# Degenerate Convection-Diffusion Equations and Implicit Monotone Difference Schemes

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**Abstract.** We analyse implicit monotone finite difference schemes for nonlinear, possibly strongly degenerate, convection-diffusion equations in one spatial dimension. Since we allow strong degeneracy, solutions can be discontinuous and are in general not uniquely determined by their data. We thus choose to work with weak solutions that belong to the $BV$ (in space and time) class and, in addition, satisfy an entropy condition. The difference schemes are shown to converge to the unique $BV$ entropy weak solution of the problem. This paper complements our previous work [8] on explict monotone schemes.

## 1. Degenerate Convection-Diffusion Equations

We are interested in finite difference schemes for nonlinear, possibly strongly degenerate, convection-diffusion problems of the form

$$\partial_t u + \partial_x f(u) = \partial_x(k(u)\partial_x u), \qquad u(x,0) = u_0(x), \tag{1}$$

where $(x,t) \in Q_T = \mathbb{R} \times (0,T)$ and $u_0, f, k$ are given, sufficiently smooth functions. For later use, we need a conservative-form version of (1),

$$\partial_t u + \partial_x f(u) = \partial_x^2 K(u), \qquad K(u) = \int_0^u k(\xi)\, d\xi. \tag{2}$$

By the term 'strongly degenerate' we mean that there are two numbers $\alpha$ and $\beta$ such that $k(u) = 0$ for all $u \in [\alpha, \beta]$. Hence, the class of equations under consideration is very large and contains, to mention a few, the heat equation, the porous medium equation, the two-phase flow equation and conservation laws. Strongly degenerate equations will in general possess discontinuous – shock wave – solutions. Furthermore, discontinuous weak solutions are not uniquely determined by their data. In fact, an entropy condition is needed to single out the physically relevant weak solution of the problem. We call a bounded measurable function $u(x,t)$ an entropy weak solution if

$$\partial_t|u-c| + \partial_x\big[\mathrm{sgn}(u-c)(f(u)-f(c))\big] + \partial_x^2|K(u)-K(c)| \le 0 \quad \text{(weakly)}.$$

It is not difficult to construct an entropy weak solution of (1), even in several space dimensions, see [12]. To the authors knowledge, the main open question seems to be the uniqueness of such solutions, even in one space dimension. This has motivated us to seek solutions in the (significantly) smaller class containing the $BV$ entropy weak solutions. Before introducing this notion of a solution, we recall that uniqueness of weak solutions for the purely parabolic case (no convection term) in the class of bounded integrable functions has been proved by Brezis and Crandall [1], and that uniqueness of entropy weak solutions for hyperbolic problems (no diffusion term) is a classical result due to Kruzkov [9].

**Definition 1.1.** *A bounded measurable function $u(x,t)$ is said to be a $BV$ entropy weak solution of the initial value problem* (1) *if*

*(a) $u(x,t) \in BV(Q_T)$ and $K(u) \in C^{1,\frac{1}{2}}(Q_T)$.*

*(b) For all non-negative $\phi \in C_0^\infty(Q_T)$ and any $c \in \mathbb{R}$,*

$$\iint\limits_{Q_T} \Big( |u-c|\partial_t \phi + \operatorname{sgn}(u-k)(f(u)-f(c)-\partial_x K(u))\partial_x \phi \Big)\, dt\, dx + \int\limits_{\mathbb{R}} |u_0 - c|\phi(x,0)\, dx \geq 0. \tag{3}$$

What makes this definition interesting is that uniqueness of $BV$ entropy solutions follows from the work of Wu and Yin [13] (actually, instead of (a), they require $u \in BV(Q_T)$ and only $\partial_x K(u) \in L^1_{\mathrm{loc}}(Q_T)$). We mention here that the jump conditions proposed by Volpert and Hudjaev [12] are in general not correct, and thus the uniqueness proof presented there is incomplete, see [13] for more details. Roughly speaking, entropy weak solutions that are of bounded variation in both space and time are solutions in the sense of Wu and Yin. One should note that it is rather restrictive to require $BV$ (in space and time) regularity of solutions to parabolic equations. In particular, for $\partial_t u$ to be a (locally) finite measure on $Q_T$, $\partial_x u$ and $\partial_x^2 K(u)$ need to be (locally) finite measures on $Q_T$. This fact immediately implies that the diffusion term $K(u(\cdot,t))$ needs to possess a certain amount of smoothness, which in turn indicates that it should be harder (than for conservation laws) to establish the analog of the Crandall and Majda theory [5] for strongly degenerate parabolic equations. The convergence of a scheme to the desired $BV$ solution is *not* an immediate consequence of a $BV$ estimate (in space), as is the case with hyperbolic conservation laws.

