# Blowup Asymptotics for Scalar Conservation Laws with a Source

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**Abstract**

We study asymptotic properties of solutions of scalar conservation laws with a positive superlinear source and with blowup at a single point. In a neighbourhood of the blowup point the asymptotic profile turns out to be different depending on whether blowup occurs along a shock curve or in a region where the solution is smooth. The profile is explicitly calculated in the cases with a powerlike flux and an exponential or powerlike source.

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# 1 Introduction

We study the blowup for nonnegative solutions of the scalar conservation law with source

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = g(u(x,t)), \qquad x \in \mathbb{R},\ t \geq 0 \tag{1.1}$$

with initial value

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{1.2}$$

Here $f$ is assumed convex and the source term $g$ has superlinear growth. Because of the presence of the source, the solution of this problem may become unbounded as time approaches a critical value $T$. This behaviour is called *blowup* in finite time and any point $(X,T)$ such that

$$\limsup_{x \to X, t \uparrow T} u(x,t) = +\infty$$

is called a blowup point. Our aim is to study the asymptotic profile of the solution near such a point. The study of the scalar equation (1.1) is a first step towards the understanding of blowup for some physically relevant systems (for instance the reactive Euler system in gasdynamics, see e.g. [2], [6]).

There exists a wide literature concerning the blowup of solutions of nonlinear evolution problems (see e.g. the recent monographs [1], [11] or the survey paper [12], and the references therein). A first issue is to determine classes of equations for which there is blowup in finite time, and to single out conditions on the data which are necessary or sufficient for the blowup. The next step is to analyze the structure of the solutions in a neighbourhood of a blowup point and to find their asymptotic profile. Up to now researchers have considered mainly nonlinear perturbations of the wave equation or of the heat equation, but in some cases also other types of problems.

The case of conservation laws of the form (1.1) was considered by Natalini and Tesei [10], who constructed explicitly the solution for a special class of initial data. It was then shown by Natalini, Sinestrari and Tesei [9] that solutions of (1.1) admit a natural continuation after the blowup time, in the region delimited by a free boundary called blowup curve. In these papers it is proved that a solution of (1.1) can blow up in finite time whenever $g$ has superlinear growth. However, the existence of a blowup point is guaranteed only if $g$ grows faster than $f$; if this is not the case, and if the initial value has compact support, then the blowup occurs "at $x = +\infty$" and the solution remains bounded on compact sets.

The aim of this paper is to carry on the study of the blowup for equation (1.1) by finding the asymptotic profile of the solutions near the blowup point. Our work is inspired by the similar analysis done by Bressan [3] for a semilinear reactive–Euler system. A major difficulty when dealing with conservation laws of the form (1.1) is the lack of regularity: even if the initial data are smooth, the gradient of the solution in general becomes unbounded at some finite time, and one has to consider discontinuous generalized solutions. Such a formation of singularities (usually called *shocks*) is not a consequence of the blowup of the solution and can occur before the blowup time. Therefore, a basic step in our analysis will be to study how "bad" the discontinuity set can be near a blowup point. Having done this, the blowup profile of the solution can be analyzed by using the classical method of characteristics in the regions of smoothness.

We give a general procedure to study the asymptotic profile of solutions of (1.1) near a blowup point. We assume for simplicity that the initial value changes monotonicity only once, having a single maximum point with strictly negative second derivative. Our analysis is performed in the original variables and thus it does not require the rescaling procedures which are usually employed in this kind of problems. We use instead Dafermos' theory of generalized characteristics [4] which provides a precise description of the structure of discontinuous solutions. We divide the analysis of the blowup patterns in different cases. Roughly speaking, we have to consider separately the case when the solution becomes unbounded along a curve of discontinuity and the case when the blowup occurs in a region of smoothness. These two situations will be called *shock case* and *smooth case* respectively. In the shock case we prove that the solution converges to an unbounded traveling wave on one side of the shock, while it remains bounded on the other side (see Theorem 3.2). In the smooth case the solution tends asymptotically to a function constant on parabolas about the blowup point (see Theorem 4.7). We also prove that the smooth case can occur only if the source term grows fast enough (see Corollary 4.3); this property had already been observed in a particular case by Alinhac [1, Sect. I.C]. As an application of our results we discuss two specific examples where the source term $g$ is either exponential or powerlike. We compute the explicit expression of the blowup profile and we give conditions on the growth of $g$ and on the initial value under which the solution is smooth up to the blowup time.

# 2 Preliminaries

We consider the Cauchy problem (1.1)–(1.2) assuming that the following assumptions hold for some $k \geq 3$.

**(H1)** $f$ belongs to $C^1([0,\infty)) \cap C^{k+1}((0,\infty))$, is strictly convex and $f'(0) = 0$.

**(H2)** $g$ belongs to $C^1([0,\infty)) \cap C^k((0,\infty))$ and $g(u) > 0$ for any $u > 0$. In addition

$$\int_1^\infty \frac{f'(v)}{g(v)} dv < \infty. \tag{2.1}$$

**(H3)** $u_0$ is nonnegative and of class $C^k$, is increasing in $(-\infty, x^*]$ and decreasing in $[x^*, \infty)$ for some $x^* \in \mathbb{R}$.

**(H4)** For any $x < x^*$ such that $u_0(x) > 0$ we have $u_0'(x) > 0$; moreover $u_0''(x^*) < 0$.

We say that $u \in L^\infty(\mathbb{R} \times [0,\tau]) \cap C([0,\tau], L^1_{loc}(\mathbb{R}))$ is a solution of our equation if $u$ satisfies (1.1) in the sense of distributions, if $u(\cdot,t) \in BV_{loc}(\mathbb{R})$ for every $t$ and if the *entropy condition*

$$u(x+,t) \leq u(x-,t) \tag{2.2}$$

holds for all $x \in \mathbb{R}, t > 0$ (with $u(x+,t)$, $u(x-,t)$ we denote the one-sided limits of $u(\cdot\,,t)$ ). After possibly redefining the solution on a zero measure set, we can assume that it is continuous from the left at each point, i.e. $u(x-,t) = u(x,t)$.

It is known (see [8]) that the above assumptions ensure the existence and uniqueness of a solution of problem (1.1)–(1.2), at least locally in time. The solution is nonnegative everywhere by the comparison principle and by assumption (H2). By standard continuation arguments one obtains that the solution has a maximal interval of existence $[0,T)$, with $0 < T \le +\infty$. This means that $u$ solves problem (1.1)–(1.2) in every strip $\mathbb{R} \times [0,\tau]$ with $\tau < T$, and that either $T = +\infty$ or $T < +\infty$ and

$$\limsup_{\tau \to T} \|u\|_{L^\infty(\mathbb{R}\times[0,\tau])} = +\infty. \tag{2.3}$$

In the latter case we say that $u$ blows up at time $T$. We always assume in the following $T < +\infty$.

*Remark.* From properties (H1) and (H2) we easily obtain that

$$\int_1^\infty \frac{dv}{g(v)} < \infty. \tag{2.4}$$

A standard argument based on comparison with the solutions of the o.d.e. $u' = g(u)$ shows that (2.4) is a necessary condition for the blowup for the conservation law (1.1). The reason why we assume the stronger hypothesis (2.1) is that it ensures also the existence of a blowup point (see Theorem 2.4 below). In fact, it was shown in [9] that, if $g$ satisfies (2.4) but not (2.1), then the solution remains bounded on every compact space interval up to the blowup time (roughly speaking, the blowup occurs at $x = +\infty$). A situation of this kind has been recently considered by Escobedo, Herrero and Velazquez [7] as an intermediate step for the analysis of a Fokker–Planck equation.

An important role in our analysis is played by the theory of generalized characteristics (see [4]). We recall that a (generalized) *characteristic* of equation (1.1) associated with a solution $u$ is a locally Lipschitz function $\zeta : I \to \mathbb{R}$, with $I$ subinterval of $[0,T)$, which satisfies

$$\zeta'(s) \in [f'(u(\zeta(s)+,s)), f'(u(\zeta(s),s))] \tag{2.5}$$

for almost every $s \in I$.

From the theory of ordinary differential equations with discontinuous right-hand side (see [5]) it follows that through any point $(x,t) \in \mathbb{R} \times [0,T)$ there is at least one trajectory of (2.5). All solutions are confined between a minimal and a maximal one (not necessarily distinct). Since $u$ is bounded on $\mathbb{R} \times [0,\tau]$ for any $\tau < T$, any characteristic can be continued in $[0,T)$.

By a *backward* (*forward*) characteristic through a point $(x,t) \in \mathbb{R} \times [0,T)$ we mean a characteristic $\zeta$ defined in $[0,t]$ (resp. in $[t,T)$ ) with $\zeta(t) = x$. The characteristics associated with the solution of problem (1.1)–(1.2) have the following properties, whose proofs can be found in [4].

