Q_U(U) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} \equiv A(U)A_0(U).$$

Obviously, $A(U)$ is symmetric negative semidefinite and $A_0(U)$ is positive definite for $U \in G$.

The three-dimensional Broadwell model: $n = 6$ and

$$\begin{aligned} Q_1(U) = Q_2(U) = &\ u_3u_4 + u_5u_6 - 2u_1u_2, \\ Q_3(U) = Q_4(U) = &\ u_1u_2 + u_5u_6 - 2u_3u_4, \\ Q_5(U) = Q_6(U) = &\ u_1u_2 + u_3u_4 - 2u_5u_6. \end{aligned}$$

Set $X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$Q_U(U) = \begin{pmatrix} -2X & X & X \\ X & -2X & X \\ X & X & -2X \end{pmatrix} \mathrm{diag}(u_2, u_1, u_4, u_3, u_6, u_5) \equiv A(U)A_0(U).$$

It is easy to verify that the eigenvalues of the symmetric matrix $A(U)$ are 0 and -6 with the multiplicity 4 and 2, respectively. And $A_0(U)$ is positive definite for $U \in G$.

The one-dimensional coplanar model: $n = 4$, $Q_1(U) = Q_2(U) = u_3u_4 - u_1u_2$ and $Q_3(U) = Q_4(U) = -Q_1(U)$. With the X defined above, we see

$$Q_U(U) = \begin{pmatrix} -X & X \\ X & -X \end{pmatrix} \mathrm{diag}(u_2, u_1, u_4, u_3) \equiv A(U)A_0(U).$$

It is easy to verify that the eigenvalues of the symmetric matrix $A(U)$ are -4 and 0 with the multiplicity 3. And $A_0(U)$ is positive definite for $U \in G$.

Similarly we can show that the one-, two-dimensional Broadwell models both have the formal structure in (3.2).

3.2. A Simplified Model for Reacting Flow

In order to develop numerical methods for chemically reacting flow problems, LeVeque and others in [12] (see also [17]) presented the following simplified mathematical model for reacting flow

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1}p_x &= 0, \\ s_t + us_x &= \frac{s_e(\rho) - s}{\epsilon}, \end{aligned} \tag{3.3}$$

where ρ and u are the respective density and fluid velocity, s is the mass fracion of one mode of a two-mode gas, $p = p(\rho, s)$ is the pressure, and $s_e(\rho)$ is a given equilibrium distribution function of ρ.

Here we show that this model system admits the stability condition. To this end, set $U = (\rho, u, s)^*$ and $Q(U) = (0, 0, s_e(\rho) - s)^*$. Then we have

$$U_t + A_1(U)U_x = \frac{Q(U)}{\epsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 \\ p_\rho\rho^{-1} & u & p_s\rho^{-1} \\ 0 & 0 & u \end{pmatrix}.$$

Since

$$Q_U(U) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ s_{e\rho} & 0 & -1 \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) of the stability condition is satisfied with some P. By Lemma 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' of the weaker stability condition holds, that is,

$$A_0Q_U + Q_U^*A_0 \le 0.$$

To do so, let $s_{e\rho} \equiv \dfrac{ds_e}{d\rho} \neq 0$ and define

$$A_0(U) = \begin{pmatrix} p_\rho & 0 & p_s \\ 0 & \rho^2 & 0 \\ p_s & 0 & \dfrac{-p_s}{s_{e\rho}} \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. Moreover, if $\rho \neq 0$, $p_\rho + p_s s_{e\rho} > 0$ and $p_s s_{e\rho} < 0$, then $A_0(U)$ is positive definite. Thus, we take

$$G = \Big\{U \in \mathbf{R}^3 : \rho \neq 0, p_\rho(\rho, s) > -p_s(\rho, s) s_{e\rho}(\rho) > 0\Big\}. \tag{3.4}$$

A simple calculation indicates that $A_0 A_1$ is symmetric and (iii)' is satisfied. Hence the above model system admits the stability condition.

It is shown in [22] that the constraint $U \in G$ guarantees the asymptotic stability of relaxation shock profiles for the above model system. Note that $p_\rho + p_s s_{e\rho} > 0$ implies that the corresponding equilibrium system

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1} p_x &= 0, \\ s &= s_e(\rho) \end{aligned}$$

is strictly hyperbolic.

In [12], the pressure function is taken to be $p = \rho c^2(1 - s + \beta s)$ with c and β being constants. Thus,

$$G = \Big\{U \in \mathbf{R}^3 : \rho \neq 0, 1 - s + \beta s > \rho(1-\beta) s_{e\rho}(\rho) > 0\Big\}.$$

The constraint $\rho(1-\beta)s_{e\rho}(\rho) > 0$ is also derived in [12] but with a different approach.

For the system in (3.3), it is obvious that the corresponding limiting inner system

$$\tilde{\rho}_\tau = 0, \qquad \tilde{u}_\tau = 0, \qquad \tilde{s}_\tau = s_e(\tilde{\rho}) - \tilde{s}$$

has unique continuous solutions whose limits at infinity exist.

3.3. Relativistic Magnetohydrodynamics

For a one-dimensional conducting fluid in a magnetic field, the usual equations of mass, momentum and energy in fluid dynamics are modified by the inclusion of the magnetic force in the momentum equation and the Joule heat term in the entropy equation. These, together with Maxwell's equations and Ohm's law, give the following system (see [19])

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1} p_x &= \frac{B(E - uB)}{\epsilon\rho}, \\ p_t + up_x + \gamma p u_x &= \frac{(\gamma-1)(E-uB)^2}{\epsilon}, \\ B_t + E_x &= 0, \\ E_t + c^2 B_x &= \frac{-\kappa(E - uB)}{\epsilon}, \end{aligned} \tag{3.5}$$

where ρ, p and u are the respective fluid density, pressure and velocity, B is the magnetic induction, and E is the electric field; γ is the ratio of specific heats, c is the velocity of light, $1/\kappa$ is the dielectric constant, and $1/\epsilon$ is the electric conductivity.

Unlike Whitham in [19], we consider the relativistic effect and therefore do not omit the displacement current E_t/κ.

Now we show that the above system satisfies our stability condition. To this end, set $U = \big(\rho, u, p, B, E\big)^*$ and $Q(U) = \big(0, \rho^{-1}B, (\gamma-1)(E-uB), 0, -\kappa\big)^*(E-uB)$. Then we have

$$U_t + A_1(U)U_x = \frac{Q(U)}{\epsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 & 0 & 0 \\ 0 & u & \rho^{-1} & 0 & 0 \\ 0 & \gamma p & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & c^2 & 0 \end{pmatrix}.$$

Since for $U \in \mathcal{E}$,

$$Q_U(U) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho^{-1}B^2 & 0 & -\rho^{-1}Bu & \rho^{-1}B \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \kappa B & 0 & \kappa u & -\kappa \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) is satisfied with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \rho\kappa & 0 & 0 & B \\ 0 & \sqrt{\kappa}B & 0 & \sqrt{\kappa}u & -\sqrt{\kappa} \end{pmatrix}. \tag{3.6}$$

By Lemma 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' holds. To do so, define

$$A_0(U) = \begin{pmatrix} 1 & 0 & -\dfrac{\rho}{\gamma p} & 0 & 0 \\ 0 & \kappa\rho & 0 & 0 & 0 \\ -\dfrac{\rho}{\gamma p} & 0 & \dfrac{\rho^2}{\gamma^2 p^2} + \dfrac{\kappa}{\gamma p} & 0 & 0 \\ 0 & 0 & 0 & c^2 & -u \\ 0 & 0 & 0 & -u & 1 \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. By a direct calculation, we know that $A_0(U)$ is positive definite and $A_0(U)A_1(U)$ is symmetric for U in

$$G = \Big\{U \in \mathbf{R}^5 : |u| < c, p > 0, \rho > 0\Big\}. \tag{3.7}$$

Moreover, $A_0 Q_U + Q_U^* A_0 \leq 0$. Hence the stability condition is satisfied.

For the system in (3.5), the corresponding limiting inner system is

$$\begin{aligned} &\tilde\rho_\tau = 0, && \tilde u_\tau = \tilde\rho^{-1}\tilde B(\tilde E - \tilde u\tilde B), && \tilde p_\tau = (\gamma-1)(\tilde E - \tilde u\tilde B)^2, \\ &\tilde B_\tau = 0, && \tilde E_\tau = -\kappa(\tilde E - \tilde u\tilde B). \end{aligned}$$

Obviously, $\tilde{\rho}$ and $\tilde{B}$ are independent of τ. Thus we deduce from the second and the last equations that

$$(\tilde{E} - \tilde{u}\tilde{B})_\tau = -(\kappa + \tilde{\rho}^{-1}\tilde{B}^2)(\tilde{E} - \tilde{u}\tilde{B}).$$

Therefore, $(\tilde{E} - \tilde{u}\tilde{B})(\tau)$ decays exponentially to zero as τ goes to infinity since $\kappa + \tilde{\rho}^{-1}\tilde{B}^2 \geq \kappa > 0$. Substituting $(\tilde{E} - \tilde{u}\tilde{B})(\tau)$ into the second, the third and the last equations, we easily know that the limits of $\tilde{u}(\tau), \tilde{p}(\tau)$ and $\tilde{E}(\tau)$ exist as τ goes to infinity and they converge exponentially to the limits.