It is possible to use the theory developed in [13] to treat strongly degenerate boundary value problems as well, see Bürger and Wendland [2] (and the references therein). In [2] the authors analyse their recently proposed model for the settling and consolidation of a flocculated suspension under the influence of gravity. We refer to Concha and Bürger [4] for an overview of the activity centring around this and related sedimentation models. Cockburn and Gripenberg [3] have recently shown that solutions of degenerate equations also depend continuously on the nonlinear fluxes of the problem, see [3, 8] for more details.

It is important to realize that solutions of strongly degenerate parabolic equations (1) in general have a more complex structure than solutions of conservation laws. The following example demonstrates this. Let $f(u) = u^2$ (referred to as the

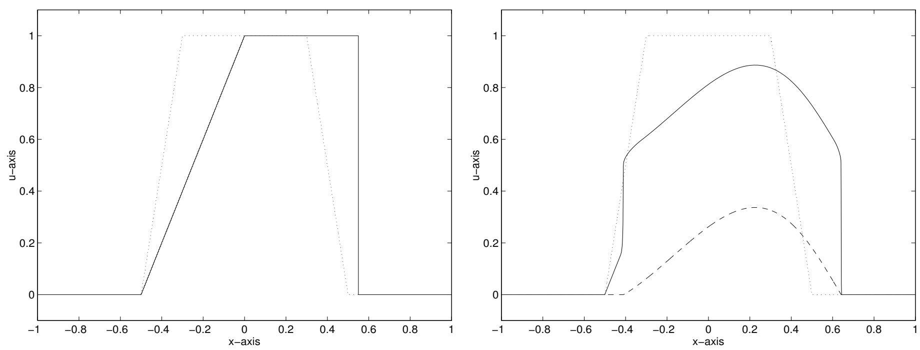


FIGURE 1. Left: The solution (solid) of the inviscid Burgers' equation. Right: The solution (solid) of Burgers' equation with a strongly degenerate diffusion term and the corresponding diffusion function $K(u(\cdot,t))$ (dashed). The initial function is shown as dotted.

Burgers flux), and let $k(u) = 0$ for $u \in [0, 0.5]$, $2.5u - 1.25$ for $u \in (0.5, 0.6)$ and $0.25$ for $u \in [0.6, 1.0]$. Note that $k(u)$ is continuous and degenerates on the interval $[0, 0.5]$. In Fig. 1 we have plotted the initial function, the solution of the corresponding conservation law, i.e., $k \equiv 0$ in (1), and the solution of (1) at time $T = 0.15$. An interesting observation is that the solution of (1) has a 'new' increasing jump, despite of the fact that $f$ is convex. Thus the solution is not bounded in the so-called $Lip^+$ norm, as opposed to the solution of the conservation law. Moreover, while the speed of a jump in the conservation law solution is determined solely by $f$, the speed of a jump in the solution of (1) is in general determined by the jumps in both $f(u)$ and $\partial_x K(u)$, see [13] for precise statements of these jump conditions. Here it suffices to that say the speed $s$ of a jump is

$$s = \frac{f(u^+) - f(u^-) - \left(\lim_{x \to x_0+} \partial_x K(u) - \lim_{x \to x_0-} \partial_x K(u)\right)}{u^+ - u^-}, \tag{4}$$

where $u^-$ and $u^+$ denote the usual left and right limits (taken along the unit normal to the shock curve) of $u$ respectively. Furthermore, the entropy condition requires that the following inequalities hold for all $c \in \operatorname{int}(u^-, u^+)$:

$$\frac{f(u^+) - f(c) - \lim_{x \to x_0+} \partial_x K(u)}{u^+ - c} \leq s \leq \frac{f(u^-) - f(c) - \lim_{x \to x_0-} \partial_x K(u)}{u^- - c}. \tag{5}$$

See Fig. 2 for an illustration of (5). Finally, we mention here that the techniques developed by Kruzkov (stability) and Kuznetsov (error estimates) for first order equations are not straight on adaptable to second order problems such as (1).