**Theorem 2.1** *Let $\zeta : I \to \mathbb{R}$ be a characteristic. Then for almost every $s \in I$*

$$\zeta'(s) = \begin{cases} f'(u(\zeta(s), s)) & \text{if } u(\zeta(s), s) = u(\zeta(s)+, s) \\[1em] \dfrac{f(u(\zeta(s), s)) - f(u(\zeta(s)+, s))}{u(\zeta(s), s) - u(\zeta(s)+, s)} & \text{if } u(\zeta(s), s) > u(\zeta(s)+, s). \end{cases}$$

**Theorem 2.2** *Let $\zeta_-$, $\zeta_+ : [0, t] \to \mathbb{R}$ be the minimal and maximal backward characteristics through a point $(x, t) \in \mathbb{R} \times (0, T)$. Then there exist $v_-$, $v_+ : [0, t] \to \mathbb{R}$ with the following properties.*

**(i)** *$u(\zeta_-(s), s) = u(\zeta_-(s)+, s) = v_-(s)$ and $u(\zeta_+(s), s) = u(\zeta_+(s)+, s) = v_+(s)$ for all $s \in [0, t)$.*

**(ii)** *The pairs $(\zeta_-, v_-)$ and $(\zeta_+, v_+)$ are solutions of the problem*

$$\begin{cases} \zeta'(s) = f'(v(s)) \\ v'(s) = g(v(s)) \end{cases} \qquad s \in [0, t]. \tag{2.6}$$

*with terminal conditions*

$$\zeta_\pm(t) = x, \qquad v_\pm(t) = u(x\pm, t). \tag{2.7}$$

*Consequently, $\zeta_-$ and $\zeta_+$ are of class $C^{k+1}$ and coincide if and only if $u(x, t) = u(x+, t)$.*

**Theorem 2.3** *The following properties hold.*

**(i)** *From any point $(x, t) \in \mathbb{R} \times [0, T)$ starts a unique forward characteristic.*

**(ii)** *If $\zeta : [a, b] \to \mathbb{R}$ is a characteristic and $u(\zeta(a), a) > u(\zeta(a)+, a)$, then $u(\zeta(s), s) > u(\zeta(s)+, s)$ for every $s \in (a, b]$ (such a characteristic is called a* shock*).*

*Remark.* The uniqueness of the forward characteristic stated above implies that characteristic curves cannot cross. By this we mean the following property: if $\zeta_1$ and $\zeta_2$ are characteristics defined in the intervals $I_1$ and $I_2$ respectively, and $\zeta_1(t) < \zeta_2(t)$ for some $t \in I_1 \cap I_2$, then $\zeta_1(s) \leq \zeta_2(s)$ for all $s \in I_1 \cap I_2$ (with strict inequality for $s \leq t$).

It is convenient to introduce some notations for the trajectories of system (2.6)–(2.7). For any $v \in \mathbb{R}$, let us denote by $U(v, \cdot)$ the solution of

$$\begin{cases} U'(s) = g(U(s)) \\ U(0) = v. \end{cases} \tag{2.8}$$

Then the solution of (2.6)–(2.7) has the form

$$\begin{cases} v_\pm(s) = U(u(x\pm, t), s - t), \\ \zeta_\pm(s) = x - F(u(x\pm, t), t - s), \end{cases} \qquad s \in [0, t], \tag{2.9}$$

where we have set

$$F(v,t) \doteq \int_0^t f'(U(v,-\tau))d\tau = \int_{U(v,-t)}^{v} \frac{f'(w)}{g(w)}dw. \tag{2.10}$$

It is also convenient to define, for any $t > 0$,

$$U(\infty,-t) \doteq \lim_{v\to+\infty} U(v,-t), \qquad F(\infty,t) \doteq \lim_{v\to+\infty} F(v,t). \tag{2.11}$$

The above quantities are finite by (2.1), (2.4) and satisfy

$$\int_{U(\infty,-t)}^{\infty} \frac{dv}{g(v)} = t, \qquad F(\infty,t) = \int_{U(\infty,-t)}^{\infty} \frac{f'(v)}{g(v)}dv. \tag{2.12}$$

Let us denote by $\xi(\cdot)$ the forward characteristic starting from the point $(x^*,0)$. Since characteristics cannot cross, any backward characteristic from a point $(x,t)$ with $x < \xi(t)$ hits the $x$–axis at a point belonging to the half–line $(-\infty,x^*)$ where $u_0$ is increasing. A symmetric property holds for the points lying on the right of $\xi$. Using Theorem 2.2 it is then easy to prove that $u(\cdot,t)$ is increasing in $(-\infty,\xi(t)]$ and decreasing in $(\xi(t),+\infty)$ for any $t > 0$. By the well–known property that a solution of (1.1) with convex $f$ preserves its regularity in the regions where it is increasing in $x$, we also deduce that $u$ is of class $C^k$ in the region to the left of $\xi$.

We remark that both $U(\cdot,t)$ and $F(\cdot,t)$ are strictly increasing for any fixed $t \neq 0$, as a consequence of the definitions and of the convexity of $f$. This monotonicity property will be often used in the sequel.

**Theorem 2.4** *The curve $\xi(t)$ tends to a limit $X < +\infty$ as $t \to T^-$. Moreover, there exists a classical characteristic $\zeta$ ending at $(X,T)$ along which $u$ tends to $+\infty$. More precisely, setting*

$$\zeta(t) \doteq X - F(\infty, T-t), \qquad t \in [0,T] \tag{2.13}$$

*we have that*

$$u(\zeta(t),t) = U(\infty, t-T), \qquad t \in [0,T). \tag{2.14}$$

*In addition, we have*

$$\lim_{\substack{(x,t)\to(X,T)\\ x\le\xi(t),t<T}} u(x,t) = +\infty. \tag{2.15}$$

**Proof.** Since $u(\cdot,t)$ attains its maximum at the point $\xi(t)$, we deduce from (2.3) that there exists an increasing sequence $\{t_n\}$ such that

$$\lim_{n\to\infty} t_n = T, \qquad \lim_{n\to\infty} u(\xi(t_n),t_n) = +\infty. \tag{2.16}$$

For any $n$, we set for simplicity $w_n \doteq u(\xi(t_n),t_n)$ and define, for $t \in [0,t_n]$,

$$v_n(t) \doteq U(w_n, t-t_n), \qquad \zeta_n(t) \doteq \xi(t_n) - F(w_n, t_n - t). \tag{2.17}$$

Then $\zeta_n$ is the minimal backward characteristic starting from $(\xi(t_n), t_n)$ and $u(\zeta_n(t), t) \equiv v_n(t)$. Since characteristics cannot cross, it is easily checked that

$$\zeta_{n+1}(t) \le \zeta_n(t) \le \xi(t) \tag{2.18}$$

for any $n$ and for any $t \le t_n$. Therefore the characteristics $\zeta_n$ tend pointwise to some limit $\zeta \le \xi$ (possibly $-\infty$). Since $\xi$ is increasing, $\xi(t_n)$ tends to some limit $X \in (x^*, +\infty]$. Letting $n \to \infty$ in (2.17) we obtain that $v_n$ also converges to some function $v$ and that

$$v(t) = U(\infty, t - T), \qquad \zeta(t) = X - F(\infty, T - t), \qquad t \in [0, T). \tag{2.19}$$

In particular we obtain that $X - \zeta(t) < +\infty$ for all $t < T$, which implies $X < +\infty$ and $\zeta(t) > -\infty$. Since $u$ is smooth on the left of $\xi$, we also have

$$u(\zeta(t), t) = \lim_{n\to\infty} u(\zeta_n(t), t) = \lim_{n\to\infty} v_n(t) = v(t).$$

Finally, let $(y_n, s_n)$ be any sequence converging to $(X, T)$, such that $y_n \le \xi(s_n)$. We have to show that $u(y_n, s_n) \to +\infty$ as $n \to \infty$. Since $u(\cdot, s_n)$ is increasing in $(-\infty, \xi(s_n)]$, it is not restrictive to assume that $y_n \le \zeta(s_n)$. Let us denote by $\gamma_n$ the backward characteristic from $(y_n, s_n)$. Since characteristics cannot cross, we have $\gamma_n \le \zeta$ for any $n$. Therefore

$$\liminf_{n\to\infty} F(u(y_n, s_n), s_n) = \liminf_{n\to\infty} (y_n - \gamma_n(0)) \ge X - \zeta(0) = F(\infty, T),$$

which implies $\lim_{n\to\infty} u(y_n, s_n) = +\infty$. ∎

*Remark.* Let us suppose that $\xi$ is a classical characteristic in $[0, T)$. Then, by Theorem 2.2, we have equality in (2.18) for all $n, t$; this implies that $\zeta$ coincides with $\xi$ and in particular that $\zeta(0) = x^*$. Conversely, if $\xi$ is not a classical characteristic in $[0, T)$ then $\xi$ is a shock for $t$ in some interval of the form $[T_0, T)$ by Theorem 2.3–(ii). Thus, we have strict inequality in (2.18) for $n$ large enough and $\zeta(t) < \xi(t)$ for all $t < T$.

**Theorem 2.5** *If the characteristic curve $\xi$ is a shock in some interval $[T_0, T)$, then $u$ is uniformly bounded in the region $\{(x, t) \ : \ t < T,\ x > \xi(t)\}$.*

**Proof.** Given $t \in [T_0, T)$, let us denote by $l_+(t)$ the point where the maximal backward characteristic starting from the point $(\xi(t), t)$ hits the $x$-axis. Using the property that characteristics cannot cross we easily obtain that $x^* \le l_+(t)$ and that $l_+$ is strictly increasing. Therefore $l_+(t)$ converges to some limit $\bar{x}_+ > x^*$ as $t \to T^-$.

On the other hand by (2.9) we know that

$$l_+(t) = \xi(t) - F(u(\xi(t)+, t), t).$$

It follows that $u(\xi(t)+, t) \to u_+$ as $t \to T^-$ where $u_+ \in (0, \infty]$ is the unique value such that

$$\bar{x}_+ = X - F(u_+, T).$$

Since $\bar{x}_+ > x^* > \zeta(0) = X - F(\infty, T)$, we have $u_+ = \lim_{t\to T^-} u(\xi(t)+, t) < \infty$. But $u(\cdot, t)$ is decreasing in $(\xi(t), +\infty)$, and therefore $u$ is uniformly bounded for $x > \xi(t)$. ∎

**Theorem 2.6** *If $\mathcal{N}$ is a neighbourhood of $(X,T)$, then $u$ is uniformly bounded in* $(\mathbb{R}\times[0,T))\setminus\mathcal{N}$.

