DISCRETE APPROXIMATIONS OF BV SOLUTIONS TO DOUBLY NONLINEAR DEGENERATE PARABOLIC EQUATIONS

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Abstract. In this paper we present and analyse certain discrete approximations of solutions to scalar, doubly nonlinear degenerate, parabolic problems of the form

$$\text{(P)} \qquad \partial_t u + \partial_x f(u) = \partial_x A\left(b(u)\partial_x u\right), \qquad u(x,0) = u_0(x), \qquad A(s) = \int_0^s a(\xi)\,d\xi,\ a(s) \geq 0,\ b(s) \geq 0,$$

under the very general structural condition $A(\pm\infty) = \pm\infty$. To mention only a few examples: the heat equation, the porous medium equation, the two-phase flow equation, hyperbolic conservation laws and equations arising from the theory of non-Newtonian fluids are all special cases of (P). Since the diffusion terms $a(s)$ and $b(s)$ are allowed to degenerate on intervals, shock waves will in general appear in the solutions of (P). Furthermore, weak solutions are not uniquely determined by their data. For these reasons we work within the framework of weak solutions that are of bounded variation (in space and time) and, in addition, satisfy an entropy condition. The well-posedness of the Cauchy problem (P) in this class of so-called BV entropy weak solutions follows from a work of Yin [18]. The discrete approximations are shown to converge to the unique BV entropy weak solution of (P).

Contents

§1. Introduction.

In this paper we present and analyse certain finite difference schemes for a class of scalar, doubly nonlinear degenerate, parabolic equations in one spatial dimension. Nonlinear parabolic evolution equations arise in a variety of applications, ranging from models of turbulence, via traffic flow, finanical modelling and flow in porous media, to models for various sedimentation processes. The problem we study here is of the form

$$\text{(1)} \qquad \begin{cases} \partial_t u + \partial_x f(u) = \partial_x A(b(u)\partial_x u), & (x,t) \in Q_T = \mathbb{R} \times \langle 0, T\rangle\,, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$A(s) = \int_0^s a(\xi)\,d\xi, \qquad a(s) \geq 0, \qquad b(s) \geq 0.$$

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We assume that $a(s)$, $b(s)$, $f(s)$ and $u_0(x)$ are appropiately smooth functions. The functions $a(s)$ and $b(s)$ are allowed to have infinite number of degenerate intervals in $\mathbb{R}$. Difficulties arise because of this *double degeneracy* as well as the *double nonlinearity* represented by the nonlinear functions a and b. By defining B as

$$B(u) = \int_0^u b(\xi)\, d\xi,$$

we may write (1) as

$$\partial_t u + \partial_x f(u) = \partial_x A\left(\partial_x B(u)\right). \tag{2}$$

Examples of such equations include the heat equation, the porous medium equation and, more generally, convection-diffusion equations of the form

$$\partial_t u + \partial_x f(u) = \partial_x^2 B(u). \tag{3}$$

Included are also hyperbolic conservation laws

$$\partial_t u + \partial_x f(u) = 0, \tag{4}$$

as well as certain equations arising from the theory of non-Newtonian fluids,

$$\partial_t u = \partial_x\left(\partial_x u^m \left|\partial_x u^m\right|^{n-1}\right), \qquad n \geq 1,\ m \geq 1, \tag{5}$$

which corresponds to the case $A(v) = v|v|^{n-1}$ and $B(u) = u^m$.

Kalashnikov [10] has established the existence of *continuous solutions* of the Cauchy problem for (2) when $f = 0$ under some smoothness and boundedness conditions on the initial data u_0 and some structural conditions on $a(s)$ and $b(s)$. In particular, these conditions imply that $a(s)$ and $b(s)$ may have degeneracy at and only at the origin $s = 0$. We also refer to some recent work by Lu [14] for results concerning the regularity of solutions when the equations are degenerate at points at which u and $\partial_x u$ vanish.

The more interesting cases are those in which $a(s)$ and $b(s)$ may have infinite or uncountable points of degeneracy. A striking feature of such nonlinear strongly degenerate parabolic equations is that the solution will generally develop discontinuities in finite time, even with smooth initial data. This feature can reflect the physical phenomenon of breaking of waves and the development of shock waves. Consequently, due to the loss of regularity, one needs to work with weak solutions. However, for the class of equations under consideration, weak solutions are in general not uniquely determined by their data. Therefore an additional condition, the so-called entropy condition (see (b) below), is needed to single out the physically relevant weak solution. Hence attention focuses on finding a physically reasonable framework which incorporates *discontinuous solutions* and at the same time guarantees uniqueness. The concept of a (weak) solution, which we adopt to the Cauchy problem (1) in this paper, is that of *BV entropy weak solutions* as formulated by Yin [18] for the initial-boundary value problem. We shall say that $u(x,t)$ is a BV entropy weak solution (see §2 for precise statements) if

(a) $u(x,t)$ is in $BV(Q_T)$ and $B(u)$ is uniformly Hölder continuous on Q_T.

(b) $\partial_t |u-c| + \partial_x \left[\mathrm{sign}(u-c)\left(f(u) - f(c) - A\left(\partial_x B(u)\right)\right)\right] \leq 0$ (weakly).

Letting $k \to \pm\infty$ in (b), we see that (1) holds in the usual weak sense. Yin [18] has shown well-posedness of the initial-boundary value problem assuming only the (very general) structural condition

$$A(+\infty) = +\infty \qquad \text{and} \qquad A(-\infty) = -\infty. \tag{6}$$

The well-posedness for the Cauchy problem (1) in the class of functions satisfying the conditions (a) and (b) follows by a similar analysis, see §2. Here we should also note, as pointed out by Yin, that the assumption (6) on A is needed only for the existence result. Under the additional assumption that $B(s)$ is *strictly increasing*, which permits $b(s)$ to become zero in some set of measure zero, BV solutions are continuous. Esteban and Vazquez [7] studied the occurrence of finite velocity of propagation for the solutions of the special case (5). In particular, they showed that the interface of the equation is nondecreasing and Lipschitz continuous. Wang and Yin [16] have investigated the properties of the interface of the solution for the general problem (2) when $f = 0$.

Since the diffusion term $\partial_x A(b(u)\partial_x u)$ can degenerate both in a and b, different kinds of interactions between nonlinear convection and nonlinear diffusion will take place. The (lack of) smoothness of the solution is a result of the (lack of) balance between the convective and diffusive fluxes. In the following we will briefly discuss some simple numerical examples whose purpose is to demonstrate the effect of the degeneracy in a and b on intervals. As long as the diffusion term is nondegenerate ($a, b > 0$), there is a perfect balance between the convective and diffusive fluxes and the equation then has a classical smooth solution. The degeneracy which may occur in a or/and b, implies that there is a loss of regularity in the solution.

First we discuss the effect of degeneracy in a. For this purpose, let us consider the equation (1) when $b(u) = 1$. We then have equations of the form

$$\partial_t u + \partial_x f(u) = \partial_x A(\partial_x u). \tag{7}$$

Let f be the Burgers flux $f(s) = s^2$ and A the continuous function

$$A(s) = \begin{cases} s+4, & \text{for } s \in \langle -\infty, -5\rangle, \\ -1, & \text{for } s \in [-5, -1\rangle, \\ s, & \text{for } s \in [-1, 1], \\ +1, & \text{for } s \in \langle +1, +5], \\ s-4, & \text{for } s \in \langle +5, +\infty\rangle. \end{cases} \tag{8}$$

Hence A satisfies (6) and degenerates on the two intervals $[-5, -1]$ and $[1, 5]$. In Figure 1 (left) we have plotted the solution at time $T = 0.15$. The degeneracy introduces only a 'mild' loss of regularity in the solution due to the fact that the convective and diffusive fluxes will be in balance for large gradients. Hence no jumps will arise in the solution.

Next we consider the general problem (1). When $b(s)$ is zero on an interval, jumps will in general occur in the solution. Let f be the Burgers flux function as before, while A is the function given by (8) and b is the continuous function given by

$$b(s) = \begin{cases} 0, & \text{for } s \in [0, 0.5\rangle, \\ 2.5s - 1.25, & \text{for } s \in [0.5, 0.6\rangle, \\ 0.25, & \text{for } s \in [0.6, 1]. \end{cases}$$

In Figure 1 we have plotted the solution of this degenerate parabolic problem (right) at time $T = 0.15$. It is instructive to compare this solution with the solution of the corresponding conservation law (4), see Figure 1 (middle). In particular, we observe that the solution of the degenerate problem has a 'new' increasing jump, despite the fact that f is convex. In that sense the solution of the degenerate problem has a more complex structure than the solution of the conservation law (4), as well as the solution of the problem (7). Moreover, while the speed of the jump of the conservation law solution is determined solely by f (Rankine-Hugoniot condition), the speeds of the jumps in the solution of the degenerate problem are determined by both f and $A(\partial_x B(u))$, see §2 for precise statements of the jump conditions.