By using the explicit expression of P in (3.6), we can write out the corresponding equilibrium system as follows

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ B_t + uB_x + Bu_x &= 0, \\ p_t + up_x + \gamma p u_x &= 0, \\ \rho\kappa(u_t + uu_x + \rho^{-1}p_x) + B(E_t + c^2 B_x) &= 0, \\ E &= uB. \end{aligned}$$

Note that there is no a constant annihilator $C \in \mathbf{R}^{4\times 5}$ such that

$$CQ(U) \equiv 0.$$

Such an assumption is required in [2].

3.4. Inviscid gas dynamics with relaxation

In changing flow the internal energy may lag behind the equilibrium value corresponding to the ambient pressure and density. This is the so-called relaxation effect and the equations of motion take the following form (see [19])

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1}p_x &= 0, \\ p_t + up_x + \gamma p u_x &= \frac{(\gamma-1)\rho(E - \mu p\rho^{-1})}{\epsilon}, \\ E_t + uE_x &= -\frac{E - \mu p \rho^{-1}}{\epsilon}, \end{aligned} \tag{3.8}$$

where ρ, p and u are the respective fluid density, pressure and velocity, and E is the energy in the lagging degrees of freedom; $\gamma > 1$ and $\mu > 0$ are two constants related to the degrees of freedom, and $1/\epsilon$ is the relaxation time.

Here we show that this relaxation system admits the stability conditoin. To this end, set $U = \big(\rho, u, p, E\big)^*$ and $Q(U) = \big(0, 0, (\gamma-1)\rho, -1\big)^*(E - \mu p \rho^{-1})$. Then we have

$$U_t + A_1(U)U_x = \frac{Q(U)}{\epsilon}$$

with

$$A_1(U) = \begin{pmatrix} u & \rho & 0 & 0 \\ 0 & u & \rho^{-1} & 0 \\ 0 & \gamma p & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}.$$

Since for $U \in \mathcal{E}$,

$$Q_U(U) = \frac{1}{\rho^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ (\gamma-1)\mu\rho p & 0 & -(\gamma-1)\mu\rho^2 & (\gamma-1)\rho^3 \\ -\mu p & 0 & \mu\rho & -\rho^2 \end{pmatrix},$$

its rank $r(=1)$ is equal to the number of its non-zero eigenvalues. Thus, (i) is satisfied with

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \rho(\gamma-1) \\ \mu\rho^{-1}p & 0 & -\mu & \rho \end{pmatrix}. \tag{3.9}$$

By Lemma 2.2, it suffices to find a positive definite symmetrizer A_0 such that (iii)' holds. To do so, define

$$A_0(U) = \frac{1}{(\gamma-1)\rho^2}\begin{pmatrix} \gamma\mu p^2 & 0 & -\mu\rho p & 0 \\ 0 & \mu(\gamma-1)\rho^3 p & 0 & 0 \\ -\mu\rho p & 0 & \mu\rho^2 & 0 \\ 0 & 0 & 0 & (\gamma-1)\rho^4 \end{pmatrix}.$$

Obviously, $A_0(U)$ is symmetric. By a direct calculation, we know that $A_0(U)$ is positive definite and $A_0(U)A_1(U)$ is symmetric for U in

$$G = \Big\{U \in \mathbf{R}^4 : p > 0, \rho > 0\Big\}. \tag{3.10}$$

Moreover, $A_0Q_U + Q_U^*A_0 \le 0$. Hence the stability condition is satisfied.

For the system in (3.8), the corresponding limiting inner system is

$$\tilde{\rho}_\tau = 0, \qquad \tilde{u}_\tau = 0, \qquad \tilde{p}_\tau = (\gamma-1)(\tilde{\rho}\tilde{E} - \mu\tilde{p}), \qquad \tilde{E}_\tau = -(\tilde{E} - \mu\tilde{\rho}^{-1}\tilde{p}).$$

Obviously, $\tilde{\rho}$ and $\tilde{u}$ are independent of τ. Thus we deduce from the last two equations that

$$(\tilde{E} - \mu\tilde{\rho}^{-1}\tilde{p})_\tau = -(1 + \mu(\gamma-1))(\tilde{E} - \mu\tilde{\rho}^{-1}\tilde{p}).$$

Therefore, $(\tilde{E} - \mu\tilde{\rho}^{-1}\tilde{p})(\tau)$ decays exponentially to zero as τ goes to infinity. Substituting $(\tilde{E} - \mu\tilde{\rho}^{-1}\tilde{p})(\tau)$ into the last two equations, we easily know that the limits of $\tilde{p}(\tau)$ and $\tilde{E}(\tau)$ exist as τ goes to infinity and they converge exponentially to the limits.

By using the explicit expression of P in (3.9), we can write out the corresponding equilibrium system as follows

$$\begin{aligned} \rho_t + u\rho_x + \rho u_x &= 0, \\ u_t + uu_x + \rho^{-1}p_x &= 0, \\ p_t + up_x + \frac{\gamma + \mu(\gamma-1)}{1 + \mu(\gamma-1)}pu_x &= 0, \\ E &= \mu\rho^{-1}p. \end{aligned}$$

4. Composite Asymptotic Expansions

In this section we construct a formal asymptotic expansion for the solution U^ϵ to the nonlinear problem in (1.1). This will be achieved with the composite expansion method in [5]. Thus, the expected m-th order expansion U_ϵ is of the following form

$$U_\epsilon = U(x,t,\epsilon) + I(x,t,\epsilon)$$

with the *outer expansion*

$$U(x,t,\epsilon) = \sum_{k=0}^{m} \epsilon^k U_k(x,t) \tag{4.1}$$

and the *initial-layer correction*

$$I(x,t,\epsilon) = \sum_{k=0}^{m} \epsilon^k I_k(x,\tau) \tag{4.2}$$

whose terms decay to zero as the "fast time" variable $\tau = t/\epsilon$ goes to infinity.

Define

$$\Phi(V) = V_t + \sum_{j=1}^{d} A_j(V) V_{x_j} - \frac{Q(V)}{\epsilon}.$$

Then

$$\begin{aligned} \Phi(U_\epsilon) =& \Phi(U) + [\Phi(U+I) - \Phi(U)], \\ U_\epsilon(x,0) =& \sum_{k=0}^{m} \epsilon^k [U_k(x,0) + I_k(x,0)]. \end{aligned} \tag{4.3}$$

Experience suggests that the outer expansion approximates the exact solution away from $t = 0$, and thereby we establish some relations among the coefficients of the expansion in (4.1). From now on we take $m = 1$ for simplicity. Expanding $\Phi(U)$ into power series in ϵ and equating the coefficients of ϵ^{-1} and ϵ^0 to zero, we obtain

$$Q(U_0(x,t)) = 0, \tag{4.4}$$

$$U_{0t} + \sum_{j=1}^{d} A_j(U_0) U_{0x_j} = Q_U(U_0) U_1. \tag{4.5}$$

And the coefficient of ϵ^1 is

$$F_1 \equiv U_{1t} + \sum_{j=1}^{d} A_j(U_0) U_{1x_j} + \sum_{j=1}^{d} A_j(U_0)_{x_j} U_1 - \frac{1}{2}(Q_{UU}(U_0)U_1)U_1. \tag{4.6}$$

Because of the equation in (4.4), U_0 lies on the equilibrium manifold defined by $Q(U) = 0$. In view of the stability condition, the invertible matrix $P = P(U_0)$ is well defined and diagonalizes $Q_U(U_0)$, that is,

$$PQ_U(U_0) = \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}(U_0) \end{pmatrix} P.$$

Let P^I, P^{II} be the matrices composed of the first $(n-r)$ and last r rows of P, respectively. Multiplying the equation in (4.5) with P from the left we get

$$P^I(U_0)\Big(U_{0t} + \sum_{j=1}^d A_j(U_0)U_{0x_j}\Big) = 0. \tag{4.7}$$

$$P^{II}(U_0)\Big(U_{0t} + \sum_{j=1}^d A_j(U_0)U_{0x_j}\Big) = \hat{S}(U_0)P^{II}(U_0)U_1. \tag{4.8}$$

It remains to find equations for P^IU_1. To this end, we observe the equation in (4.6) and let

$$P^IF_1 = 0. \tag{4.9}$$

Then

$$P^IU_{1t} + \sum_{j=1}^d P^IA_j(U_0)U_{1x_j} = -\sum_{j=1}^d P^IA_j(U_0)_{x_j}U_1 + \frac{1}{2}P^I(Q_{UU}(U_0)U_1)U_1. \tag{4.10}$$

Under the stability condition, the system in (4.10) can be converted to a symmetrizable (semilinear) hyperbolic system for P^IU_1. In fact, (4.6) is a symmetrizable hyperbolic system for U_1 provided that F_1 is known. Thus, it is easy to see that we have a symmetrizable hyperbolic system for PU_1 if PF_1 is given. Since $P^{-*}A_0(U_0)P^{-1}$ is of the block-diagonal form (see Lemma 2.2), the system in (4.10) is a symmetrizable hyperbolic system for P^IU_1 if $P^{II}U_1$ is provided by the relation in (4.8).