**Proof.** If the assertion was false, we could find a sequence $(x_n,t_n)$ such that

$$u(x_n,t_n)\to+\infty,\quad t_n\to T,\quad x_n\to\bar{X}\neq X.$$

Let us set $\bar{\zeta}(t)=\bar{X}-F(\infty,T-t)$. Then, arguing as in the proof of Theorem 2.4, we obtain

$$u(\bar{\zeta}(t),t)=U(\infty,t-T)=u(\zeta(t),t). \tag{2.20}$$

Consider first the case when $\bar{X}>X$. Then the above equality shows that $u$ is not bounded on the right of $\xi$. By Theorem 2.5, $\xi$ is a classical characteristic and coincides with $\zeta$. Since $u(\cdot,t)$ is decreasing in $[\zeta(t),+\infty)$, we obtain from (2.20) that $u(\cdot,t)$ is constant in $[\zeta(t),\bar{\zeta}(t)]$. But then $u_0$ must be constant in $[\zeta(0),\bar{\zeta}(0)]$, in contradiction with the assumption that the point $x^*=\zeta(0)$ is a strict maximum point for $u_0$. We obtain a similar contradiction if we assume $\bar{X}<X$. This proves our assertion. ■

**Corollary 2.7** *There is a unique way to define $u(\cdot,T):\mathbb{R}\to[0,+\infty]$, such that*

**(i)** *$u(\cdot,T)$ is continuous from the left, $u(X,T)=+\infty$, $u(x,T)<+\infty$ for $x\neq X$;*

**(ii)** *we have $u(\cdot,t)\to u(\cdot,T)$ as $t\to T$ in the $L^1_{loc}(\mathbb{R}\setminus\{X\})$ norm.*

*Moreover, the result in Theorem 2.2 still holds for $t=T$, provided the solution of system* (2.6)–(2.7) *is defined by means of equalities* (2.9)–(2.11) *when $x=X$.*

We refer to $(X,T)$ as the *blowup point.* Given $\delta>0$, let us denote by $N_\delta$ the punctured half–disk

$$N_\delta\doteq\{(x,t)\ :\ (x,t)\in B_\delta(X,T)\setminus\{(X,T)\},\ t\leq T\}, \tag{2.21}$$

where $B_\delta(X,T)$ is the open disk with center $(X,T)$ and radius $\delta>0$. Any set of the form $U\cap(\mathbb{R}\times[0,T]\setminus\{(X,T)\})$, where $U$ is a neighbourhood of $(X,T)$ is refered to as a backward neighbourhood of $(X,T)$. We now single out three possible behaviours of the solution for which we will describe the asymptotic profile of $u(x,t)$ near the blowup point.

**(I)** The characteristic $\xi$ is a shock for $t$ close enough to $T$, i.e. $u(\xi(t),t)>u(\xi(t)+,t)$ for all $t$ greater than some $T_0<T$.

**(II)** The solution $u$ is smooth in some backward neighbourhood of the blowup point $(X,T)$ and

$$\lim_{(x,t)\to(X,T)}u(x,t)=+\infty, \tag{2.22}$$

whenever the limit is taken within $N_\delta$.

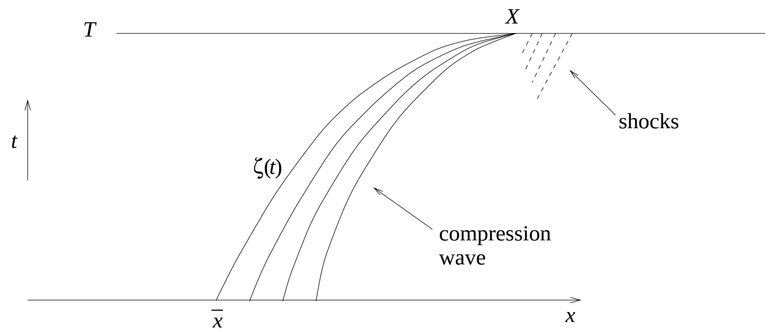


Figure 1: Case III, possible behaviours

**(III)** The characteristic $\xi$ is classical in $[0,T)$ but either any neighbourhood of $(X,T)$ contains points where $u$ is discontinuous, or $u$ tends to different values along different characteristics ending at $(X,T)$. This last possibility occurs when the blowup point is the center of a centered rarefaction wave.

By the remark after Theorem 2.4, the three cases above cover all possible behaviours of the solution. In the rest of the paper, when we speak of case (I), (II), or (III), or of solutions of type (I), (II), or (III), we will refer to the above definitions. Cases (I) and (II) will be also called the **shock case** and the **smooth case**, respectively.

The paper is organized as follows. Section 3 is devoted to the shock case. We see from Theorem 2.5 that in this case only the region on the left of the curve $\xi(\cdot)$ is relevant for the study of the blowup profile of $u$. Since $u$ is smooth in that region, we can obtain the profile by the method of characteristics (see Theorem 3.2). We then apply this result to obtain the explicit form of the profile in the case when $f$ is power–like and $g$ is either exponential (Theorem 3.3) or power–like (Theorem 3.4).

In Section 4 we treat case (II) and (III). We show that both these cases can occur only if the source grows fast enough. To this end, we give the definition of critical growth for a source $g$ (Definition 4.1) and show that if the source is subcritical then the blowup can only be of type (I) (Corollary 4.3 and Theorem 4.4). If $g$ is supercritical, all cases are possible. In the smooth case we can again use the method of characteristics. We obtain for the solution a blowup profile which is different from the shock case (Theorem 4.7) because the starting point of the characteristic $\zeta$ now coincides with the stationary point $x^*$ for the initial value. As for the shock case we derive the explicit form of the profile for an exponential or a power–like source (Corollaries 4.8 and 4.9).

Case (III) appears at a first sight the most difficult one, since the solutions could have in principle a large variety of possible behaviours. However, we show that for a supercritical source solutions of type (III) have a unique possible structure (Theorem 4.10) and this allows us to obtain the profile with arguments similar to the two cases previously considered (Theorem 4.11). We conclude the paper with some remarks about source terms with critical growth, without treating this case in full generality.

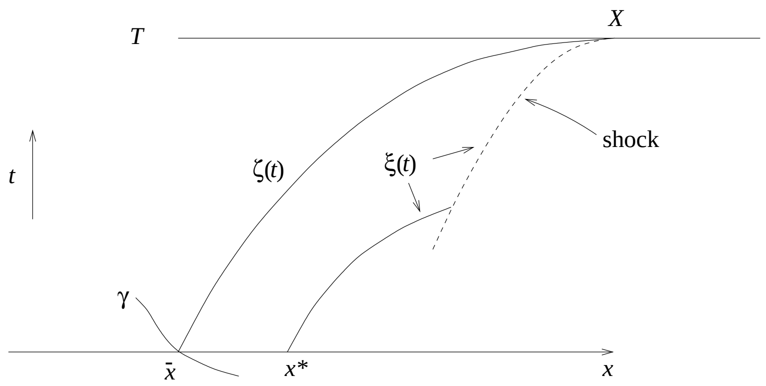


Figure 2: Case I, shock formation before blowup

# 3 Shock Case

In this section we consider the case where the characteristic $\xi(\cdot)$ starting at the point $(x^*,0)$ and ending at the blowup point $(X,T)$, is a shock in some interval $[T_0,T)$. We have proved in the previous section that there is a classical characteristic $\zeta(\cdot)$ ending at $(X,T)$ and that $\zeta(t)<\xi(t)$ for $t\in[0,T)$. Let us set $\bar{x}\doteq\zeta(0)=X-F(\infty,T)$, $\bar{u}\doteq u_0(\bar{x})$, and $\bar{p}\doteq u_0'(\bar{x})$; then we have $\bar{x}<x^*$ and $\bar{p}>0$ by assumption (H4).

We can assume that the solution $u$ of problem (1.1)–(1.2) is defined in a neighbourhood of $\bar{x}$ also for negative times $t\in[-\varepsilon',0]$ for some $\varepsilon'>0$. In fact, such a problem has a local $C^k$ solution both forward and backward in time. Let us set $\bar{u}=u_0(\bar{x})$. Using the fact that $u$ is a solution of (1.1) we obtain that there exists $\varepsilon>0$ and a function $\gamma\in C^k(-\varepsilon,\varepsilon)$ such that

$$\gamma(0)=0,\quad u(\bar{x}+\gamma(t),t)\equiv\bar{u},\quad \gamma'(t)=f'(\bar{u})-\frac{g(\bar{u})}{u_x(\bar{x}+\gamma(t),t)}$$

for any $t\in(-\varepsilon,\varepsilon)$. Let us set

$$c\doteq\gamma'(0)=f'(\bar{u})-g(\bar{u})\bar{p}^{-1}. \tag{3.1}$$

We recall that $u$ is smooth in the region on the left of $\xi$ and therefore this region is covered univalently by the characteristics associated with $u$. Then, given $\tau\in(-\varepsilon,\varepsilon)$, we easily obtain that the characteristic curve through the point $(\bar{x}+\gamma(\tau),\tau)$ has the form $\sigma\to(\zeta(\sigma)+\gamma(\tau),\sigma+\tau)$ and that

$$u(\zeta(\sigma)+\gamma(\tau),\sigma+\tau)=u(\zeta(\sigma),\sigma) \tag{3.2}$$

provided $\zeta(\sigma)+\gamma(\tau)\le\xi(\sigma+\tau)$.

