MONOTONE DIFFERENCE APPROXIMATIONS OF BV SOLUTIONS TO DEGENERATE CONVECTION-DIFFUSION EQUATIONS

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Abstract. We consider consistent, conservative-form, monotone finite difference schemes for nonlinear convection-diffusion equations in one space dimension. Since we allow the diffusion term to be strongly degenerate, solutions can be discontinuous and are in general not uniquely determined by their data. Here we choose to work with weak solutions that belong to the BV (in space and time) class and, in addition, satisfy an entropy condition. A recent result of Wu and Yin [30] states that these so-called BV entropy weak solutions are unique. The class of equations under consideration is very large and contains, to mention only a few, the heat equation, the porous medium equation, the two phase flow equation and hyperbolic conservation laws. The difference schemes are shown to converge to the unique BV entropy weak solution of the problem. In view of the classical theory for monotone difference approximations of conservation laws, the main difficulty in obtaining a similar convergence theory in the present context is to show that the approximations are L^1 Lipschitz continuous in the time variable (this is trivial for conservation laws). This continuity result is in turn intimately related to the regularity properties possessed by the (strongly degenerate) discrete diffusion term. We provide the necessary regularity estimates on the diffusion term by deriving and carefully analysing a linear difference equation satisfied by the numerical flux of the difference schemes.

§1. Introduction.

We are interested in monotone finite difference approximations of nonlinear, possibly strongly degenerate, convection-diffusion problems of form

$$(1)\qquad \begin{cases} \partial_t u + \partial_x f(u) = \partial_x(k(u)\partial_x u), & (x,t) \in Q_T \equiv \mathbb{R} \times \langle 0, T\rangle, \qquad k(u) \geq 0, \\ u(x,0) = u_0(x), \end{cases}$$

where the initial condition $u_0(x)$, the convection flux $f(u)$ and the diffusion flux $k(u) \geq 0$ are given, sufficiently regular functions. Convection-diffusion equations arise in a variety of applications, among others turbulence, traffic flow, financial modelling, front propagation, two phase flow in oil reservoirs, and in models describing certain sedimentation processes.

When (1) is *non-degenerate*, i.e., $k(u) > 0$, it is well known that (1) admits a unique classical solution [21]. This contrasts with the degenerate case where $k(u)$ may vanish for some values of u. A simple example of a degenerate equation is the porous medium equation,

$$(2)\qquad \partial_t u = \partial_x (u^m), \qquad m > 1,$$

1991 *Mathematics Subject Classification.* 65M12, 35K65, 35L65.

Key words and phrases. Degenerate convection-diffusion equations, BV solutions, entropy condition, monotone finite difference schemes, convergence.

Karlsen has been supported by VISTA, a research cooperation between the Norwegian Academy of Science and Letters and Den norske stats oljeselskap a.s. (Statoil).

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

which degenerates at $u = 0$. In general, a manifestation of the degeneracy in (2) is the finite speed of propagation of disturbances; that is, if at some fixed time the solution u has compact support, then it will continue to have compact support for all later times. The transition from a region where $u > 0$ to one where $u = 0$ is not smooth and it is therefore necessary to deal with (continuous) weak solutions rather than classical solutions. We refer to the book [24] for a nice overview of the theory of degenerate equations.

An essential condition for uniqueness of weak solutions in the class of bounded and measurable functions is that the function

$$K(u) = \int_0^u k(\xi)\, d\xi$$

is strictly increasing in u, which is also sufficient for the existence of continuous solutions, see Zhao [32]. A sufficient condition for $K(u)$ to be strictly increasing is that

$$\operatorname{meas}\{u : k(u) = 0\} = 0, \tag{3}$$

which does not rule out the possibility that $k(u)$ has an infinite number of zero points. Accordingly, we refer to the problem (1) as *degenerate* if the condition (3) holds.

If the condition (3) is not satisfied, i.e., if there exists a least one interval $[\alpha, \beta]$ such that

$$k(u) = 0, \qquad \text{for all } u \in [\alpha, \beta],$$

we say that the parabolic problem (1) is *strongly degenerate*. A simple example of a strongly degenerate equation is a hyperbolic conservation law

$$\partial_t u + \partial_x f(u) = 0. \tag{4}$$

Strongly degenerate equations will in general possess discontinuous solutions. Furthermore, discontinuous weak solutions are not uniquely determined by their data. In fact, an additional condition is needed to single out the physically relevant weak solution of the problem. We call a bounded measurable function $u(x,t)$ an *entropy weak solution* if

(a) $\partial_t |u - c| + \partial_x \left[\operatorname{sgn}(u - c)\big(f(u) - f(c)\big)\right] + \partial_x^2 |K(u) - K(c)| \le 0 \quad \text{(weakly)}.$

Letting $c \to \pm\infty$ in (a), it is clear that entropy weak solutions are also weak solutions. It is not difficult to construct an entropy weak solution of (1), even in several space dimensions, see Volpert and Hudjaev [29]. An entropy weak solution can also be constructed as the limit of monotone difference approximations. However, the main open question seems to be the uniqueness of such solutions, even in one space dimension. On the other hand, uniqueness of weak solutions for the purely parabolic case (no convection term) in the class of bounded integrable functions has been proved by Brezis and Crandall [2], while uniqueness of entropy weak solutions for conservation laws is a classical result due to Kruzkov [18]. Since a general uniqueness result for mixed hyperbolic-parabolic equations is lacking, we have chosen to seek solutions in the smaller class containing the BV entropy weak solutions. We call a bounded measurable function $u(x,t)$ a *BV entropy weak solution* if

(b) $u(x,t) \in BV(Q_T)$ and $\partial_x K(u) \in L^1_{\text{loc}}(Q_T)$,

(c) $\partial_t |u - c| + \partial_x \left[\operatorname{sgn}(u - c)\big(f(u) - f(c) - \partial_x K(u)\big)\right] \le 0 \quad \text{(weakly)}.$

What makes this class interesting is that a uniqueness result for solutions in the sense of (b) and (c) has recently been proved by Wu and Yin [30]; see §2 for a precise statement the result. Their proof depends heavily on the theory of BV functions of several variables and geometric measures. Here one should note that the jump conditions proposed by Volpert and Hudjaev [29] are in general not correct, and thus the uniqueness proof presented there is incomplete, see [30] for more details. The theory developed in [30] has also been used to treat various boundary value problems, see [4,31]. Particularly interesting is the problem analysed by Bürger and Wendland [3,4], which is used to model the settling and consolidation of a flocculated suspension under the influence of gravity (a certain sedimentation process).

It seems to be a common opinion that by adding a 'diffusion' term to a conservation law, one obtains an equation that is (in some sense) 'easier' than the conservation law itself. This is indeed true if the diffusion term is non-degenerate. However, if the diffusion term is allowed to strongly degenerate, the solution of the resulting convection-diffusion equation has a more complex structure than the solution of the conservation law. The following example demonstrates this. Let $f(u) = u^2$ and let $k(u)$ be the continuous function given by

$$k(u) = \begin{cases} 0, & \text{for } u \in [0, 0.5], \\ 2.5u - 1.25 & \text{for } u \in \langle 0.5, 0.6 \rangle, \\ 0.25 & \text{for } u \in [0.6, 1.0]. \end{cases} \tag{5}$$

Note that $k(u)$ degenerates on the interval $[0, 0.5]$. In Figure 1 we have plotted the solution of the conservation law (4) and the solution of (1) at time $T = 0.15$.

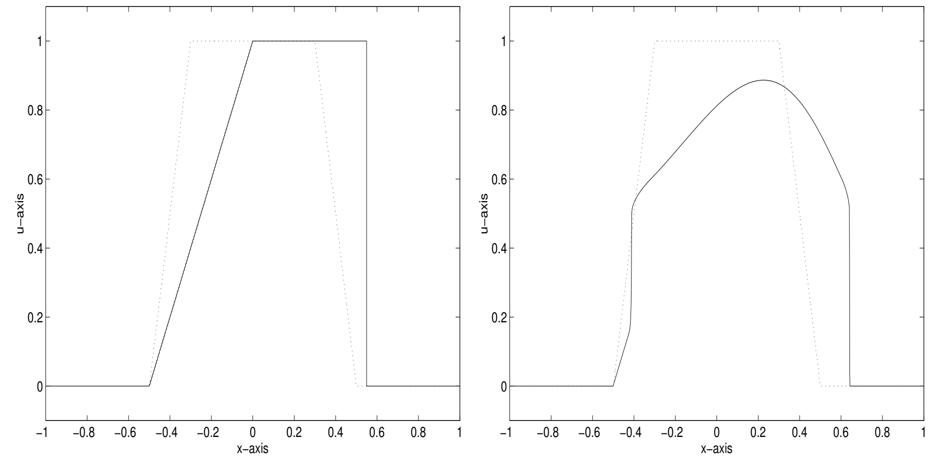

Figure 1. Left: The solution of Burgers' equation (solid). Right: The solution of Burgers' equation with a strongly degenerate diffusion term (solid). The initial function (dotted) is the same in both plots.

