High-Resolution Finite Volume Methods for Acoustics in a Rapidly-Varying Heterogeneous Medium*

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Abstract

Behavior of acoustic waves in a rapidly-varying heterogeneous medium may be investigated analytically via various homogenization theories. Alternatively, one can solve the heterogeneous equations directly with a numerical method. Although a numerical solution can be difficult with standard methods, we have found the equations can be solved directly using the high-resolution multi-dimensional finite volume methods of CLAWPACK, which are based on solving "Riemann Problems".

The acoustics equations in a heterogeneous medium with density $\rho(x)$ and bulk modulus $\mu(x)$ may be written as the variable-coefficient hyperbolic system

$$q_t + A(x)q_x = 0, \tag{1}$$

where

$$q = \begin{bmatrix} p \\ u \end{bmatrix}, \quad A = \begin{bmatrix} 0 & \mu(x) \\ 1/\rho(x) & 0 \end{bmatrix}. \tag{2}$$

CLAWPACK (Conservation LAWs PACKage) is a high-resolution finite volume method based on the solving of a Riemann problem at each grid point in order to calculate the left and right going fluctuations [2], [3].

Let Q_i denote the cell average of q over the i'th cell. Let ρ_i, μ_i be the material parameters in this cell and $c_i = \sqrt{\mu_i/\rho_i}$ the wave speed. The "Riemann problem" at the interface between gird cells $i-1$ and i, say, consists of the equation (2) with piecewise constant initial data given by the cell averages Q_{i-1} and Q_i in the adjacent cells. The solution is found by decomposing

$$Q_i - Q_{i-1} = \alpha^1 \begin{bmatrix} -\rho_{i-1}c_{i-1} \\ 1 \end{bmatrix} + \alpha^2 \begin{bmatrix} \rho_i c_i \\ 1 \end{bmatrix} \equiv \mathcal{W}^1 + \mathcal{W}^2$$

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into a left-going wave $\mathcal{W}^1$ propagating with speed $-c_{i-1}$ and a right-going wave $\mathcal{W}^2$ propagating with speed c_i. These are simply eigenvectors of A_{i-1} and A_i respectively. The cell averages are updated on the basis of these waves, using "limiter functions" to eliminate nonphysical oscillations and numerical dispersion.

This decomposition into waves automatically handles reflection and transmission at interfaces between regions of different impedence ρc. Hence this method produces high-quality results for problems with discontinuous coefficients. A simple example of acoustics in a region with a single discontinuity in ρ was given in [2].

Recently we have investigated the question of whether this method can be applied successfully when there are many discontinuities, in a rapidly varying or even random medium. The answer appears to be affirmative and we show two sample calculations.

Example 1. Figure 1 shows a Gaussian pulse that has propagated through a periodic medium for 10 time units. The oscillating square wave shows the variation in $\rho = \mu$ (which varied between 1 and 3). This example is taken from [4], where homogenization theory based on a Bloch wave expansion is used to determine an effective dispersion relation. The effective wave speed for this medium is found to be 0.866 (even though $c \equiv 1$). Figure 1 shows computed results along with the "exact" solution, obtained by the method of characteristics. The two lie on top of one another and the correct wave speed and dispersion are seen.

These results were obtained only after modifying the limiting procedure, as the default procedure used in CLAWPACK led to an interesting numerical instability in this periodic medium. This issue has been studied in [1], where more details and additional examples can be found.

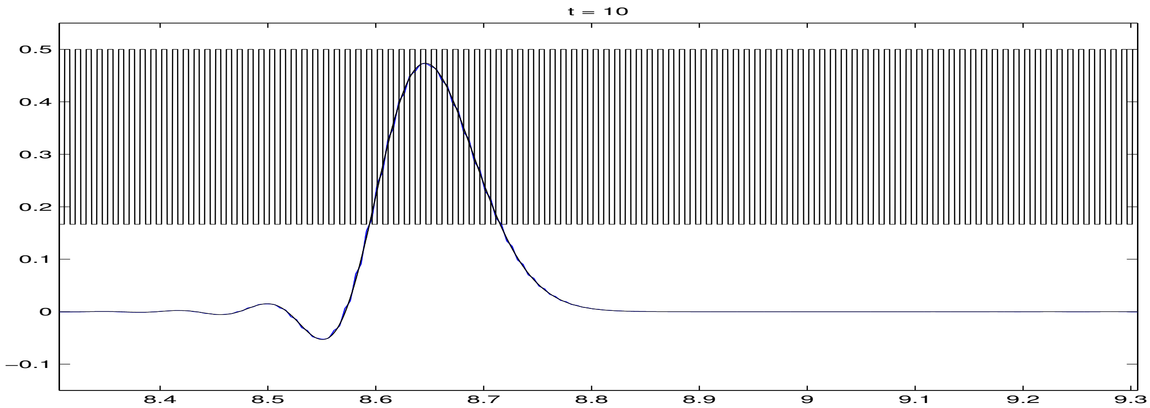

FIG. 1. *comparison of "exact" solution with $\triangle t = \triangle x$ and "numerical" solution with $\triangle t < \triangle x$, right-going pulses, modified limiter, time=10, N=4000. The square-wave density profile is also shown, scaled by 1/6.*