**Lemma 3.1** *There exists a neighbourhood $I$ of the blowup point $(X,T)$ with the following property: setting*

$$\tilde{I}\doteq\{(x,t)\in I\ :\ t<T, x<\xi(t)\},$$

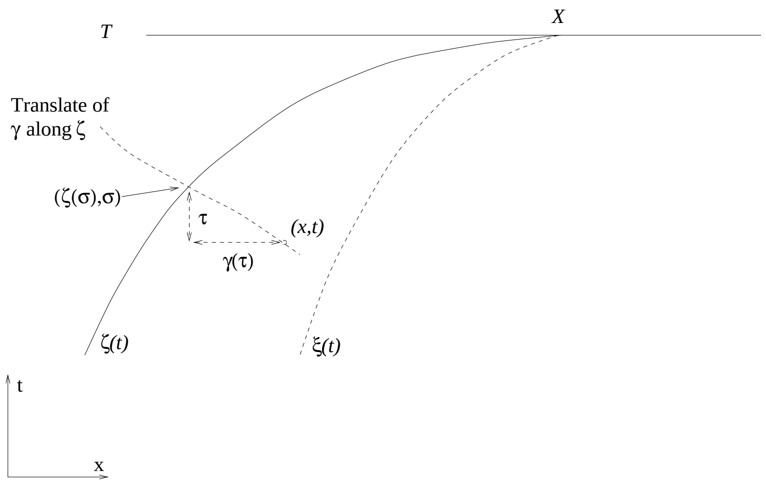


Figure 3: Asymptotics to the left of the shock curve

*then for any* $(x,t) \in \tilde{I}$ *there exists a unique pair* $(\sigma,\tau) \in (T-\varepsilon,T) \times (-\varepsilon,\varepsilon)$ *such that*

$$(x,t) = (\zeta(\sigma) + \gamma(\tau), \sigma + \tau). \tag{3.3}$$

*In addition there exists* $C > 0$ *such that*

$$|t - \sigma| < C(|T - t - c(X - x)|), \tag{3.4}$$

*where* $c$ *is the constant defined in* (3.1)*, see Figure 3.*

**Proof.** Let us set $p^* = \lim_{u\to\infty} f'(u)$. Then, by Theorem 2.1, we have $\zeta'(t) \to p^*$, $\xi'(t) \to p^*$ as $t \to T^-$. Let us define $h(y)$ as the inverse of $\zeta$ if $y \in [\bar{x}, X]$ and let us extend it linearly for $y > X$ in a way such that $h \in C^1$. Then we set, for $(y,\tau)$ close to $(X,0)$,

$$\Phi(y,\tau) \doteq (y + \gamma(\tau), h(y) + \tau).$$

Then $\Phi(X,0) = (X,T)$ and $D\Phi$ is non degenerate at that point, since $p^* > c$. It follows that, for $(x,t)$ near $(X,T)$, there exists a unique point $(y,\tau)$ near $(X,0)$ such that $(x,t) = \Phi(y,\tau)$. Moreover, there is a constant $K$ such that

$$|y - X| + |\tau| \leq K(|X - x| + |T - t|).$$

Now consider a pair $(x,t)$ near $(X,T)$ such that $x < X + \gamma(t-T)$. Then the associated pair $(y,\tau)$ satisfies

$$y + \gamma(t - h(y)) = x < X + \gamma(t - h(X)).$$

Since $p^* > c$, the function $z \to z + \gamma(t - h(z))$ is increasing in a neighbourhood of $z = X$ provided $t$ is close enough to $T$. Thus, the above computation shows that, if $x < X+\gamma(t-T)$ then $y < X$. Since $\xi(t) \leq X + \gamma(t-T)$, we conclude that, if the pair $(x,t)$ is close to $(X,T)$

and satisfies $x < \xi(t)$, then there exists a unique pair $(y,\tau)$ close to $(X,0)$ with $y < X$ and $(x,t) = \Phi(y,\tau)$. By definition of $\Phi$ and of $h$, and by the fact that $\zeta'$ is bounded from below by a positive constant we conclude, setting $\sigma = h(y)$, that we can associate to $(x,t)$ as above a unique pair $(\sigma,\tau)$ close to $(T,0)$, with $\sigma < T$ and such that

$$x = \zeta(\sigma) + \gamma(\tau), \qquad t = \sigma + \tau$$

$$|T-\sigma| + |\tau| \leq K_1(|X-x| + |T-t|).$$

To conclude, let us take $q \in (c, p^*)$ and let us assume that $t$ is close enough to $T$ to have $\xi'(s) \geq q$ for $s \geq t$ a.e.. Then, if $x < \xi(t)$, we have $X - x > q(T-t) > c(T-t)$. It follows that

$$\begin{aligned}
|X - x - c(T-t)| &= X - x - c(T-t) \\
&= \frac{q-c}{q}(X-x) + \frac{c}{q}((X-x) - q(T-t)) \\
&> \frac{q-c}{q}(X-x) = \frac{q-c}{q+1}\,\frac{q+1}{q}(X-x) \\
&\geq \frac{q-c}{q+1}((X-x) + (T-t)) \geq K_2|\tau|.
\end{aligned}$$

Since $\tau = t - \sigma$ we obtain inequality (3.4). ∎

We are now able to describe the asymptotic profile of the solutions in a neighbourhood of the blowup point in the shock case.

**Theorem 3.2** *Let assumptions* (H1)–(H4) *be satisfied. Suppose that the solution $u$ of problem* (1.1)–(1.2) *blows up at time $T$ and that the forward characteristic $x = \xi(t)$ from the point $(x^*, 0)$ is a shock for $t$ close to $T$. Then, defining $c$ as in* (3.1) *and $I$ as in Lemma 3.1, we have*

$$\int_{u(x,t)}^{+\infty} \frac{f'(v) - c}{g(v)}\,dv = X - x - c(T-t) + O(|X - x - c(T-t)|^2), \tag{3.5}$$

*for $(x,t) \in I$ such that $x \leq \xi(t)$, while $u(x,t)$ is uniformly bounded for $x > \xi(t)$.*

**Proof.** Let us take $(x,t) \in \tilde{I}$ and let $\sigma, \tau$ be as in the previous lemma. We have, by (3.2) and (2.14),

$$u(x,t) = u(\zeta(\sigma), \sigma) = U(\infty, \sigma - T).$$

Thus, by (2.12), (2.13), (3.1), (3.3) and (3.4) we deduce, for $(x,t) \in \tilde{I}$,

$$\begin{aligned}
\int_{u(x,t)}^{+\infty} \frac{f'(v) - c}{g(v)}\,dv &= F(\infty, T-\sigma) - c(T-\sigma) = X - \zeta(\sigma) - c(T-\sigma) \\
&= X - x - c(T-\sigma) + \gamma(t-\sigma) \\
&= X - x - c(T-t) + O(|t-\sigma|^2), \\
&= X - x - c(T-t) + O(|X - x - c(T-t)|^2).
\end{aligned}$$

The last assertion follows from Theorem 2.5. ∎

Observe that, up to lower order terms, formula (3.5) defines implicitly $u$ as a traveling wave with speed $c$. Thus, in this sense one has convergence towards an unbounded traveling wave. For specific choices of $f$ and $g$ it is possible to derive from (3.5) a more explicit description of the profile of $u$, and to compute the asymptotics of the shock curve as well. This will be worked out in the rest of the section in the case of a power–like flux function $f$ and a source term $g$ which is either exponential or power–like.

**Exponential case**. Let us first consider equation (1.1) with $f(u) = u^m/m$ for some $m > 1$ and $g(u) = e^u$. In this case we have

$$\int_u^\infty \frac{f'(v) - c}{g(v)}\, dv \;=\; \int_u^\infty \frac{v^{m-1} - c}{e^v}\, dv = e^{-u} u^{m-1}(1 + o(1)).$$

We obtain from (3.5) and (2.15)

$$\lim_{\substack{(x,t)\to(X,T)\\(x,t)\in\tilde{I}}} \frac{e^{-u(x,t)} u(x,t)^{m-1}}{X - x - c(T - t)} = 1,$$

which implies

$$-u(x,t) + (m-1)\ln(u(x,t)) = \ln(X - x - c(T-t)) + o(1). \tag{3.6}$$

We deduce

$$\frac{-\ln(X - x - c(T-t))}{u(x,t)} = 1 - (m-1)\frac{\ln(u(x,t))}{u(x,t)} + o(1) = 1 + o(1),$$

so that

$$\ln(u(x,t)) = \ln(-\ln(X - x - c(T-t))) + o(1).$$

Substituting the last equality into (3.6) we find the following expression for the asymptotic profile of $u(x,t)$ as $(x,t) \to (X,T)$ in $\tilde{I}$:

$$u(x,t) = -\ln(X - x - c(T-t)) + (m-1)\ln(-\ln(X - x - c(T-t))) + o(1). \tag{3.7}$$

We now compute the asymptotics of the shock curve $\xi(\cdot)$ near the blowup time. We first observe that $\xi(T) = X$ and that, by the Rankine–Hugoniot conditions (Theorem 2.1) and by Theorem 2.5,

$$\xi'(t) = \frac{u^m(\xi(t),t) - u^m(\xi(t)+,t)}{m[u(\xi(t),t) - u(\xi(t)+,t)]} = \frac{u^{m-1}(\xi(t),t)}{m}(1 + o(1)).$$

Here $o(1)$ is meant with respect to the limit $t \to T^-$. Let us set for simplicity $\beta(t) \doteq X - \xi(t) - c(T-t)$. Then, recalling (3.7), we obtain,

$$\beta'(t) = -\frac{(-\ln\beta(t))^{m-1}}{m}(1 + o(1)). \tag{3.8}$$

Since

$$\int_0^z \frac{dy}{(-\ln y)^{m-1}} \quad = \quad \frac{z}{(-\ln z)^{m-1}}(1+o(1)) \quad \text{as } z \to 0^+,$$

we find, integrating (3.8) in $[t,T]$,

$$\frac{\beta(t)}{(-\ln \beta(t))^{m-1}} = \frac{T-t}{m}(1+o(1)). \tag{3.9}$$

This implies that

$$\beta(t) = \frac{T-t}{m}\left(-\ln\left(\frac{T-t}{m}\right)\right)^{m-1}(1+o(1)).$$

Let us summarize the above results.

**Theorem 3.3** *Let $f(u) = u^m/m$, with $m > 1$, let $g(u) = e^u$, and let $u_0$ satisfy assumptions* (H3) *and* (H4). *Suppose that the solution $u$ of problem* (1.1)–(1.2) *blows up at $(X,T)$ and that the forward characteristic $\xi(\cdot)$ from the point $(x^*,0)$ is a shock for $t$ close to $T$. Let $c$ be given by* (3.1) *and set*

$$\phi(z) \doteq -\ln(z) + (m-1)\ln(-\ln(z)), \qquad z \in (0,1).$$

*Then we have, as $(x,t) \to (X,T)$ with $t < T$,*

$$u(x,t) = \begin{cases} \phi(X - x - c(T-t)) + o(1) & \text{if } x \le \xi(t) \\ O(1) & \text{if } x > \xi(t). \end{cases}$$

*Moreover*

$$\lim_{t \to T^-} \frac{X - \xi(t)}{\frac{T-t}{m}\left(-\ln\left(\frac{T-t}{m}\right)\right)^{m-1}} = 1.$$

We note that $c$ influences the speed of the traveling wave giving the profile of $u$ but not its shape.