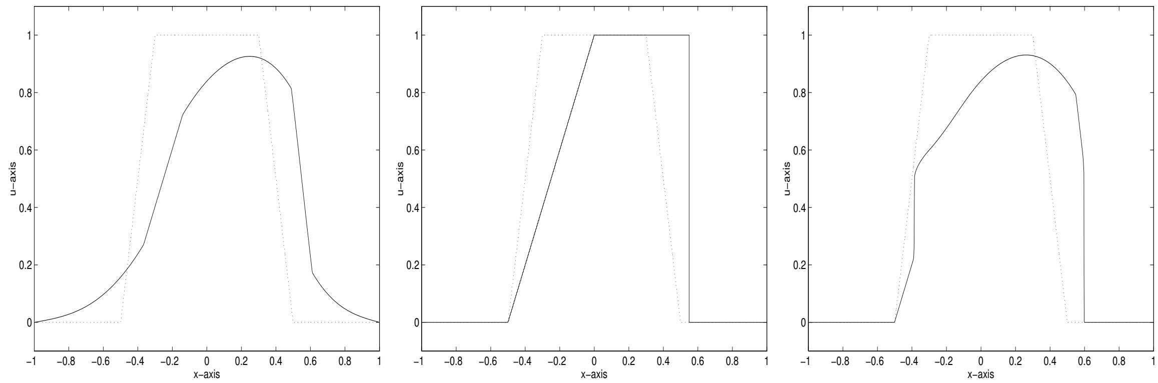

Figure 1. Left: The solution of Burgers' equation with a diffusion term A which degenerates on intervals. Middle: The solution of the inviscid Burgers' equation. Right: The solution of Burgers' equation with diffusion terms A and B which degenerate on intervals.

Convergence of explicit monotone finite difference schemes has been established recently [8] for the special case $A(s) = s$. To the best of our knowledge, for the general case no convergence results for discrete approximations are available. The analysis presented here follows along the lines of [8]. Both works were inspired by the theory developed by Crandall and Majda [4]. However, due to the double degeneracy as well as the double nonlinearity, the analysis in the present case is significantly more involved than in [4,8].

In what follows, we restrict our attention to implicit three-point difference schemes. That is, we consider discretizations of (2) of the following form (see §3 for more details)

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+D_-\left(h(U_j^{n+1},U_{j+1}^{n+1})-A(D_+B(U_j^{n+1}))\right)=0, \tag{9}$$

where h denotes a monotone and consistent numerical flux function, $\Delta x, \Delta t$ are the mesh sizes and D_+, D_- are the usual forward and backward difference operators respectively. Extension to general p-point monotone schemes follows easily. Note here that we choose to discretize the diffusion term written on its conservative form. In [8] we observed that this seems to be essential in order to ensure that the scheme is consistent with the entropy condition. In this paper we show that (9) satisfies a cell entropy inequality consistent with the entropy inequality (b). In addition we establish several regularity estimates for the approximate solutions which are sufficient to guarantee convergence (of a subsequence) to a limit. The main difficulty here is to show that the discrete diffusion term possesses the regularity properties which ensure that the approximate solutions are in $BV(Q_T)$. This is obtained by deriving and carefully analysing a linear difference equation satisfied by the numerical flux of the difference scheme (9). In addition it turns out that due to the double nonlinearity the interpolants must be chosen carefully when constructing the approximate solutions. As a by-product of our analysis, we also establish the existence and regularity properties of solutions of the Cauchy problem (1), and in that respect complement the work of Yin [18] on the intial-boundary value problem.

We should emphasise that this paper and the companion papers [8,9] (on strongly degenerate convection-diffusion equations) are intended as preliminary theoretical thrusts at the numerical approximation of non-classical solutions of degenerate parabolic equations, and they utilise discrete approximations which could be somewhat 'too crude' for practical applications. Having said this, we are currently looking into the issue of devising higher order difference schemes for degenerate parabolic equations. Another important issue that is under investigation is the problem of deriving rigorous error estimates for our schemes. We also mention that our interest in degenerate parabolic equations is *partially* motivated by the recent efforts made in developing mathematical models for the settling and consolidation of a flocculated suspensions in solid-liquid separation vessels (so-called thickeners). We refer to Bürger and Wendland [1] and Concha and Bürger [2] for an overview of the activity centring around these sedimentation models, whose main ingredients are degenerate parabolic equations.

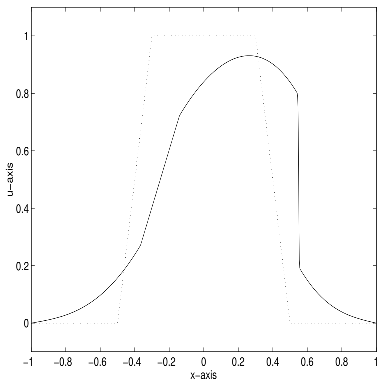

Figure 2. The solution of the inviscid Burgers' equation with a bounded diffusion term A and $b(s)=1$.

Before ending this introduction, we should make some comments concerning the structural condition (6) on A. Let us for a moment return to equation (7). Such equations have been studied more recently by Kurganov, Levy, Rosenau [13,12] under the condition that A is bounded. In particular, they observe by numerical experiments and analysis how the solutions develop infinite spatial derivatives in finite time from smooth initial conditions. For an example of this phenomenon, see Figure 2 where we have plotted the solution of the problem (7), but now with A bounded. Intuitively it is obvious what happens. In this case the equation imposes an upper bound on the amount of the diffusive flux while the convective flux may be as large as desired. When the fluxes are no longer in balance, smooth upstream-downstream transit becomes impossible and a subshock is formed. The importance of (6) used in this paper, is that under this condition it is possible to obtain an estimate $|\partial_x B(u(x,t))| \leq \text{Const}$ from the estimate $|A(\partial_x B(u(x,t))| \leq \text{Const}$. This is obviously not true if A is bounded.

The rest of this paper is organised as follows: In §2 we give a brief summary of the theory of doubly nonlinear degenerate parabolic equations. We also recall some classical results needed from the Crandall and Liggett theory [3]. In §3 we present and discuss the discrete approximations. In section §4 we derive a number of regularity estimates satisfied by the discrete approximations. In §5 we exploit these estimates to prove the convergence (compactness) of the approximate solutions to the unique solution of (1).

§2. Mathematical Preliminaries.

In this section we recall the known mathematical theory of double nonlinear degenerate parabolic equations. To this end, let Ω be an open subset of $\mathbb{R}^d$ ($d > 1$). The space $BV(\Omega)$ of functions of bounded variation consists of all $L^1_{\mathrm{loc}}(\Omega)$ functions $u(y)$ whose first order partial derivatives $\frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_d}$ are represented by (locally) finite Borel measures. The total variation $|u|_{BV(\Omega)}$ is by definition the sum of the total masses of these Borel measures. Moreover, $BV(\Omega)$ is a Banach space when equipped with the norm $||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + |u|_{BV(\Omega)}$. It is well known that the inclusion $BV(\Omega) \subset L^{d/(d-1)}(\Omega)$ holds for $d > 1$ and that $BV(\Omega) \subset L^\infty(\Omega)$ for $d = 1$. Furthermore, $BV(\Omega)$ is compactly imbedded into the space $L^q(\Omega)$ for $1 \le q < d/(d-1)$. Finally, we will also need the Hölder space $C^{1,\frac{1}{2}}(Q_T)$ consisting of bounded functions $z(x,t)$ on $\mathbb{R} \times [0,T]$ which satisfies

$$|z(y,\tau) - z(x,t)| \le L\big(|y-x| + |\tau - t|^{\frac{1}{2}}\big), \qquad \forall\ x, t, y, \tau,$$

for some constant $L > 0$ (not depending on x, y, t, τ).

In what follows, we shall always assume, if not otherwise stated, that the structural condition (6) holds. Due to possibly strong degeneracy, we seek solutions of the Cauchy problem (1) in the following sense.

Definition 2.1. *A bounded measurable function $u(x,t)$ is said to be a BV entropy weak solution of (1) provided the following two requirements hold:*

1. *$u \in BV(Q_T)$ and $B(u) \in C^{1,\frac{1}{2}}(Q_T)$.*
2. *For all test functions $\phi \ge 0$ with support in $\mathbb{R} \times [0,T\rangle$ and any $c \in \mathbb{R}$,*

$$\iint_{Q_T} \Big(|u-c|\partial_t\phi + \mathrm{sign}(u-c)\big(f(u) - f(c) - A(\partial_x B(u))\big)\partial_x\phi\Big)\,dt\,dx + \int_{\mathbb{R}} |u_0 - c|\,dx \ge 0. \tag{10}$$

Definition 2.1 is similar to the one used by Yin [18] who studied the initial-boundary value problem. The uniqueness proof for the Cauchy problem follows from the analysis of the corresponding initial boundary value problem. In fact, the Cauchy problem is simpler since the BV solutions of the boundary value problem must satisfy some extra conditions on the boundary. The following characterization of the set of discontinuity points (jumps) of u can be proved along the lines of Yin [18].