In this paper we are interested in implicit monotone difference schemes for (1). A convergence analysis of explicit monotone schemes was given recently in [8]. In view of the classical monotone theory for conservation laws [5], the main difficulty in obtaining a convergence theory for (1) is to show that the approximations are $L^1$ Lipschitz continuous in the time variable, i.e., that they are in $BV$ (in space

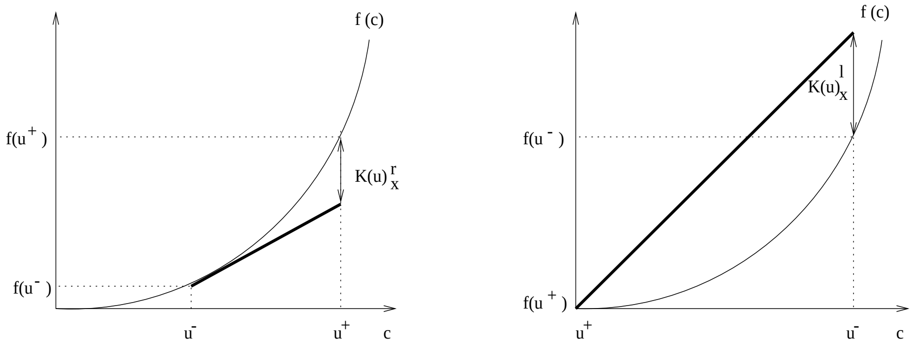


FIGURE 2. Geometric interpretation of the entropy condition (5) for the solution shown in Fig. 1 (right). Left (the left jump): Note that $K(u)_x^l = 0$, $K(u)_x^r > 0$, see Fig. 1 (right). Condition (5) requires that the graph of $f$ restricted to the interval $[u^-, u^+]$ lies above or equals the straight line between $(u^-, f(u^-))$ and $(u^+, f(u^+) - K(u)_x^r)$. Right (the right jump): Note that $K(u)_x^l < 0$, $K(u)_x^r = 0$. Condition (5) requires that the graph of $f$ restricted to $[u^+, u^-]$ lies below or equals the straight line between $(u^+, f(u^+))$ and $(u^-, f(u^-) - K(u)_x^l)$.

and time), see Lemmas 2.3 and 2.4 in this paper. To the authors knowledge, there exists no general finite difference theory for strongly degenerate parabolic equations, except for [8]. The main purpose of this paper is show that the theory developed in [8] can be easily extended, using the theory of Crandall and Liggett [6], to implicit schemes as well. An accurate and efficient operator splitting scheme for (1) has been proposed and analysed in [7]. However, for this approximation it is in general impossible to prove $L^1$ Lipschitz in time regularity, see [7] for details. Finally, we are currently looking into the issue of devising higher order difference schemes for strongly degenerate parabolic equations. In particular, we are investigating to what extent the 'higher order' theory/schemes developed for hyperbolic conservation laws can be taken over to strongly degenerate equations.

## 2. Implicit Monotone Difference Schemes

We will follow the work [8] closely and refer the reader to it for details not found here. Introduce the difference operators $D_- U_j = \frac{U_j - U_{j-1}}{\Delta x}$ and $D_+ = \frac{U_{j+1} - U_j}{\Delta x}$. We then consider the three-point monotone difference schemes of the form

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + D_-\left(F(U_j^{n+1}, U_{j+1}^{n+1}) - D_+ K(U_j^{n+1})\right) = 0, \tag{6}$$

where $F(u,u) = f(u)$ and $\{U_j^0\}$ is some discretization of $u_0$. In what follows, we assume, without loss of generality, that $u_0$ has compact support and $f(0) = 0$. The