Note that (4.4), together with the equations in (4.7), is just the equilibrium system. Thus, U_0 and U_1 will be completely determined provided that we specify the initial values for P^IU_0 and P^IU_1. Once U_0 and U_1 are determined, we have

$$\Phi(U) = \epsilon F_1(x,t) + \epsilon^2\mathcal{R}_1. \tag{4.11}$$

To find the initial data, we write

$$\Phi(U+I) - \Phi(U) = c_{-1}(x,\tau)\epsilon^{-1} + c_0(x,\tau)\epsilon^0 + \epsilon^1\mathcal{R}_2 \tag{4.12}$$

by using the expansion for I in (4.2) and the power series expansion

$$U = U_0(x,0) + \epsilon[\tau U_{0t}(x,0) + U_1(x,0)] + \epsilon^2\int_0^1 \big[\tau^2(1-\theta)U_{0tt}(x,\theta t) + \tau U_{1t}(x,\theta t)\big]d\theta.$$

Setting $\tilde{I}_k(x,\tau) = I_k(x,\tau) + U_k(x,0)(k=0,1)$ and letting $c_{-1}(x,\tau) = c_0(x,\tau) = 0$ yields

$$\frac{d\tilde{I}_0}{d\tau} = \frac{dI_0}{d\tau} = Q(\tilde{I}_0(x,\tau)). \tag{4.13}$$

$$\frac{d\tilde{I}_1}{d\tau} = Q_U(\tilde{I}_0(x,\tau))\tilde{I}_1 - Q_U(U_0(x,0))U_1(x,0) + R(x,\tau).$$

Here

$$R(x,\tau) = \tau[Q_U(\tilde{I}_0) - Q_U(U_0(x,0))]U_{0t}(x,0) - \sum_{j=1}^d[A_j(\tilde{I}_0)\tilde{I}_{0x_j} - A_j(U_0(x,0))U_{0x_j}(x,0)].$$

By setting $S(x,\tau) = P(U_0(x,0))Q_U(\tilde{I}_0)P^{-1}(U_0(x,0))$ and $V(x,\tau) = P(U_0(x,0))\tilde{I}_1$, we derive from the last equation that

$$\frac{dV}{d\tau} = S(x,\tau)V - \mathrm{diag}(0, \hat{S}(U_0(x,0)))P(U_0(x,0))U_1(x,0) + P(U_0(x,0))R(x,\tau). \tag{4.14}$$

We also expand the prescribed initial value $\bar{U}_0(x,\epsilon)$ into power series in ϵ

$$\bar{U}(x,\epsilon) = \bar{U}(x) + \epsilon\bar{U}_1(x) + O(\epsilon^2). \tag{4.15}$$

Now we briefly show how to determine the coefficients of the expansions. A further discussion will be given in the next section. First, we take $\tilde{I}_0(x,0) = \bar{U}(x)$. This, together with the equation in (4.13), is precisely the limiting inner problem, which will be assumed to have a unique continuous solution converging to $\tilde{I}_0(x,\infty) \in \mathcal{E}$ as τ goes to infinity. Knowing $\tilde{I}_0(x,\infty)$, we consider the equilibrium system (4.4) and (4.7) with $U_0(x,0) = \tilde{I}_0(x,\infty)$, which is the reduced problem. Moreover, $P^{II}U_1$ is also obtained by (4.8). Then we solve (4.14) with $V(x,0) = P(U_0(x,0))\bar{U}_1(x)$. We will show in the next section that $V(x,\tau)$ exists uniquely, decays exponentially to $V(x,\infty)$ and $V^{II}(x,\infty) = (P^{II}U_1)(x,0)$. Thus, we take $(P^I U_1)(x,0) = V^I(x,\infty)$ and solve the symmetrizable hyperbolic system in (4.10) to get $P^I U_1$.

In conclusion, we have determined all coefficients in expansions (4.1) and (4.2). Furthermore, we see from (4.3), (4.11), (4.12) and (4.15) that

$$\begin{aligned} U_{\epsilon t} + \sum_{j=1}^{d} A_j(U_\epsilon)U_{\epsilon x_j} &= \frac{Q(U_\epsilon)}{\epsilon} + \epsilon F_1 + \epsilon F_2, \\ U_\epsilon(x,0) &= U^\epsilon(x,0) + O(\epsilon^2). \end{aligned} \tag{4.16}$$

Here $F_2 = \epsilon\mathcal{R}_1 + \mathcal{R}_2$.

Finally, we point out that higher-order approximations can be constructed similarly. Moreover, the higher-order terms $U_k(x,t)$ with $k \geq 2$ in the outer expansion satisfy linear equations with successively known nonhomogeneous terms.

5. Estimates of the Expansions

In this section we show some regularity properties and estimates of U_ϵ, F_1 and F_2 constructed in the previous section.

First of all, let us agree on some notations. For vectors $U, V \in \mathbf{C}^k$ and matrix $A \in \mathbf{C}^{l\times k}$ our basic inner product and norm are

$$< U, V >= U^*V, \qquad |U| =< U, U >^{1/2} \qquad \text{and} \qquad |A| = \max\{|AU| : |U| = 1\}.$$

Let $\Omega = (0,1]^d$. L^2 is the space of square integrable (vector- or matrix-valued) functions on Ω. $||A||$ and (U,V) denote its norm and inner product, respectively. In case U, V and A are functions of another variable t as well as $x \in \Omega$, we write $||A(t)||$ and $(U(t), V(t))$ to remind the reader that the norm and the inner product are computed with respect to x while t is viewed as a parameter. Similar notations will be adopted for the function spaces introduced below. For a nonnegative integer s, the Sobolev space H^s is defined as the space of functions which and their distribution derivatives of order $\leq s$ are all in L^2. We use $||A||_s$ and $(U,V)_s$ to denote the norm and the inner product of H^s.

Let R be an open subset of some real Euclid space. We denote by $C(R)$ (resp. $C^k(R)$ with k being a positive integer or ∞) the space of continuous (resp. k-times continuously differentiable) functions on R. $C_b(R)$ (resp. $C_b^k(R)$) is a subspace of $C(R)$ (resp. $C^k(R)$) whose elements (resp. and all derivatives of order $\leq k$) are bounded on R. $C_b(R)$ and $C_b^k(R)$ are both Banach spaces respectively for the norms

$$|A|_{0,R} = \sup\big\{|A(x)|, x \in R\big\} \quad \text{and} \quad |A|_{k,R} = \sup\big\{|\partial^\alpha A|_{0,R}, |\alpha| \leq k\big\}.$$

We also adopt the multi-index notations in [9] and denote them by Greek alphabets α, β and so on. When it can be inferred from the context, the subscript R will frequently be omitted from the above notations. In addition, we denote by $C^k\big([0,T], \mathbf{X}\big)$ the k-times continuously differentiable functions on $[0,T]$ with values in the Banach space $\mathbf{X}$.

We need the following well-known calculus inequalities in Sobolev spaces. In what follows, C_s denotes a generic constant depending only on s and d.

Lemma 5.1. *Let s, s_1 and s_2 be three non-negative integers.*

(a). If $s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\} \geq 0$, then

$$H^{s_1} H^{s_2} \subset H^{s_3}.$$

Here $s_0 = [d/2] + 1$ and the inclusion symbol $\subset$ implies the continuity of the embedding.