An interesting observation is that the solution of (1) has a 'new' increasing jump (shock), despite of the fact that f is convex. Thus the solution is not bounded in the Lip^+ norm, as opposed to the solution of the conservation law. We refer to Tadmor [26] (and the references therein) for a discussion of the Lip^+ norm and the importance of this norm in the theory of conservation laws. Moreover, while the speed of a jump in the conservation law solution is determined solely by $f(u)$ through the Rankine-Hugoniot condition, the speed of a jump in the solution of (1) is in general determined by the jumps in both $f(u)$ and $\partial_x K(u)$; see §2 for precise statements of the jump conditions for (1). Finally, let us mention that techniques developed by Kruzkov [18] (stability) and later Kuznetsov [19] (error estimates) do not apply to (BV) entropy weak solutions of problems such as (1).

The analysis of numerical schemes for problems such as (1) has so far mainly been concerned with one or two point degenerate equations and often only the 'convection free' case. We refer to [10,13,15,16,22,23] for analysis of some finite element and difference schemes within this context. In this paper we present a rather general convergence theory for a large class of difference schemes, which also applies to strongly degenerate problems.

Selecting a mesh size $\Delta x > 0$, a time step $\Delta t > 0$ and an integer N so that $N\Delta t = T$, the value of our difference approximation at $(j\Delta x, n\Delta t)$ will be denoted by U_j^n. Capital letters U, V etc. will always denote functions on the mesh $\{j\Delta x : j \in \mathbb{Z}\}$. To simplify the notation, we introduce the finite difference operators

$$D_- U_j = \frac{1}{\Delta x}(U_j - U_{j-1}), \qquad D_+ U_j = \frac{1}{\Delta x}(U_{j+1} - U_j).$$

A novel feature of our difference schemes is that they will be based on differencing the conservative-form equation

$$\partial_t u + \partial_x \big(f(u) - \partial_x K(u) \big) = 0. \tag{6}$$

We consider consistent, conservative, monotone finite difference schemes of the form

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + D_- \left(F(U_{j-p+1}^n, \dots, U_{j+p}^n) - D_+ K(U_j^n) \right) = 0, \tag{7}$$

The main purpose of this paper is to show that (7) converges to the unique BV entropy weak solution of the strongly degenerate problem (1). By combining the arguments developed in this paper with the Crandall and Liggett theory [8] it is possible to give an elegant treatment of implicit schemes as well, see [12] for details. To put this work in a proper perspective, let us make some comments about the hyperbolic case (4). Harten, Hyman and Lax [14] proved that if the monotone difference approximations converge as $\Delta x, \Delta t \to 0$, they converge to the unique entropy weak solution of the conservation law. Kuznetsov [19] proved that monotone schemes for conservation laws converge to the entropy solution in several space dimensions and provided suitable error estimates. Later, Crandall and Majda [7] proved a similar result without the error estimates. Sanders [25] proved convergence (with error estimates) for certain three-point monotone schemes with variable spatial differencing.

The class of functions in which we seek solutions in this paper (see Definition 2.1), which represents a slight modification of the class used in [30], is significantly smaller than the class of entropy weak solutions, see (a) above. From this point of view, we stress that it is non-trivial to show that the monotone difference schemes produce solutions contained in this class. To complement this claim, entropy weak solutions constructed by viscous operator splitting are not in this class, since they are only L^1 Hölder continuous in time and thus not contained in $BV(Q_T)$, see [12]. Our main source of inspiration is the theory developed by Crandall and Majda [7]. However, compared with their theory, the main difficulty in obtaining a similar convergence theory in the present context is indeed to show that the approximations are L^1 Lipschitz continuous in the time variable. This continuity result is in turn intimately related to the regularity properties possessed by the discrete diffusion term. We obtain the necessary regularity estimates on the discrete diffusion term by analysing a certain linear difference equation which governs the behaviour of the total numerical flux of the schemes, see Lemmas 3.4 and 3.6 for details.

For completeness, let us give an example of a (three-point) monotone scheme. For a monotone flux f, the upwind scheme is defined by

$$F\left(U_j^n, U_{j+1}^n\right) = f\left(U_j^n\right) \text{ if } f' \geq 0, \qquad F\left(U_j^n, U_{j+1}^n\right) = f\left(U_{j+1}^n\right) \text{ if } f' < 0. \tag{8}$$

More generally, for a non-monotone flux f, the generalised upwind scheme of Engquist and Osher is defined by

$$F\left(U_j^n, U_{j+1}^n\right) = f^+(U_j^n) + f^-(U_{j+1}^n),$$

where

$$f^+(u) = f(0) + \int_0^u \max\left(f'(s), 0\right) ds, \qquad f^-(u) = \int_0^u \min\left(f'(s), 0\right) ds$$

A simple calculation reveals that the upwind scheme and the generalised upwind scheme for (1) are monotone provided the following CFL condition holds

$$\max|f'|\frac{\Delta t}{\Delta x} + 2\max|k|\frac{\Delta t}{\Delta x^2} \leq 1.$$

The monotone schemes devised in this paper are based on differencing the conservation-form equation (6) and not the equation in its original form. Of course, one can devise schemes based on differencing (1) directly, yielding, for example, schemes of the form

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + D_-\left(F(U_{j-p+1}^n, \ldots, U_{j+p}^n) - k(U_{j+1/2}^n)D_+U_j^n\right) = 0, \tag{9}$$

where $U_{j+1/2}^n = \frac{1}{2}\left(U_j^n + U_{j+1}^n\right)$. Although it is possible to prove that (9) converges to a limit, we have not been able to show that this limit satisfies an entropy condition. In fact, we do not believe that (9) will converge to the physically correct solution in the case of strong degeneracy. To support this view we now present a simple numerical example with fluxes $\tilde{f}(u) = \frac{1}{4}u^2$ and $\tilde{k}(u) = 4k(u)$, where k is given in (5). In Figure 2 we have plotted the initial function and the solutions produced (using very small discretization parameters) by the schemes (7) and (9) at three different times. In these calculations the upwind flux (8) was used as the convective numerical flux in the schemes (7) and (9). Clearly, the non-conservative scheme (9) produces a wrong solution. Moreover, the 'difference' between this solution and the correct solution produced by (7) seems to increase with time. We are currectly investigating this phenomenon and will come back to it in a separate report.

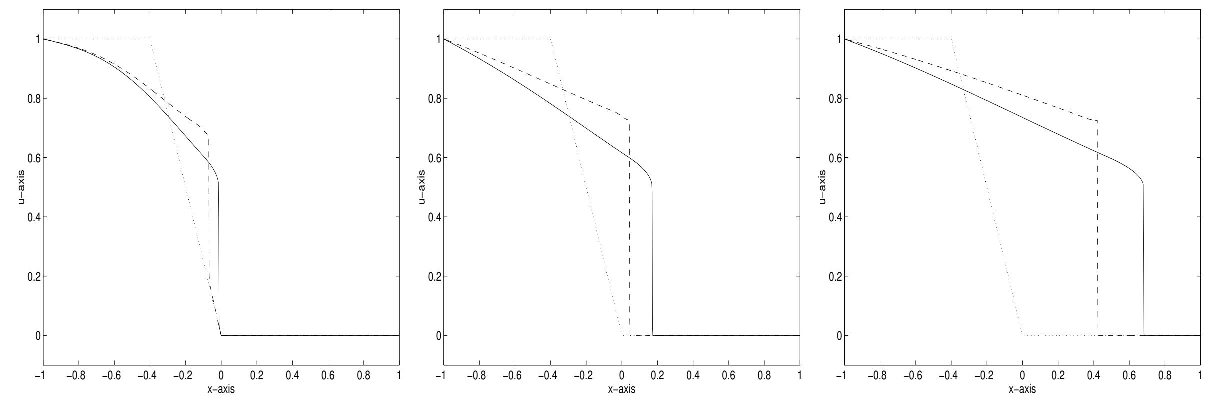

Figure 2. The solutions produced by the schemes (7) (solid) and (9) (dashed) plotted at the three different times $T_1 = 0.0625$, $T_2 = 0.25$ and $T = 1.0$. The initial function is shown as dotted.

The rest of this paper is organized as follows: In §2 we give a brief survey of the known mathematical theory of one-dimensional strongly degenerate equations, while in §3 we present the convergence analysis of the monotone schemes (7).

§2. Mathematical Preliminaries.

We shall here briefly recall the known mathematical theory of nonlinear strongly degenerate parabolic equations. Let Ω be an open subset of $\mathbb{R}^d$ ($d > 1$). The space $BV(\Omega)$ of functions of bounded variation consists of all $L^1_{\mathrm{loc}}(\Omega)$ functions $u(y)$ whose first order partial derivatives $\frac{\partial u}{\partial y_1}, \dots, \frac{\partial u}{\partial y_d}$ are represented by (locally) finite Borel measures. The total variation $|u|_{BV(\Omega)}$ is by definition the sum of the total masses of these Borel measures. Moreover, $BV(\Omega)$ is a Banach space when equipped with the norm $||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + |u|_{BV(\Omega)}$. It is well known that the inclusion $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$ holds for $d > 1$ and that $BV(\Omega) \subset L^\infty(\Omega)$ for $d = 1$. Furthermore, $BV(\Omega)$ is compactly imbedded into the spaces $L^q(\Omega)$ for $1 \leq q < d/(d-1)$. Finally, we will also need the Hölder space $C^{1,\frac{1}{2}}(Q_T)$ consisting of bounded functions $z(x,t)$ on $\mathbb{R} \times [0,T]$ that satisfy

$$|z(y,\tau) - z(x,t)| \leq L\big(|y-x| + \sqrt{|\tau - t|}\big), \qquad \forall x, y, t, \tau,$$

for some constant $L > 0$ (not depending on x, y, t, τ).