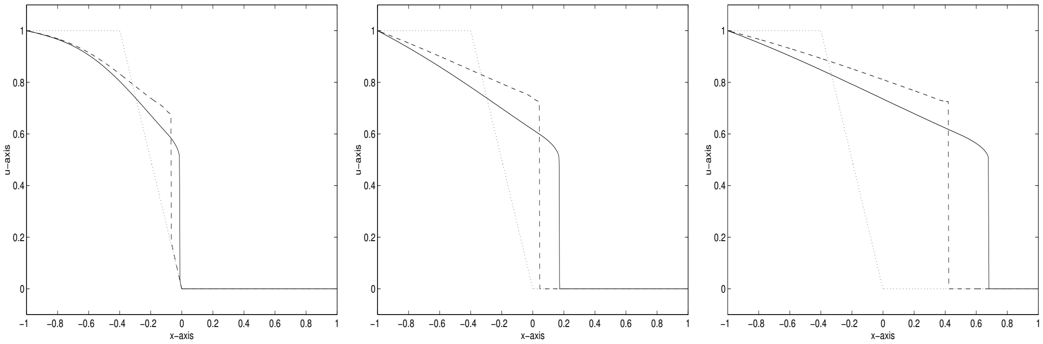


FIGURE 3. The solutions produced by the conservative scheme (6) (solid) and the non-conservative scheme (8) (dashed) plotted at the times $T_1 = 0.0625$, $T_2 = 0.25$ and $T = 1.0$. The initial function is shown as dotted; see the text for a further description of the problem.

assumption of monotonicity in the case of implicit schemes reads

$$F_u(r_1, r_2) + \frac{1}{\Delta x} k(r_3) \geq 0, \qquad \frac{1}{\Delta x} k(r_3) - F_v(r_1, r_2) \geq 0, \qquad \forall (r_1, r_2, r_3). \quad (7)$$

Note that a sufficient condition for (7) to hold is that $F_u \geq 0$ and $F_v \leq 0$. Consequently, any monotone numerical flux $F$ for conservation laws will produce a monotone scheme for (1). To keep the notation simple and making the arguments more transparent, we consider only three-point schemes in this paper.

The monotone schemes (6) are based on differencing the conservative-form equation (2), and not the equation in its original form. One can also devise schemes based on differencing (1) directly, yielding, for example, schemes of the form

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + D_-\big(F(U_j^{n+1}, U_{j+1}^{n+1}) - k(U_{j+1/2}^{n+1}) D_+ U_j^{n+1}\big) = 0, \quad (8)$$

where $U_{j+1/2}^{n+1} = \frac{1}{2}\big(U_j^{n+1} + U_{j+1}^{n+1}\big)$. Although it is possible to prove that (8) converges to a limit, we have not been able to show that this limit satisfies an entropy condition. In fact, we do not believe that (8) will converge to the physically correct solution in the case of strong degeneracy. We now present a simple numerical example intended to support this view. For this purpose we use fluxes $\tilde{f}(u) = \frac{1}{4}u^2$ and $\tilde{k}(u) = 4k(u)$, where $k$ is the one used above. In Fig. 3 we have plotted the solution produced (using small grid parameters) by (6) and (8) at three different times. The convective numerical flux was the upwind flux $F(U_j^{n+1}, U_{j+1}^{n+1}) = f(U_j^{n+1})$ in these calculations. Clearly, the non-conservative scheme (8) produces an incorrect solution. We are currently investigating this phenomenon and will come to back to it in a separate report.

As an aid in the following analysis we shall view the equation (6) in terms of an m-accretive operator and an associated contraction solution operator, i.e., we shall use the Crandall and Liggett theory [6]. A similar treatment of implicit difference schemes for conservation laws has been given earlier by Lucier [11]. If $X$

is a Banach space, a duality mapping $J : X \to X^*$ has the properties that for all $x \in X$, $\|J(x)\|_{X^*} = \|x\|_X$ and $J(x)(x) = \|x\|_X^2$. A possibly multi-valued operator $\mathcal{A}$, defined on some subset $D(\mathcal{A})$ of $X$, is said to be accretive if for every pair of elements $(x, \mathcal{A}(x))$ and $(y, \mathcal{A}(y))$ in the graph of $\mathcal{A}$, and for every duality mapping $J$ on $X$, $J(x-y)(\mathcal{A}(x) - \mathcal{A}(y)) \geq 0$. If, in addition, for all positive $\lambda$, $\mathcal{I} + \lambda\mathcal{A}$ is a surjection, then $\mathcal{A}$ is m-accrective. For a fixed $n$, let us now rewrite the difference equation (6) as (suppressing the $\Delta x$ dependence)