(b). Suppose $s \geq s_0 + 1, A \in H^s$ and $U \in H^{s-1}$. Then for all multi-indices α with $|\alpha| \leq s, \partial^\alpha(AU) - A\partial^\alpha U \in L^2$ and

$$||\partial^\alpha(AU) - A\partial^\alpha U|| \leq C_s ||A||_s ||U||_{|\alpha|-1}.$$

(c). Suppose $s \geq s_0, A \in C_b^s(G)$ and $V \in H^s(\Omega, G)$. Then $A(V(\cdot)) \in H^s$ and

$$||A(V(\cdot))||_s \leq C_s |A|_s (1 + ||V||_s^s).$$

The proof of this lemma can be found in [10] and [20]. In fact, (a) is a slight modification of Lemma 2.1 in [10], while (b) and (c) follow directly from (a) (see [20]). See also Proposition 2.1 in [14].

Using the above lemma, we can easily prove (see [20])

Lemma 5.2. *Let $s \geq s_0$ be an integer.*

(a). Both $C\big([0,T], H^s\big)$ and $C\big([0,T], H^s\big) \cap C^1\big([0,T], H^{s-1}\big)$ are algebra.

(b). Assume $F \in C^s(G)$ and $V \in C\big([0,T], H^s\big)$ with values in a compact subset of G. Then $F(V) \in C\big([0,T], H^s\big)$. Moreover, $F(V) \in C\big([0,T], H^s\big) \cap C^1\big([0,T], H^{s-1}\big)$ if so is V and $F \in C^{s+1}(G)$.

Our further arguments are based on the following assumption on the two solutions respectively to the limiting inner problem and the reduced problem.

Assumption: (1). The limiting inner problem (4.13) with $\tilde{I}_0(x,0) = \bar{U}(x,0)$ has a unique solution $\tilde{I}_0(x,\tau)$ in $C\big([0,+\infty), H^{s+2}\big)$, which decays exponentially to $\tilde{I}_0(x,+\infty) \in \mathcal{E}$ in H^{s+2} as τ goes to infinity. Here $s \geq s_0 + 1$.

(2). The reduced problem (4.4)-(4.7) with $U(x,0) = \tilde{I}_0(x,+\infty)$ has a unique solution $U_0(x,t)$ in $C\big([0,T_0], H^{s+2}\big) \cap C^1\big([0,T_0], H^{s+1}\big) \cap C^2\big([0,T_0], H^s\big)$.

(3). $\Big\{U_0(x,t) + \theta I_0(x,\tau) : (x,t,\tau,\theta) \in \Omega \times [0,T_0] \times [0,+\infty) \times [0,1]\Big\} \subset\subset G.$

We also assume that $A_j (j = 1, 2, \cdots, d), Q, \hat{S}$ and P are in $C^{s+2}(G)$ and $\bar{U}_1(x)$ in (4.15) belongs to H^s.

Let us first analyse U_1. On the basis of the above assumption, we use Lemma 5.2 to see that

$$A_j(U_0), \hat{S}(U_0), P(U_0) \in C\big([0,T_0], H^{s+1}\big) \cap C^1\big([0,T_0], H^s\big).$$

Thus, it follows from (4.8) and Lemma 5.2 that

$$P^{II}(U_0)U_1 \in C\big([0,T_0], H^{s+1}\big) \cap C^1\big([0,T_0], H^s\big). \tag{5.1}$$

On the other hand, recall from (4.10) that $w \equiv P^I(U_0)U_1$ satisfies a semilinear symmetrizable hyperbolic system

$$w_t + \sum_{j=1}^{d} a_j(x,t) w_{x_j} = f(w,x,t), \tag{5.2}$$

where each component of $f(w,x,t)$ is a polynomial of w. Using Lemma 5.2 and (5.1) we can easily show that the given functions in (5.2) satisfy the conditions of Theorem III.3 in [20]. Thus, we assume $w(x,0) \in H^s$ for the moment and conclude from the theorem that there is a positive constant T_1 such that the above problem has a unique classical solution

$$w \equiv P^I(U_0)U_1 \in C\big([0,T_1], H^s\big) \cap C^1\big([0,T_1], H^{s-1}\big).$$

In view of (5.1), we have

$$U_1 \in C\big([0,T_1], H^s\big) \cap C^1\big([0,T_1], H^{s-1}\big). \tag{5.3}$$

Furthermore, it follows from (4.6) that

$$F_1 \in C\big([0,T_1], H^{s-1}\big). \tag{5.4}$$

Next, we consider the equation in (4.14) related to $I_1 = P^{-1}(U_0(x,0))V - U_1(x,0)$ and choose $w(x,0) \equiv P^I(U_0(x,0))U_1(x,0) \in H^s$. To this end, we rewrite the equation as

$$\begin{aligned} \frac{dV(x,\tau)}{d\tau} =& S(x,\tau)V(x,\tau) - S(x)P(U_0(x,0))U_1(x,0) + F(x,\tau), \\ V(x,0) =& P(U_0(x,0))\bar{U}_1(x,0), \end{aligned} \tag{5.5}$$

where

$$\begin{aligned} S(x) =\ & \mathrm{diag}(0, \hat{S}(U_0(x,0))), \\ F(x,\tau) =\ & \tau P(U_0(x,0))[Q_U(\tilde{I}_0) - Q_U(U_0(x,0))]U_{0t}(x,0) - P(U_0(x,0)) \textstyle\sum_{j=1}^{d} A_j(\tilde{I}_0) I_{0x_j} \\ & -P(U_0(x,0)) \textstyle\sum_{j=1}^{d} [A_j(\tilde{I}_0) - A_j(U_0(x,0))]U_{0x_j}(x,0). \end{aligned}$$

Since $\tilde{I}_0(x,\tau) \in C\big([0,\infty), H^{s+1}\big)$, it follows from Lemma 5.2 that $Q_U(\tilde{I}_0(x,\tau))$ and thereby $S(x,\tau) \equiv P(U_0(x,0))Q_U(\tilde{I}_0(x,\tau))P^{-1}(U_0(x,0))$ is in $C\big([0,\infty), H^{s+1}\big)$. Moreover, by using the assumption, we can also show $F(x,\tau) \in C\big([0,\infty), H^{s+1}\big)$. Then, it is easy to see that the above problem has a unique solution in $C^1\big([0,+\infty), H^{s+1}\big)$ if an initial value in H^{s+1} is prescribed. In view of (5.3), we have $I_1 \in C^1\big([0,+\infty), H^s\big)$. Thus, combining (5.3) and the assumption shows that for each $\epsilon > 0$,

$$U_\epsilon \in C\big([0,T_1], H^s\big) \cap C^1\big([0,T_1], H^{s-1}\big). \tag{5.6}$$

Moreover, from (4.16) and (5.4) it follows that for each $\epsilon > 0$,

$$F_2 \in C\big([0, T_1], H^{s-1}\big). \tag{5.7}$$

Now we show that $||I_1(\tau)||_s$ decays exponentially to zero as τ goes to infinity. To do so, we first estimate $||S(\cdot,\tau) - S(\cdot)||_s$ for fixed τ. Since $Q_{UU} \in C^s(G)$ and $U_0(x,0) + \theta I_0(x,\tau) \in C\big([0,1], H^s\big)$, we have $Q_{UU}\big(U_0(x,0) + \theta I_0(x,\tau)\big) \in C\big([0,1], H^s\big)$. Thus, $\int_0^1 Q_{UU}\big(U_0(x,0) + \theta I_0(x,\tau)\big) d\theta \in H^s$ and from Lemma 5.1.c

$$\Big|\Big| \int_0^1 Q_{UU}\big(U_0(x,0) + \theta I_0(x,\tau)\big) d\theta \Big|\Big|_s \leq C_s |Q_{UU}|_s \big(1 + ||U_0(0)||_s^s + ||I_0(\tau)||_s^s\big).$$

Because of the assumption, there exist positive constants μ and C such that $||I_0(\tau)||_s \leq Ce^{-\mu\tau}$. Thus, we obtain from Lemma 5.1.a

$$\begin{aligned} &||S(\cdot,\tau) - S(\cdot)||_s \equiv ||P(U_0(0))[Q_U(\tilde{I}_0(\tau)) - Q_U(U_0(0))]P^{-1}(U_0(0))||_s \\ \leq& C_s ||P(U_0(0))||_s ||P^{-1}(U_0(0))||_s \Big|\Big| \int_0^1 Q_{UU}\big(U_0(x,0) + \theta I_0(x,\tau)\big) d\theta \Big|\Big|_s ||I_0(\tau)||_s \\ \leq& Ce^{-\mu\tau}. \end{aligned} \tag{5.8}$$

Similarly, we can show that $||F(\tau)||_s$ decays exponentially to zero as τ goes to infinity.

