Rate of convergence for the zero relaxation limit

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Abstract. Some recent results on the rate of convergence towards equilibrium for some 2×2 systems of conservation laws which include stiff relaxation terms are presented. We focus our attention on certain specific model problems. We will discuss the possibility of deriving properties like bounds on the total variation, stability with respect the initial data and error estimates for finite difference schemes which are independent of the relaxation parameters, and explain how such bounds can be used to obtain estimates on the rate of convergence for the zero relaxation limit.

1. Introduction

The main objective of this paper is to study systems of conservation laws which include stiff relaxation terms. The systems have the property that as the relaxation parameters tend to zero, the systems formally reduce to simpler equilibrium models. Properties of such relaxation systems will be illustrated by considering certain examples.

For a general introduction to systems of conservation laws with relaxation effects we refer to the book of Whitham [15]. Such models arise in traffic modeling, chromatography, water wave and gas dynamics. A more mathematical discussion of the relaxation effects was given in Liu [5], where the so-called *sub-characteristic condition* was derived. The sub–characteristic condition is considered to be a necessary condition for convergence to equilibrium.

In order to explain this condition consider the following triangular 2×2 system with a relaxation term:

$$\begin{cases} u_t + f(u,v)_x = 0, \\ v_t + g(v)_x = \dfrac{1}{\delta}(a(u) - v), \end{cases} \tag{1}$$

where the relaxation time $\delta > 0$ is a small parameter. The characteristic speeds for (1) are

$$\lambda_1 = f_u(u,v), \quad \lambda_2 = g'(v).$$

As $\delta \to 0^+$, we formally obtain the equilibrium relation $v = a(u)$, and the equilibrium equation

$$u_t + f(u, a(u))_x = 0.$$

The characteristic speed for this scalar equation is $\lambda_* = f_u + f_v a'$.

A first order asymptotic expansion w.r.t. δ for the system (1) gives the following approximation for v:

$$v \approx a(u) - \delta a'(\lambda_2 - \lambda_*)u_x,$$

which implies the higher order scalar equation

$$u_t + f(u, a(u))_x \approx \delta((\lambda_* - \lambda_1)(\lambda_2 - \lambda_*)u_x)_x.$$

Therefore, the "diffusion coefficient", $(\lambda_* - \lambda_1)(\lambda_2 - \lambda_*)$, is positive if the following sub-characteristic condition holds:

$$\lambda_1 < \lambda_* < \lambda_2. \tag{2}$$

The purpose of this paper is to discuss the zero relaxation limit for certain model problems. We are particularly interested in the rate of convergence towards equilibrium for the relaxation systems. An adsorption model in chromatography and a model in viscoelasticity will be studied in detail. For these systems we will discuss the possibility of deriving properties like TV–bounds, stability with respect to initial data and error estimates for finite difference schemes which are independent of the relaxation parameters δ. In particular, we will derive rates for the convergence of the solutions of the nonequilibrium model to the solutions of the corresponding equilibrium model.

For a more general discussion on the zero relaxation limit, we refer to Chen, Levermore and Liu [2], Natalini [7], and references given in these papers.

In Section 2 we review the results on the adsorption model in chromatography, and in Section 3 we present the results for the viscoelasticity model.

2. An adsorption model

Our first example is an adsorption model in chromatography studied in Tveito & Winther [14] and in Schroll, Tveito & Winther [10]. Here we will present the main results of these papers.