**Power case**. Let us now consider the case when $f(u) = u^m/m$, $g(u) = u^p$. We assume $p > m > 1$, so that assumptions (H1) and (H2) are satisfied. Then we have

$$\int_u^\infty \frac{f'(v) - c}{g(v)}\, dv \quad = \quad \frac{1}{(p-m)u^{p-m}} - \frac{c}{(p-1)u^{p-1}}, \tag{3.10}$$

and so (3.5) and (2.15) imply

$$u(x,t) = ((p-m)(X - x - c(T-t)))^{\frac{1}{m-p}}(1+o(1)), \quad (x,t) \in \tilde{I}. \tag{3.11}$$

In the case when $p - 1 > m \ge 2$ we can obtain a more refined estimate. Let us set for simplicity

$$z \doteq X - x - c(T-t), \qquad k \doteq (p-m)^{1/(m-p)}. \tag{3.12}$$

We find, by (3.5) and (3.10)

$$u(x,t) = kz^{\frac{1}{m-p}}\left(1+\frac{c}{(p-1)zu^{p-1}}+O(z)\right)^{\frac{1}{m-p}}.$$

Now, using (3.11) and the property

$$(1+y)^{\frac{1}{m-p}} = 1-\frac{y}{p-m}+o(y) \quad \text{as } y\to 0,$$

we obtain

$$\begin{aligned}
u(x,t) &= kz^{\frac{1}{m-p}}\left(1+\frac{ck^{1-p}}{p-1}z^{\frac{m-1}{p-m}}(1+o(1))+O(z)\right)^{\frac{1}{m-p}}\\
&= kz^{\frac{1}{m-p}}\left(1-\frac{ck^{1-m}}{p-1}z^{\frac{m-1}{p-m}}(1+o(1))+O(z)\right)\\
&= kz^{\frac{1}{m-p}}-\frac{ck^{2-m}}{p-1}z^{\frac{m-2}{p-m}}(1+o(1))\\
&= \begin{cases} kz^{\frac{1}{m-p}}+o(1) & \text{if } m>2\\[2ex] kz^{\frac{1}{2-p}}-\dfrac{c}{p-1}+o(1) & \text{if } m=2.\end{cases}
\end{aligned}$$

We now compute the asymptotics of the shock curve. Defining $\beta(\cdot)$ as in the exponential case we obtain

$$\beta'(t) = -\frac{k^{m-1}}{m}\beta(t)^{\frac{m-1}{m-p}}(1+o(1)),$$

and therefore

$$\beta(t)^{\frac{p-1}{p-m}} = -\int_T^t \frac{d}{ds}\left(\beta(s)^{\frac{p-1}{p-m}}\right)ds = \frac{(p-1)k^{m-1}}{(p-m)m}(T-t)(1+o(1)).$$

We conclude, recalling the definition of $k$,

$$\beta(t) = \frac{1}{p-m}\left(\frac{p-1}{m}\right)^{\frac{p-m}{p-1}}(T-t)^{\frac{p-m}{p-1}}(1+o(1)).$$

**Theorem 3.4** *Let $f(u)=u^m/m$, $g(u)=u^p$, with $p>m>1$, and let $u_0$ satisfy assumptions* (H3) *and* (H4). *Suppose that the solution $u$ of problem* (1.1)–(1.2) *blows up at $(X,T)$ and that the forward characteristic $\xi(\cdot)$ from the point $(x^*,0)$ is a shock for $t$ close to $T$. With $c$ given by* (3.1) *and $k$ by* (3.12) *we have, as $(x,t)\to(X,T)$ with $t<T$,*

$$u(x,t) = \begin{cases} k(X-x-c(T-t))^{\frac{1}{m-p}}(1+o(1)) & \text{if } x\le\xi(t)\\ O(1) & \text{if } x>\xi(t).\end{cases}$$

*Moreover*

$$\lim_{t\to T} \frac{X-\xi(t)}{\frac{1}{p-m}\left(\frac{p-1}{m}\right)^{\frac{p-m}{p-1}}(T-t)^{\frac{p-m}{p-1}}} = 1.$$

*If we assume that* $p-1 > m \geq 2$*, we have the following sharper estimate as* $(x,t) \to (X,T)$ *with* $t < T$ *and* $x < \xi(t)$*,*

$$u(x,t) = \begin{cases} k(X-x-c(T-t))^{\frac{1}{m-p}} + o(1) & \text{if } m > 2 \\ k(X-x-c(T-t))^{\frac{1}{m-p}} - \frac{ck^{2-m}}{p-1} + o(1) & \text{if } m = 2. \end{cases}$$

Let us mention briefly that the approach of this section can be also applied to study the continuation of solutions after blowup time as introduced in [9]. To fit in the framework of that paper we need slightly stronger hypotheses on $f, g$ than above, namely we have to assume, in addition to (H1) and (H2), that $f'(u) \to \infty$ as $u \to \infty$ and that $g(0) = 0$. The initial value $u_0$ is supposed to be nonnegative, of bounded variation, and with support bounded from below. Then it was shown in [9] that any solution of (1.1) admits a continuation $V$ defined for all $t > 0$; if $t \geq T$ the continuation is not defined in the whole real line but only in the region on the left of a curve $\chi(\cdot)$ such that $\chi(T) = X$. The continuation $V$ solves the equation in its domain of definition, coincides with $u$ in $\mathbb{R} \times [0,T)$ and satisfies $V(x,t) \to +\infty$ as $x \to \chi(t)-$ for any $t \geq T$. Therefore $\chi$ represents the curve along which the solution "becomes infinite". It was proved in [9] that such a continuation is uniquely defined, that is, that there exists a unique pair $V$, $\chi$ satisfying the above properties.

Given a continuation $V$, one can study its profile not only in a neighbourhood of the "first" blowup point $(X,T)$, but also near any point $(\chi(t_0), t_0)$ with $t_0 > T$. It is easily checked that the methods of this section can be applied also in this case. More precisely, assuming that the initial value satisfies (H3) and (H4) and has support bounded from below, one can prove the following properties. The curve $\chi$ is of class $C^k$ and the continuation $V$ is also of class $C^k$ in the region $\{(x,t) \ : \ t \geq T, x < \chi(t)\}$. Given $t_0 > T$, there exists a classical characteristic ending at $(\chi(t_0), t_0)$ along which $V$ becomes infinite. Arguing as in the proof of Theorem 3.2 (with $\chi$ taking the role of $\xi$) one can obtain the profile of $V$ near the point $(\chi(t_0), t_0)$. The only difference with the situation considered in this section is that the asymptotic expansion does not apply to the whole region to the left of the curve $\chi$, but only to slightly smaller cones. We state without proof the result for the exponential case.

**Theorem 3.5** *Suppose* $f(u) = u^m/m$, $g(u) = e^u$*, let* $V$ *be the unique continuation after the blowup time of the solution of* (1.1)–(1.2) *(in the sense of [9]) and let* $\chi$ *be the blowup curve associated to* $V$*. Given* $t_0 > T$ *and* $\varepsilon > 0$*, let us set* $c_0 \doteq \chi'(t_0)$ *and*

$$I \doteq \{(x,t) \ : \ \chi(t) - x > c_0(t_0)(t-t_0) + \varepsilon|t-t_0|\}.$$

*Then we have*

$$\begin{aligned}\lim_{\substack{(x,t)\to(\chi(t_0),t_0)\\(x,t)\in I}} & V(x,t)+\ln(\chi(t_0)-x-c_0(t_0)(t_0-t)) \\ & +(m-1)\ln(-\ln(\chi(t_0)-x-c_0(t_0)(t_0-t))) \;=\; 0.\end{aligned}$$

An analogous results holds for the case of a power–like source term.

# 4 Smooth Case

In this section we study the blowup profile of $u$ in the cases (II) and (III) according to the terminology of Section 2. A common feature of these two cases is that the forward characteristic $\xi$ through the point $(x^*, 0)$ is classical for $t \in [0, T)$. By the remark after Theorem 2.4 we have that $\xi$ coincides with $\zeta$; in other words, $\xi$ is a classical characteristic. We will show that the possibility of having solutions of the types under consideration is related to the growth of the source term $g$. Let us first give a definition.

**Definition 4.1** *Let the functions $f$ and $g$ be as in assumptions* (H1)–(H2) *and set* $p^* \doteq \lim_{u\to\infty} f'(u)$. *We say that $g$ has* subcritical *growth with respect to $f$ if $p^* = +\infty$ and*

$$\lim_{v\to+\infty} \frac{f''(v)g(v)}{f'(v)^3} = 0. \tag{4.1}$$

*If the limit in (4.1) is $+\infty$ or if $p^* < +\infty$ we say that $g$ has* supercritical *growth. In the other cases we call the growth of $g$* critical.

For instance, if we take $f(u) = u^m$ and $g(u) = u^p$, then the growth of $g$ is subcritical if $p < 2m-1$ and supercritical if $p > 2m-1$. The role of the quantity in (4.1) will be clear later (see in particular the proofs of Lemmas 4.2 and 4.5). Let us now set, for $(t, \theta) \in [0, T] \times \mathbb{R}$,

$$x(t;\theta) \doteq \theta + \int_{u_0(\theta)}^{U(u_0(\theta),t)} \frac{f'(v)}{g(v)} dv.$$

Then $x(\cdot;\theta)$ coincides with the forward characteristic from $(\theta, 0)$ as long as the latter remains classical. By Theorem 2.2, any classical characteristic associated with $u$ is of the form $x(\cdot;\theta)$ for some $\theta \in \mathbb{R}$. In particular, $\xi(t) = \zeta(t) = x(t; x^*)$ for any $t$. From the definition it follows that $x(\cdot;\cdot)$ is continuous and that it is smooth except possibly at the point $(t, \theta) = (T, x^*)$. The next lemma shows that the behaviour of $x$ at that point is related to the growth of $g$. Here and in the following we use the notation $\bar{u} \doteq u_0(x^*)$ as in the previous section.