$$U_j^{n+1} + \Delta t \mathcal{A}(U^{n+1}; j) = U_j^n, \tag{9}$$

where $\mathcal{A}(U; j) = D_-\big(F(U_j^{n+1}, U_{j+1}^{n+1}) - D_+ K(U_j)\big)$. Let $(\Omega, d\mu)$ be a measure space. Recall that, since the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$, any duality mapping $J$ in $L^1(\Omega)$ is of the form $J(u)(v) = \int_\Omega \hat{J}(u)(x) v(x)\, d\mu$, where

$$\hat{J}(u)(x) = \|u\|_{L^1(\Omega)} \begin{cases} 1, & \text{if } u(x) > 0, \\ -1, & \text{if } u(x) < 0, \\ \alpha(x), & \text{if } u(x) = 0, \end{cases} \tag{10}$$

where $\alpha(x)$ is any measurable function with $|\alpha(x)| \leq 1$ for almost every $x \in \Omega$. We shall rely heavily on the following well-known results (see e.g. [6, 11]):

**Theorem 2.1.** *Let $(\Omega, d\mu)$ be a measure space. Suppose that the nonlinear and possibly multi-valued operator $\mathcal{A} : L^1(\Omega) \to L^1(\Omega)$ is m-accretive. Then for any $\lambda > 0$ and any $u \in L^1(\Omega)$ the equation $\mathcal{T}(u) + \lambda\mathcal{A}(\mathcal{T}(u)) = u$ has a unique solution $\mathcal{T}(u)$. If $\mathcal{A}$ satisfies $\int_\Omega \mathcal{A}(u)\, d\mu = 0$ and commutes with translations, then the solution operator $\mathcal{T} : L^1(\Omega) \to L^1(\Omega)$ possesses the following properties: (a) $\int_\Omega \mathcal{T}(u)\, d\mu = \int_\Omega u\, d\mu$, (b) $\|\mathcal{T}(u) - \mathcal{T}(v)\|_{L^1(\Omega)} \leq \|u - v\|_{L^1(\Omega)}$, (c) $|\mathcal{T}(u)|_{BV(\Omega)} \leq |u|_{BV(\Omega)}$, (d) $u \leq v \Rightarrow \mathcal{T}(u) \leq \mathcal{T}(v)$, (e) $\|\mathcal{T}(u)\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$.*

The following lemma deals with the question of existence, uniqueness and properties of the solution of the (nonlinear) system (6).

**Lemma 2.2.** *If (7) is satisfied, then for any $U$ there exists a unique $U^*$ satisfying the equation $\frac{U_j^* - U_j}{\Delta t} + D_-\big(F(U_j^*, U_{j+1}^*) - D_+ K(U_j^*)\big) = 0$, $\forall j \in \mathbb{Z}$. Furthermore, we have the properties: (a) $U_j \leq V_j\ \forall j \in \mathbb{Z} \Rightarrow U_j^* \leq V_j^*\ \forall j \in \mathbb{Z}$, (b) $\big\|U^*\big\|_{L^\infty(\mathbb{Z})} \leq \big\|U\big\|_{L^\infty(\mathbb{Z})}$, (c) $\big\|U^* - V^*\big\|_{L^1(\mathbb{Z})} \leq \big\|U - V\big\|_{L^1(\mathbb{Z})}$, (d) $\big|U^*\big|_{BV(\mathbb{Z})} \leq \big|U\big|_{BV(\mathbb{Z})}$.*

*Proof.* We will first show that the operator $\mathcal{A}$ is accretive. As a first step to achieve this goal, we observe that

$$\begin{aligned}\sum_{j\in\mathbb{Z}} &\operatorname{sgn}(U_j - V_j)\big(\mathcal{A}(U;j) - \mathcal{A}(V;j)\big) \\ &\geq -\sum_{j\in\mathbb{Z}} \big|cW_j - \big(\mathcal{A}(U;j) - \mathcal{A}(V;j)\big)\big| + c\sum_{j\in\mathbb{Z}} \big|W_j\big|,\end{aligned} \tag{11}$$

where $W_j$ denotes $U_j - V_j$ and $c = c(\Delta x) > 0$ is a number chosen so that $c \geq \frac{1}{\Delta x}(F_u(r_1, r_2) - F_v(r_3, r_1)) + \frac{2}{\Delta x^2} k(r_4)$, $\forall (r_1, r_2, r_3, r_4)$. Next, we write