Let u denote the density of some specie contained in a fluid flowing through a fixed bed, and let v be the density of the specie absorbed on the material in the bed. If we assume that the reaction time is positive, then, following [15], a model for this process is given by *relaxation model*,

$$\begin{cases} (u+v)_t + f(u)_x = 0, \\ \\ v_t = \dfrac{1}{\delta}(A(u) - v). \end{cases} \tag{3}$$

Here the first equation expresses the conservation of mass, while the second equation models the effect of adsorption. The parameter δ denotes the reaction time. In the present study we assume that the flux $f = f(u)$ satisfies

$$f(0) = 0, \quad \text{and} \quad f'(u) \geq 0 \quad \text{for } u \in [0, 1],$$

and the adsorbtion function $A = A(u)$ satisfies

$$A(0) = 0,\ A(1) = 1, \quad \text{and } A'(u) \geq 0, \quad \text{for all } u.$$

The characteristic speeds for (3) are $\lambda_1 = 0$ and $\lambda_2 = f'(u)$. As the reaction time δ tends to zero in (3), we obtain the equilibrium relation $v = A(u)$, and the following equilibrium model:

$$(w + A(w))_t + f(w)_x = 0, \tag{4}$$

with the characteristic speed $\lambda_* = \frac{f'(w)}{1+A'(w)}$. It is easy to see that the subcharacteristic condition (2) is satisfied under the assumptions $f' > 0$ and $A' > 0$.

For the relaxation system (3), the existence of entropy solutions with bounded total variation independent of δ is established. A finite difference scheme for the system is also studied, and the convergence of the numerical solutions towards the entropy solutions is derived. Furthermore, error estimate for the numerical solutions, which are independent of δ, will be given. These results can then be used to prove convergence towards equilibrium for the relaxation model. In particular, an estimate for the rate of convergence is established.

Before we give a precise description of the main results for this model, we give formal arguments for the bounds on the total variation and the continuous dependence of the solutions with respect to perturbations of the initial data. These formal results can be established rigorously for a finite difference scheme introduced below. If we rewrite (3) on the form

$$\begin{cases} u_t + f(u)_x \;=\; -\dfrac{1}{\delta}(A(u) - v), \\[2ex] v_t = \dfrac{1}{\delta}(A(u) - v), \end{cases}$$

and differentiate this system w.r.t. x, we obtain

$$\begin{cases} [u_x]_t + [f(u)_x]_x \;=\; -\dfrac{1}{\delta}\left[A(u)_x - v_x\right], \\[2ex] [v_x]_t = \dfrac{1}{\delta}\left[A(u)_x - v_x\right]. \end{cases}$$

By multiplying the first equation by $\operatorname{sgn}(u_x)$, the second equation by $\operatorname{sgn}(v_x)$ and by integration w.r.t. x we get the estimate

$$\frac{d}{dt}\int_{\mathcal{R}} \Big(|u_x| + |v_x|\Big)\, dx = \frac{1}{\delta}\int_{\mathcal{R}} \left[A(u)_x - v_x\right] \Big[\operatorname{sgn}(v_x) - \operatorname{sgn}(u_x)\Big]\, dx \;\leq\; 0,$$

which is the desired bound on the total variation. A formal continuous dependence result can be drived in a similar manner. Let (u^1, v^1) and (u^2, v^2) be two pairs of solutions. We have

$$\begin{cases} (u^1-u^2)_t + \big[f(u^1)-f(u^2)\big]_x \;=\; -\dfrac{1}{\delta}\Big[\, \big(A(u^1)-v^1\big) - \big(A(u^2)-v^2\big)\,\Big], \\[2ex] (v^1-v^2)_t \;=\; \dfrac{1}{\delta}\Big[\, \big(A(u^1)-v^1\big) - \big(A(u^2)-v^2\big)\,\Big], \end{cases}$$

which leads to the continuous dependence estimate

$$\begin{aligned} &\Big[|u^1-u^2| + |v^1-v^2|\Big]_t + |f(u^1)-f(u^2)|_x \\ &= \frac{1}{\delta}\Big[\, \big(A(u^1)-v^1\big) - \big(A(u^2)-v^2\big)\,\Big]\cdot\Big[\mathrm{sgn}(v^1-v^2) - \mathrm{sgn}(u^1-u^2)\Big] \;\le\; 0. \end{aligned} \tag{5}$$