Here we seek generalised solutions to the problem (1) in the following sense:

Definition 2.1. *A bounded measurable function $u(x,t)$ is said to be a BV entropy weak solution of the initial value problem (1) provided the following two conditions hold:*

1. *$u(x,t) \in BV(Q_T)$ and $K(u) \in C^{1,\frac{1}{2}}(Q_T)$.*

2. *For all non-negative $\phi \in C_0^\infty(Q_T)$ with $\phi|_{t=T} = 0$ and any $c \in \mathbb{R}$, the following entropy inequality holds*

$$\begin{aligned}(10)\qquad &\iint\limits_{Q_T} \Big(|u-c|\partial_t\phi + \mathrm{sgn}(u-c)\big(f(u) - f(c) - \partial_x K(u)\big)\partial_x\phi\Big)\,dt\,dx \\ &\qquad + \int\limits_{\mathbb{R}} |u_0 - c|\phi(x,0)\,dx \geq 0.\end{aligned}$$

Remark. *First, in the context of hyperbolic equations ($k \equiv 0$), the entropy condition (10) coincides with the celebrated entropy condition due to Volpert* [28], *see also Kruzkov* [18]. *Secondly, note that $K(u) \in C^{1,\frac{1}{2}}(Q_T)$ implies that the weak derivative $\partial_x K(u)$ is in $L^\infty(Q_T)$, which in turn implies that $\partial_x K(u) \in L^1_{\mathrm{loc}}(Q_T)$. Hence, solutions in the sense of Definition 2.1 are also solutions in the sense of Wu and Yin* [30].

As pointed out earlier, entropy weak solutions of (1) can in general be discontinuous. The jump conditions take the following (correct!) form [30]:

Theorem 2.2 [30]**.** *Let Γ_u be the set of jumps of $u(x,t)$; $\nu = (\nu_t, \nu_x)$ the unit normal to Γ_u; $u^-(x_0,t_0)$ and $u^+(x_0,t_0)$ the approximate limits of u at $(x_0,t_0) \in \Gamma_u$ from the sides of the half-planes $(t-t_0)\nu_t + (x-x_0)\nu_x < 0$ and $(t-t_0)\nu_t + (x-x_0)\nu_x > 0$ respectively; $u^l(x,t)$ and $u^r(x,t)$ denote the left and right approximate limits of $u(\cdot,t)$ respectively. Introduce the notations* $\mathrm{sgn}^+ := \mathrm{sgn}$ *and* $\mathrm{sgn}^- := \mathrm{sgn}^+ - 1$, *and let $int(a,b)$ denote the closed interval* *bounded by a and b. Finally, let H_1 denote the one-dimensional Hausdorff measure. Then H_1 - almost everywhere on Γ_u,*

$$k(u) = 0, \qquad \forall u \in int(u^-, u^+), \qquad \nu_x \neq 0 \tag{11}$$

$$(u^+ - u^-)\nu_t + \big(f(u^+) - f(u^-)\big)\nu_x - \big(\partial_x K(u)^r - \partial_x K(u)^l\big)|\nu_x| = 0, \tag{12}$$

(13)
$$\begin{aligned}&\big|u^+ - c\big|\nu_t + \operatorname{sgn}(u^+ - c)\big[f(u^+) - f(c) - \big(\partial_x K(u)^r \operatorname{sgn}^+ \nu_x - \partial_x K(u)^l \operatorname{sgn}^- \nu_x\big)\big]\nu_x \\ &\quad \le \big|u^- - c\big|\nu_t + \operatorname{sgn}(u^- - c)\big[f(u^-) - f(c) - \big(\partial_x K(u)^l \operatorname{sgn}^+ \nu_x - \partial_x K(u)^r \operatorname{sgn}^- \nu_x\big)\big]\nu_x.\end{aligned}$$

These jump conditions are essential ingredients in the proof of the following L^1 stability theorem, which is proved in [30]:

Theorem 2.3 [30]. *Let u_1 and u_2 be BV entropy weak solutions of (1) with initial data $u_{0,1}$ and $u_{0,2}$ respectively. Then for any $t > 0$*

$$\int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)|\,dx \le \int_{\mathbb{R}} |u_{0,1}(x) - u_{0,2}(x)|\,dx.$$

The uniqueness of BV entropy weak solutions of the problem (1) is an immediate consequence of the above theorem. Cockburn and Gripenberg [6] have used the theory of Crandall and Liggett [8] to construct semigroup (generalised) solutions of multi-dimensional degenerate convection-diffusion equations. Furthermore, they have proved that these semigroup solutions depend continuously on the nonlinear fluxes of the problem (see below). Now observe that since 'parabolic regularizations' are smooth the semigroup solution of (1) coincides with the viscosity solution of (1). Moreover, it turns out that the viscosity solution of (1) is also a solution in the sense of Definition 2.1 (this follows from [29] and Theorem 3.11 in this paper). Hence, the semigroup solution conincides with the unique BV entropy weak solution in the case of one-dimensional equations and we have:

Theorem 2.4 [6]. *Let u_1, u_2 be BV entropy weak solutions of (1) with initial data $u_{0,1}, u_{0,2}$, convective fluxes f_1, f_2 and diffusive fluxes k_1, k_2 respectively. Furthermore, suppose that $m \le u_{0,1}, u_{0,2} \le M$ and put $C = \min\big(|u_{0,1}|_{BV(\mathbb{R})}, |u_{0,2}|_{BV(\mathbb{R})}\big)$. Then for any $t > 0$*

$$\begin{aligned}\int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)|\,dx &\le \int_{\mathbb{R}} |u_{0,1}(x) - u_{0,2}(x)|\,dx \\ &\quad + C\Big(t \sup_{u\in[m,M]} \big|f_1'(u) - f_2'\big| + 4\sqrt{t} \sup_{u\in[m,M]} \big|\sqrt{k_1} - \sqrt{k_2}\big|\Big).\end{aligned}$$

Finally, we note that the jump conditions in Theorem 2.2 can be more instructively stated as follows:

Corollary 2.5. *Assume that $k(u) = 0$ for $u \in [u_*, u^*]$ for some $u_*, u^* \in [m, M]$. Let $u(x,t)$ be a BV entropy weak solution of (1) and let Γ_u be a smooth discontinuity curve of $u(x,t)$. A jump between two values u^- and u^+ of the solution $u(x,t)$, which we refer to as a shock, can occur only for $u^-, u^+ \in [u_*, u^*]$. This shock must satisfy the following two conditions:*

1. *The shock speed s is given by*

$$s = \frac{f(u^+) - f(u^-) - \Big(\lim_{x\to x_0+} \partial_x K(u) - \lim_{x\to x_0-} \partial_x K(u) \Big)}{u^+ - u^-}.$$

2. *For all $u \in int(u^-, u^+)$, the following entropy condition holds*

$$\frac{f(u^+) - f(u) - \lim_{x\to x_0+} \partial_x K(u)}{u^+ - u} \le s \le \frac{f(u^-) - f(u) - \lim_{x\to x_0-} \partial_x K(u)}{u^- - u}.$$

Proof. In the following we have scaled $\nu = (\nu_t, \nu_x)$ so that $\nu_x > 0$. The first assertion follows directly from (12) since $s = -\frac{\nu_t}{\nu_x}$. For the second assertion, we introduce the symmetric means

$$\bar{u} = \frac{u^+ + u^-}{2}, \quad \overline{f(u)} = \frac{f(u)^+ + f(u)^-}{2}, \quad \widetilde{\partial_x K(u)} = \frac{(\partial_x K(u))^l + (\partial_x K(u))^r}{2}, \tag{14}$$

and then note that by using (12) we can change (13) into the form

$$\big(\mathrm{sgn}(u^+ - c) - \mathrm{sgn}(u^- - c)\big)\big[(\bar{u} - c)\nu_t + \big(\overline{f(u)} - f(c)\big)\nu_x - \widetilde{\partial_x K(u)}\nu_x\big] \le 0. \tag{15}$$

Clearly, $\mathrm{sgn}(u^+ - c) = -\,\mathrm{sgn}(u^- - c)$ for any $c \in \mathrm{int}(u^-, u^+)$. Inserting this identity into (15) and using the definition of the symmetric means (14), we get

$$\begin{aligned} \mathrm{sgn}(u^+ - c)\big[(u^+ + u^- - 2c)\nu_t + \big(f(u^+) + f(u^-) - 2f(c)\big)\nu_x \\ - \big(\partial_x K(u)^r + \partial_x K(u)^l\big)\nu_x\big] \le 0. \end{aligned}$$

From this inequality we obtain

$$\begin{aligned} &\mathrm{sgn}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\nu_x - \partial_x K(u)^r \nu_x\big] \\ &\quad \le -\,\mathrm{sgn}(u^+ - c)\big[(u^- - c)\nu_t + \big(f(u^-) - f(c)\big)\nu_x - \partial_x K(u)^l \nu_x\big] \\ &\quad = -\,\mathrm{sgn}(u^+ - c)\big[(u^- - u^+)\nu_t + \big(f(u^-) - f(u^+)\big)\nu_x - \big(\partial_x K(u)^l - \partial_x K(u)^r\big)\nu_x\big] \\ &\qquad - \mathrm{sgn}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\nu_x - \partial_x K(u)^r \nu_x\big]. \end{aligned}$$

The first term on the right-hand side of the last equality is zero due to the jump condition (12), and we therefore have

$$\mathrm{sgn}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\nu_x - \partial_x K(u)^r \nu_x\big] \le 0.$$

From this we easily get

$$\frac{f(u^+) - f(c) - (\partial_x K(u))^r}{u^+ - c} \le -\frac{\nu_t}{\nu_x} = s,$$

and the first inequality is proved. The second inequality follows similarly. □

Remark. *Note that in general* $\lim_{x\to x_0\pm}\partial_x K(u)$ *is unknown a priori, which implies that the propagation of a shock cannot be predicted a priori. This contrasts with what is known from the theory of hyperbolic conservation laws (the Rankine-Hugoniot condition).*

§3. Convergence Analysis.