$$\begin{aligned}
\mathcal{A}(U;j) - \mathcal{A}(V;j) = \frac{1}{\Delta x}\big((F_u(\alpha_j, U_{j+1})W_j + F_v(V_j, \alpha_{j+1})W_{j+1}) \\
- (F_u(\alpha_{j-1}, U_j)W_{j-1} + F_v(V_{j-1}, \alpha_j)W_j)\big) \\
- \frac{1}{\Delta x^2}\big(k(\beta_{j-1})W_{j-1} - 2k(\beta_j)W_j + k(\beta_{j+1})W_{j+1}\big),
\end{aligned}$$

for some numbers $\alpha_j, \beta_j$ between $U_j$ and $V_j$. Inserting this into (11) yields

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}} \operatorname{sgn}(U_j - V_j)\big(\mathcal{A}(U;j) - \mathcal{A}(V;j)\big) \\
&\quad \geq c\sum_{j\in\mathbb{Z}}|W_j| - \sum_{j\in\mathbb{Z}}\Big[\frac{1}{\Delta x}F_u(\alpha_{j-1}, U_j) + \frac{1}{\Delta x^2}k(\beta_{j-1})\Big]|W_{j-1}| \\
&\qquad - \sum_{j\in\mathbb{Z}}\Big[c - \frac{1}{\Delta x}(F_u(\alpha_j, U_{j+1}) - F_v(V_{j-1}, \alpha_j)) - \frac{2}{\Delta x^2}k(\beta_j)\Big]|W_j| \\
&\qquad - \sum_{j\in\mathbb{Z}}\Big[\frac{1}{\Delta x^2}k(\beta_{j+1}) - \frac{1}{\Delta x}F_v(V_j, \alpha_{j+1})\Big]|W_{j+1}| \equiv 0,
\end{aligned} \tag{12}$$

which shows that $\mathcal{A}$ is accretive. Observe that $\mathcal{A}$ is Lipschitz continuous, which implies that $\mathcal{A}$ is not only accretive but also m-accretive. We can now invoke Theorem 2.1 to conclude the existence of a unique solution operator $\mathcal{S}$ of (6), i.e., $U_j^* = \mathcal{S}(U;j)$, which proves the first part of the lemma. Since $\sum_{j\in\mathbb{Z}}\mathcal{A}(U;j) = 0$ and $\mathcal{A}$ commutes with translations, the second part of the lemma follows. □

The next lemma plays a key role in our analysis and has no counterpart in the theory of monotone difference approximations for conservation laws.

**Lemma 2.3.** *If* (7) *is satisfied, we have*

$$\begin{aligned}
\big\|F(U_j^{n+1}, U_{j+1}^{n+1}) - D_+K(U_j^{n+1})\big\|_{L^\infty(\mathbb{Z})} \leq \big\|F(U_j^0, U_{j+1}^0) - D_+K(U_j^0)\big\|_{L^\infty(\mathbb{Z})}, \\
\big|F(U_j^{n+1}, U_{j+1}^{n+1}) - D_+K(U_j^{n+1})\big|_{BV(\mathbb{Z})} \leq \big|F(U_j^0, U_{j+1}^0) - D_+K(U_j^0)\big|_{BV(\mathbb{Z})}.
\end{aligned} \tag{13}$$

*Proof.* From the equation (6), it follows that $V_j^n = \Delta x \sum_{i=-\infty}^{j}\left(\frac{U_i^n - U_i^{n-1}}{\Delta t}\right)$ satisfies

$$V_j^{n+1} = -\big(F(U_j^{n+1}, U_{j+1}^{n+1}) - D_+K(U_j^n)\big). \tag{14}$$

Next, we derive an equation for the quantity $\{V_j^n\}$. For this purpose consider the difference equation (6) evaluated at $i\Delta x$ and subtract the corresponding equation at time $t = n\Delta t$. Multiplying the resulting equation by $\Delta x$ and then summing over $i = -\infty, \ldots, j$, yields the following equation

$$\big(V_j^{n+1} - V_j^n\big) + \big(F(U_j^{n+1}, U_{j+1}^{n+1}) - F(U_j^n, U_{j+1}^n)\big) - D_+\big(K(U_j^{n+1}) - K(U_j^n)\big) = 0.$$