With the above μ, we define $\tilde{V} = Ve^{-\mu\tau}$ and deduce from (5.5) that $\tilde{V}$ satisfies

$$\frac{d\tilde{V}}{d\tau} = [S(x,\tau) - \mu I]\tilde{V} - e^{-\mu\tau} S(x) P(U_0(0,x)) U_1(x,0) + e^{-\mu\tau} F(x,\tau). \tag{5.9}$$

Since $\hat{S}(x) \equiv \hat{S}(U_0(x,0))$ is a stable matrix, there is a positive definite Hermitian matrix $\hat{E}(x)$ such that

$$\hat{E}(x)\hat{S}(x) + \hat{S}^*(x)\hat{E}(x) \leq -I_r. \tag{5.10}$$

Thus, with $E(x) = \mathrm{diag}(I_{n-r}, \hat{E}(x))$, we have

$$E(x)[S(x) - \mu I] + [S(x) - \mu I]^* E(x) \leq -\mathrm{diag}(-2\mu I_{n-r}, -I_r - 2\mu \hat{E}(x)).$$

On the other hand, note that $|S(\cdot,\tau) - S(\cdot)|_0$ tends to zero as τ goes to infinity. Then for any $\eta > 0$, there is $\tau_0 > 0$ such that

$$E(x)[S(x,\tau) - S(x)] + [S(x,\tau) - S(x)]^* E(x) \leq \eta I$$

for $\tau \geq \tau_0$ and all x. Thus, we have

$$E(x)[S(x,\tau) - \mu I] + [S(x,\tau) - \mu I]^* E(x) \leq -I \tag{5.11}$$

if η is small enough.

At this point we establish the following

Lemma 5.3. *Suppose $f(x,\tau) \in C\big([0,\infty), L^2\big)$, $||f(\tau)||$ decays exponentially to zero as τ goes to infinity, and $E(x) \in L^\infty$ is a uniformly positive definite Hermitian matrix such that for all sufficiently large τ and all x,*

$$E(x)A(x,\tau) + A^*(x,\tau)E(x) \leq -I.$$

If $V(x,\tau)\in C\big([0,\infty),L^2\big)$ *satisfies*

$$\frac{dV}{d\tau}=A(x,\tau)V+f(x,\tau), \tag{5.12}$$

then $||V(\tau)||$ *decays exponentially to zero as* τ *goes to infinity. Moreover, if* $V(x,\tau)$, $f(x,\tau)\in C\big([0,\infty),H^s\big)$ *and* $||f(\tau)||_s$ *decays exponentially to zero as* τ *goes to infinity, then* $||V(\tau)||_s$ *decays exponentially to zero as* τ *goes to infinity.*

Proof. According to the conditions, there exist positive constants μ,τ_0 such that

$$E(x)A(x,\tau)+A^*(x,\tau)E(x)\le -I,\qquad ||f(\tau)||^2\le\tau_0e^{-\mu\tau},\qquad |E|_0\le\tau_0 I$$

for $\tau\ge\tau_0$ and all x. Multiplying the equation in (5.12) with $V^*(x,\tau)E(x)$, taking the real parts and integrating over Ω, we obtain, for $\tau\ge\tau_0$,

$$\begin{aligned}\frac{d}{d\tau}(V(\tau),EV(\tau))\le&\ -(V(\tau),V(\tau))+2\mathrm{Re}(V(\tau),Ef(\tau))\\ \le&\ -\frac{(V(\tau),V(\tau))}{2}+2|E|_0^2||f(\tau)||^2\\ \le&\ -\frac{1}{2\tau_0}(V(\tau),EV(\tau))+2\tau_0^2\tau_0e^{-\mu\tau}.\end{aligned}$$

Thus,

$$(V(\tau),EV(\tau))\le\exp\big(\frac{-\tau}{2\tau_0}\big)\sqrt{e}(V(\tau_0),EV(\tau_0))+2\tau_0^3\exp\big(\frac{-\tau}{2\tau_0}\big)\int_{\tau_0}^{\tau}\exp\big(\frac{\sigma}{2\tau_0}-\mu\sigma\big)d\sigma.$$

This and the uniform positiveness of $E(x)$ together imply the exponential decay of $||V(\tau)||$ to zero.

For the other statement, note that for all multi-indices α such that $|\alpha|\le s$,

$$||A(\tau)V_\alpha(\tau)-\big(A(\tau)V(\tau)\big)_\alpha||\le C_s||A(\tau)||_s||V(\tau)||_{|\alpha|-1}$$

due to Lemma 5.1.b. Therefore, we may use induction on $|\alpha|$ to prove the exponential decay of $||V(\tau)||_s$ to zero. This completes the proof. $\square$

Because of the inequality in (5.11), we apply Lemma 5.3 to the equation in (5.9) to see that $||\tilde V(\tau)||_s$ decays exponentially to zero as τ goes to infinity. Set $W=V-P(U_0(x,0))U_1(x,0)$ and rewrite the equation in (5.5) as

$$\frac{dW(x,\tau)}{d\tau}=S(x)W(x,\tau)+\underbrace{[S(x,\tau)-S(x)]V(x,\tau)+F(x,\tau)}_{f(x,\tau)}.$$

Since the first $(n-r)$ rows of $S(x)$ vanishes and $||f(\tau)||_s$ decays exponentially to zero as τ goes to infinity, $\Big|\Big|\dfrac{dV^I(\tau)}{d\tau}\Big|\Big|_s$ decays exponentially to zero as τ goes to infinity. Here we have used the superscript I (II) to denote the first $(n-r)$ (last r) components of a vector. Therefore, it is clear that $V_\infty^I(x)\equiv\lim\limits_{\tau\to\infty}V^I(x,\tau)$ exists in H^s. Thus, we take $P^I(U_0(x,0))U_1(x,0)=V_\infty^I(x)$. Moreover,

$$\frac{dW^{II}(x,\tau)}{d\tau}=\hat S(x)W^{II}(x,\tau)+f^{II}(x,\tau).$$

In view of (5.10), we apply Lemma 5.3 to the last equation to see that $||W^{II}(\tau)||_s$ decays exponentially to zero as τ goes to infinity. Consequently, we arrive at

$$(5.13) \qquad ||I_1(\tau)||_s \equiv ||P^{-1}(U_0(0))W(\tau)||_s \leq Ce^{-\mu\tau}.$$

Having (5.3), (5.6) and the estimate on I_1, we write out the expressions of F_2. Then as in obtaining (5.8), we get

$$(5.14) \qquad ||F_2(t,\epsilon)||_{s-1} \leq C\epsilon + Ce^{-\mu\tau}$$

for sufficiently small ϵ.

In conclusion, we have estimates in (5.6) for U_ϵ, (4.9) + (5.4) for F_1 and (5.7) + (5.14) for F_2.

6. Validity of Expansions and Existence

Having constructed a sufficiently regular asymptotic solution U_ϵ to the initial value problem in (1.1), we prove here the existence of a smooth solution in the ϵ-independent time interval, where U_ϵ exists, and the validity of the expansion solution under the stability condition and similar regularity assumptions as that in the previous section.

For the sake of exactness, we refer to the results in the previous section and make the following assumption for U_ϵ, F_1 and F_2 in (4.16) throughout this section. To this end, let s be an integer such that $s \geq s_0 + 1$ and denote by $\mathcal{R}(V)$ the range of $V = V(x,t,\epsilon)$ for $(x,t,\epsilon) \in \Omega \times [0,T_1] \times [0,\epsilon_0]$. Here ϵ_0 is a small positive constant and T_1 is such that the asymptotic solution U_ϵ exists in $[0,T_1]$ (see (5.3)).

Assumption A: (1). $A_j (j = 0,1,2,\cdots,d), Q \in C^\infty(G)$ and $\bar{U}(\cdot,\epsilon) \in H^s$ is periodic in x with period $(1,1,\cdots,1) \in \mathbf{R}^d$.

(2). The reduced problem has a unique solution $U_0 \in C\big([0,T_1],H^s\big)$ such that $P(U_0) \in C\big([0,T_1],H^{s+1}\big) \cap C^1\big([0,T_1],H^s\big)$, $P^{-1}(U_0) \in C\big([0,T_1],H^s\big)$ and $F_1 \in C\big([0,T_1],H^s\big)$.

(3). $U_\epsilon \in C\big([0,T_1],H^{s+1}\big)$ for each ϵ, $||\bar{U}(\cdot,\epsilon) - U_\epsilon(0)||_s = O(\epsilon^{3/2})$ and $||U_\epsilon(t)||_{s+1}$ is uniformly bounded with respect to ϵ and t.

(4). There is a convex open set G_0 such that $\mathcal{R}(U_0) \cup \mathcal{R}(U_\epsilon) \subset G_0 \subset\subset G$.

(5). $F_2 \in C\big([0,T_1],H^s\big)$ for each ϵ and $||F_2(t,\epsilon)||_s \leq C\epsilon + Ce^{-\mu\tau}$ with μ being a positive constant.