In this section we analyse the monotone difference schemes. Implicit versions of these schemes are analysed in [12]. In the following treat the case where u_0 has compact support and f, K are locally C^1. Then towards the end of this section we will briefly discuss the general case where u_0 is not necessarily compactly supported and f, K are locally Lipschitz continuous. If not otherwise stated, we will always assume, without loss of generality, that $f(0) = 0$. The function space that contains u_0 will be taken as

$$\mathcal{B}(f,K) = \left\{ z \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : |f(z) - \partial_x K(z)|_{BV(\mathbb{R})} < \infty \right\}. \tag{16}$$

Letting $F(U;j)$ denote the convective numerical flux, i.e,

$$F(U;j) := F(U^n_{j-p+1}, \dots, U^n_{j+p}),$$

the schemes under consideration takes the form

$$\begin{cases} \dfrac{U^{n+1}_j - U^n_j}{\Delta t} + D_-\left(F(U^n;j) - D_+K(U^n_j)\right) = 0, & (j,n) \in \mathbb{Z} \times \{0,\dots,N\}, \\ U^0_j = \dfrac{1}{\Delta x}\displaystyle\int_{j\Delta x}^{(j+1)\Delta x} u_0(x)\,dx, & j \in \mathbb{Z}. \end{cases} \tag{17}$$

To make the schemes (17) consistent with the convection-diffusion equation (1) it is sufficient to require that

$$F(u,\dots,u) = f(u).$$

The assumption of monotoncity guarantees that (17), when viewed as an algorithm of the form (supressing the Δx and Δt dependency)

$$U^{n+1}_j = \mathcal{S}(U^n_{j-p+1}, \dots, U^n_{j+p}) =: \mathcal{S}(U^n;j), \tag{18}$$

has the property that $\mathcal{S}$ is a non-decreasing function of all its arguments.

For later use, recall that the $L^\infty(\mathbb{Z})$ norm, the $L^1(\mathbb{Z})$ norm and the $BV(\mathbb{Z})$ semi-norm of a lattice function U are defined respectively as

$$\begin{aligned} \|U\|_{L^\infty(\mathbb{Z})} &= \sup_{j\in\mathbb{Z}} |U_j|, \\ \|U\|_{L^1(\mathbb{Z})} &= \sum_{j\in\mathbb{Z}} |U_j|, \\ |U|_{BV(\mathbb{Z})} &= \sum_{j\in\mathbb{Z}} |U_j - U_{j-1}| \equiv \|D_-U\|_{L^1(\mathbb{Z})}. \end{aligned}$$

If not specified, i, j will always denote integers from $\mathbb{Z}$; m, n, l integers from $\{0,\dots,N\}$; x, y, c real numbers from $\mathbb{R}$ and t, τ real numbers from $[0,T]$. Furthermore, C will denote a generic positive constant that can depend on the data of the problem but not on $\Delta x, \Delta t$.

We shall need the following lemma due to Crandall and Tartar [9]:

Lemma 3.1 [9]**.** *Let $(\Omega, d\mu)$ be a measure space. If the operator $\mathcal{T} : L^1(\Omega) \to L^1(\Omega)$ satisfies $\int_\Omega \mathcal{T}(u)\, d\mu = \int_\Omega u\, d\mu$, then $\mathcal{T}$ is a contraction on $L^1(\Omega)$ if and only if $\mathcal{T}$ is monotone.*

We shall also need the following lemma, which is due to Lucier [20].

Lemma 3.2 [20]**.** *If $\mathcal{T}$ maps $L^1(\mathbb{Z})$ or $L^1(\mathbb{R})$ to itself, preserves the integral and commutes with translations, $\mathcal{T}$ satisfies the minimum principle and the maximum principles, that is,*

$$\liminf \mathcal{T}(u)(x) \geq \liminf u(x), \qquad \limsup \mathcal{T}(u)(x) \leq \limsup u(x).$$

In a series of lemmas we will provide uniform (in $\Delta x, \Delta t$) a priori estimates on the difference approximations. The first lemma gives the classical L^∞ and BV (in space) estimates.

Lemma 3.3. *We have*

$$\left\|U^n\right\|_{L^\infty(\mathbb{Z})} \leq \left\|U^0\right\|_{L^\infty(\mathbb{Z})}, \qquad \left|U^n\right|_{BV(\mathbb{Z})} \leq \left|U^0\right|_{BV(\mathbb{Z})}. \tag{19}$$

Proof. Recall that we can rewrite the difference approximation (17) as $U^{n+1} = \mathcal{S}(U^n)$ where $\mathcal{S} : L^1(\mathbb{Z}) \to L^1(\mathbb{Z})$ maps sequences $U = \{U_j\}$ to sequences according to the formula

$$\mathcal{S}(U;j) = U_j - \Delta t D_- \left(F(U^n;j) - D_+ K(U_j^n)\right).$$

Since the difference approximation has compact support, we get $\sum_{j\in\mathbb{Z}} \mathcal{S}(U;j) = \sum_{j\in\mathbb{Z}} U_j$. Thanks to Lemmas 3.1 and 3.2, the lemma now follows since $\mathcal{S}$ is monotone and obviously commutes with translations. $\square$

The next lemma (see also Lemma 3.6), which eventually will lead to the desired regularity properties possessed by the diffusion term $K(u)$, plays a key role in our convergence analysis and has no counterpart in the theory of monotone schemes for conservation laws as developed by Harten et al. [14], and later by Crandall and Majda [7]. Let us for the moment assume that (1) is non-degenerate. Following Tadmor and Tassa [27], by differentiating (6) with respect to t and subsequently integrating with respect to x, we find that

$$\partial_t v + a(x,t)\partial_x v = \partial_x(b(x,t)\partial_x v), \qquad v(x,t) = \int_{-\infty}^{x} \partial_t u(\xi,t)\, d\xi, \tag{20}$$

where $a = f'(u)$ and $b = k(u)$. This is a non-degenerate linear parabolic equation with smooth bounded coefficients, which has a unique smooth solution $v(x,t)$ satisfying

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|v(\cdot,0)\|_{L^\infty(\mathbb{R})}, \qquad \|v(\cdot,t)\|_{BV(\mathbb{R})} \leq \|v(\cdot,0)\|_{BV(\mathbb{R})}.$$

Thus, since $v = f(u) - \partial_x K(u)$, we get uniform $L^\infty(\mathbb{R})$ and $BV(\mathbb{R})$ estimates on $\partial_x K(u(\cdot,t))$. However, this is merely formalism since the solution to (1) in general only exists in a weak sense, but these calculations clearly motivate similar results for the finite difference approximations (see also Theorem 3.11).

Lemma 3.4. *We have*

$$\left\| F(U^n;j) - D_+K(U_j^n) \right\|_{L^\infty(\mathbb{Z})} \leq \left\| F(U^0;) - D_+K(U_j^0) \right\|_{L^\infty(\mathbb{Z})}, \tag{21}$$

$$\left| F(U^n;j) - D_+K(U_j^n) \right|_{BV(\mathbb{Z})} \leq \left| F(U^0;j) - D_+K(U_j^0) \right|_{BV(\mathbb{Z})}. \tag{22}$$

Proof. To make the calculations more transparent and the notation simpler, we are going to write out the proof of this lemma only for the three-point schemes

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + D_- \left(F(U_j^n, U_{j+1}^n) - D_+K(U_j^n) \right) = 0. \tag{23}$$

The proof in the general case (17) is similar to the three-point case, but the notation is messier. Let F_u and F_v denote the partial derivatives of $F = F(u,v)$ with respect to the first argument and the second argument respectively. A simple calculation will reveal that (23) is monotone provided the following conditions hold:

$$F_u(r_1,r_2) + \frac{1}{\Delta x}k(r_3) \geq 0, \qquad \frac{1}{\Delta x}k(r_3) - F_v(r_1,r_2) \geq 0, \qquad \forall (r_1,r_2,r_3), \tag{24}$$

$$1 - \frac{\Delta t}{\Delta x}\left(F_u(r_1,r_2) - F_v(r_3,r_1)\right) - 2\frac{\Delta t}{\Delta x^2}k(r_4) \geq 0, \qquad \forall (r_1,r_2,r_3,r_4). \tag{25}$$