This discussion motivates, in particular, the following *weak entropy formulation*: For any constants (k,q), a solution pair (u,v) of (3) is required to satisfy

$$\begin{aligned} &\Big[|u-k| + |v-q|\Big]_t + |f(u)-f(k)|_x \\ &\le \frac{1}{\delta}(A(u)-v)\Big[\mathrm{sgn}(v-q) - \mathrm{sgn}(u-k)\Big]. \end{aligned} \tag{6}$$

The main tool for our study is the semi-implicit finite difference approximation

$$\begin{cases} \dfrac{(u_j^{n+1}+v_j^{n+1}) - (u_j^n+v_j^n)}{\Delta t} + \dfrac{f(u_j^n)-f(u_{j-1}^n)}{\Delta x} = 0, \\[2ex] \dfrac{v_j^{n+1}-v_j^n}{\Delta t} = \dfrac{1}{\delta}(A(u_j^{n+1}) - v_j^{n+1}). \end{cases} \tag{7}$$

Here u_j^n and v_j^n are approximations of $u(x,t)$ and $v(x,t)$ over the grid blocks

$$B_j^n = \Big[x_{j-1/2}, x_{j+1/2}\Big) \times \Big[t_n, t_{n+1}\Big),$$

where $(x_j = j\Delta x,\ t_n = n\Delta t)$, and $(\Delta x,\ \Delta t)$ are constants. We assume that the CFL-condition

$$\frac{\Delta t}{\Delta x}\max_u(f'(u)) \le 1$$

is satisfied.

The results for the solutions of the system (3) is obtained by deriving appropriate a–priori estimates on the numerical solutions. The following result is derived as a consequence of these a–priori estimates:

Theorem 2.1. *As the grid parameters $(\Delta x, \Delta t)$ tend to zero, the family $\{(u_\Delta, v_\Delta)\}$ of approximate solutions generated by the finite difference scheme (7) converges in $(L^1_{loc}(\mathcal{R}\times\mathcal{R}_0^+))^2$ towards a weak solution (u,v) of (3) which satisfies the entropy formulation in (6). Furthermore, the entropy solution (u,v) satisfies*

$$\|A(u(\cdot,t)) - v(\cdot,t)\|_1 \le M\delta \quad \textit{for } t \ge 0,$$

where M is a finite constant independent of δ. If $(\bar{u}^0, \bar{v}^0)$ is another pair of initial data, there is a unique entropy solution $(\bar{u}, \bar{v})$ such that

$$\|u(\cdot,t) - \bar{u}(\cdot,t)\|_1 + \|v(\cdot,t) - \bar{v}(\cdot,t)\|_1 \leq \|u^0 - \bar{u}^0\|_1 + \|v^0 - \bar{v}^0\|_1 \quad \text{for } t \geq 0.$$

An error estimate for the finite difference solutions was proved in [10], where a bound of order $\mathcal{O}(\Delta t)^{1/2}$ measured in L^1 was derived.

Theorem 2.2. *Let (u, v) be the entropy solution of (3) and (u^N, v^N) be a piecewise constant representation of the data $\{u_j^N, v_j^N\}$ generated by the scheme (7). Then, for any fixed $T = N\Delta t > 0$ there is a finite constant M independent of Δt, Δx and δ such that*

$$\|u(\cdot,T) - u^N\|_1 + \|v(\cdot,T) - v^N\|_1 \leq M\sqrt{\Delta t}.$$

The rate of convergence towards equilibrium for the relaxation model (3) is derived in both [14] and [10], using two independent arguments. The first one, established in [14], is based on a parabolic regularization of the system (3). The second one, introduced in [10], utilize the error estimate for the finite difference solutions stated in Theorem 2.2. The following theorem was proved.

Theorem 2.3. *Let (u^0, v^0) and w^0 be the initial data satisfying $w^0 = u^0$ and $\|A(u^0) - v^0\|_1 \leq M\delta$, and let (u, v) and w be solutions of (3) and (4) respectively. Then, for any finite $T > 0$, there is a finite constant M such that*

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \leq M\delta^{1/3} \quad \text{for all } 0 \leq t \leq T.$$

For details of the proofs for these results, we refer to [14, 10].