After observing that $\frac{U_j^{n+1}-U_j^n}{\Delta t} = D_- V_j^{n+1}$, we can write

$$F(U_j^{n+1}, U_{j+1}^{n+1}) - F(U_j^n, U_{j+1}^n) = \Delta t a_{u,j}^n D_- V_j^{n+1} + \Delta t a_j^v D_- V_{j+1}^{n+1}, \qquad (15)$$

where $a_{u,j}^n = F_u(\alpha_j^n, U_{j+1}^{n+1})$, $a_j^v = F_v(U_j^n, \tilde{\alpha}_{j+1}^n)$ and $\alpha_j^n, \tilde{\alpha}_j^n$ are some numbers between $U_j^n$ and $U_j^{n+1}$. Similarly, we can write

$$K(U_j^{n+1}) - K(U_j^n) = \Delta t b_j D_- V_j^{n+1},$$

where $b_j^n = k(\beta_j^n)$ and $\beta_j$ is a number between $U_j^n$ and $U_j^n$. Summing up, the sequence $\{V_j^n\}$ satisfies the following linear system of equations

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + \big(a_{u,j}^n D_- V_j^{n+1} + a_{v,j}^n D_- V_{j+1}^{n+1}\big) = D_+\big(b_j^n D_- V_j^{n+1}\big). \qquad (16)$$

Observe that this system can be written as

$$A_j^n V_{j-1}^{n+1} + B_j^n V_j^{n+1} + C_j^n V_{j+1}^{n+1} = V_j^n, \qquad (17)$$

where $A_j^n = -\Big[\frac{\Delta t}{\Delta x} a_{u,j}^n + \frac{\Delta t}{\Delta x^2} b_j^n\Big]$, $B_j^n = \Big[1 + \frac{\Delta t}{\Delta x}\big(a_{u,j}^n - a_{v,j}^n\big) + \frac{\Delta t}{\Delta x^2}\big(b_j^n + b_{j+1}^n\big)\Big]$ and $C_j^n = -\Big[\frac{\Delta t}{\Delta x^2} b_{j+1}^n - \frac{\Delta t}{\Delta x} a_{v,j}^n\Big]$. Because of (7), the linear system (17) is strictly diagonal dominant. Hence, there exists a unique solution $V^{n+1}$ of (17) satisfying $\|V^{n+1}\|_{L^\infty(\mathbb{Z})} \leq \|V^n\|_{L^\infty(\mathbb{Z})}$. An argument similar to the one in [8] will also reveal that $|V^{n+1}|_{BV(\mathbb{Z})} \leq |V^n|_{BV(\mathbb{Z})}$. The lemma now follows by induction. □

A direct consequence of (13) is that the approximations are $L^1$ Lipschitz continuous in the time variable, and thus in $BV$ in space and time.

**Lemma 2.4.** *If* (7) *is satisfied, we have*

$$\|U^m - U^n\|_{L^1(\mathbb{Z})} \leq \big|F(U_j^0, U_{j+1}^0) - D_+ K(U_j^0)\big|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x} |m-n|.$$

*Proof.* The result follows directly from (6) and (13), see also [8]. □

**Lemma 2.5.** *If* (7) *is satisfied, we have*

$$\big|K(U_i^m) - K(U_j^n)\big| = \mathcal{O}(1)\big(|(i-j)\Delta x| + \sqrt{|(m-n)\Delta t|}\big).$$

*Proof.* First, $\big|K(U_i^m) - K(U_j^n)\big| \leq Q_1 + Q_2$, where $Q_1 = \big|K(U_i^m) - K(U_j^m)\big|$ and $Q_2 = \big|K(U_j^m) - K(U_j^n)\big|$. In view of (13), $\|D_+ K(U^m)\|_{L^\infty(\mathbb{Z})} = \mathcal{O}(1)$ and thus $Q_1 = \mathcal{O}(1)|(i-j)\Delta x|$. Kruzkov [10] has developed a technique for deriving a modulus of continuity in time from a known modulus of continuity in space of certain parabolic equations. To estimate $Q_2$ we apply a discrete version of this technique to the parabolic difference equation (16). To this end, let $\phi(x)$ be a test function, put $\phi_j = \phi(j\Delta x)$ and let $m < n$. Using the difference equation (16) and summation by parts (on the right-hand side of (16)), we easily find that