Note that a sufficient condition for (24) to hold is that $F_u(r_1,r_2) \geq 0$ and $F_v(r_1,r_2) \leq 0$ for all (r_1,r_2). Let us begin with proving (21). To this end, we define the quantity

$$V_j^n = \Delta x \sum_{i=-\infty}^{j} \left(\frac{U_i^n - U_i^{n-1}}{\Delta t} \right). \tag{26}$$

Multiplying the difference equation (23) evaluated at $i\Delta x$ by Δx and subsequently summing over $i = -\infty, \dots, j$, we get the relation

$$V_j^{n+1} = -\left(F(U_j^n, U_{j+1}^n) - D_+K(U_j^n) \right). \tag{27}$$

Next we derive an equation for the quantity $\{V_j^n\}$. For this purpose consider the difference equation (23) evaluated at $i\Delta x$ and subtract the corresponding equation at time $n\Delta t$, yielding

$$\begin{aligned} &\frac{(U_i^{n+1} - U_i^n)}{\Delta t} - \frac{(U_i^n - U_i^{n-1})}{\Delta t} \\ &\qquad + D_-\left(\left(F(U_i^n, U_{i+1}^n) - F((U_i^{n-1}, U_{i+1}^n) \right) - D_+\left(K(U_i^n) - K(U_i^{n-1}) \right) \right) = 0. \end{aligned}$$

Multiplying this equality by Δx and then summing over $i = -\infty, \dots, j$, yields

$$\left(V_j^{n+1} - V_j^n \right) + \left(F(U_j^n, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^n) \right) - D_+\left(K(U_j^n) - K(U_j^{n-1}) \right) = 0.$$

Observe that

$$\frac{U_j^n - U_j^{n-1}}{\Delta t} = \frac{1}{\Delta x}\left[\Delta x \sum_{i=-\infty}^{j}\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}\right) - \Delta x \sum_{i=-\infty}^{j-1}\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}\right)\right] = D_- V_j^n.$$

Having this identity in mind, we can rewrite as follows

$$\begin{aligned}
&F(U_j^n, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^{n-1}) \\
&\quad \equiv \big(F(U_j^n, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^n)\big) + \big(F(U_j^{n-1}, U_{j+1}^n) - F(U_j^{n-1}, U_{j+1}^{n-1})\big) \\
&\quad = F_u(\alpha_j^n, U_{j+1}^n)\big(U_j^n - U_j^{n-1}\big) + F_v(U_j^{n-1}, \tilde{\alpha}_{j+1}^n)\big(U_{j+1}^n - U_{j+1}^{n-1}\big) \\
&\quad = \Delta t a_{u,j}^n D_- V_j^n + \Delta t a_{v,j}^n D_- V_{j+1}^n,
\end{aligned}$$

where

$$a_{u,j}^n = F_u(\alpha_j^n, U_{j+1}^n), \qquad a_{v,j}^n = F_v(U_j^{n-1}, \tilde{\alpha}_{j+1}^n), \qquad \alpha_j^n, \tilde{\alpha}_j^n \in \operatorname{int}(U_j^{n-1}, U_j^n). \tag{28}$$

Similarly, we can write

$$K(U_j^n) - K(U_j^{n-1}) = k(\beta_j^n)\big(U_j^n - U_j^{n-1}\big) = \Delta t b_j^n D_- V_j^n,$$

where

$$b_j^n = k(\beta_j^n), \qquad \beta_j^n \in \operatorname{int}(U_j^{n-1}, U_j^n). \tag{29}$$

Summing up, we see that the sequence $\{V_j^n\}$ satisfies the linear difference equation

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + \big(a_{u,j}^n D_- V_j^n + a_{v,j}^n D_- V_{j+1}^n\big) = D_+\big(b_j^n D_- V_j^n\big). \tag{30}$$

We will now show that the solution of (30) satisfies a maximum principle. To this end, observe that (30) can be written as

$$V_j^{n+1} = A_j^n V_{j-1}^n + B_j^n V_j^n + C_j^n V_{j+1}^n, \tag{31}$$

where

$$\begin{aligned}
A_j^n &= \Big[\frac{\Delta t}{\Delta x} a_{u,j}^n + \frac{\Delta t}{\Delta x^2} b_j^n\Big], \\
B_j^n &= \Big[1 - \frac{\Delta t}{\Delta x}\left(a_{u,j}^n - a_{v,j}^n\right) - \frac{\Delta t}{\Delta x^2}\left(b_j^n + b_{j+1}^n\right)\Big], \\
C_j^n &= \Big[\frac{\Delta t}{\Delta x^2} b_{j+1}^n - \frac{\Delta t}{\Delta x} a_{v,j}^n\Big].
\end{aligned}$$

Since (24) and (25) are assumed to hold,

$$A_j^n,\ B_j^n,\ C_j^n \geq 0, \qquad A_j^n + B_j^n + C_j^n \equiv 1.$$

Consequently, we obtain from (31) that

$$\sup_{j\in\mathbb{Z}}\big|V_j^{n+1}\big| \leq \sup_{j\in\mathbb{Z}}\big|V_j^n\big| \leq \cdots \leq \sup_{j\in\mathbb{Z}}\big|V_j^1\big|.$$

In view of the relation (27), we can immediately conclude that (21) is true.

Next, we prove that the solution of (30) has bounded variation on $\mathbb{Z}$. Introduce the quantity $Z_j^n = V_j^n - V_{j-1}^n$ and observe that

$$\frac{Z_j^{n+1} - Z_j^n}{\Delta t} + D_-\big(a_{u,j}^n Z_j^n + a_{v,j}^n Z_{j+1}^n\big) = D_- D_+\big(b_j^n Z_j^n\big).$$

Similarly to (31), we can write this equation as

$$Z_j^{n+1} = \bar{A}_j^n Z_{j-1}^n + \bar{B}_j^n Z_j^n + \bar{C}_j^n Z_{j+1}^n, \tag{32}$$

where

$$\begin{aligned}
\bar{A}_j^n &= \Big[\frac{\Delta t}{\Delta x} a_{u,j-1}^n + \frac{\Delta t}{\Delta x^2} b_{j-1}^n\Big],\\
\bar{B}_j^n &= \Big[1 - \frac{\Delta t}{\Delta x}\left(a_{u,j}^n - a_{v,j-1}^n\right) - 2\frac{\Delta t}{\Delta x^2} b_j^n\Big],\\
\bar{C}_j^n &= \Big[\frac{\Delta t}{\Delta x^2} b_{j+1}^n - \frac{\Delta t}{\Delta x} a_{v,j}^n\Big].
\end{aligned}$$

Since (24) and (25) are again assumed to hold,

$$\bar{A}_j^n,\ \bar{B}_j^n,\ \bar{C}_j^n \geq 0, \qquad \bar{A}_{j+1}^n + \bar{B}_j^n + \bar{C}_{j-1}^n \equiv 1.$$

We can thus derive from (32) that

$$\sum_{j\in\mathbb{Z}} \big|Z_j^{n+1}\big| \leq \sum_{j\in\mathbb{Z}} \big(\bar{A}_{j+1}^n + \bar{B}_j^n + \bar{C}_{j-1}^n\big)\big|Z_j^n\big| \equiv \sum_{j\in\mathbb{Z}} \big|Z_j^n\big| \leq \cdots \leq \sum_{j\in\mathbb{Z}} \big|Z_j^1\big|,$$

which immediately implies (22). This concludes the proof of the lemma. $\square$

A direct consequence of the previous lemma is that the difference approximations are L^1 Lipschitz continuous in the time variable (and thus in BV in both space and time).

Lemma 3.5. *We have*

$$\big\|U^m - U^n\big\|_{L^1(\mathbb{Z})} \leq \big|F(U^0; j) - D_+ K(U_j^0)\big|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x} |m - n|. \tag{33}$$

Proof. Suppose that $m > n$. Using (17), we readily calculate that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \big|U_j^m - U_j^n\big| &\leq \sum_{l=n}^{m-1} \sum_{j\in\mathbb{Z}} \big|U_j^{l+1} - U_j^l\big| \leq \Delta t \sum_{l=n}^{m-1} \sum_{j\in\mathbb{Z}} \big|D_-\big(F(U^l; j) - D_+ K(U_j^l)\big)\big|\\
&\leq \big|F(U^0; j) - D_+ K(U_j^0)\big|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x}(m - n).
\end{aligned}$$

where the BV estimate (22) has been used. This concludes the proof of the lemma. $\square$

Let us now return to the formal discussion which led to the uniform L^∞ and BV estimates on $\partial_x K(u(\cdot, t))$ in the case of non-degeneracy. As we will see, it is possible to use the BV estimate to derive a result concerning also the continuity of $K(u)$ with respect to the time variable. To this end, we shall employ a technique introduced by Kruzkov [17] to derive a modulus of continuity in time from a known modulus of continuity in space of certain parabolic equations. Let $\phi(x)$ be a test function on $\mathbb{R}$. Multiplying (20) by ϕ, integrating the result in space and subsequently doing integrating by parts on one of the terms, yields

$$\left|\int_{\mathbb{R}} \phi(x)\partial_t v\, dx\right| \leq \left(\|a\|_{L^\infty(Q_T)}\|\phi\|_{L^\infty(\mathbb{R})} + \|b\|_{L^\infty(Q_T)}\|\phi'\|_{L^\infty(\mathbb{R})}\right)\int_{\mathbb{R}} |\partial_x v(\cdot,t)|\, dx.$$