Example 2. Figure 2 shows a plane wave propagating through a two-dimensional random medium. In this case $\mu \equiv 1$ and ρ varies about a piecewise constant mean state $\rho_0(x, t)$ which has a discontinuity from $\rho_0 = 1$ to $\rho_0 = 4$ along a line at an angle to the Cartesian grid. The random noise added to this piecewise constant density is uniformly distributed over $[-\frac{1}{2}\rho_0, \frac{1}{2}\rho_0]$. The density ρ_{ij} in each grid cell is shown in Figure 2(a). The initial plane wave shown in Figure 2(b) strikes the interface, splitting into transmitted and reflected waves as shown in Figures 2(c) and 2(d) at later times (cf. Figure 9 in [2].) Note that in the present case there is reflection at every cell interface, but this noise mostly cancels out and the expected gross behavior is observed along with smaller scale fluctuations.

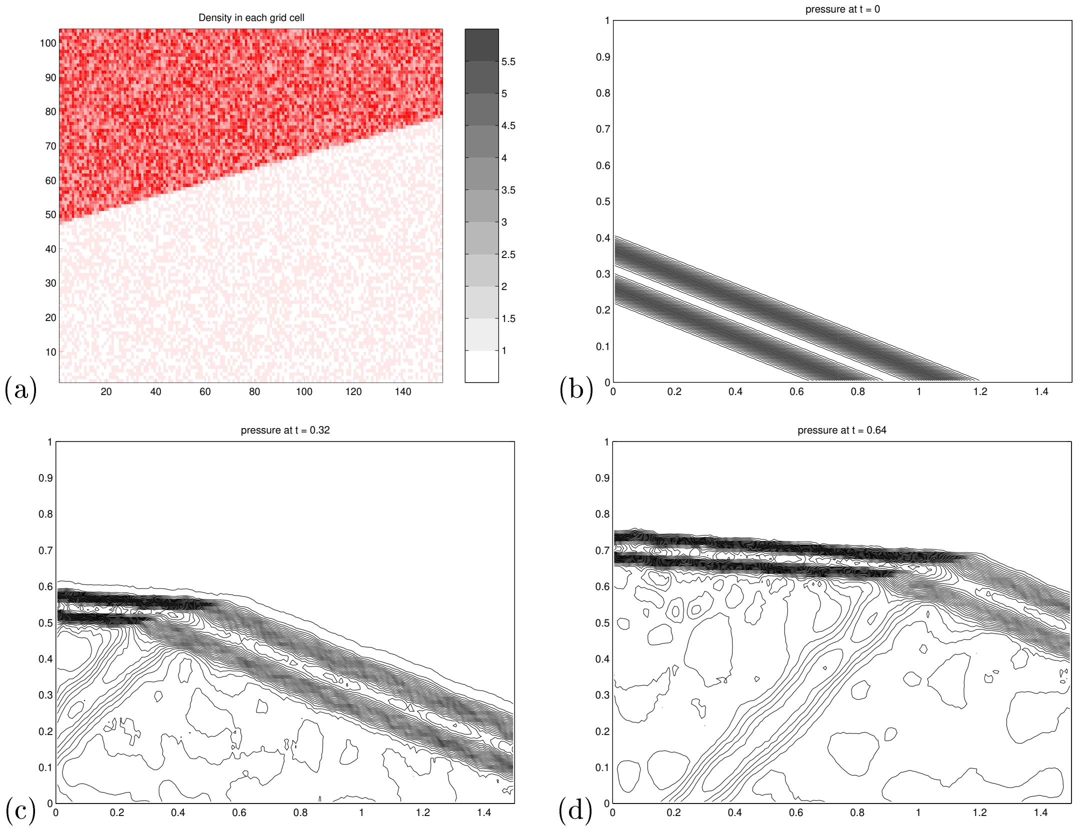

FIG. 2. *An acoustic pulse moving through a medium where the density varies randomly about a piecewise constant state. (a) The density in each grid cell on a* 150×100 *grid. (b) Inital pressure pulse corresponding to a plane wave. (c) Pulse at time* $t = 0.32$. *(d) Pulse at time* $t = 0.64$. *Reflection and refraction at the jump in mean density is clearly visible, in addition to dispersion caused by the random variations. Compare to Figure 9 of [2] which shows the same test problem for a piecewise constant density with no random component.*

References

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