Remark 1. Some results on an improved rate of convergence can be found in a recent work of Kurganov and Tadmor [4]. If $f(u)$ and $A(u)$ are convex, they establish that

$$\|u(\cdot,t) - w(\cdot,t)\|_1 \leq M\delta^{1/2}.$$

This result is based on Tadmor's results on *Lip'*–stability (cf. [13, 4]) and therefore some convexity assumption is needed. However, for the case with nonconvex flux, it seems that $\mathcal{O}(\delta^{1/3})$ is still the best convergence result which is known.

Remark 2. A more general 2×2 system of conservation laws with relaxation terms in two space dimensions was studied in Shen, Tveito & Winther [11], where an error bound of order $\mathcal{O}(\delta^{1/3})$ is proved. This bound is derived by similar technique as in [14, 10], combined with dimensional splitting.

3. A model in viscoelasticity

Our second example is the Cauchy problem for a model in viscoelasticity

$$\begin{cases} u_t + \sigma_x = 0, \\ (\sigma - f(u))_t + \dfrac{1}{\delta}(\sigma - \mu f(u)) = 0, \end{cases} \tag{8}$$

where $u(x,t), \sigma(x,t)$ are the unknown functions and μ, δ are parameters such that $\mu \in (0,1)$ and $0 < \delta << 1$. As the relaxation parameter δ tends to zero we formally obtain the equilibrium model

$$\bar{u}_t + \mu f(\bar{u})_x = 0. \tag{9}$$

The system (8) arises in modeling of motions of a viscoelastic solid, where the relaxation phenomenon presents the strength of memory. The Riemann problem for the system with $\delta = 1$ was studied by Greenberg & Hsiao [3]. The zero relaxation limit of this viscoelasticity model with vanishing memory was analyzed in the fundamental paper of Chen & Liu [1], where nonlinear stability in the zero relaxation limit was established for the model. This is achieved by first deriving energy estimates from proper construction of entropy pairs, and then applying the theory of compensated compactness. More recent results can be found in the paper by Chen, Levermore & Liu [2].

In a recent paper [12], we have established similar results, but in the BV-framework. For any positive value of the relaxation parameter, we proved the existence of a BV-solution of the system. The bound on the total variation of the solutions, and a suitable stability estimate with respect to perturbations of the initial data in L^1, are both independent of the relaxation parameter. Furthermore, a uniform Lip^+–bound, similar to Oleinik's entropy condition, is obtained. By following the approach of Kurganov and Tadmor [4], this bound is used to establish an $\mathcal{O}(\sqrt{\delta})$ estimate for the L^1–difference between the solutions of the relaxation system (8) and the the solutions of the equilibrium model (9). The zero relaxation limit for the model (8) has also been studied by Yong [16] and Luo & Natalini [6]. However, these papers do not derive a rate for the convergence to equilibrium.

In the rest of this section we present the results from [12] in more detail. The flux function $f = f(u)$ in (8) is a smooth function with the following properties:

$$f(0) = 0, \quad f'(u) > 0,\ f''(u) \geq 0 \quad \text{for all } u \geq 0.$$

We introduce the variable $v = f(u) - \sigma$, such that $u = g(\sigma + v)$, where the function $g = f^{-1}$. Under the assumption that $u \geq 0$, we obtain a reformulation of the system (8)

$$\begin{cases} g(\sigma + v)_t + \sigma_x = 0, \\ \\ v_t = \dfrac{1}{\delta} R(\sigma, v), \end{cases} \tag{10}$$

where $R(\sigma, v) = ((1-\mu)\sigma - \mu v)$, and R satisfies

$$R(\sigma, v)(\mathrm{sgn}(\sigma) - \mathrm{sgn}(v)) \geq 0.$$

The associated equilibrium model is

$$g(\frac{\bar{\sigma}}{\mu})_t + \bar{\sigma}_x = 0. \tag{11}$$

Again, our main tool in analyzing the system is a finite difference scheme derived from the formulation (10). Let Δt and Δx denote the step lengths in the t and x directions, respectively. We consider a semi-implicit difference scheme of the form