Theorem 2.2 [Yin]. *Let Γ_u be the set of jumps of u; $\nu = (\nu_x, \nu_t)$ the unit normal to Γ_u; $u^-(x_0,t_0)$ and $u^+(x_0,t_0)$ the approximate limits of u at $(x_0,t_0) \in \Gamma_u$ from the sides of the half-planes $(t-t_0)\nu_t + (x-x_0)\nu_x < 0$ and $(t-t_0)\nu_t + (x-x_0)\nu_x > 0$ respectively; $u^l(x,t)$ and $u^r(x,t)$ denote the left and right approximate limits of $u(\cdot,t)$ respectively. Let int(α,β) denote the closed interval bounded by α and β. Furthermore, define*

$$\mathrm{sign}^+(s) = \begin{cases} 1, & \textit{if } s > 0; \\ 0, & \textit{if } s \le 0, \end{cases} \qquad \mathrm{sign}^-(s) = \begin{cases} 0, & \textit{if } s \ge 0; \\ -1, & \textit{if } s < 0. \end{cases}$$

Finally, let H_1 be the one-dimensional Hausdorff measure. Then H_1 - almost everywhere on Γ_u

$$b(u) = 0,\ \forall u \in \mathit{int}(u^-, u^+) \qquad \textit{and} \qquad \nu_x \ne 0, \tag{11}$$

$$(u^+ - u^-)\nu_t + (f(u^+) - f(u^-))\nu_x - \big(A(\partial_x B(u))^r - A(\partial_x B(u))^l\big)|\nu_x| = 0, \tag{12}$$

$$\begin{aligned} &|u^+ - c|\nu_t + \mathrm{sign}(u^+ - c)\big[f(u^+) - f(c) - \big(A(\partial_x B(u))^r\,\mathrm{sign}^+\,\nu_x - A(\partial_x B(u))^l\,\mathrm{sign}^-\,\nu_x\big)\big]\nu_x \\ &\quad \le |u^- - c|\nu_t + \mathrm{sign}(u^- - c)\big[f(u^-) - f(c) - \big(A(\partial_x B(u))^l\,\mathrm{sign}^+\,\nu_x - A(\partial_x B(u))^r\,\mathrm{sign}^-\,\nu_x\big)\big]\nu_x. \end{aligned} \tag{13}$$

By explicitly making use of the above jump conditions, the following stability result, from which uniqueness follows, can be obtained along the lines of Yin [18].

Theorem 2.3 [Yin]. *Let u_1 and u_2 be BV entropy weak solutions of (1) with initial functions $u_{0,1}$ and $u_{0,2}$ respectively. Then for any $t > 0$,*

$$\int_{\mathbb{R}} |u_1(x,t) - u_2(x,t)|\,dx \le \int_{\mathbb{R}} |u_{0,1}(x) - u_{0,2}(x)|\,dx.$$

Finally, we note that the jump conditions in Theorem 2.2 can be more instructively stated as follows.

Corollary 2.4. *Assume that $b(u) = 0$ for $u \in [u_*, u^*]$ for some $u_*, u^* \in \mathbb{R}$. Let u be a piecewise smooth solution of (1) and let Γ_u be a smooth discontinuity curve of u. A jump between two values u^l and u^r of the solution u, which we refer to as a shock, can occur only for $u^l, u^r \in [u_*, u^*]$. This shock must satisfy the following two conditions:*

1. *The shock speed s is given by*

$$s = \frac{\big[f(u^r) - A(\partial_x B(u))^r\big] - \big[f(u^l) - A(\partial_x B(u))^l\big]}{u^r - u^l}. \tag{14}$$

2. *For all $c \in int(u^l, u^r)$, the following entropy condition holds*

$$\frac{\big[f(u^r) - A(\partial_x B(u))^r\big] - f(c)}{u^r - c} \le s \le \frac{\big[f(u^l) - A(\partial_x B(u))^l\big] - f(c)}{u^l - c}. \tag{15}$$

Proof. For $u \in L^\infty(Q_T) \cap BV(Q_T)$ it can be shown that the following relation between u^+,u^-,u^r and u^l holds H_1 almost everywhere on $\Gamma_u^* = \{(x,t) \in \Gamma_u : \ \nu_x \neq 0\}$

$$\begin{aligned} u^+(x,t) &= u^r(x,t)\,\mathrm{sign}^+\,\nu_x - u^l(x,t)\,\mathrm{sign}^-\,\nu_x \\ u^-(x,t) &= u^l(x,t)\,\mathrm{sign}^+\,\nu_x - u^r(x,t)\,\mathrm{sign}^+\,\nu_x. \end{aligned} \tag{16}$$

These identities are non-trivial and we refer to [17] for a proof. Since $|\nu_x| = (\mathrm{sign}^+\,\nu_x + \mathrm{sign}^-\,\nu_x)\nu_x$, (12) can be written as

$$(u^+ - u^-)\nu_t + \big(f(u^+) - f(u^-)\big)\,\nu_x - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x + \big(w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x = 0, \tag{17}$$

where $w_u^r = A(\partial_x B(u))^r$ and $w_u^l = A(\partial_x B(u))^l$. For $c \in \mathrm{int}(u^-, u^+) = \mathrm{int}(u^l, u^r)$ (by (16)) we have the relation $\mathrm{sign}(u^+ - c) = -\,\mathrm{sign}(u^- - c)$. In light of this and (17), we now use (13) and perform the following calculation.

$$\begin{aligned} &\mathrm{sign}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\,\nu_x - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big] \\ &\quad \le -\,\mathrm{sign}(u^+ - c)\big[(u^- - c)\nu_t + \big(f(u^-) - f(c)\big)\,\nu_x - \big(w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big] \\ &\quad = -\,\mathrm{sign}(u^+ - c)\big[(u^- - u^+)\nu_t + \big(f(u^-) - f(u^+)\big)\,\nu_x + \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x \\ &\qquad\qquad - \big(w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big] \\ &\qquad - \mathrm{sign}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\,\nu_x - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x \\ &\qquad\qquad + \big(w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x - \big(w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big] \\ &\quad = -\,\mathrm{sign}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\,\nu_x - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big]. \end{aligned}$$

Hence

$$\mathrm{sign}(u^+ - c)\big[(u^+ - c)\nu_t + \big(f(u^+) - f(c)\big)\,\nu_x - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)\,\nu_x\big] \le 0.$$

Dividing by $|u^+ - c|$ yields

$$\nu_t + \frac{\big(f(u^+) - f(c)\big) - \big(w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x\big)}{u^+ - c}\nu_x \le 0$$

or

$$\frac{\big[f(u^+) - (w_u^r\,\mathrm{sign}^+\,\nu_x - w_u^l\,\mathrm{sign}^-\,\nu_x)\big] - f(c)}{u^+ - c}\nu_x \le -\nu_t. \tag{18}$$

Similarly, we can show that

$$-\nu_t \le \frac{\big[f(u^-) - (w_u^l\,\mathrm{sign}^+\,\nu_x - w_u^r\,\mathrm{sign}^-\,\nu_x)\big] - f(c)}{u^- - c}\nu_x. \tag{19}$$

Combining (18) and (19) we have for $c \in \operatorname{int}(u^l, u^r)$ that

(20)
$$\frac{\left[f(u^+) - (w_u^r \operatorname{sign}^+ \nu_x - w_u^l \operatorname{sign}^- \nu_x)\right] - f(c)}{u^+ - c}\nu_x \le -\nu_t \le \frac{\left[f(u^-) - (w_u^l \operatorname{sign}^+ \nu_x - w_u^r \operatorname{sign}^- \nu_x)\right] - f(c)}{u^- - c}\nu_x.$$

Invoking (16) it is not difficult to see that (12) can be written on the form

$$(u^r - u^l)\nu_t + \left(f(u^r) - f(u^l)\right)\nu_x - \left(w_u^r - w_u^l\right)\nu_x = 0.$$

Let $s = -\frac{\nu_t}{\nu_x}$, then (14) follows. Finally, in view of (16), we see that (20) is equivalent to (15). Hence the proof is completed. □

The jump conditions (14) and (15) represent a generalization of the Rankine-Hugoniot condition and Oleinik's entropy condition for conservation laws. The geometric interpretation of (14) and (15) is as follows:

Corollary 2.5. *Let (u^l, u^r) be a jump which satisfies the jump condition (14). Then the entropy condition (15) holds if and only if*

(i) *in case $u^r < u^l$:*
The graph of $y = f(u)$ over $\left[u^r, u^l\right]$ lies below or equals the chord connecting the point $(u^r, f(u^r))$ to $(u^l, f(u^l) - A(\partial_x B(u))^l)$;

(ii) *in case $u^l < u^r$:*
The graph of $y = f(u)$ over $\left[u^l, u^r\right]$ lies above or equals the chord connecting the point $(u^l, f(u^l))$ to $(u^r, f(u^r) - A(\partial_x B(u))^r)$.

We close this section by briefly recalling a few key results from the Crandall and Liggett theory, since it will be used later in the discussion of properties of the difference schemes. If X is a Banach space, a duality mapping $J : X \to X^*$ has the properties that for all $x \in X$, $\|J(x)\|_{X^*} = \|x\|_X$ and $J(x)(x) = \|x\|_X^2$. A possibly multi-valued operator $\mathcal{A}$, defined on some subset $D(\mathcal{A})$ of X, is said to be accretive if for every pair of elements $(x, \mathcal{A}(x))$ and $(y, \mathcal{A}(y))$ in the graph of $\mathcal{A}$, and for every duality mapping J on X,

$$J(x - y)\left(\mathcal{A}(x) - \mathcal{A}(y)\right) \ge 0.$$

If, in addition, for all positive λ, $\mathcal{I} + \lambda\mathcal{A}$ is a surjection, then $\mathcal{A}$ is m-accrective.