**Lemma 4.2** *Suppose that* $\lim_{v\to+\infty} f'(v) = +\infty$ *and that*

$$\lim_{v\to+\infty} \frac{f''(v)g(v)}{f'(v)^3} = l \in [0, +\infty].$$

*Then*

$$\lim_{\theta\to x^*\pm}\frac{\partial x}{\partial\theta}(T;\theta)=\begin{cases}1 & \text{if } l=+\infty\\ 1\mp\sqrt{\frac{-u_0''(x^*)}{g(\bar u)l}} & \text{if } 0<l<+\infty\\ \mp\infty & \text{if } l=0.\end{cases}\tag{4.2}$$

*If* $\lim_{v\to+\infty}f'(v)<+\infty$ *the limit in (4.2) is* 1.

**Proof.** We have

$$\begin{aligned}\lim_{\theta\to x^*}\frac{\partial x}{\partial\theta}(T;\theta) &= \lim_{\theta\to x^*}\left(1+\frac{f'(U(u_0(\theta),T))-f'(u_0(\theta))}{g(u_0(\theta))}u_0'(\theta)\right)\\ &= 1+\frac{1}{g(\bar u)}\left(\lim_{\theta\to x^*}f'(U(u_0(\theta),T))u_0'(\theta)\right).\end{aligned}\tag{4.3}$$

Let us now define

$$\tau(\theta)\doteq\int_{u_0(\theta)}^{\bar u}\frac{dw}{g(w)}.\tag{4.4}$$

Then by definition $U(u_0(\theta),T)\equiv U(\infty,-\tau(\theta))$. In addition, by (H3) and (H4),

$$\lim_{\theta\to x^*}\frac{\tau(\theta)}{(\theta-x^*)^2}=\frac{-u_0''(x^*)}{2g(\bar u)}.\tag{4.5}$$

It follows that

$$\begin{aligned}&\lim_{\theta\to x^*}[f'(U(u_0(\theta),T))u_0'(\theta)]^2\\ &= \lim_{\theta\to x^*}\left(f'(U(\infty,-\tau(\theta)))^2\tau(\theta)\frac{(\theta-x^*)^2}{\tau(\theta)}\left(\frac{u_0'(\theta)}{\theta-x^*}\right)^2\right)\\ &= -2g(\bar u)u_0''(x^*)\cdot\left(\lim_{\theta\to x^*}f'(U(\infty,-\tau(\theta)))^2\tau(\theta)\right)\\ &= -2g(\bar u)u_0''(x^*)\cdot\left(\lim_{v\to+\infty}f'(v)^2\int_v^\infty\frac{dw}{g(w)}\right).\end{aligned}\tag{4.6}$$

If $\lim_{v\to+\infty}f'(v)<+\infty$ the above quantity is zero and the limit in (4.3) is 1. Otherwise we have, by L'Hôpital's rule,

$$\lim_{v\to+\infty}\frac{\int_v^\infty\frac{dw}{g(w)}}{f'(v)^{-2}}=\lim_{v\to+\infty}\frac{f'(v)^3}{2g(v)f''(v)}.$$

Assertion (4.2) follows from the last relation, together with (4.3) and (4.6) and from the fact that $u_0'(\theta)$ has same sign as $(x^*-\theta)$, while $f'$ is positive everywhere. ∎

**Corollary 4.3** *Let assumptions* (H1)–(H3) *hold. If $g$ has subcritical growth, no initial value $u_0$ exists which satisfies* (H4) *and such that the corresponding solution $u$ is smooth in a neighbourhood of the blowup point.*

**Proof.** If $u$ is smooth in a neighbourhood of the blowup point, then such a neighbourhood is covered univalently by the characteristics starting in a neighbourhood of $x^*$. Then we have $\partial x/\partial\theta(t;\theta) \geq 0$ for all $(\theta, t)$ close to $(x^*, T)$. This cannot happen in the subcritical case by the previous lemma. ∎

*Remark.* On the other hand, if $g$ has supercritical growth, it is easily checked that there exist data satisfying (H4) and such that the solution is of type (II). Indeed, since the characteristic speeds are positive by (H1), a *sufficient* condition on the initial data which guarantees that such a property holds is that

$$\frac{\partial x(t;\theta)}{\partial\theta} > 0, \qquad \text{for all } t \in [0, T] \text{ and for all } \theta \in [x^*, X]. \tag{4.7}$$

By computations similar to those of the previous lemma, we obtain that, if $g$ has supercritical growth, then there exist data satisfying both conditions (4.7) and (H4).

We point out that the above corollary relies on assumption (H4). If we relax our assumptions on the data, we can construct a large class of smooth solutions regardless of the growth of the source, as the next example shows.

**Example.** Let $f$, $g$ be given functions satisfying (H1) and (H2) and let $\bar{u} > 0$ be given. Set

$$T = \int_{\bar{u}}^{+\infty} \frac{dv}{g(v)}, \qquad X = \int_{\bar{u}}^{+\infty} \frac{f'(v)}{g(v)} dv.$$

We want to find a solution $u \in C((\mathbb{R} \times [0, T]) \setminus \{(X, T)\})$ such that

$$u(x, 0) = \phi(x) \ \forall x \leq 0, \qquad u(x, T) = \psi(x) \ \forall x > X,$$

where $\phi$ and $\psi$ are given $C^2$ functions with the following properties.

**(i)** $\phi : (-\infty, 0] \to \mathbb{R}$ is increasing, $\phi(0) = \bar{u}$;

**(ii)** $\psi : (X, +\infty) \to \mathbb{R}$ is decreasing, $\psi(x) \to +\infty$ as $x \to X^+$.

A solution satisfying the above requirements is easily constructed by the method of characteristics. Setting $\zeta(t) = X - F(\infty, T - t)$, we can solve the equation in the region on the left (respectively, on the right) of $\zeta$ by tracing the forward (resp. backward) characteristics from the half–line where the initial datum $\phi$ (resp. final datum $\psi$) is prescribed. Our requirements on $\phi$ and $\psi$ are such that the characteristics fill the whole strip $\mathbb{R} \times [0, T]$ without self–intersections. We obtain in this way a solution $u$ which is continuous in $\mathbb{R} \times [0, T]$ except at the point $(X, T)$ where it tends to $+\infty$. However, the initial value assumed by $u$ does not satisfy in general assumptions (H3) and (H4). It surely satisfies the monotonicity requirements but, in general, either it is not twice differentiable at zero, or it has zero second derivative.

The example shows also that there exist solutions of (1.1) taking arbitrary values at the right of the blowup point. Thus, there is no hope to classify the blowup patterns of the solutions without some restriction on the initial data like in assumption (H4).

We want now to prove that, if the source has subcritical growth, then the blowup is necessarily of type (I). Actually, we have been able to prove this property only under an additional assumption on the flux function $f$.

**Theorem 4.4** *Suppose that assumptions* (H1)–(H4) *hold and that $g$ has subcritical growth. Suppose in addition that there exists $\delta > 0$ such that, for any $v_2 > v_1 > 0$,*

$$\frac{f(v_2) - f(v_1)}{v_2 - v_1} \leq (1-\delta) f'(v_2) + \delta f'(v_1). \tag{4.8}$$

*Then the characteristic $x = \xi(t)$ starting at $(x^*, 0)$ is a shock for $t$ close to the blowup time.*

*Remark.* If $f(u) = u^m/m$ for some $m \geq 2$, then it is easily seen that property (4.8) holds with $\delta = 1/m$.

The proof of Theorem 4.4 requires the following lemma.

**Lemma 4.5** *Suppose that $g$ has subcritical growth and that $f$ satisfies* (4.8). *Then we have*

$$\lim_{v\to\infty} \left( \int_v^\infty \frac{f'(w)}{g(w)} dw \right)^2 \left( \int_v^\infty \frac{dw}{g(w)} \right)^{-1} = +\infty.$$

*On the other hand, if $g$ has supercritical growth, then the above limit is* 0, *regardless of the validity of* (4.8).

**Proof.** The result in the subcritical case follows applying twice L'Hôpital's rule:

$$\lim_{v\to\infty} \frac{\left( \int_v^\infty \frac{f'(w)}{g(w)} dw \right)^2}{\int_v^\infty \frac{dw}{g(w)}} = \lim_{v\to\infty} \frac{2\int_v^\infty \frac{f'(w)}{g(w)} dw}{f'(v)^{-1}} = \lim_{v\to\infty} \frac{2 f'(v)^3}{g(v) f''(v)}. \tag{4.9}$$

The same computation proves our result if $g$ is supercritical and $\lim f'(v) = +\infty$. If $\lim f'(v) < +\infty$ then the result follows already after the first application of L'Hôpital's rule in (4.9). ∎

**Proof of Theorem 4.4.** We construct a suitable family of supersolutions of our problem. We define, for $\varepsilon > 0$,

$$w_{\varepsilon,0}(x) \doteq \begin{cases} \bar{u} & \text{if } x \leq x^* + \varepsilon \\ u_0(x^* + \varepsilon) & \text{if } x > x^* + \varepsilon. \end{cases}$$

Let us denote by $w_\varepsilon$ the solution of equation (1.1) with initial value $w_{\varepsilon,0}$; then $w_\varepsilon$ has the form

$$w_\varepsilon(x,t) = \begin{cases} U(\bar{u},t) & \text{if } x \le \alpha_\varepsilon(t) \\ U(\bar{u},t-\tau(x^*+\varepsilon)) & \text{if } x > \alpha_\varepsilon(t), \end{cases}$$

where $\tau(\cdot)$ is defined as in (4.4), and

$$\alpha_\varepsilon(t) \doteq x^* + \varepsilon + \int_0^t \frac{f(U(\bar{u},s)) - f(U(\bar{u},s-\tau(x^*+\varepsilon)))}{U(\bar{u},s) - U(\bar{u},s-\tau(x^*+\varepsilon))} dt.$$