From the construction of U_ϵ, we can see that there is a *boundary-layer* function $B_\epsilon(t)$, a function satisfying

$$\sup_\epsilon \int_0^{T_1} B_\epsilon(t)dt < \infty,$$

such that

$$(6.1) \qquad \frac{||U_\epsilon(t) - U_0(t)||_s}{\epsilon} \leq B_\epsilon(t).$$

On the other hand, because of the well-known embedding inequality

$$(6.2) \qquad |\cdot|_0 \leq C_{s_0}||\cdot||_{s_0},$$

U_0 and U_ϵ are continuously differentiable due to the above assumption and the corresponding equations.

Fix $\epsilon > 0$ and assume $\mathcal{R}(\bar{U}) \subset\subset G_0$. Then, according to Theorem 2.1 in [14], there exists $T_\epsilon > 0$ such that the initial value problem in (1.1) for the symmetrizable hyperbolic system has a unique classical solution U^ϵ satisfying $U^\epsilon(x,t) \in \bar{G}_0$ for $(x,t) \in \Omega \times [0, T_\epsilon]$ and

$$U^\epsilon \in C([0,T_\epsilon], H^s) \cap C^1([0,T_\epsilon], H^{s-1}).$$

Namely,

$$\begin{aligned} U_t^\epsilon + \sum_{j=1}^{d} A_j(U^\epsilon)U_{x_j}^\epsilon =& \frac{Q(U^\epsilon)}{\epsilon}, \\ U(x,0) =& \bar{U}(x,\epsilon). \end{aligned} \tag{6.3}$$

Without loss of generality, we assume that $[0, T_\epsilon]$ is the *maximal time interval* where the H^s-solution U^ϵ exists and $U^\epsilon(x,t) \in \bar{G}_0$ for $(x,t) \in \Omega \times [0, T_\epsilon]$. Note that T_ϵ may tend to zero as so does ϵ.

In order to show $T_\epsilon \geq T_1$, we first prove

Theorem 6.1. *Under the stability condition and the assumption A, there exists a constant K, depending only on the assumption, such that*

$$||U^\epsilon(t) - U_\epsilon(t)||_s \leq K\epsilon^{3/2}$$

for ϵ sufficiently small and $t \in [0, \min\{T_\epsilon, T_1\}]$.

Before proving this theorem, we show that one of its consequences is $T_\epsilon \geq T_1$ if $\mathcal{R}(U_\epsilon) \subset\subset G_0$. In fact, if $T_\epsilon < T_1$, then Theorem 6.1 gives

$$||U^\epsilon(T_\epsilon) - U_\epsilon(T_\epsilon)||_s \leq K\epsilon^{3/2}.$$

Thus, it follows from the embedding inequality in (6.2) and $\mathcal{R}(U_\epsilon(T_\epsilon)) \subset\subset G_0$ that $\mathcal{R}(U^\epsilon(T_\epsilon)) \subset\subset G_0$ if ϵ is small enough. Now we could apply Theorem 2.1 in [14], beginning at the time T_ϵ, to continue this solution beyond T_ϵ. This is a contradiction. Consequently, we have proved

Theorem 6.2. *Under the stability condition and the assumption A, assume $\mathcal{R}(U_\epsilon) \subset\subset G_0$. Then the initial value problem in (1.1) has a unique classical solution U^ϵ, defined in $\Omega \times [0, T_1]$, satisfying $U^\epsilon(x,t) \in G_0$ for $(x,t) \in \Omega \times [0, T_1]$ and*

$$U^\epsilon \in C([0,T_1], H^s) \cap C^1([0,T_1], H^{s-1}).$$

Another consequence of Theorem 6.1 is that we can write

$$U^\epsilon(x,t) = U_0(x,t) + I_0(x,t/\epsilon) + \epsilon U_1(x,t) + \epsilon I_1(x,t/\epsilon) + O(\epsilon^{3/2}) \tag{6.4}$$

due to the form of U_ϵ. This is just the asymptotic expansion of $U^\epsilon(x,t)$.

Now we turn to prove Theorem 6.1 and begin with the following nonlinear Gronwall-type inequality.

Lemma 6.3. *Suppose $\psi(t)$ is a positive C^1-function of $t \in [0,T)$ with $T \leq \infty$, $m > 1$ and $b_1(t), b_2(t)$ are integrable on $[0,T)$. If*

$$\psi'(t) \leq b_2(t)\psi^m(t) + b_1(t)\psi(t),$$

then there exists $\delta > 0$, depending only on m, C_{1b} and C_{2b}, such that

$$\sup_{t\in[0,T)} \psi(t) \le e^{C_{1b}},$$

whenever $\psi(0) \in (0,\delta]$. Here

$$C_{1b} = \sup_{t\in[0,T)} \int_0^t b_1(t')dt' \quad \text{and} \quad C_{2b} = \int_0^T \max\{b_2(t), 0\}dt.$$

Proof. Set $\Phi(t) = \psi^{1-m}(t)$ and compute that $\Phi(t)$ satisfies

$$\Phi'(t) \ge (1-m)(b_2(t) + b_1(t)\Phi(t)) \ge (1-m)(\max\{b_2(t), 0\} + b_1(t)\Phi(t)).$$

Therefore,

$$\begin{array}{rl} & \Phi(t)\exp\left((m-1)\int_0^t b_1(t')dt'\right) - \Phi(0) \\ \ge & (1-m)\int_0^t \max\{b_2(t'), 0\}dt' \exp\left((m-1)\int_0^{t'} b_1(t'')dt''\right). \end{array}$$

Since

$$\int_0^t b_1(t')dt' \le C_{1b} \quad \text{and} \quad \int_0^t \max\{b_2(t'), 0\}dt' \le C_{2b},$$

we deduce that

$$e^{(m-1)C_{1b}}\Phi(t) \ge \psi^{1-m}(0) - (m-1)C_{2b}e^{(m-1)C_{1b}} \tag{6.5}$$

By choosing δ so small that $\delta^{1-m} - (m-1)C_{2b}e^{(m-1)C_{1b}} \ge 1$, (6.5) directly leads to the conclusion of the lemma and the proof is complete. □

The Proof of Theorem 6.1

Recall the equation in (4.16) and set $\tilde{W} = U_\epsilon - U^\epsilon$. Then we deduce from (6.3) and (4.16) that

$$\begin{aligned} \tilde{W}_t + \sum_{j=1}^d A_j(U^\epsilon)\tilde{W}_{x_j} =& \ \frac{Q(U_\epsilon) - Q(U^\epsilon)}{\epsilon} + \epsilon F_1 + \epsilon F_2 + \sum_{j=1}^d [A_j(U^\epsilon) - A_j(U_\epsilon)]U_{\epsilon x_j}, \\ \tilde{W}(x,0) =& \ U_\epsilon(x,0) - \bar{U}(x,\epsilon). \end{aligned}$$

Define $W = P(U_0)\tilde{W} \equiv P\tilde{W}$. Then W satisfies

$$\begin{aligned} W_t \ &+ \sum_{j=1}^d PA_j(U^\epsilon)P^{-1}W_{x_j} = \frac{P[Q(U_\epsilon) - Q(U^\epsilon)]}{\epsilon} + \epsilon PF_1 + \epsilon PF_2 \\ &+ P\sum_{j=1}^d [A_j(U^\epsilon) - A_j(U_\epsilon)]U_{\epsilon x_j} + [P_t + \sum_{j=1}^d PA_j(U^\epsilon)P^{-1}P_{x_j}]P^{-1}W, \\ &W(x,0) = P[U_\epsilon(x,0) - \bar{U}(x,\epsilon)]. \end{aligned}$$

Differentiating the last equation with ∂^α for $|\alpha| \le s$ and setting $W_\alpha = \partial^\alpha W$, we get

$$\begin{aligned} W_{\alpha t} + \sum_{j=1}^d PA_j(U^\epsilon)P^{-1}W_{\alpha x_j} &= \frac{PQ_U(U_0)P^{-1}}{\epsilon}W_\alpha + F_1^\alpha + F_2^\alpha, \\ W_\alpha(x,0) &= \left\{P[U_\epsilon(x,0) - \bar{U}(x,\epsilon)]\right\}_\alpha. \end{aligned} \tag{6.6}$$

Here

$$
\begin{aligned}
F_1^\alpha = &\ \epsilon(PF_1)_\alpha + \frac{[PQ_U(U_0)P^{-1}W]_\alpha - PQ_U(U_0)P^{-1}W_\alpha}{\epsilon},\\
F_2^\alpha = &\ \epsilon(PF_2)_\alpha + \frac{\{P[Q(U_\epsilon) - Q(U^\epsilon)] - PQ_U(U_0)P^{-1}W\}_\alpha}{\epsilon} +\\
&+ \big\{P\sum_{j=1}^d [A_j(U^\epsilon) - A_j(U_\epsilon)]U_{\epsilon x_j}\big\}_\alpha + \big\{[P_t + \sum_{j=1}^d PA_j(U^\epsilon)P^{-1}P_{x_j}]P^{-1}W\big\}_\alpha\\
&+ \sum_{j=1}^d \big\{PA_j(U^\epsilon)P^{-1}W_{\alpha x_j} - [PA_j(U^\epsilon)P^{-1}W_{x_j}]_\alpha\big\}\\
\equiv &\ f_1^\alpha + f_2^\alpha + f_3^\alpha + f_4^\alpha + f_5^\alpha.
\end{aligned}
$$

To be easy of understanding, we group the following arguments into lemmas.