Let $(\Omega, d\mu)$ be a measure space. Then recall that, since the dual of $L^1(\Omega)$ is $L^\infty(\Omega)$, any duality mapping J in $L^1(\Omega)$ is of the form $J(u)(v) = \int_\Omega \hat{J}(u)(x)v(x)\,d\mu$, where

$$\hat{J}(u)(x) = \|u\|_{L^1(\Omega)}\begin{cases} 1, & \text{if } u(x) > 0, \\ -1, & \text{if } u(x) < 0, \\ \alpha(x), & \text{if } u(x) = 0, \end{cases}$$

where $\alpha(x)$ is any measurable function with $|\alpha(x)| \le 1$ for almost every $x \in \Omega$. Later we shall rely heavily on the following well-known results (see [3,5,15]) about m-accretive operators on $X = L^1(\Omega)$:

Theorem 2.6. *Let $(\Omega, d\mu)$ be a measure space. Suppose that the nonlinear and possibly multi-valued operator $\mathcal{A} : L^1(\Omega) \to L^1(\Omega)$ is m-accretive. Then for any $\lambda > 0$ and any $u \in L^1(\Omega)$ the equation*

$$\mathcal{T}(u) + \lambda\mathcal{A}(\mathcal{T}(u)) = u,$$

has a unique solution $\mathcal{T}(u)$. Furthermore, suppose that $\mathcal{A}$ satisfies $\int_\Omega \mathcal{A}(u)\,d\mu = 0$ and commutes with translations. Then $\mathcal{T} : L^1(\Omega) \to L^1(\Omega)$ possesses the following properties:

(1) $\int_\Omega \mathcal{T}(u)\,d\mu = \int_\Omega u\,d\mu$,
(2) $\|\mathcal{T}(u) - \mathcal{T}(v)\|_{L^1(\Omega)} \le \|u - v\|_{L^1(\Omega)}$,
(3) $\|\mathcal{T}(u)\|_{BV(\Omega)} \le \|u\|_{BV(\Omega)}$,
(4) $u \le v$ *a.e. implies that* $\mathcal{T}(u) \le \mathcal{T}(v)$ *a.e.*,
(5) $\|\mathcal{T}(u)\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}$.

§3. The Discrete Approximations.

Selecting mesh sizes $\Delta x > 0$, $\Delta t > 0$, the value of our difference approximation at $(x_j, t^n) = (j\Delta x, n\Delta t)$ will be denoted by U_j^n. Capital letters U, V etc. will denote functions on the lattice $\Delta = \{j\Delta x : j \in Z\}$. The value of U at (x_j, t^n) will be written U_j^n. Thus U^n is a function on Δ with values U_j^n. The following notations will be used on occasions:

$$\lambda = \frac{\Delta t}{\Delta x}, \qquad \mu = \frac{\Delta t}{\Delta x^2},$$

$$\Delta_- U_j = U_j - U_{j-1}, \qquad D_- = \frac{1}{\Delta x}\Delta_-, \qquad \Delta_+ U_j = U_{j+1} - U_j, \qquad D_+ = \frac{1}{\Delta x}\Delta_+.$$

For later use, we introduce the following two constants:

$$a_\infty = \sup_{\min u_0 \le \xi \le \max u_0} |a(\xi)| < \infty, \qquad b_\infty = \sup_{\min u_0 \le \xi \le \max u_0} |b(\xi)| < \infty.$$

To approximate (1) we consider three-point implicit difference schemes of the form

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + D_-\big(h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))\big) = 0, & (j,n) \in \mathbb{Z} \times \{0, \dots, N-1\}, \\ U_j^0 = \dfrac{1}{\Delta x}\displaystyle\int_{j\Delta x}^{(j+1)\Delta x} u_0(x)\, dx, & j \in \mathbb{Z}. \end{cases} \tag{21}$$

We assume that the numerical flux $h(u,v)$ satisfies the consistency condition

$$h(u,u) = f(u) \tag{22}$$

and the monotonicity conditions

$$\partial_u h(u,v) \ge 0, \qquad \partial_v h(u,v) \le 0. \tag{23}$$

We will see later that (23) ensures that the solution operator of (21) is monotone. An example of a scheme which satisfies these conditions is provided by a variant of the Engquist-Osher scheme where the numerical flux $h(u,v)$ is given by

$$h(u,v) = f^+(u) + f^-(v)$$

where

$$f^+(u) = f(0) + \int_0^u \max(f'(s), 0)\, ds, \qquad f^-(u) = \int_0^u \min(f'(s), 0)\, ds.$$

For another example, assume that β, γ are strictly increasing and nondecreasing respectively, and consider the numerical flux h given by

$$h(u,v) = \frac{f(u) + f(v)}{2} - \frac{\Delta x}{2\lambda}\beta\Big(\frac{\gamma(v) - \gamma(u)}{\Delta x}\Big).$$

This corresponds to a central (space) differencing of

$$\partial_t u + \partial_x f(u) = \partial_x A(\partial_x B(u)) + \varepsilon \partial_x \beta(\partial_x \gamma(u)),$$

where ε is chosen as $\frac{\Delta x^2}{2\Delta t}$. Notice that this scheme is monotone provided that $\pm\lambda f'(u) + \beta'(v)\gamma'(u) \ge 0$ for all u, v. When the problem is nondegenerate $(a, b > 0)$ we can use the numerical flux given by

$$h(u,v) = \frac{f(u) + f(v)}{2},$$

which corresponds to central (space) differencing of (1). In this case the monotonicity assumptions are given by the weaker assumptions (compared to (23))

$$\frac{1}{\Delta x} a(r_4) b(r_3) + \partial_u h(u,v)|_{(r_1, r_2)} \ge 0, \qquad \frac{1}{\Delta x} a(r_4) b(r_3) - \partial_v h(u,v)|_{(r_1, r_2)} \ge 0, \tag{24}$$

where r_1, r_2, r_3, r_4 are arbitrary numbers in $\mathbb{R}$. It follows that (24) is satisfied provided $\Delta x |f'| \ge 2a_\infty b_\infty$.

§4. Regularity Estimates.

In this section we establish the regularity estimates which will be needed later for showing convergence of the discrete approximations. In the following we treat the case where u_0 has compact support and f, A, B are locally C^1. Then at the end of section §5 we briefly discuss the general case where u_0 is not necessarily compactly supported and f, A, B are locally Lipschitz continuous. If not otherwise stated, we will always assume, without loss of generality, that $f(0) = 0$. The function space that contains u_0 will be taken as

$$\mathcal{B}(f, A, B) = \{ z \in L^1(\mathbb{R}) \cap BV(\mathbb{R}) : |f(z) - A(\partial_x B(z)|_{BV(\mathbb{R})} < \infty \}. \tag{25}$$

Convergence in L^1_{loc} of a subsequence of the family u_Δ of approximate solutions generated from (21) is obtained by establishing three estimates for $\{U_j^n\}$:

(a) a uniform L^∞ bound,
(b) a uniform total variation bound,
(c) L^1 Lipschitz continuity in the time variable,

and two estimates for the discrete total flux term $h(U_j^{n+1}, U_{j+1}^{n+1}) - A\left(D_+ B(U_j^{n+1})\right)$:

(d) a uniform L^∞ bound,
(e) a uniform total variation bound.

The estimates (d) and (e) play a main role in that we utilize estimate (e) to obtain estimate (c), while (d) is used to obtain the Hölder continuity in time and space of the discrete diffusion term $B(U_j^n)$.

For later use, recall that the $L^\infty(\mathbb{Z})$ norm, the $L^1(\mathbb{Z})$ norm and the $BV(\mathbb{Z})$ semi-norm of a lattice function U is defined respectively as

$$\|U\|_{L^\infty(\mathbb{Z})} = \sup_{j\in\mathbb{Z}} |U_j|,$$
$$\|U\|_{L^1(\mathbb{Z})} = \sum_{j\in\mathbb{Z}} |U_j|,$$
$$|U|_{BV(\mathbb{Z})} = \sum_{j\in\mathbb{Z}} |U_j - U_{j-1}| \equiv \Delta x \|D_- U\|_{L^1(\mathbb{Z})}.$$

If not specified, i, j will always denote integers from $\mathbb{Z}$; m, n, l integers from $\{0, \dots, N\}$; x, y, c real numbers from $\mathbb{R}$ and t, τ real numbers from $[0, T]$. Throughout this paper C will denote a positive constant, not necessarily the same at different occurrences, which is independent of the discretization parameters involved.

The following lemma deals with the question of existence, uniqueness and properties of the solution of the (nonlinear) system (21).

Lemma 4.1. *If (23) is satisfied, then for any U there is a unique U^* satisfying the following equation*

$$\frac{U_j^* - U_j}{\Delta t} + D_-\left(h(U_j^*, U_{j+1}^*) - A(D_+ B(U_j^*))\right) = 0, \qquad j \in \mathbb{Z}. \tag{26}$$

Furthermore, the solution U^* of (26) possesses the following properties:

(a) $U_j \le V_j \ \forall j \in \mathbb{Z}$ *implies that* $U_j^* \le V_j^* \ \forall j \in \mathbb{Z}$,
(b) $\|U^*\|_{L^\infty(\mathbb{Z})} \le \|U\|_{L^\infty(\mathbb{Z})}$,
(c) $\|U^* - V^*\|_{L^1(\mathbb{Z})} \le \|U - V\|_{L^1(\mathbb{Z})}$,
(d) $|U^*|_{BV(\mathbb{Z})} \le |U|_{BV(\mathbb{Z})}$.

Proof. As an aid in the analysis we shall view the equation (21) in terms of an m-accretive operator and an associated contraction solution operator, i.e., we shall use the Crandall and Liggett theory [3]. A similar treatment of implicit difference schemes for conservation laws has been given earlier by Lucier [15] and for strongly degenerate convection-diffusion equations in [9].