We argue by contradiction and suppose that the characteristic $\xi$ associated with $u$ is classical in $[0,T)$. Then we have $u(x,t) \le u(\xi(t),t) = U(\bar{u},t)$ for any $(x,t)$; this implies that both $u$ and $w_\varepsilon$ blow up at time $T$, where $T$ is the unique value such that $U(\infty,-T) = \bar{u}$. By (H3) we have $w_{\varepsilon,0} \ge u_0$, and so the comparison principle (see [8]) implies that $w_\varepsilon \ge u$. Therefore for any $t \in [0,T)$ we have $w_\varepsilon(\xi(t),t) \ge u(\xi(t),t) = U(\bar{u},t)$, which implies that $\alpha_\varepsilon(t) \ge \xi(t)$. Let us write for simplicity $\tau_\varepsilon = \tau(x^*+\varepsilon)$ in the following formulas. Using (4.8), (4.5) and (H1), we find, for $\varepsilon > 0$ small enough,

$$\begin{aligned} 0 &\le \lim_{t\to T} \{\alpha_\varepsilon(t) - \xi(t)\} \\ &= \varepsilon + \int_0^T \left\{ \frac{f(U(\bar{u},t)) - f(U(\bar{u},t-\tau_\varepsilon))}{U(\bar{u},t) - U(\bar{u},t-\tau_\varepsilon)} - f'(U(\bar{u},t)) \right\} dt \\ &\le \varepsilon - \delta \int_0^T \{f'(U(\bar{u},t)) - f'(U(\bar{u},t-\tau_\varepsilon))\} dt \\ &= \varepsilon - \delta \int_{T-\tau_\varepsilon}^T f'(U(\bar{u},t))dt + \delta \int_{-\tau_\varepsilon}^0 f'(U(\bar{u},t))dt \\ &< \left( \frac{3g(\bar{u})\tau_\varepsilon}{-u_0''(x^*)} \right)^{\frac{1}{2}} - \delta \int_{U(\infty,-\tau_\varepsilon)}^{+\infty} \frac{f'(w)}{g(w)} dw + \delta f'(\bar{u})\tau_\varepsilon. \end{aligned}$$

This implies that there exists a constant $C > 0$ such that, for any $\varepsilon > 0$ small enough,

$$\left( \int_{U(\infty,-\tau_\varepsilon)}^{+\infty} \frac{f'(w)}{g(w)} dw \right)^2 \le C\tau_\varepsilon = C \int_{U(\infty,-\tau_\varepsilon)}^{+\infty} \frac{dw}{g(w)}.$$

Since $U(\infty,-\tau_\varepsilon) \to +\infty$ as $\varepsilon \to 0$, the above inequality contradicts Lemma 4.5 if $g$ is subcritical. ∎

We now analyze the blowup profile of the solutions when the source is supercritical. We consider first the smooth case and assume that the solution $u$ is smooth on a backward neighbourhood of $(X,T)$ of the form $N_\delta$, with $N_\delta$ defined as in (2.21). Note that $N_\delta$ contains points $(x,t)$ with $t = T$. From our assumptions it follows that $u(x,t)$ has a level curve $\gamma : (-\varepsilon,\varepsilon) \to \mathbb{R}$, parametrized by $\eta = x - x^*$, with $\gamma(0) = x^*$, and with strictly positive curvature $\alpha$ at $\eta = 0$. We have

$$\alpha = \gamma''(0) = -\frac{u_0''(x^*)}{g(\bar{u})}. \tag{4.10}$$

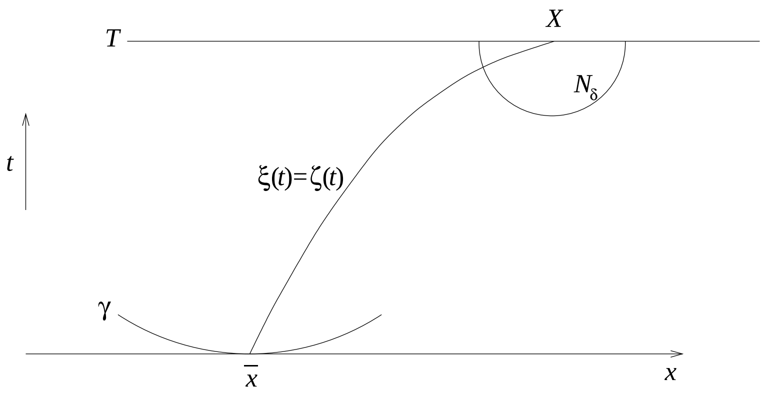


Figure 4: Case II, smooth blowup

The key property of the solution in this case is the following.

**Lemma 4.6** *If $g$ is supercritical and the solution $u$ of* (1.1) *is smooth on some backward neighbourhood $N_\delta$ of $(X, T)$, then the translates of $\gamma$ along $\zeta$ (i.e. the curves parametrized by $\eta \mapsto (\zeta(\tilde{t}) + \eta, \tilde{t} + \gamma(\eta))$, with $|\eta|$ sufficiently small) are again level curves of $u$ and cover univalently $N_\delta$. As a consequence we have that*

$$\lim_{(x,t)\to(X,T)} u(x,t) = +\infty, \tag{4.11}$$

*whenever the limit is taken within $N_\delta$.*

**Proof.** If $g$ is supercritical and $u$ is smooth near $(X, T)$, then Lemma 4.2 implies that a neighbourhood of $(X, T)$ is covered univalently by the classical characteristics starting near $x^*$. Since $u$ takes the same values along two characteristics crossing $\gamma$, and since the flux $f$ depends on $u$ only, it follows that two such characteristics are translates of each other. In turn, this implies that if $u$ remains smooth close to $\zeta$, then the translates of $\gamma$ along $\zeta$ are again level curves of $u$. As for the limit, when $(x, t) \to (X, T)$ in $N_\delta$, the level curves of $u$ are crossed. Since $\zeta$ meets all of these level curves and $u \to +\infty$ along $\zeta$, the result follows. ∎

**Theorem 4.7** *Let assumptions* (H1)–(H4) *be satisfied and let $g$ have supercritical growth. Suppose that the solution $u$ of problem* (1.1)–(1.2) *blows up at time $T$ and that it is smooth in a backward neighbourhood $N$ of the blowup point $(X, T)$. Then, defining $\alpha$ as in* (4.10)*, we have, as $(x, t) \to (X, T)$,*

$$\int_{u(x,t)}^{+\infty} \frac{dv}{g(v)} = (T - t + \frac{\alpha}{2}(X - x)^2)(1 + o(1)). \tag{4.12}$$

**Proof.** We take $(x,t) \in N \doteq N_\delta$ for a sufficiently small $\delta > 0$. Denote by $(\tilde{x}, \tilde{t})$ the point where the level curve through $(x,t)$ intersects $\zeta$, such that $\tilde{x} = \zeta(\tilde{t})$. Let $\eta = x - \tilde{x}$, such that $t - \tilde{t} = \gamma(\eta)$, and put $x' \doteq X - x$, $t' \doteq T - t$. We then have

$$\eta = -x' + \int_{u(x,t)}^{+\infty} \frac{f'(v)}{g(v)}\, dv, \tag{4.13}$$

and

$$\gamma(\eta) = \frac{\alpha}{2}\eta^2(1 + O(\eta)) = -t' + \int_{u(x,t)}^{+\infty} \frac{dv}{g(v)}, \tag{4.14}$$

where $\alpha$ is given by (4.10). We use these relations to obtain the asymptotic profile of the solution. By (4.11), $\eta$ is $o(1)$ as $(x,t) \to (X,T)$ in $N$. Substituting (4.13) into (4.14), we obtain

$$\frac{\alpha}{2}\Big(\int_u^{+\infty} \frac{f'(v)}{g(v)}dv - x'\Big)^2(1 + o(1)) = \int_u^{+\infty} \frac{dv}{g(v)} - t'. \tag{4.15}$$

By (4.14) we have

$$|\eta|^2 = O\Big(\int_u^{+\infty} \frac{dv}{g(v)}\Big),$$

and so we obtain from (4.13)

$$|x'| \int_u^{+\infty} \frac{f'(v)}{g(v)} \le O\Big(\Big(\int_u^{+\infty} \frac{dv}{g(v)}\Big)^{\frac{1}{2}}\Big) \int_u^{+\infty} \frac{f'(v)}{g(v)}dv + \Big(\int_u^{+\infty} \frac{f'(v)}{g(v)}dv\Big)^2.$$

Recalling also Lemma 4.5 we obtain from (4.15)

$$(1 + o(1)) \int_u^{+\infty} \frac{dv}{g(v)} = t' + (\frac{\alpha}{2}x'^2)(1 + o(1))$$

which implies our assertion. ∎

We now use formula (4.12) to compute explicitly the asymptotic profile of the solution for the two choices of $f, g$ considered in the previous section.

**Exponential case**. Consider again the equation

$$u_t + \Big(\frac{u^m}{m}\Big)_x = e^u, \tag{4.16}$$

with initial data $u_0$. We note that the source term has supercritical growth and that $T$, $X$, and $\alpha$ are given by

$$T = e^{-\bar{u}}, \qquad X = x^* + \int_{\bar{u}}^{+\infty} v^{m-1}e^{-v}\, dv, \qquad \alpha = -u_0''(x^*)e^{-\bar{u}}. \tag{4.17}$$

In this case (4.12) yields

**Corollary 4.8** *Assume that* (H3) *and* (H4) *are satisfied, and let* $T$*,* $X$*, and* $\alpha$ *be given by* (4.17)*. If the solution of* (4.16) *blows up at time* $T$ *and is smooth in some backward neighbourhood* $N_\delta$ *of* $(X,T)$*, then* $u$ *satisfies*

$$u(x,t) = -\ln\Big((T-t) + \frac{\alpha}{2}(X-x)^2\Big) + o(1),$$

*when* $(x,t)$ *tends to* $(X,T)$ *in* $N_\delta$*.*

*Remark.* In the exponential case the sufficient condition (4.7) takes the following form. By solving for $u$ along the characteristic $x(t;\theta)$ we obtain

$$u(t;\theta) = -\ln(e^{-u_0(\theta)} - t).$$

Furthermore we have

$$\frac{\partial x(t;\theta)}{\partial\theta} = u_0'(\theta)e^{-u_0(\theta)}(u(t;\theta)^{m-1} - u_0(\theta)^{m-1}) + 1.$$

Solving $\frac{\partial x(t;\theta)}{\partial\theta} = 0$ for $t$ we get that (4.7) is satisfied provided

$$\exp\Big[-\Big(u_0(\theta)^{m-1} - \frac{e^{u_0(\theta)}}{u_0'(\theta)}\Big)^{\frac{1}{m-1}}\Big] < e^{-u_0(\theta)} - e^{-\bar{u}}, \tag{4.18}$$

holds for all $\theta \in (x^*, X]$. This will be the case if $u_0$ decreases slowly enough in $(x^*, X]$.