$$\Big|\Delta x \sum_{j\in\mathbb{Z}} \phi_j \big(V_j^m - V_j^n\big)\Big| = \mathcal{O}(1)\big(\|\phi\|_{L^\infty(\mathbb{R})} + \|\phi'\|_{L^\infty(\mathbb{R})}\big)\Delta t(m-n),$$

since, for all $l$, $a^l_{u,j}$, $a^l_{v,j}$ and $\left|V^{l+1}\right|_{BV(\mathbb{Z})}$ are uniformly bounded quantities. From this weak estimate and the $BV$ regularity of $V^n$, it now follows that

$$\Delta x \sum_{j\in\mathbb{Z}} \left|V_j^m - V_j^n\right| = \mathcal{O}(1)\sqrt{|m-n|\Delta t},$$

see [8] for details. On the other hand, from (14) and Lemma 2.4, we also have

$$\Delta x \sum_{j\in\mathbb{Z}} \left|V_j^m - V_j^n\right| = \mathcal{O}(1)(m-n)\Delta t + \Delta x \sum_{j\in\mathbb{Z}} \left|D_+K(U_j^m) - D_+K(U_j^n)\right|.$$

We thus conclude that $\Delta x \sum_{j\in\mathbb{Z}} \left|D_+K(U_j^m) - D_+K(U_j^n)\right| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}$. From this the desired Hölder estimate in time follows, since

$$Q_2 = \left|K(U_j^m) - K(U_j^n)\right| \le \Delta x \sum_{i\in\mathbb{Z}} \left|D_+K(U_i^m) - D_+K(U_i^n)\right| = \mathcal{O}(1)\sqrt{(m-n)\Delta t}.$$

This concludes the proof of the lemma. □

**Lemma 2.6.** *If* (7) *is satisfied, then the following cell entropy inequality holds*

$$\begin{aligned}\frac{\left|U_j^{n+1}-c\right| - \left|U_j^n - c\right|}{\Delta t} &+ D_-\left(F(U_j^{n+1}\vee c, U_{j+1}^{n+1}\vee c) - F(U_j^{n+1}\wedge c, U_{j+1}^{n+1}\wedge c)\right.\\ &\left. - D_+\left|K(U_j^{n+1}) - K(c)\right|\right) \le 0.\end{aligned}$$

*Proof.* The proof is similar to the one presented in [8], see also [5, 11]. □

Let $u_\Delta$ (where $\Delta = (\Delta x, \Delta t)$) be the interpolate of degree one associated with the discrete data points $\{U_j^n\}$, see [8]. Note that $u_\Delta$ is continuous everywhere and differentiable almost everywhere. In view of Lemmas 2.2 - 2.4, we conclude that there is a constant $C = C(T) > 0$ such that

$$\begin{aligned}&\|u_\Delta\|_{L^\infty(Q_T)} + |u_\Delta|_{BV(Q_T)} \le C,\\ &|K(u_\Delta(y,\tau)) - K(u_\Delta(x,t))| \le C\left(|y-x| + \sqrt{|\tau - t|} + \Delta x + \sqrt{\Delta t}\right).\end{aligned}$$

for all $(x,t),(y,\tau) \in \mathbb{R}\times[0,T]$. Consequently, since $BV$ is compactly imbedded into $L^1$ on compacta, there is a subsequence of discretization parameters and a function $u \in L^\infty(Q_T)\cap BV(Q_T)$ such that $u_{\Delta_j} \to u$ a.e. in $Q_T$. Furthermore, via the Ascoli-Arszela theorem, $K(u_{\Delta_j}) \to K(u)$ uniformly on compact sets $\mathcal{K}\subset Q_T$, and $K(u) \in C^{1,\frac{1}{2}}(Q_T)$. Repeating the proof of the Lax-Wendroff theorem (with Lemma 2.6 in mind), it follows that $u$ satisfies the entropy condition (3).

Summing up, we have proven the following main theorem:

**Theorem 2.7.** *The sequence* $\{u_\Delta\}$, *which is built from the implicit monotone difference schemes* (6), *converges a.e. to the* $BV$ *entropy weak solution of* (1).

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