For a fixed n, let us now rewrite the difference equation (21) as (supressing the Δx dependence)

$$U_j^{n+1} + \Delta t \mathcal{A}(U^{n+1}; j) = U_j^n, \qquad j \in \mathbb{Z}, \tag{27}$$

where the operator $\mathcal{A} : L^1(\mathbb{Z}) \to L^1(\mathbb{Z})$ is defined by

$$\mathcal{A}(U; j) = D_-\left(h(U_j, U_{j+1}) - A\left(D_+ B(U_j)\right)\right).$$

We first show that the operator $\mathcal{A}$ is accretive. To this end, it is sufficient to establish that for any U, V with $U - V \in L^1(\mathbb{Z})$,

$$\sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j - V_j)\left(\mathcal{A}(U; j) - \mathcal{A}(V; j)\right) \ge 0.$$

As a first step to achieve this goal, we perform the following calculation

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j - V_j)\big(\mathcal{A}(U;j) - \mathcal{A}(V;j)\big) \\
&\qquad = \sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j - V_j)\left(\mathcal{A}(U;j) - \mathcal{A}(V;j) - c\big(U_j - V_j\big)\right) + c\sum_{j\in\mathbb{Z}}\left|U_j - V_j\right|, \\
&\qquad \geq -\sum_{j\in\mathbb{Z}}\left|cW_j - \left(\mathcal{A}(U;j) - \mathcal{A}(V;j)\right)\right| + c\sum_{j\in\mathbb{Z}}\left|W_j\right|,
\end{aligned}
\tag{28}
$$

where W_j denotes $U_j - V_j$ and $c = c(\Delta x) > 0$ is a number chosen so that

$$
c \geq \frac{1}{\Delta x}\left(\max_{(u,v)} \partial_u h(u,v) - \min_{(u,v)} \partial_v h(u,v)\right) + \frac{2}{\Delta x^2} a_\infty b_\infty. \tag{29}
$$

Next, we observe that

$$
\begin{aligned}
&\mathcal{A}(U;j) - \mathcal{A}(V;j) \\
&\quad = \frac{1}{\Delta x}\Big(h_u(\alpha_j, U_{j+1})W_j + h_v(V_j, \tilde{\alpha}_{j+1})W_{j+1} - h_u(\alpha_{j-1}, U_j)W_{j-1} - h_v(V_{j-1}, \tilde{\alpha}_j)W_j\Big) \\
&\qquad - \frac{1}{\Delta x^2}\Big(a(\gamma_j)\big(b(\beta_{j+1})W_{j+1} - b(\beta_j)W_j\big) - a(\gamma_{j-1})\big(b(\beta_j)W_j - b(\beta_{j-1})W_{j-1}\big)\Big),
\end{aligned}
\tag{30}
$$

where $\alpha_j, \tilde{\alpha}_j, \beta_j \in \operatorname{int}(U_j, V_j)$ and $\gamma_j \in \operatorname{int}(D_+B(U_j), D_+B(V_j))$. Inserting this into inequality (28) yields the desired result:

$$
\begin{aligned}
&\sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j - V_j)\big(\mathcal{A}(U;j) - \mathcal{A}(V;j)\big) \\
&\quad \geq c\sum_{j\in\mathbb{Z}}|W_j| - \sum_{j\in\mathbb{Z}}\bigg|\Big[\frac{1}{\Delta x}h_u(\alpha_{j-1}, U_j) + \frac{1}{\Delta x^2}a(\gamma_{j-1})b(\beta_{j-1})\Big]W_{j-1} \\
&\qquad\qquad + \Big[c - \frac{1}{\Delta x}\left(h_u(\alpha_j, U_{j+1}) - h_v(V_{j-1}, \tilde{\alpha}_j)\right) - \frac{1}{\Delta x^2}b(\beta_j)\left(a(\gamma_j) + a(\gamma_{j-1})\right]\Big)W_j \\
&\qquad\qquad + \Big[\frac{1}{\Delta x^2}a(\gamma_j)b(\beta_{j+1}) - \frac{1}{\Delta x}h_v(V_j, \tilde{\alpha}_{j+1})\Big]W_{j+1}\bigg| \\
&\quad \geq c\sum_{j\in\mathbb{Z}}|W_j| - \sum_{j\in\mathbb{Z}}\Big[\frac{1}{\Delta x}h_u(\alpha_{j-1}, U_j) + \frac{1}{\Delta x^2}a(\gamma_{j-1})b(\beta_{j-1})\Big]\left|W_{j-1}\right| \\
&\qquad\qquad - \sum_{j\in\mathbb{Z}}\Big[c - \frac{1}{\Delta x}\left(h_u(\alpha_j, U_{j+1}) - h_v(V_{j-1}, \tilde{\alpha}_j)\right) - \frac{1}{\Delta x^2}b(\beta_j)\left(a(\gamma_j) + a(\gamma_{j-1})\right)\Big]\left|W_j\right| \\
&\qquad\qquad - \sum_{j\in\mathbb{Z}}\Big[\frac{1}{\Delta x^2}a(\gamma_j)b(\beta_{j+1}) - \frac{1}{\Delta x}h_v(V_j, \tilde{\alpha}_{j+1})\Big]\left|W_{j+1}\right| \equiv 0,
\end{aligned}
$$

due to the monotonicity conditions (23) and the choice of c given by (29). From (30) we observe that the operator $\mathcal{A}$ is Lipschitz continuous,

$$
\left\|\mathcal{A}(U) - \mathcal{A}(V)\right\|_{L^1(\mathbb{Z})} \leq \left(\frac{2}{\Delta x}L(h) + \frac{4}{\Delta x^2}L(A,B)\right)\left\|U - V\right\|_{L^1(\mathbb{Z})},
$$

where $L(h) = \max|h_u| + \max|h_v|$ and $L(A,B) = a_\infty b_\infty$. This implies that $\mathcal{A}$ is not only accretive but also m-accretive, see [6]. We can now invoke Theorem 2.4 to conclude the existence of a unique monotone solution operator $\mathcal{S}$ associated with (21) such that

$$
U_j^* = \mathcal{S}(U;j),
$$

which proves the first part of the lemma. Since $\sum_{j\in\mathbb{Z}}\mathcal{A}(U;j) = 0$ and $\mathcal{A}$ commutes with translations, the second part of the lemma also follows from Theorem 2.4. □

As a direct consequence of Lemma 4.1 the following lemma is established.

Lemma 4.2. *We have*

$$\left\|U^{n+1}\right\|_{L^\infty(\mathbb{Z})} \leq \left\|U^0\right\|_{L^\infty(\mathbb{Z})}, \qquad \left|U^{n+1}\right|_{BV(\mathbb{Z})} \leq \left|U^0\right|_{BV(\mathbb{Z})}. \tag{31}$$

Next we establish a regularity property for the total flux $h(U_j^{n+1}, U_{j+1}^{n+1}) - A\left(D_+ B(U_j^{n+1})\right)$. As mentioned, this regularity property is of fundamental importance when proving convergence of the scheme (21). Let us first indicate how this regularity estimate can be derived at the continuous level in the case of classical solutions. To this end, consider the uniformly parabolic equation

$$\partial_t u + \partial_x f(u) = \partial_x A\left(\partial_x B(u)\right), \qquad A', B' > 0, \tag{32}$$

and recall that this equation has a unique classical solution u. By differentiating (32) with respect to t and subsequently integrating with respect to x, we find that the quantity

$$v(x,t) = \int_{-\infty}^{x} \partial_t u(\xi, t) d\xi$$

satisfies the linear, variable coefficients, uniformly parabolic cequation

$$\partial_t v + f'(u)\partial_x v = a(\partial_x B(u))\partial_x(b(u)\partial_x v). \tag{33}$$

From the maximum principle for this equation it follows that

$$\|v(\cdot,t)\|_{L^\infty(\mathbb{R})} \leq \|v_0\|_{L^\infty(\mathbb{R})}.$$

From (32) and the definition of v we see that $v = -f(u) + A(\partial_x B(u))$, which implies that

$$\|A\left(\partial_x B(u(\cdot,t))\right)\|_{L^\infty(\mathbb{R})} \leq C,$$

where $C = 2\max|f| + \|A\left(\partial_x B(u(\cdot,0))\right)\|_{L^\infty(\mathbb{R})}$. This is merely formalism since the solution of (1) in general only exists in a weak sense. However, these calculations clearly motivate the next lemma whose content is a uniform L^∞ bound as well as a BV bound for the discrete total flux $h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))$.