**Power case**. We next consider the case of a power source,

$$u_t + \Big(\frac{u^m}{m}\Big)_x = u^p, \tag{4.19}$$

with $m > 1$ and $p > 2m-1$ in order to have a source term with supercritical growth. We note that $T$, $X$, and $\alpha$ are now given by

$$T = \frac{\bar{u}^{1-p}}{p-1}, \qquad X = x^* + \frac{\bar{u}^{m-p}}{p-m}, \qquad \alpha = -u_0''(x^*)\bar{u}^{-p}. \tag{4.20}$$

With $w^* = (p-1)^{\frac{1}{1-p}}$ , we obtain from (4.12) the following

**Corollary 4.9** *Assume that* (H3) *and* (H4) *are satisfied. Also let* $T$*,* $X$*, and* $\alpha$ *be given by* (4.20)*. Then, if the solution of* (4.19) *has smooth blowup at time* $T$*, then one must have* $p > 2m-1$ *and*

$$u(x,t) = w^*\Big((T-t) + \frac{\alpha}{2}(X-x)^2\Big)^{\frac{1}{1-p}}(1 + o(1)),$$

*when* $(x,t)$ *tends to the blowup point* $(X,T)$ *in some backward neighbourhood* $N_\delta$*.*

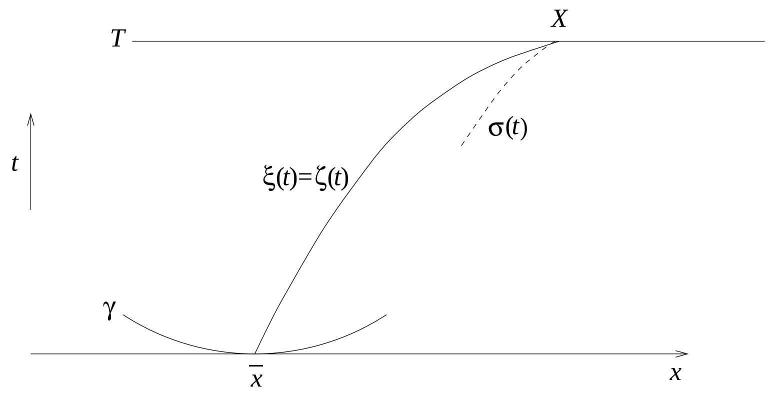


Figure 5: Case III, supercritical source

*Remark.* In the power case the sufficient condition (4.7) is satisfied provided

$$\Big(\frac{u_0(\theta)}{\bar u}\Big)^{p-1}+\Big(1-\frac{u_0^{p-m+1}}{u_0'(\theta)}\Big)^{\frac{1-p}{m-1}}<1,$$

for all $\theta\in(x^*,X]$. As in the exponential case, this will be the case provided $u_0$ decreases slowly enough in $(x^*,X]$.

We now turn our attention to type (III) blowup for supercritical sources. We first show that in this case the solution has necessarily a special structure.

**Theorem 4.10** *Let assumptions* (H1)–(H4) *hold, let $g$ be supercritical and let $u$ be of type* (III). *Then there exists a shock curve $\sigma:[T-d,T)\to\mathbb{R}$ such that $\sigma(t)>\xi(t)$ for any $t\in[T-d,T)$, $\sigma(t)\to X$ as $t\to T$ and that $u$ is smooth in $\{(x,t)\;:\;t\in[T-d,T),x<\sigma(t)\}$.*

**Proof.** From Lemma 4.2 we deduce that there exists $a>0$ such that

$$\frac{\partial x}{\partial\theta}(T;\theta)>0,\qquad\text{for all }\theta\in[x^*-a,x^*+a].\tag{4.21}$$

We recall that

$$\frac{\partial x}{\partial\theta}(T;\theta)=1+\frac{f'(U(u_0(\theta),t))-f'(u_0(\theta))}{g(u_0(\theta))}u_0'(\theta),$$

and therefore $\frac{\partial x}{\partial\theta}(t;\theta)$ is positive for all $t$ if $\theta\le x^*$, while it is a decreasing function of $t$ if $\theta>x^*$. Then we deduce from (4.21) that

$$\frac{\partial x}{\partial\theta}(t;\theta)>0,\qquad\text{for all }(\theta,t)\in(-\infty,x^*+a]\times[0,T].\tag{4.22}$$

To prove our result we will show that, if $u$ is not of type (I), then either it is of type (II) or it satisfies the property of the theorem. Let us first consider the case when $x(\cdot;x^*{+}a)$ is a classical characteristic in the whole interval $[0, T)$. Then inequality (4.22) implies that $u$ is smooth in the region $A \setminus \{(X, T)\}$, where $A = \{(x, t) \ : \ t \in [0, T], x < x(t; x^* + a)\}$. In fact, suppose that $u$ is discontinuous at some point in $A$. Then there exist two distinct backward characteristics $x(\cdot; \theta_1)$ and $x(\cdot; \theta_2)$ starting at that point. Since these characteristics cannot cross $x(\cdot; x^* + a)$, we have $\theta_1, \theta_2 < x^* + a$. But two such curves cannot intersect by (4.22). The contradiction shows that $u$ is smooth in $A$, which is a backward neighbourhood of $(X, T)$, and so we are in the smooth case.

Suppose instead that $x(\cdot; x^* + a)$ is not a characteristic in the whole interval $[0, T)$. This means that the forward generalized characteristic $\sigma(\cdot)$ starting from $(x^* + a, 0)$ coincides with $x(\cdot; x^* + a)$ in some time interval $[0, T - d]$ and is a shock in $(T - d, T)$. Since $u$ is not of type (I), $\xi$ is classical in $[0, T)$ and so $\sigma(t) > \xi(t)$ for any $t \in [T - d, T)$. Using the argument of the first part of the proof, we see that $u$ is smooth in the region $B \setminus \{(X, T)\}$, where $B = \{(x, t) \ : \ t \in [0, T], x < \sigma(t)\}$. Let us call $Y = \lim_{t \to T} \sigma(t)$. If $Y > X$ the region $B$ is again a backward neighbourhood of $(X, T)$, while if $Y = X$ the assertion of the theorem holds. The case $Y < X$ cannot occur since $\sigma(0) > \xi(0)$ and therefore $Y \geq \xi(T) = X$. ∎

We now study the blowup profile for solutions described by the previous theorem. The analysis in this case is similar to the shock case in the region to the right of $\sigma$, whereas the asymptotic profile to the left of $\sigma$ can be studied as in Theorem 4.7. We state without proof the following results.

**Theorem 4.11** *Let assumptions* (H1)–(H4) *be satisfied and let $g$ have supercritical growth. Suppose that $u$ is of type* (III) *and let $\sigma : [T - d, T) \to \mathbb{R}$ be a shock curve such that $\sigma(t) \to X$ as $t \to T$ and that $u$ is smooth in $\{(x, t) \ : \ t \in [T - d, T], x < \sigma(t)\}$. Then there exists a backward neighbourhood $N$ of $(X, T)$ such that $u$ satisfies* (4.12) *in the region $\{(x, t) \in N \ : \ x < \sigma(t)\}$ and is uniformly bounded in $\{(x, t) \in N \ : \ x > \sigma(t)\}$.*

We can then compute the asymptotics of the shock curve $\sigma(\cdot)$ using (4.12) and the Rankine–Hugoniot relation. When $f(u) = u^m/m$ and $g$ is exponential or power–like, we obtain the following result, after computations similar to those of the shock case. If $g(u) = e^u$, then

$$\lim_{t \to T^-} \frac{X - \sigma(t)}{(T - t)[-\ln(T - t)]^{m-1}} = 1;$$

whereas if $g(u) = u^p$ with $p > 2m - 1$, then

$$\lim_{t \to T^-} \frac{X - \sigma(t)}{\frac{w^{* \, p-m}}{m(p-m)}(T - t)^{\frac{p-m}{p-1}}} = 1.$$

We summarize our results in the following

**Corollary 4.12** *Suppose that assumptions* (H1)–(H4), (4.8) *hold and let the solution of* (1.1)–(1.2) *blow up at time $T$. If the source $g$ has subcritical growth with respect to $f$ (see Definition 4.1), then the solution is of type* (I) *and the blowup pattern is given by Theorem 3.2. If $g$ has supercritical growth, then the solution can be of type* (I), (II) *or* (III)*, and its asymptotic profile is given by Theorems 3.2, 4.7 and 4.11 respectively.*

We conclude this section with some remarks on the critical case. Suppose that $f'(u) \to +\infty$ as $u \to +\infty$ and that the functions $f, g$ in the equation are such that the quantity in (4.1) tends to some limit $l \in (0, +\infty)$ (a typical example is provided by $f(u) = u^m$, $g(u) = u^{2m-1}$, with $m > 1$). Then we see from Lemma 4.2 that the value of $u_0''(x^*)$ gives some information about the possible structure of $u$ near the blowup point. For instance, if $-u_0''(x^*) > g(u_0(x^*))l$ (i.e. the initial value is too "spiky" near $x^*$) then we see as in Corollary 4.3 that no smooth blowup is possible. Conversely, if we have $-u_0''(x^*) < g(u_0(x^*))l$ then smooth blowup is possible and we can compute the asymptotic pattern of the solution as in Theorem 4.7. The analysis in this case is more complex because Lemma 4.5 does not apply and we have to take into account some terms that are negligible in the supercritical case. On the other hand, the result of Theorem 4.10 is still valid and so the analysis of case (III) can be obtained from the smooth case as in Theorem 4.11. The case when $-u_0''(x^*) = g(u_0(x^*))l$ is again a borderline case where all types of blowup are possible and will not be treated here.

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