From this estimate we get the following weak continuity result

$$\left|\int_{\mathbb{R}} \phi(x)\big(v(x,\tau)-v(x,t)\big)\, dx\right| = \mathcal{O}(1)\big(\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}\big)(\tau-t),$$

where we have taken into account that $|v(\cdot,t)|_{BV(\mathbb{R})} \leq |v(\cdot,0)|_{BV(\mathbb{R})} < \infty$ and the uniform boundedness of $a = a(x,t)$, $b = b(x,t)$. It is not difficult, using a suitable approximation of $\operatorname{sgn}(v(\cdot,\tau)-v(\cdot,t))$ (see below), to conclude from this that

$$\|v(\cdot,\tau)-v(\cdot,t)\|_{L^1(\mathbb{R})} = \mathcal{O}(1)\sqrt{|\tau-t|}.$$

Now, since u is L^1 Lipschitz continuous in the time variable and $v = f(u) - \partial_x K(u)$, it follows that

$$\|\partial_x K(u(\cdot,\tau)) - \partial_x K(u(\cdot,t))\|_{L^1(\mathbb{R})} = \mathcal{O}(1)\sqrt{|\tau-t|}.$$

Observe that

$$K(u(x,t)) = \int_{-\infty}^{x} \partial_x K(u(\xi,t))\, d\xi,$$

which implies the desired Hölder result

$$\|K(u(\cdot,\tau)) - K(u(\cdot,t))\|_{L^\infty(\mathbb{R})} \leq \|\partial_x K(u(\cdot,\tau)) - \partial_x K(u(\cdot,t))\|_{L^1(\mathbb{R})} = \mathcal{O}(1)\sqrt{|\tau-t|}.$$

Again this is merely formalism since the solution of (1) is in general non-smooth. However, our next lemma states that a Hölder estimate on the discrete diffusion term is indeed true.

Lemma 3.6. *We have*

$$\left|K(U_i^m) - K(U_j^n)\right| \leq C\left(|(i-j)\Delta x| + \sqrt{|(m-n)\Delta t|}\right). \tag{34}$$

Proof. We will write out the proof of this lemma only for three-point schemes, for which the proof is essentially to apply a discrete version of Kruzkov's technique [17] to the parabolic difference equation (30). Again, the proof in the general case (17) is similar to the three-point case, but the notation is messier. First, notice that

$$\left|K(U_i^m) - K(U_j^n)\right| \leq \left|K(U_i^m) - K(U_j^m)\right| + \left|K(U_j^m) - K(U_j^n)\right| =: I_1 + I_2.$$

In view of Lemma 3.4, $\left\|D_+K(U^m)\right\|_{L^\infty(\mathbb{Z})} = \mathcal{O}(1)$ and therefore $I_1 = \mathcal{O}(1)|(i-j)\Delta x|$.

Next, we wish to bound I_2. To this end, let $\phi(x)$ be a test function, put $\phi_j = \phi(j\Delta x)$ and let $m < n$. Using the difference equation (30) and summation by parts, we get

$$
\begin{aligned}
\Big|\Delta x \sum_{j\in\mathbb{Z}} \phi_j\big(V_j^m - V_j^n\big)\Big| &= \Delta x \Big|\sum_{l=n}^{m-1}\sum_{j\in\mathbb{Z}} \phi_j\big(V_j^{l+1} - V_j^l\big)\Big| \\
&\le \Delta x\Delta t\Big|\sum_{l=n}^{m-1}\sum_{j\in\mathbb{Z}} \phi_j\big(a_{u,j}^l D_- V_j^l + a_{v,j}^l D_- V_{j+1}^l\big)\Big| + \Delta x\Delta t\Big|\sum_{l=n}^{m-1}\sum_{j\in\mathbb{Z}} D_-\phi_j\big(b_j^l D_- V_j^l\big)\Big| \qquad (35)\\
&\le \Big(\|\phi\|_{L^\infty(\mathbb{R})}\big(\sup_{j,l}\big|a_{u,j}^l\big| + \sup_{j,l}\big|a_{v,j}^l\big|\big) + \|\phi'\|_{L^\infty}\sup_{j,l}\big|b_j^l\big|\Big)\sup_l \big|V^l\big|_{BV(\mathbb{Z})}\,\Delta t(m-n) \\
&= \mathcal{O}(1)\big(\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}\big)\Delta t(m-n),
\end{aligned}
$$

since $a_{u,j}^l$, $a_{v,j}^l$ and $\big|V^l\big|_{BV(\mathbb{Z})}$ are all uniformly bounded quantities. Next, introduce the function

$$
\beta(x) = \begin{cases} \operatorname{sgn}\Big(\sum_{j\in\mathbb{Z}}\big(V_j^m - V_j^n\big)\chi_j(x)\Big), & \text{for } |x| \le J-\rho, \\ 0, & \text{for } |x| > J-\rho, \end{cases}
$$

where χ_j denotes the characteristic function of $[j\Delta x, (j+1)\Delta x\rangle$ and $J \in \mathbb{Z}$. Let $\omega_\rho(x)$ be a standard C_0^∞ - mollifier given by $\omega_\rho(x) = \frac{1}{\rho}\omega(\frac{x}{\rho})$, where

$$
\omega(x) \in C_0^\infty(\mathbb{R}), \quad \omega(x) \ge 0, \quad \omega(x) = 0 \ \text{ for } |x| \ge 1, \quad \int_{\mathbb{R}} \omega(x)dx = 1.
$$

Let $\beta^\rho = \omega_\rho * \beta$ and observe that

$$
\beta^\rho \in C_0^\infty(-J,J), \quad \|\beta^\rho\|_{L^\infty(\mathbb{R})} \le 1, \quad \|\left(\beta^\rho\right)'\|_{L^\infty(\mathbb{R})} = \mathcal{O}(1/\rho), \quad \|\beta^\rho\|_{L^1(\mathbb{R})} = 1.
$$

Notice that

$$
\begin{aligned}
\Delta x \sum_{j=-J}^{J} \big|V_j^m - V_j^n\big| \le \Delta x \sum_{j=-J}^{J} \big||V_j^m - V_j^n| - \beta^\rho(j\Delta x)\big(V_j^m - V_j^n\big)\big| \\
+ \Big|\Delta x \sum_{j=-J}^{J} \beta^\rho(j\Delta x)\big(V_j^m - V_j^n\big)\Big| =: Q_1 + Q_2.
\end{aligned}
$$

To simplify the notation, let $E_j = V_j^m - V_j^n$ and observe that $|E|_{BV(\mathbb{Z})}$ is bounded. Choose two integers $l = l(j,\rho)$ and $r = r(j,\rho)$ such that $[l\Delta x, r\Delta x]$ becomes the smallest interval containing $[j\Delta x - \rho, j\Delta x + \rho]$. Then

$$
\begin{aligned}
Q_1 = \Delta x \sum_{j=-J}^{J} \big||E_j| - \beta^\rho(j\Delta x)E_j\big| &= \Delta x \sum_{j=-J}^{J} \Big|\int_{\mathbb{R}} w_\rho(j\Delta x - y)\big(|E_j| - \beta(y)E_j\big)\Big|\, dy \\
&\le \Delta x \sum_{j=-J}^{J}\sum_{i=l}^{r}\int_{x_i}^{x_{i+1}} w_\rho(j\Delta x - y)\big||E_j| - \operatorname{sgn}(E_i)E_j\big|\, dy \\
&\le 2\Delta x \sum_{j=-J}^{J}\sum_{i=l}^{r}\int_{x_i}^{x_{i+1}} w_\rho(j\Delta x - y)\big|E_j - E_i\big|\, dy \\
&\le 2\sum_{j=-J}^{J}(r-l)\Delta x \max_{i\in\{l,\dots,r\}}\big|E_j - E_i\big| = \mathcal{O}(1)\rho\sum_{j\in\mathbb{Z}}\max_{i\in\{l,\dots,r\}}\big|E_j - E_i\big| = \mathcal{O}(1)\rho,
\end{aligned}
$$

where we have used that $\left|E\right|_{BV(\mathbb{Z})} < \infty$. Next, using (35) with $\phi_j = \beta^\rho(j\Delta x)$, we have

$$Q_2 = \mathcal{O}(1)(m-n)\Delta t/\rho.$$

Hence, for some constants C_1 and C_2 not depending on $\Delta x, \Delta t$, it follows that

$$\Delta x \sum_{j=-J}^{J} \left|V_j^m - V_j^n\right| \leq C_1\rho + C_2(m-n)\Delta t/\rho.$$

Choosing $\rho = \sqrt{(m-n)\Delta t}$ and letting $J \to \infty$, we obtain

$$\Delta x \sum_{j\in\mathbb{Z}} \left|V_j^m - V_j^n\right| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.$$

On the other hand, from the relation (27) and Lemma 3.5, we also have

$$\begin{aligned}\Delta x \sum_{j\in\mathbb{Z}} \left|V_j^m - V_j^n\right| &= \mathcal{O}(1)\Delta x \sum_{j\in\mathbb{Z}} \left|U_j^m - U_j^n\right| + \Delta x \sum_{j\in\mathbb{Z}} \left|D_+K(U_j^m) - D_+K(U_j^n)\right| \\ &= \mathcal{O}(1)(m-n)\Delta t + \Delta x \sum_{j\in\mathbb{Z}} \left|D_+K(U_j^m) - D_+K(U_j^n)\right|.\end{aligned}$$

We thus conclude that

$$\Delta x \sum_{j\in\mathbb{Z}} \left|D_+K(U_j^m) - D_+K(U_j^n)\right| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.$$

From this the desired Hölder estimate in time follows,

$$\begin{aligned}I_2 = \left|K(U_j^m) - K(U_j^n)\right| &= \Delta x \left| \sum_{i=-\infty}^{j} D_+K(U_i^m) - \sum_{i=-\infty}^{j} D_+K(U_i^n)\right| \\ &\leq \Delta x \sum_{i\in\mathbb{Z}} \left|D_+K(U_i^m) - D_+K(U_i^n)\right| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.\end{aligned}$$

This concludes the proof of (34). $\square$

In what follows, we need the standard notations $u \vee v = \max(u,v)$ and $u \wedge v = \min(u,v)$.