Lemma 6.4. *Under the stability condition and the assumption A, we have*

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega e(W_\alpha(x,t))dx + \frac{||W_\alpha^{II}(t)||^2}{2\epsilon} \le C\epsilon^3 + \frac{C||W^{II}(t)||^2_{|\alpha|-1}}{\epsilon} + C||W_\alpha(t)||||F_2^\alpha(t)||\\
+C\int_\Omega \Big\{1 + |U_t^\epsilon(x,t)| + \sum_{j=1}^d |U_{x_j}^\epsilon(x,t)| + \frac{|U^\epsilon(x,t) - U_0(x,t)|}{\epsilon}\Big\}|W_\alpha(x,t)|^2 dx.
\end{aligned}
$$

Here $e(W_\alpha) = W_\alpha^* P^{-*}A_0(U^\epsilon)P^{-1}W_\alpha$, $||\cdot||_{-1} = 0$ *and* C *is a generic constant depending only on the assumption A.*

Proof. Recall that $A_0(U^\epsilon)$ and $A_0(U^\epsilon)A_j(U^\epsilon)(j = 1, 2, \cdots, d)$ are all symmetric. Multiplying the equation in (6.6) with $W_\alpha^* P^{-*}A_0(U^\epsilon)P^{-1}$ from the left, we get

$$
\begin{aligned}
&e(W_\alpha)_t + \sum_{j=1}^d \big\{W_\alpha^* P^{-*}A_0(U^\epsilon)A_j(U^\epsilon)P^{-1}W_\alpha\big\}_{x_j}\\
&= \frac{2}{\epsilon}\mathrm{Re}W_\alpha^* P^{-*}A_0(U_0)Q_U(U_0)P^{-1}W_\alpha + 2\mathrm{Re}W_\alpha^* P^{-*}A_0(U^\epsilon)P^{-1}F_1^\alpha\\
&\quad + \frac{2}{\epsilon}\mathrm{Re}W_\alpha^* P^{-*}[A_0(U^\epsilon) - A_0(U_0)]Q_U(U_0)P^{-1}W_\alpha + 2\mathrm{Re}W_\alpha^* P^{-*}A_0(U^\epsilon)P^{-1}F_2^\alpha\\
&\quad + W_\alpha^*\Big\{\frac{\partial[P^{-*}A_0(U^\epsilon)P^{-1}]}{\partial t} + \sum_{j=1}^d \frac{\partial[P^{-*}A_0(U^\epsilon)A_j(U^\epsilon)P^{-1}]}{\partial x_j}\Big\}W_\alpha.
\end{aligned}
\tag{6.7}
$$

Now let us analyse the right-hand side of (6.7) term-by-term.

$$
\frac{\partial[P^{-*}A_0(U^\epsilon)P^{-1}]}{\partial t} + \sum_{j=1}^d \frac{\partial[P^{-*}A_0(U^\epsilon)A_j(U^\epsilon)P^{-1}]}{\partial x_j} \le C(1 + |U_t^\epsilon| + \sum_{j=1}^d |U_{x_j}^\epsilon|).
\tag{6.8}
$$

$$
2\mathrm{Re}W_\alpha^* P^{-*}A_0(U^\epsilon)P^{-1}F_2^\alpha \le C|W_\alpha||F_2^\alpha|.
\tag{6.9}
$$

$$
2\mathrm{Re}W_\alpha^* P^{-*}[A_0(U^\epsilon) - A_0(U_0)]Q_U(U_0)P^{-1}W_\alpha \le C|U^\epsilon - U_0||W_\alpha|^2.
\tag{6.10}
$$

Since

$$
A_0(U_0)Q_U(U_0) + Q_U^*(U_0)A_0(U_0) \le -P^*\begin{pmatrix} 0 & 0\\ 0 & I_r\end{pmatrix}P,
$$

we have

$$2\mathrm{Re}W_\alpha^* P^{-*} A_0(U_0) Q_U(U_0) P^{-1} W_\alpha \le -|W_\alpha^{II}|^2. \tag{6.11}$$

In addition, since $P^I F_1 \equiv 0$ (see (4.9)) and

$$PQ_U(U_0)P^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \hat{S}(U_0) \end{pmatrix},$$

the first $(n-r)$ components F_1^α vanish. Moreover, $P^{-*}A_0(U_0)P^{-1}$ is of the block-diagonal form (Lemma 2.2). Thus, we have

$$\begin{aligned}
&2\mathrm{Re}W_\alpha^* P^{-*} A_0(U^\epsilon) P^{-1} F_1^\alpha \\
=&2\mathrm{Re}W_\alpha^* P^{-*} A_0(U_0) P^{-1} F_1^\alpha + 2\mathrm{Re}W_\alpha^* P^{-*} [A_0(U^\epsilon) - A_0(U_0)] P^{-1} F_1^\alpha \\
\le&\frac{|W_\alpha^{II}|^2}{2\epsilon} + C\epsilon |F_1^\alpha|^2 + C|W_\alpha||U^\epsilon - U_0||F_1^\alpha| \\
\le&\frac{|W_\alpha^{II}|^2}{2\epsilon} + C\epsilon |F_1^\alpha|^2 + \frac{C|U^\epsilon - U_0|^2}{\epsilon}|W_\alpha|^2 \\
\le&\frac{|W_\alpha^{II}|^2}{2\epsilon} + C\epsilon |F_1^\alpha|^2 + \frac{C|U^\epsilon - U_0|}{\epsilon}|W_\alpha|^2
\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}
||F_1^\alpha(t)|| \le& \epsilon ||(PF_1)_\alpha(t)|| + \frac{1}{\epsilon} ||[PQ_U(U_0)P^{-1}W]_\alpha - PQ_U(U_0)P^{-1}W_\alpha|| \\
\le& C\epsilon + \frac{C||W^{II}(t)||_{|\alpha|-1}}{\epsilon}.
\end{aligned} \tag{6.13}$$

Here we have used Lemma 5.1 and the assumption A.

Having the estimates in (6.8)-(6.13) and using the periodicity, we integrate the two sides of the relation in (6.7) over $x \in \Omega \equiv (0,1]^d$ to conclude the lemma. □

Lemma 6.5. *Let*

$$\Delta(t) = \frac{||U_\epsilon(t) - U^\epsilon(t)||_s}{\epsilon}.$$

Then

$$\begin{aligned}
&|U_t^\epsilon(t)|_0 + \sum_{j=1}^d |U_{x_j}^\epsilon(t)|_0 + \frac{|U^\epsilon(t) - U_0(t)|_0}{\epsilon} \le C + CB_\epsilon(t) + C\Delta(t), \\
&||F_2^\alpha(t)|| \le C\epsilon^2 + C\epsilon e^{-\mu\tau} + C(1 + B_\epsilon(t))(1 + \Delta^{s+1}(t))||W||_{|\alpha|}.
\end{aligned} \tag{6.14}$$

Proof. It follows from the equation for U^ϵ and $Q(U_0) = 0$ that

$$|U_t^\epsilon| \le \sum_{j=1}^d |A_j(U^\epsilon)U_{x_j}^\epsilon| + \frac{|Q(U^\epsilon) - Q(U_0)|}{\epsilon} \le C \sum_{j=1}^d |U_{x_j}^\epsilon| + \frac{C|U^\epsilon - U_0|}{\epsilon}.$$

On the other hand, because $s \ge s_0 + 1$, we use the embedding inequality in (6.2) to obtain

$$\begin{aligned}
|U_{x_j}^\epsilon(x,t)| \le& \ C_{s_0}||U^\epsilon(t)||_s \le C_{s_0}||U^\epsilon(t) - U_\epsilon(t)||_s + C_{s_0}||U_\epsilon(t)||_s \\
\le& \ C_{s_0}\epsilon\Delta(t) + C, \\
|U^\epsilon(x,t) - U_0(x,t)| \le& \ C_{s_0}||U^\epsilon(t) - U_0(t)||_s \le C_{s_0}||U^\epsilon(t) - U_\epsilon(t)||_s + C_{s_0}||U_\epsilon(t) - U_0(t)||_s \\
\le& \ C_{s_0}\epsilon\Delta(t) + C_{s_0}\epsilon B_\epsilon(t).
\end{aligned}$$

Thus we have the first line in (6.14).