Lemma 4.3. *We have*

$$\left\|h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))\right\|_{L^\infty(\mathbb{Z})} \leq \left\|h(U_j^0, U_{j+1}^0) - A(D_+ B(U_j^0))\right\|_{L^\infty(\mathbb{Z})}, \tag{34}$$

$$\left|h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))\right|_{BV(\mathbb{Z})} \leq \left|h(U_j^0, U_{j+1}^0) - A(D_+ B(U_j^0))\right|_{BV(\mathbb{Z})}. \tag{35}$$

Proof. To prove these regularity properties for the approximate solutions, we introduce two auxiliary sequences $\{W_j^n\}$ and $\{V_j^n\}$ given by

$$W_j^{n+1} = \frac{U_j^{n+1} - U_j^n}{\Delta t}, \qquad V_j^{n+1} = \Delta x \sum_{k=-\infty}^{j} W_k^{n+1}.$$

Using the finite difference scheme (21) we observe

$$W_k^{n+1}\Delta x = -\Delta_-\left(h(U_k^{n+1}, U_{k+1}^{n+1}) - A(D_+ B(U_k^{n+1}))\right). \tag{36}$$

Summing over all $k = -\infty, \dots, j$ and having in mind that $U_k^n = 0$ for sufficiently large k and $h(0,0) = f(0) = 0$, we get

$$V_j^{n+1} = -\left(h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))\right). \tag{37}$$

From this relation it is clear that it is sufficient to establish L^∞ and BV estimates for V^n. As a first step toward that end, we derive an equation for the auxiliary sequence $\{V_j^n\}$. For this purpose, consider the difference equation given by (21) and subtract the corresponding equation at time t^n. Then we obtain

$$\begin{aligned} W_k^{n+1}\Delta x = W_k^n \Delta x &- \Delta_-\left(h(U_k^{n+1}, U_{k+1}^{n+1}) - h(U_k^n, U_{k+1}^n)\right) \\ &+ \Delta_-\left(A(D_+ B(U_k^{n+1})) - A(D_+ B(U_k^n))\right). \end{aligned}$$

Again we sum over all $k=-\infty,\dots,j$, yielding

$$V_j^{n+1} = V_j^n - \left(h(U_j^{n+1},U_{j+1}^{n+1}) - h(U_j^n,U_{j+1}^n)\right) + \left(A(D_+B(U_j^{n+1})) - A(D_+B(U_j^n))\right). \tag{38}$$

We now rewrite the two last terms. To this end, we first we observe that

$$\Delta x \frac{U_j^{n+1}-U_j^n}{\Delta t} = \Delta x W_j^n = \Delta x \sum_{k=-\infty}^{j} W_k^n - \Delta x \sum_{k=-\infty}^{j-1} W_k^n = V_j^n - V_{j-1}^n. \tag{39}$$

Then we have

$$\begin{aligned}
&h(U_j^{n+1},U_{j+1}^{n+1}) - h(U_j^n,U_{j+1}^n)\\
&\quad = \left(h(U_j^{n+1},U_{j+1}^{n+1}) - h(U_j^{n+1},U_{j+1}^n)\right) + \left(h(U_j^{n+1},U_{j+1}^n) - h(U_j^n,U_{j+1}^n)\right)\\
&\quad = \partial_v h(U_j^{n+1},\tilde\alpha_{j+1}^{n+\frac12})\left(U_{j+1}^{n+1}-U_{j+1}^n\right) + \partial_u h(\alpha_j^{n+\frac12},U_{j+1}^n)\left(U_j^{n+1}-U_j^n\right)\\
&\quad = \lambda\left(\partial_v h(U_j^{n+1},\tilde\alpha_{j+1}^{n+\frac12})[V_{j+1}^{n+1}-V_j^{n+1}] + \partial_u h(\alpha_j^{n+\frac12},U_{j+1}^n)[V_j^{n+1}-V_{j-1}^{n+1}]\right)\\
&\quad = \Delta t\left(h_{v,j}^{n+1}D_+V_j^{n+1} + h_{u,j}^{n+1}D_+V_{j-1}^{n+1}\right),
\end{aligned} \tag{40}$$

where

$$h_{v,j}^{n+1} = \partial_v h(U_j^{n+1},\tilde\alpha_{j+1}^{n+\frac12}), \qquad h_{u,j}^{n+1} = \partial_u h(\alpha_j^{n+\frac12},U_{j+1}^n), \qquad \alpha_j^{n+\frac12},\tilde\alpha_j^{n+\frac12}\in \operatorname{int}(U_j^n,U_j^{n+1}). \tag{41}$$

Similary, we rewrite the last term of (38).

$$\begin{aligned}
&A(D_+B(U_j^{n+1})) - A(D_+B(U_j^n))\\
&\quad = a(\gamma_j^{n+\frac12})D_+\left(B(U_j^{n+1})-B(U_j^n)\right)\\
&\quad = a(\gamma_j^{n+\frac12})D_+\left(b(\beta_j^{n+\frac12})[U_j^{n+1}-U_j^n]\right)\\
&\quad = \Delta t\cdot a(\gamma_j^{n+\frac12})D_+\left(b(\beta_j^{n+\frac12})D_-V_j^{n+1}\right)\\
&\quad = \Delta t\cdot a_j^{n+1}D_+\left(b_j^{n+1}D_-V_j^{n+1}\right),
\end{aligned} \tag{42}$$

where

$$\begin{aligned}
a_j^{n+1} &= a(\gamma_j^{n+\frac12}), &\qquad \gamma_j^{n+\frac12} &\in \operatorname{int}(D_+B(U_j^n),D_+B(U_j^{n+1})),\\
b_j^{n+1} &= b(\beta_j^{n+\frac12}), &\qquad \beta_j^{n+\frac12} &\in \operatorname{int}(U_j^n,U_j^{n+1}).
\end{aligned} \tag{43}$$

From (38), (40) and (42) we obtain the following linear difference equation for $\{V_j^n\}$.

$$\frac{V_j^{n+1}-V_j^n}{\Delta t} + \left(h_{u,j}^{n+1}D_+V_{j-1}^{n+1} + h_{v,j}^{n+1}D_+V_j^{n+1}\right) = a_j^{n+1}D_+\left(b_j^{n+1}D_-V_j^{n+1}\right). \tag{44}$$

This equation can be written as

$$c_j^{n+1}V_{j-1}^{n+1} + d_j^{n+1}V_j^{n+1} + e_j^{n+1}V_{j+1}^{n+1} = V_j^n, \tag{45}$$

where

$$\begin{aligned}
c_j^{n+1} &= -\left[\lambda h_{u,j}^{n+1} + \mu a_j^{n+1}b_j^{n+1}\right],\\
d_j^{n+1} &= \left[1+\lambda\left(h_{u,j}^{n+1}-h_{v,j}^{n+1}\right) + \mu a_j^{n+1}\left(b_{j+1}^{n+1}+b_j^{n+1}\right)\right],\\
e_j^{n+1} &= -\left[\mu a_j^{n+1}b_{j+1}^{n+1} - \lambda h_{v,j}^{n+1}\right].
\end{aligned}$$

By the monotonicity assumption (23), we have

$$c_j^{n+1}+d_j^{n+1}+e_j^{n+1} = 1, \qquad c_j^{n+1},e_j^{n+1}\le 0, \qquad d_j^{n+1}\ge 0. \tag{46}$$

Thanks to (46), the linear system (45) is strictly diagonal dominant. Consequently, there exists a unique solution V^{n+1}. Furthermore, this solution satisfies a maximum principle:

$$c_j^{n+1}|V_{j-1}^{n+1}| + d_j^{n+1}|V_j^{n+1}| + e_j^{n+1}|V_{j+1}^{n+1}| \le |V_j^n| \Longrightarrow \left\|V^{n+1}\right\|_{L^\infty(\mathbb{Z})} \le \left\|V^n\right\|_{L^\infty(\mathbb{Z})}.$$

In view of (37) we can now conclude that (34) is satisfied. Next we prove that the solution of (44) has bounded variation on $\mathbb{Z}$. Introduce the quantity $Z_j^n = V_j^n - V_{j-1}^n$ and observe that

$$\frac{Z_j^{n+1} - Z_j^n}{\Delta t} + D_- \left(h_{u,j}^{n+1} Z_j^{n+1} + h_{v,j}^{n+1} Z_{j+1}^{n+1}\right) = D_- \left(a_j^{n+1} D_+ \left(b_j^{n+1} Z_j^{n+1}\right)\right).$$

Similarly to (45), we can write this equation as

$$\bar{c}_j^{n+1} Z_{j-1}^{n+1} + \bar{d}_j^{n+1} Z_j^{n+1} + \bar{e}_j^{n+1} Z_{j+1}^{n+1} = Z_j^n, \tag{47}$$

where

$$\begin{aligned}
\bar{c}_j^{n+1} &= -\left[\lambda h_{u,j-1}^{n+1} + \mu a_{j-1}^{n+1} b_{j-1}^{n+1}\right], \\
\bar{d}_j^{n+1} &= \left[1 + \lambda \left(h_{u,j}^{n+1} - h_{v,j-1}^{n+1}\right) + \mu b_j^{n+1} \left(a_{j-1}^{n+1} + a_j^{n+1}\right)\right], \\
\bar{e}_j^{n+1} &= -\left[\mu a_j^{n+1} b_{j+1}^{n+1} - \lambda h_{v,j}^{n+1}\right].
\end{aligned}$$

Again, due to the monotonicity assumption, we see that

$$\bar{c}_{j+1}^{n+1} + \bar{d}_j^{n+1} + \bar{e}_{j-1}^{n+1} = 1, \qquad \bar{c}_j^{n+1}, \bar{e}_j^{n+1} \le 0, \qquad \bar{d}_j^{n+1} \ge 0. \tag{48}$$

Therefore, from (47), it follows that

$$\bar{c}_j^{n+1}|Z_{j-1}^{n+1}| + \bar{d}_j^{n+1}|Z_j^{n+1}| + \bar{e}_j^{n+1}|Z_{j+1}^{n+1}| \le |Z_j^n| \Longrightarrow \sum_{j\in\mathbb{Z}} |Z_j^{n+1}| \le \sum_{j\in\mathbb{Z}} |Z_j^n|,$$

which immediately implies (35). This concludes the proof of the lemma. □

An immediate consequence of (35), and (21) is that the discrete approximations (21) are L^1 Lipschitz continuous in time, and thus contained in $BV(Q_T)$.