Lemma 3.7. *The following cell entropy inequality holds*

$$\frac{\left|U_j^{n+1} - c\right| - \left|U_j^n - c\right|}{\Delta t} + D_-\big(F(U^n \vee c; j) - F(U^n \wedge c; j) - D_+\left|K(U_j^n) - K(c)\right|\big) \leq 0.$$

Proof. Crandall and Majda [7] showed how to naturally get a cell entropy inequality in the purely hyperbolic case, see also [14]. As we will see, this construction applies to the mixed hyperbolic-parabolic case as well. First, a direct calculation yields the equality

$$\begin{aligned}\left|U_j^n - c\right| &- \Delta t D_-\big(F(U^n \vee c; j) - F(U^n \wedge c; j) - D_+\left|K(U_j^n) - K(c)\right|\big) \\ &= \mathcal{S}(U^n \vee c; j) - \mathcal{S}(U^n \wedge c; j),\end{aligned} \tag{36}$$

where $\mathcal{S}$ is defined by (18). Next, by monotonicity of the scheme (17),

$$\mathcal{S}(U^n \vee c; j) - \mathcal{S}(U^n \wedge c; j) \geq \mathcal{S}(U^n; j) \vee c - \mathcal{S}(U^n; j) \wedge c = \left|U_j^{n+1} - c\right|,$$

which inserted into (36) produces the desired cell entropy inequality. □

Let u_Δ (where $\Delta = (\Delta x, \Delta t)$) be the interpolant of degree one associated with the discrete data points $\{U_j^n\}$; that is, u_Δ interpolates at the vertices of each rectangle

$$R_j^n = [j\Delta x, (j+1)\Delta x] \times [n\Delta t, (n+1)\Delta t].$$

Note that u_Δ is continuous everywhere, differentiable almost everywhere, and inside each rectangle R_j^n it is explicitly given by the formula

$$\begin{aligned} u_\Delta(x,t) = U_j^n &+ \left(U_{j+1}^n - U_j^n\right)\left(\frac{x - j\Delta x}{\Delta x}\right) + \left(U_j^{n+1} - U_j^n\right)\left(\frac{t - n\Delta t}{\Delta t}\right) \\ &+ \left(U_{j+1}^{n+1} - U_j^{n+1} - U_{j+1}^n + U_j^n\right)\left(\frac{x - j\Delta x}{\Delta x}\right)\left(\frac{t - n\Delta t}{\Delta t}\right). \end{aligned} \tag{37}$$

We have the following compactness results:

Lemma 3.8. *Let $\{\Delta\}$ be a sequence of discretization parameters tending to zero. Then there exists a subsequence $\{\Delta_j\}$ such that $\{u_{\Delta_j}\}$ converges in $L^1_{\mathrm{loc}}(Q_T)$ and pointwise almost everywhere in Q_T to a limit u as $j \to \infty$,*

$$u \in L^\infty(Q_T) \cap BV(Q_T).$$

Furthermore, $\{K(u_{\Delta_j})\}$ converges uniformly on compacta $\mathcal{K} \subset Q_T$ to $K(u)$ as $j \to \infty$,

$$K(u) \in C^{1,\frac{1}{2}}(Q_T).$$

Proof. From (37) and Lemma 3.1, we get that u_Δ is uniformly bounded by $\|u_0\|_{L^\infty(\mathbb{R})}$. Using Lemma 3.3, we get that

$$\begin{aligned} \iint_{Q_T} |\partial_x u_\Delta| \, dt\, dx &\leq \sum_{j,n} \iint_{R_j^n} \frac{1}{\Delta x}\left(1 - \frac{t - n\Delta t}{\Delta t}\right)\left|U_{j+1}^n - U_j^n\right| dt\, dx \\ &\quad + \sum_{j,n} \iint_{R_j^n} \frac{1}{\Delta x}\left(\frac{t - n\Delta t}{\Delta t}\right)\left|U_{j+1}^{n+1} - U_j^{n+1}\right| dt\, dx \\ &\leq \frac{\Delta t}{2}\sum_{n,j}\left|U_{j+1}^n - U_j^n\right| + \frac{\Delta t}{2}\sum_{n,j}\left|U_{j+1}^{n+1} - U_j^{n+1}\right| \leq T\left|U^0\right|_{BV(\mathbb{Z})}. \end{aligned}$$

Similarly, from (37) and Lemma 3.5, we also obtain that

$$\begin{aligned} \iint_{Q_T} |\partial_t u_\Delta| \, dt\, dx &\leq \sum_{j,n} \iint_{R_j^n} \frac{1}{\Delta t}\left(1 - \frac{x - j\Delta x}{\Delta x}\right)\left|U_j^{n+1} - U_j^n\right| dt\, dx \\ &\quad + \sum_{j,n} \iint_{R_j^n} \frac{1}{\Delta t}\left(\frac{x - j\Delta x}{\Delta x}\right)\left|U_{j+1}^{n+1} - U_{j+1}^n\right| dt\, dx \\ &\leq \frac{\Delta x}{2}\sum_{n,j}\left|U_j^{n+1} - U_j^n\right| + \frac{\Delta x}{2}\sum_{n,j}\left|U_{j+1}^{n+1} - U_{j+1}^{n+1}\right| \\ &\leq T\left|F(U^0; j) - D_+ K(U_j^0)\right|_{BV(\mathbb{Z})}. \end{aligned}$$

Consequently, there is a finite constant $C = C(T) > 0$ (independent of Δ) such that

$$\|u_\Delta\|_{L^\infty(Q_T)} \leq C, \qquad |u_\Delta|_{BV(Q_T)} \leq C. \tag{38}$$

These estimates show that $\{u_\Delta\}$ is bounded in $BV(\mathcal{K})$ for any compact set $\mathcal{K} \subset Q_T$. Since $BV(\mathcal{K})$ is compactly imbedded into the space $L^1(\mathcal{K})$, it is possible to select a subsequence that converges in $L^1(\mathcal{K})$ and pointwise almost everywhere in $\mathcal{K}$. Furthermore, using a standard diagonal process, we can construct a sequence that converges in $L^1_{\mathrm{loc}}(Q_T)$ and pointwise almost everywhere in Q_T to a limit u,

$$u \in L^\infty(Q_T) \cap BV(Q_T).$$

Next, we analyse the sequence $\{K(u_\Delta)\}$. To this end, let $w_\Delta = K(u_\Delta)$. Thanks to (37), (19) and (34), w_Δ is continuous everywhere, uniformly bounded and satisfies

$$|w_\Delta(i\Delta x, m\Delta t) - w_\Delta(j\Delta x, n\Delta t)| = \mathcal{O}(1)\big(|(i-j)\Delta x| + \sqrt{|(m-n)\Delta t|}\big). \tag{39}$$

Let (x,t) and (y,τ) be some given coordinates and choose integers (j,n) and (i,m) such that $(x,t) \in R_j^n$ and $(y,\tau) \in R_i^m$ (here R_j^n and R_i^m may coincide). Then

$$\begin{aligned}|w_\Delta(y,\tau) - w_\Delta(x,t)| &\leq |w_\Delta(y,\tau) - w_\Delta(i\Delta x, m\Delta t)| + |w_\Delta(i\Delta x, m\Delta t) - w_\Delta(j\Delta x, n\Delta t)| \\ &\quad + |w_\Delta(j\Delta x, n\Delta t) - w_\Delta(x,t)| =: I_1 + I_2 + I_3.\end{aligned}$$

From (39) we know that $I_2 = \mathcal{O}(1)\big(|(i-j)\Delta x| + \sqrt{|(m-n)\Delta t|}\big)$. Since $K(u)$ is non-decreasing in u, we get $I_1 + I_3 = \mathcal{O}(1)\big(\Delta x + \sqrt{\Delta t}\big)$. Consequently, we have arrived at

$$|w_\Delta(y,\tau) - w_\Delta(x,t)| \leq C\big(|y-x| + \sqrt{|\tau - t|} + \Delta x + \sqrt{\Delta t}\big),$$

where $C > 0$ is a finite constant not depending on Δ, x, y, t, τ.