$$\begin{cases} \dfrac{g\left(\sigma_j^{n+1}+v_j^{n+1}\right)-g\left(\sigma_j^n+v_j^n\right)}{\Delta t}+\dfrac{\sigma_j^n-\sigma_{j-1}^n}{\Delta x}=0\ , \\ \dfrac{v_j^{n+1}-v_j^n}{\Delta t}=\dfrac{1}{\delta}R\left(\sigma_j^{n+1},v_j^{n+1}\right)\ . \end{cases} \tag{12}$$

Here σ_j^n and v_j^n denote approximations of $\sigma(x,t)$ and $v(x,t)$ over the grid-blocks

$$B_j^n=\Big[x_{j-1/2},x_{j+1/2}\Big)\times\Big[t_n,t_{n+1}\Big),$$

where $x_j=j\Delta x$ and $t_n=n\Delta t$. We assume that the CFL-condition

$$\frac{\Delta t}{\Delta x}M_f\le 1 \tag{13}$$

is satisfied, where $M_f=\max f'(u)$.

Before we give the main results for this model, we first give the formal argument for the TV-bound and the continuous dependence of the solutions of (10). Note that we can rewrite (10) on the form

$$\begin{cases} \sigma_t+\dfrac{1}{g'(\sigma+v)}\sigma_x\ =\ -v_t\ =\ -\dfrac{1}{\delta}R(\sigma,v), \\ v_t=\dfrac{1}{\delta}R(\sigma,v). \end{cases}$$

Differentiating this system w.r.t. x, we get

$$\begin{cases} (\sigma_x)_t+(\dfrac{1}{g'(\sigma+v)}\sigma_x)_x=-\dfrac{1}{\delta}R(\sigma_x,v_x), \\ (v_x)_t=\dfrac{1}{\delta}R(\sigma_x,v_x). \end{cases}$$

Then, we obtain

$$\frac{d}{dt}\int_{\mathcal{R}}(|\sigma_x|+|v_x|)dx=-\frac{1}{\delta}\int_{\mathcal{R}}R(\sigma_x,v_x)\Big[\mathrm{sgn}(\sigma_x)-\mathrm{sgn}(v_x)\Big]\ dx\ \le 0,$$

which is the TV–estimate. A rigorous argument can be given for the finite difference scheme.

To derive the continuous dependence of the solutions with respect to perturbations of the initial data we let $(\sigma^1,v^1),(\sigma^2,v^2)$ be two solutions, and define a function $G(x,t)$ as

$$G(x,t)=\frac{g(\sigma^1+v^1)-g(\sigma^2+v^2)}{(\sigma^1+v^1)-(\sigma^2+v^2)}>0.$$

From a Lip^+–estimate we obtain $G_t \leq M$, where M is independent of δ. We define the errors $\Delta\sigma = \sigma^1 - \sigma^2$ and $\Delta v = v^1 - v^2$, and write $R = R(\Delta\sigma, \Delta v)$. The following error equations are obtained:

$$\begin{cases} [G\Delta\sigma]_t + \Delta\sigma_x = -[G\Delta v]_t = -\left[G_t\Delta v + \dfrac{1}{\delta}GR\right], \\ \Delta v_t = \dfrac{1}{\delta}R. \end{cases}$$