Lemma 4.4. *We have*

$$\left\|U^m - U^n\right\|_{L^1(\mathbb{Z})} \le \left|h(U_j^0, U_{j+1}^0) - A\left(D_+ B(U_j^0)\right)\right|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x}|m-n|. \tag{49}$$

Proof. Suppose that $m > n$. Using (21), we readily calculate that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \left|U_j^m - U_j^n\right| &\le \Delta t \sum_{l=n}^{m-1} \sum_{j\in\mathbb{Z}} \left|D_- \left(h(U_j^{l+1}, U_{j+1}^{l+1}) - A\left(D_+ B(U_j^{l+1})\right)\right)\right| \\
&\le \left|h(U_j^0, U_{j+1}^0) - A\left(D_+ B(U_j^0)\right)\right|_{BV(\mathbb{Z})} \frac{\Delta t}{\Delta x}(m-n),
\end{aligned}$$

where the BV estimate (35) has been used. This concludes the proof of the lemma. □

Lemma 4.5. *If (23) is satisfied, then the following cell entropy inequality holds*

$$\begin{aligned}
\left|U_j^{n+1} - c\right| - \left|U_j - c\right| &+ \Delta t D_- \Big(h\left(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c\right) - h\left(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c\right)\Big) \\
&- \Delta t D_- \Big(\operatorname{sign}(U_j^{n+1} - c) A\left(D_+ B(U_j^{n+1})\right)\Big) \le 0.
\end{aligned} \tag{50}$$

Proof. The arguments are as follows. First, observe that

$$h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c) \le \operatorname{sign}(U_j^{n+1} - c)\big(h(U_j^{n+1}, U_{j+1}^{n+1}) - h(c,c)\big), \tag{51}$$

$$-\Big(h(U_{j-1}^{n+1} \vee c, U_j^{n+1} \vee c) - h(U_{j-1}^{n+1} \wedge c, U_j^{n+1} \wedge c)\Big) \le \operatorname{sign}(U_j^{n+1} - c)\big(h(c,c) - h(U_{j-1}^{n+1}, U_j^{n+1})\big). \tag{52}$$

These two inequalities follow from the monotonicity of h. Due to the similarity, we only show the first inequality (51). The proof is based upon examining several cases depending on whether U_{j+1}^{n+1} is larger or smaller than U_j^{n+1}. If $c \notin \operatorname{int}(U_j^{n+1}, U_{j+1}^{n+1})$, then the left hand side of (51) is equal to the right hand side.

Next, assume that $c \in \operatorname{int}(U_j^{n+1}, U_{j+1}^{n+1})$ and $U_j^{n+1} \le U_{j+1}^{n+1}$. Then

$$\begin{aligned}
&h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c) \\
&\quad = h(c, U_{j+1}^{n+1}) - h(U_j^{n+1}, c) \\
&\quad = h(c,c) - h(U_j^{n+1}, U_{j+1}^{n+1}) + \big[h(c, U_{j+1}^{n+1}) - h(c,c)\big] + \big[h(U_j^{n+1}, U_{j+1}^{n+1}) - h(U_j^{n+1}, c)\big] \\
&\quad = \operatorname{sign}(U_j^{n+1} - c)\big(h(U_j^{n+1}, U_{j+1}^{n+1}) - h(c,c)\big) + Q_j^{n+1}
\end{aligned}$$

where

$$\begin{aligned}
Q_j^{n+1} &= \big[h(c, U_{j+1}^{n+1}) - h(c,c)\big] + \big[h(U_j^{n+1}, U_{j+1}^{n+1}) - h(U_j^{n+1}, c)\big] \\
&= h_{v,j}^{n+1}(U_{j+1}^{n+1} - c) + \tilde{h}_{v,j}^{n+1}(U_{j+1}^{n+1} - c)
\end{aligned}$$

and

$$h_{v,j}^{n+1} = \partial_v h(c, \alpha_{j+1}^{n+1}), \qquad \tilde{h}_{v,j}^{n+1} = \partial_v h(U_j^{n+1}, \tilde{\alpha}_{j+1}^{n+1}), \qquad \alpha_j^{n+1}, \tilde{\alpha}_j^{n+1} \in \operatorname{int}(c, U_j^{n+1}).$$

Due to the monotonicity assumption (23) and the fact that $c \le U_{j+1}^{n+1}$, we conclude that $Q_j^{n+1} \le 0$ and the desired inequality is obtained. Similary, we can show that this inequality holds when $U_j^{n+1} \ge U_{j+1}^{n+1}$.

For the discrete diffusion term we have the following inequality.

$$\operatorname{sign}(U_{j-1}^{n+1} - c)A\left(D_+ B(U_{j-1}^{n+1})\right) \le \operatorname{sign}(U_j^{n+1} - c)A\left(D_+ B(U_{j-1}^{n+1})\right). \tag{53}$$

In order to see this, consider the relation

$$\operatorname{sign}(U_{j-1}^{n+1} - c)A\left(D_+ B(U_{j-1}^{n+1})\right) = \operatorname{sign}(U_j^{n+1} - c)A\left(D_+ B(U_{j-1}^{n+1})\right) + R_j^{n+1},$$

where

$$\begin{aligned}
R_j^{n+1} &= \left(\operatorname{sign}(U_{j-1}^{n+1} - c) - \operatorname{sign}(U_j^{n+1} - c)\right) A\left(D_+ B(U_{j-1}^{n+1})\right) \\
&= \left(\operatorname{sign}(U_{j-1}^{n+1} - c) - \operatorname{sign}(U_j^{n+1} - c)\right)\left(U_j^{n+1} - U_{j-1}^{n+1}\right) \bar{a}_{j-\frac12}^{n+1} b(\beta_{j-\frac12}^{n+1})
\end{aligned}$$

and

$$\bar{a}_{j-\frac12}^{n+1} = \int_0^1 a(\xi D_+ B(U_{j-1}^{n+1}))d\xi \ge 0, \qquad b(\beta_{j-\frac12}^{n+1}) \ge 0, \qquad \beta_{j-\frac12}^{n+1} \in \operatorname{int}(U_{j-1}^{n+1}, U_j^{n+1}).$$

Now we observe that $R_j^{n+1} = 0$ unless c is between U_{j-1}^{n+1} and U_j^{n+1}. If c is in this interval, it is easy to check that R_j^{n+1} is nonpositive. Invoking (51),(52), (53) and (21) we obtain

$$\begin{aligned}
&|U_j^{n+1} - c| + \Delta t D_-\big(h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c)\big) \\
&\qquad - \Delta t D_-\big(\operatorname{sign}(U_j^{n+1} - c)A(D_+ B(U_j^{n+1}))\big) \\
&\quad \le |U_j^{n+1} - c| + \lambda \operatorname{sign}(U_j^{n+1} - c)\big(h(U_j^{n+1}, U_{j+1}^{n+1}) - h(c,c)\big) \\
&\qquad\qquad + \lambda \operatorname{sign}(U_j^{n+1} - c)\left(h(c,c) - h(U_{j-1}^{n+1}, U_j^{n+1})\right) \\
&\qquad\qquad - \lambda \operatorname{sign}(U_j^{n+1} - c)A\left(D_+ B(U_j^{n+1})\right) + \lambda \operatorname{sign}(U_j^{n+1} - c)A\left(D_+ B(U_{j-1}^{n+1})\right) \\
&\quad = \operatorname{sign}(U_j^{n+1} - c)\big(U_j^{n+1} - c + \Delta t D_-\big(h(U_j^{n+1}, U_{j+1}^{n+1}) - A(D_+ B(U_j^{n+1}))\big)\big) \\
&\quad = \operatorname{sign}(U_j^{n+1} - c)\big(U_j^n - c\big) \\
&\quad \le |U_j^n - c|.
\end{aligned}$$

Hence the proof is complete. □

Remark. *The estimates of Lemmas 4.2, 4.3 and 4.4 have been obtained without making use of the structural assumption (6) on A. From these estimates it is not difficult to show that there is a subsequence of the approximate solutions which converges to a limit function u. However, we do not have estimates on the diffusion term which ensures that $A(D_+ B(U_j^n))$ converges in some appropiate sense to the diffusion term $A(\partial_x B(u))$.*

In the following we will discuss continuity properties of the discrete diffusion term $\{B(U_j^n)\}$. From (34) and the assumption that u_0 is contained in $\mathcal{B}(f, A, B)$ it follows that

$$\left\| A(D_+ B(U_j^n)) \right\|_{L^\infty(\mathbb{Z})} \le \tilde{C}, \tag{54}$$

where $\tilde{C}$ is a constant independent of Δ. An immediate consequence of (54) and the assumption (6) is the following lemma.