For $||F_2^\alpha(t)||$, recall that $F_2^\alpha(t) = \sum_{k=1}^5 f_k^\alpha(t)$. Since $s \geq \max\{|\alpha|, s_0+1\}$, we use Lemma 5.1.a with $s_1 = |\alpha|$ and $s_2 = s$ and the assumption on $||F_2||_s$ to obtain

$$(6.15) \quad ||f_1^\alpha(t)|| \equiv \epsilon||(PF_2)_\alpha(t)|| \leq \epsilon||(PF_2)(t)||_{|\alpha|} \leq C_s\epsilon||P||_s||F_2||_{|\alpha|} \leq C\epsilon^2 + C\epsilon e^{-\mu\tau}.$$

Furthermore, by using Lemma 5.1 we have

$$(6.16) \quad \begin{aligned} ||f_5^\alpha(t)|| \equiv & || \sum_{j=1}^d \{PA_j(U^\epsilon)P^{-1}W_{\alpha x_j} - [PA_j(U^\epsilon)P^{-1}W_{x_j}]_\alpha\}|| \\ \leq & C_s \sum_{j=1}^d ||PA_j(U^\epsilon)P^{-1}||_s||W_{x_j}||_{|\alpha|-1} \\ \leq & C_s \sum_{j=1}^d ||P||_s||A_j(U^\epsilon)||_s||P^{-1}||_s||W||_{|\alpha|} \\ \leq & C \sum_{j=1}^d |A_j|_s(1+||U^\epsilon||_s^s)||W||_{|\alpha|} \\ \leq & C(1+2^s||U_\epsilon||_s^s + 2^s||U^\epsilon - U_\epsilon||_s^s)||W||_{|\alpha|} \\ \leq & C(1+\Delta^s(t))||W||_{|\alpha|}. \end{aligned}$$

For the last two inequalities, we have used the elementary inequality $(a+b)^s \leq 2^s(a^s+b^s)$ for $a, b \geq 0$ and that $||U_\epsilon||_s$ is uniformly bounded with respect to ϵ. Moreover,

$$(6.17) \quad ||f_4^\alpha(t)|| \equiv ||\{[P_t + \sum_{j=1}^d PA_j(U^\epsilon)P^{-1}P_{x_j}]P^{-1}W\}_\alpha|| \leq C(1+\Delta^s(t))||W||_{|\alpha|}.$$

Note that $\sup_{\epsilon,t}||U_\epsilon(t)||_{s+1} < \infty$ and

$$A_j(U^\epsilon) - A_j(U_\epsilon) = -\int_0^1 A_{jU}(U_\epsilon + \theta(U^\epsilon - U_\epsilon))d\theta P^{-1}W.$$

Then

$$(6.18) \quad ||f_3^\alpha(t)|| \equiv ||\{P\sum_{j=1}^d [A_j(U^\epsilon) - A_j(U_\epsilon)]U_{\epsilon x_j}\}_\alpha|| \leq C(1+\Delta^s(t))||W||_{|\alpha|}.$$

To estimate $f_2^\alpha(t)$, set $U(\theta) = U_\epsilon - U_0 + (1-\theta)(U^\epsilon - U_\epsilon)$. Then

$$||U(\theta)||_s \leq ||U_\epsilon - U_0||_s + ||U^\epsilon - U_\epsilon||_s \leq \epsilon(B_\epsilon(t) + \Delta(t)),$$
$$P[Q(U_\epsilon) - Q(U^\epsilon)] - PQ_U(U_0)P^{-1}W = P\int_0^1\int_0^1 U(\theta)Q_{UU}(U_0 + \tau U(\theta))d\tau d\theta P^{-1}W$$

and therefore

$$(6.19) \quad \begin{aligned} ||f_2^\alpha(t)|| \equiv & \frac{||\{P[Q(U_\epsilon) - Q(U^\epsilon)] - PQ_U(U_0)P^{-1}W\}_\alpha||}{\epsilon} \\ \leq & \frac{C}{\epsilon}\int_0^1\int_0^1 ||U(\theta)||_s||Q_{UU}(U_0 + \tau U(\theta))||_s d\theta d\tau ||W||_{|\alpha|} \\ \leq & C(B_\epsilon(t) + \Delta(t))(1+\Delta^s(t))||W||_{|\alpha|}. \end{aligned}$$

Combining the estimates in (6.15)-(6.19) and using the elementary inequality $a^k \le 1+a^{s+1}$ for $a \ge 0$ and $0 \le k \le s+1$ yields the second line in (6.14). $\square$

Substituting the estimates in (6.14) into the inequality in Lemma 6.4, we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega e(W_\alpha(x,t))dx + \frac{||W_\alpha^{II}(t)||^2}{2\epsilon} \le & \quad C\epsilon^3 + C\epsilon^2 e^{-2\mu\tau} + \frac{C||W^{II}(t)||^2_{|\alpha|-1}}{\epsilon} \\ & +C(1+B_\epsilon(t))(1+\Delta^{s+1}(t))||W||^2_{|\alpha|}. \end{aligned}$$

Note that $C^{-1}|W_\alpha|^2 \le e(W_\alpha) \le C|W_\alpha|^2$, $||W_\alpha(0)||^2 \le C||\bar{U}(\cdot,\epsilon) - U_\epsilon(0)||^2_{|\alpha|} \le C\epsilon^3$ and $\int_0^T e^{-2\mu\tau}dt \le \dfrac{\epsilon}{2\mu}$. Integrating the last inequality for $t \in [0,T] \subset [0, \min\{T_\epsilon, T_1\}]$ gives

$$\begin{aligned} ||W_\alpha(T)||^2 \quad & +\frac{1}{\epsilon}\int_0^T ||W_\alpha^{II}(t)||^2 dt \le C(1+T)\epsilon^3 + \frac{C}{\epsilon}\int_0^T ||W^{II}(t)||^2_{|\alpha|-1}dt \\ & +C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))||W(t)||^2_{|\alpha|}dt. \end{aligned}$$

Let k be an integer such that $0 \le k \le s$. We sum up the last inequality for all α with $0 \le |\alpha| \le k$ to obtain

$$\begin{aligned} ||W(T)||_k^2 + \frac{1}{\epsilon}\int_0^T ||W^{II}(t)||_k^2 dt \le C(1+T)\epsilon^3 + \frac{C}{\epsilon}\int_0^T ||W^{II}(t)||^2_{k-1}dt \\ + C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))||W(t)||_k^2 dt. \end{aligned} \tag{6.20}$$

A simple iteration based on this inequality and $||\cdot||_{-1} = 0$ leads to

$$\frac{1}{\epsilon}\int_0^T ||W^{II}(t)||_k^2 dt \le C(1+T)\epsilon^3 + C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))||W(t)||_k^2 dt. \tag{6.21}$$

Thus, it follows from (6.20) with $k = s$ that

$$||W(T)||_s^2 \le C(1+T_1)\epsilon^3 + C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))||W(t)||_s^2 dt.$$

Applying Gronwall's lemma to the last inequality leads to

$$||W(T)||_s^2 \le C\epsilon^3(1+T_1)\exp\Big(C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))dt\Big). \tag{6.22}$$

Denote by $\epsilon^2\phi(T)$ the right-hand side of (6.22), that is,

$$\phi(T) = C(T_1+1)\epsilon\exp\Big(C\int_0^T (1+B_\epsilon(t))(1+\Delta^{s+1}(t))dt\Big).$$

Recall that $\epsilon\Delta(t) \equiv ||U_\epsilon(t) - U^\epsilon(t)||_s \le C||W(t)||_s$. Then $\Delta^2(t) \le C\phi(t)$ due to (6.22) and $\phi(0) = C(T_1+1)\epsilon$. Moreover,

$$\phi'(t) \le C(1+B_\epsilon(t))\phi(t) + C(1+B_\epsilon(t))\phi^{(s+3)/2}(t).$$

Recall that $\sup_\epsilon \int_0^{T_1} B_\epsilon(t)dt < \infty$. By Lemma 6.3, we have

$$\Delta^2(t) \le C\phi(t) \le C \exp\big(C\int_0^{T_1}(1+B_\epsilon(t))dt\big)$$

if ϵ is so small that $\phi(0) = C(T_1+1)\epsilon \le \delta$. Thus, $\Delta(t)$ is uniformly bounded and it follows from (6.22) that

$$\sup_{t\in[0,T]} ||W(t)||_s^2 \le C(T_1+1)\exp\big(C(T_1+1)\big)\epsilon^3.$$

This completes the proof.

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