These equations leads to the identities

$$\begin{cases} (G|\Delta\sigma|)_t + |\Delta\sigma|_x = -\left[G_t\Delta v + \dfrac{1}{\delta}GR\right]\operatorname{sgn}(\Delta\sigma), \\ (G|\Delta v|)_t = \left[G_t\Delta v + \dfrac{1}{\delta}GR\right]\operatorname{sgn}(\Delta v), \end{cases}$$

which implies

$$\begin{aligned} [G(|\Delta\sigma| + |\Delta v|)]_t + |\Delta\sigma|_x \ &= \ -\left[G_t\Delta v + \frac{1}{\delta}GR\right]\cdot[\operatorname{sgn}(\Delta\sigma) - \operatorname{sgn}(\Delta v)] \\ \leq \ G_t\left[|\Delta v| - \Delta v \operatorname{sgn}(\Delta\sigma)\right] \ &\leq \ 2M|\Delta v| \end{aligned}$$

Here the final inequality follows from the a–priori upper bound on G_t. A rigorous argument for this error bound can also be given for the finite difference solutions.

The discussion above motivates the following weak entropy formulation: For any constants (k,q), a solution pair (σ, v) of (10) satisfies

$$\begin{aligned} &\frac{\partial}{\partial t}\Big[G(|\sigma - k| + |v - q|)\Big] + \frac{\partial}{\partial x}|\sigma - k| \\ &\leq \frac{1}{\delta}GR(\sigma, v)\Big[\operatorname{sgn}(v - q) - \operatorname{sgn}(\sigma - k)\Big] + M\Big[|v - q| - (v - q)\operatorname{sgn}(\sigma - k)\Big], \end{aligned}$$

where G is defined as

$$G(\sigma, v, k, q) = \frac{g(\sigma + v) - g(k + q)}{(\sigma + v) - (k + q)} > 0.$$

Below we state the main results of [12]. These results rely on proper assumptions on the initial data, i.e. we must require the initial data to be Lip$^+$–bounded and that the initial residual, $R|_{t=0}$, is of order δ in L^1. These assumptions will not be given explicitly below, and we refer to [12] for more details.

Theorem 3.1. *The finite difference solution $(\sigma_\Delta, v_\Delta)$ generated by (12) converges in L^1_{loc} to a weak solution $\{(\sigma, v)\}$ of (10), which satisfies an entropy condition of the form*

$$Lip^+(\sigma(\cdot,t)), Lip^+(v(\cdot,t) \leq M.$$

Moreover, the solution is stable w.r.t. perturbations in initial data in the following sense: Let $(\bar\sigma, \bar v)$ be another entropy solution of (10) with initial data $(\bar\sigma^0, \bar v^0)$. Then the following bound holds for all $t > 0$:

$$\|\sigma(\cdot,t) - \bar\sigma(\cdot,t)\|_{L^1} + \|v(\cdot,t) - \bar v(\cdot,t)\|_{L^1} \leq \bar M e^{Mt}\left[\|\sigma^0 - \bar\sigma^0\|_{L^1} + \|v^0 - \bar v^0\|_{L^1}\right].$$

Here $\bar M$ and M denote finite constants independent of δ.

We recall here that $\mathrm{Lip}^+(u)$ is given by

$$\mathrm{Lip}^+(u) := \max\left(0,\ \text{ess}\ \sup_{x\neq y} \frac{u(x)-u(y)}{x-y}\right).$$

Let also $\|\cdot\|_{\mathrm{Lip}'}$ be the L^2–dual of the standard Lipschitz norm. As a consequence of the regularity result given in Theorem 3.1 the following convergence estimate is derived in the Lip'–norm.

Theorem 3.2. *Let (u,σ) and $\bar{u}$ be corresponding solutions of (8) and (9). For each $T>0$ there is a constant M, independent of δ, such that*

$$\|u(\cdot,t)-\bar{u}(\cdot,t)\|_{Lip'} \leq M\delta, \quad 0\leq t\leq T.$$

By interpolation between the Lip' error and the TV–bounds, the following L^1 result is obtained.

Corollary 3.3. *Let (u,σ) and $\bar{u}$ be as stated in Theorem 3.2. For each $T>0$ there is a constant M, independent of δ, such that*

$$\|u(\cdot,t)-\bar{u}(\cdot,t)\|_{L^1} \leq M\sqrt{\delta}, \quad 0\leq t\leq T.$$

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