Now, by repeating the proof of the Ascoli-Arzela compactness theorem, we deduce the existence of a subsequence of $\{w_\Delta\}$ converging uniformly on each compactum $\mathcal{K} \subset Q_T$ to a limit w,

$$w \in C^{1,\frac{1}{2}}(Q_T).$$

Let $\{\Delta_j\}$ be a sequence of discretization parameters tending to zero such that $u_{\Delta_j} \to u$ and $w_{\Delta_j} \to w$ as $j \to \infty$ (such a sequence can certainly be found in view of the previous discussion). Since u_{Δ_j} converges poinwise almost everywhere to u and w is continuous,

$$w = K(u).$$

This concludes the proof of the lemma. $\square$

In view of Lemma 3.8 and Theorem 2.3, we can assume that the sequences $\{u_\Delta\}$ and $\{K(u_\Delta)\}$ themselves converge to u and $K(u)$ respectively. We continue by showing that the limit u satisfies the entropy condition (10). Let $\phi(x,t)$ be a suitable test function and put $\phi_j^n = \phi(j\Delta x, n\Delta t)$. Multiplying the cell entropy inequality in Lemma 3.7 by $\phi_j^n \Delta x$, summing over all j,n and applying summation by parts, we get

$$\begin{aligned}\Delta x \Delta t \sum_{j\in\mathbb{Z}} \sum_{n=0}^{N-1} \Big(\big|U_j^{n+1} - c\big| \frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + \big(F(U^n \vee c; j) - F(U^n \wedge c; j)\big) D_+\phi_j^n \\ + \big|K(U_j^n) - K(c)\big| D_- D_+ \phi_j^n \Big) + \Delta x \sum_{j\in\mathbb{Z}} \big|U_j^0 - c\big| \phi_j^0 \geq 0.\end{aligned} \tag{40}$$

Using (19) and that F is consistent with f, we can obviously write

$$\Delta x \sum_{j\in\mathbb{Z}} \big(F(U^n \vee c; j) - F(U^n \wedge c; j)\big) D_+\phi_j^n = \Delta x \sum_{j\in\mathbb{Z}} \operatorname{sgn}(U_j^n - c)\big(f(U_j^n) - f(c)\big) D_+\phi_j^n + \mathcal{O}(\Delta x),$$

and hence replace (40) by

$$\begin{aligned} &\iint_{Q_T} \Big(|u_\Delta - c|\partial_t\phi + \operatorname{sgn}(u_\Delta - c)\big(f(u_\Delta) - f(c)\big)\partial_x\phi + |K(u_\Delta) - K(c)|\partial_x^2\phi \Big)\, dt\, dx \\ &\qquad + \int_{\mathbb{R}} |u_0 - c|\phi(x,0)\, dx \geq -C\big(\Delta x + \Delta t\big), \end{aligned} \tag{41}$$

where $C = C(T) > 0$ is a constant not depending on Δ. Finally, after passing to the limit in (41) and then doing integration by parts, we get that the limit u satisfies (10). This completes our discussion when u_0 has compact support and f, K are locally C^1.

For $u_0 \in \mathcal{B}(f,K)$ not necessarily compactly supported and f, K merely locally Lipschitz continuous, we approximate u_0 by a compactly supported function u_0^p and f, k by a smoother function f^p, k^p, compute the difference approximation of the resulting problem and then let $p \to \infty$ and $\Delta t, \Delta x \to 0$ (see [7] for more details in the hyperbolic case).

Remark. *We can also do integration by parts in (41) (before passing to the limit), so that* $I(\Delta) := \iint_{Q_T} |K(u_\Delta) - K(c)|\partial_x^2\phi\, dt\, dx$ *becomes* $-\iint_{Q_T} \operatorname{sgn}(u_\Delta - c)\partial_x K(u_\Delta)\partial_x\phi\, dt\, dx$. *Since* $\partial_x K(u_\Delta) \overset{*}{\rightharpoonup} \partial_x K(u)$ *in* $L^\infty(Q_T)$, $\lim_{\Delta\to 0} I(\Delta) = -\iint_{Q_T} \operatorname{sgn}(u - c)\partial_x K(u)\, dt\, dx$.

We are now ready to state our main result:

Theorem 3.9. *Suppose that f, K are locally Lipschitz continuous and let $u_0 \in \mathcal{B}(f,K)$ (see (16)). Then the sequence $\{u_\Delta\}$ defined by (17) and (37) converges in $L^1_{\mathrm{loc}}(Q_T)$ and pointwise almost everywhere in Q_T to a BV entropy weak solution u of the problem*

$$\partial_t u + \partial_x f(u) = \partial_x(k(u)\partial_x u), \qquad u(x,0) = u_0(x), \qquad (x,t) \in Q_T, \qquad k(u) \geq 0.$$

Furthermore, the sequence $\{K(u_\Delta)\}$ converges uniformly on compacta $\mathcal{K} \subset Q_T$ to $K(u)$.

We let $C(0,T; L^1(\mathbb{R}))$ denote the usual Bochner space consisting of all continuous functions $u : [0,T] \to L^1(\mathbb{R})$ for which the norm $\|u\|_{C(0,T;L^1(\mathbb{R}))} = \sup_{t\in[0,T]} \|u(t)\|_{L^1(\mathbb{R})}$ is finite. A closer inspection of the arguments leading to Theorem 3.9 will reveal that $\{U_\Delta(t)\}$ converges in $C(0,T; L^1(\mathbb{R}))$ to the unique BV entropy weak solution $u(t)$, with $u(0) = u_0$, of the initial value problem (1). A reexamination of the proofs leading to Theorem 3.9 shows that we have proved the following result on existence and properties of solutions of (1):

Corollary 3.10. *Let f and K be locally Lipschitz continuous. Then for any initial function $u_0 \in \mathcal{B}(f,K)$ (see (16)) there exists a BV entropy weak solution $u \in C(0,T; L^1(\mathbb{R}))$ of the initial value problem (1). Denoting this solution by $\mathcal{S}_t u_0$, we have the following properties:*

1. *$t \to \mathcal{S}_t u_0$ is Lipschitz continuous into $L^1(\mathbb{R})$ and $\|\mathcal{S}_t u_0\|_{BV(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})}$,*
2. $\big\|\mathcal{S}_t u_0 - \mathcal{S}_t v_0\big\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1\mathbb{R})}$,
3. *$u_0 \leq v_0$ implies $\mathcal{S}_t u_0 \leq \mathcal{S}_t v_0$,*
4. *$m \leq u_0 \leq M$ implies $m \leq \mathcal{S}_t u_0 \leq M$.*

Furthermore, if $\mathcal{S}_t^1$ and S_t^2 denote the solution operators associated with the two equations

$$\partial_t u + \partial_x f_1(u) = \partial_x(k_1(u)\partial_x u),$$
$$\partial_t u + \partial_x f_2(u) = \partial_x(k_2(u)\partial_x u)$$

respectively, then the following comparison result hold (see Theorem 2.4 and [6]*):*

$$(5)\quad \left\|\mathcal{S}_t^1 u_0 - \mathcal{S}_t^2 u_0\right\|_{L^1(\mathbb{R})} \le C\Big(t\left\|f_1'(u) - f_2'\right\|_{L^\infty(m,M)} + 4\sqrt{t}\left\|\sqrt{k_1} - \sqrt{k_2}\right\|_{L^\infty(m,M)}\Big),$$

where $m \le u_0 \le M$ and $C = \min\left(|u_{0,1}|_{BV(\mathbb{R})}, |u_{0,2}|_{BV(\mathbb{R})}\right)$.

Remark. *Observe that a bounded diffusion flux $k(u)$ possessing a finite number of discontinuities is allowed by Corollary 3.10. We mention that discontinuous diffusion fluxes are of interest in applications, see for example Burger and Wendland* [3].

Finally, we will make a remark concerning the viscosity solution of (1). For any $\varepsilon > 0$, let $u_\varepsilon(x,t)$ denote the classical solution of the parabolic problem (1) with a non-degenerate diffusion coefficient $k_\varepsilon(u) = k(u) + \varepsilon$. Moreover, let

$$u(x,t) = \lim_{\varepsilon \to 0} u_\varepsilon(x,t)$$

denote the viscosity solution of the strongly degenerate problem (1), see [29]. In view of the formal discussion before Lemmas 3.4 and 3.6, there is a constant $C > 0$, which is independent of ε, such that

$$|K(u_\varepsilon(y,\tau)) - K(u_\varepsilon(x,t))| \le C\left(|y-x| + \sqrt{|\tau - t|} + \varepsilon\right).$$

We can again use the Ascoli-Arzela theorem to produce a subsequence $\{K(u^{\varepsilon_j})\}$ which converges uniformly on compact sets $\mathcal{K} \subset Q_T$ to $K(u) \in C^{1,\frac{1}{2}}(Q_T)$ as $j \to \infty$. The fact that $K(u)$ is Lipschitz continuous in the space variable was first proved by Tassa [27]. This regularity is optimal as demonstrated by an example due to Barenblatt and Zeldovich [1], see [27] for more details. We have taken the (continuous) analysis in [27] a step further by showing that $K(u)$ is Hölder continuous in the time variable. A direct consequence is that the viscosity solution of (1) is also a solution of (1) in the sense of Definition 2.1.

Summing up, we have proven the following theorem, which generalises the regularity result of Tassa [27]:

Theorem 3.11 (viscosity solutions). *Let u denote the viscosity solution of (1). Then $K(u)$ is contained in the Hölder space $C^{1,\frac{1}{2}}(Q_T)$.*

Acknowledgement.
We thank Raimund Bürger and Bernardo Cockburn for interesting discussions. We also thank Knut-Andreas Lie for his reading and criticism of the manuscript.

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