Lemma 4.6. *We have*

$$\left\|D_+B(U_j^n)\right\|_{L^\infty(\mathbb{Z})} \leq C. \tag{55}$$

Remark. *The assumption (6) cannot be removed in establishing convergence to the BV entropy weak solution in the sense of Definition 2.1. In other words, the problem may not have BV entropy weak solutions if (6) is not assumed. Recall the example with A unbounded from section 1 (Figure 2). Here $B(s) = s$, but clearly $D_+B(U_j^n) = D_+U_j^n$ is not uniformly bounded because of the appearance of a discontinuity. Hence this problem cannot have a solution in the class given by Definition 2.1.*

Knowing that the discrete diffusion term $\{B(U_j^n)\}$ is Lipschitz continuous in the space variable, the question arises how to obtain information about the regularity in the time variable. One strategy would be to continue working with the linear equation for $v = f(u) - A(\partial_x B(u))$ and try to derive a result concerning the continuity of v with respect to the time variable from the known modulus of continuity in space. This technique, introduced by Kruzkov [11], was used for the simple degenerate case [8], i.e. when $A(s) = s$. To illustrate some of the added difficulties introduced by the double nonlinearity, let us see why this technique does not work in the general case. To this end, let $\phi(x)$ be a test function on $\mathbb{R}$ and multiply (33) by ϕ and integrate over $\mathbb{R}$. Then we have

$$\int_{\mathbb{R}} \phi(x)\partial_t v\, dx = -\int_{\mathbb{R}} f'(x,t)\partial_x v \cdot \phi(x)\, dx + \int_{\mathbb{R}} a(x,t)\partial_x\left(b(x,t)\partial_x v\right) \cdot \phi(x)\, dx, \tag{56}$$

where $f'(x,t), a(x,t), b(x,t)$ denote $f'(u(x,t)), a(\partial_x B(u(x,t))), b(u(x,t))$ respectively. The first term on the right hand side of (56) is bounded since v is of bounded variation. For the case when $A(s) = s$, that is $a(x,t) = 1$, the second term is bounded since one derivative can be moved over to the test function ϕ. However, in the general case $a(x,t) = a(\partial_x B(u(x,t)))$ is not constant and therefore it is not possible to bound this term. Hence we have to choose another approach to this problem. We will employ a discrete version of a technique used by Yin [18] which combines the scheme (21) and the estimate (34). For this purpose, define u_Δ as the interpolant of the discrete values $\{U_j^n\}$ given by

$$u_\Delta(x,t) = \begin{cases} U_j^n + \frac{U_{j+1}^n - U_j^n}{\Delta x}(x - x_j) + \frac{U_{j+1}^{n+1} - U_{j+1}^n}{\Delta t}(t - t^n), & (x,t) \in T_{j,n}^L, \\ U_j^n + \frac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x}(x - x_j) + \frac{U_j^{n+1} - U_j^n}{\Delta t}(t - t^n), & (x,t) \in T_{j,n}^U. \end{cases} \tag{57}$$

Here $T_{j,n}^L$ denotes the triangle with vertices (x_j,t^n),(x_{j+1},t^n) and (x_{j+1},t^{n+1}) while $T_{j,n}^U$ denotes the triangle with vertices (x_j,t^n),(x_j,t^{n+1}) and (x_{j+1},t^{n+1}). Let

$$R_j^n = [x_j, x_{j+1}] \times [t^n, t^{n+1}]$$

and note that $R_j^n = T_{j,n}^L \cup T_{j,n}^U$. Later we will use the notation $R_{x,t}$ in order to denote a rectangle R_j^n, not necessarily unique, which contains the point (x,t). In particular, we note that u_Δ is continuous everywhere and differentiable almost everywhere in Q_T.

Lemma 4.7. *We have*

$$\left|B(U_i^m) - B(U_j^n)\right| \leq \mathrm{C}\big(|x_i - x_j| + \sqrt{|t^m - t^n|} + \Delta x\big). \tag{58}$$

Proof. We have that

$$\left|B(U_i^m) - B(U_j^n)\right| \leq \left|B(U_i^m) - B(U_i^n)\right| + \left|B(U_i^n) - B(U_j^n)\right| =: I_1 + I_2.$$

Clearly $I_2 = \mathcal{O}(|x_i - x_j|)$ by using (55). Now we focus on how to estimate I_1. Consider the interval $[x_i, x_i + \alpha]$, where α will be specified later. Then for some $x^* \in [x_i, x_i + \alpha]$ (that also will be specified later) we have

(59)

$$\begin{aligned} I_1 &= |B(u_\Delta(x_i,t^m)) - B(u_\Delta(x_i,t^n))| \\ &\leq |B(u_\Delta(x_i,t^m)) - B(u_\Delta(x^*,t^m))| + |B(u_\Delta(x^*,t^m)) - B(u_\Delta(x^*,t^n))| + |B(u_\Delta(x^*,t^n)) - B(u_\Delta(x_i,t^n))| \\ &\leq 2C\left(|x_i - x^*| + \Delta x\right) + |B(u_\Delta(x^*,t^m)) - B(u_\Delta(x^*,t^n))| \\ &\leq 2C\left(\alpha + \Delta x\right) + |B(u_\Delta(x^*,t^m)) - B(u_\Delta(x^*,t^n))|, \end{aligned}$$

where the estimate of the first and third term of the second line follow from the monotonicity of $B(s)$. Next we describe how $|B(u_\Delta(x^*,t^n)) - B(u_\Delta(x^*,t^m))|$ can be estimated. For this purpose, we introduce the quantity

$$Q(x) = \int_{-\infty}^{x} \big(u_\Delta(\xi,t^m) - u_\Delta(\xi,t^n)\big)\,d\xi.$$

Since u_Δ is continuous, $Q(x)$ is differentiable everywhere. Hence, there is a number x^* in $[x_i, x_i+\alpha]$ such that

$$Q'(x^*)\alpha = Q(x_i+\alpha) - Q(x_i) = \int_{x_i}^{x_i+\alpha} \big(u_\Delta(\xi,t^m) - u_\Delta(\xi,t^n)\big)\,d\xi.$$

We then have the following relation

$$\begin{aligned} &|B(u_\Delta(x^*,t^n)) - B(u_\Delta(x^*,t^m))| \le b_\infty |u_\Delta(x^*,t^n) - u_\Delta(x^*,t^m)| \\ &= b_\infty |Q'(x^*)| = \frac{b_\infty}{\alpha}\cdot \Big|\int_{x_i}^{x_i+\alpha} \big(u_\Delta(\xi,t^m) - u_\Delta(\xi,t^n)\big)\,d\xi\Big|. \end{aligned} \tag{60}$$

Since u_Δ is differentiable in time almost everywhere on Q_T we have

$$\begin{aligned} &\int_{x_i}^{x_i+\alpha} \big(u_\Delta(\xi,t^m) - u_\Delta(\xi,t^n)\big)\,d\xi \\ &= \int_{x_i}^{x_i+\alpha}\int_{t^n}^{t^m} \partial_t u_\Delta\,dt\,dx = \int_{x_i}^{x_j}\int_{t^n}^{t^m} \partial_t u_\Delta\,dt\,dx + \int_{x_j}^{x_i+\alpha}\int_{t^n}^{t^m} \partial_t u_\Delta\,dt\,dx =: J_1 + J_2, \end{aligned} \tag{61}$$

where j is the integer such that $0 < (x_i+\alpha) - x_j < \Delta x$. Now, in view of (59), (60) and (61) we want to show that $|J_1|, |J_2| \le \alpha^2$ and then choose α equal to $\sqrt{|m-n|\Delta t}$. We have

$$\begin{aligned} J_1 &= \int_{x_i}^{x_j}\int_{t^n}^{t^m} \partial_t u_\Delta\,dt\,dx \\ &= \sum_{k=i}^{j-1}\sum_{l=n}^{m-1}\Big(\iint_{T^U_{k,l}} \partial_t u_\Delta\,dt\,dx + \iint_{T^L_{k,l}} \partial_t u_\Delta\,dt\,dx\Big) \\ &= \frac{1}{2}\Delta x\Delta t \sum_{k=i}^{j-1}\sum_{l=n}^{m-1}\Big(\frac{U_k^{l+1}-U_k^l}{\Delta t} + \frac{U_{k+1}^{l+1}-U_{k+1}^l}{\Delta t}\Big). \end{aligned}$$

Using the finite difference scheme (21) and estimate (34) of Lemma 4.3, we obtain the following estimate

$$\begin{aligned} |J_1| &= \frac{1}{2}\Delta x\Delta t \Big|\sum_{k=i}^{j-1}\sum_{l=n}^{m-1}\Big(\frac{U_k^{l+1}-U_k^l}{\Delta t} + \frac{U_{k+1}^{l+1}-U_{k+1}^l}{\Delta t}\Big)\Big| \\ &\le 4\cdot\frac{1}{2}\Delta t|m-n|\,\big\|h(U_j^0,U_{j+1}^0) - A\left(D_+B(U_j^0)\right)\big\|_{L^\infty(\mathbb{Z})} \\ &= 2C_0|m-n|\Delta t = 2C_0\alpha^2, \end{aligned}$$

where

$$C_0 = \big\|h(U_j^0,U_{j+1}^0) - A\left(D_+B(U_j^0)\right)\big\|_{L^\infty(\mathbb{Z})}, \tag{62}$$

and we have set α equal to $\sqrt{|m-n|\Delta t}$. Repeating the arguments for J_2 we also deduce that $|J_2| \le 2C_0\alpha^2$. From (60) and (61) we now conclude that

$$|B(u_\Delta(x^*,t^n)) - B(u_\Delta(x^*,t^m))| \le 4C_0\alpha,$$

and hence, from (59), we obtain

$$I_1 = |B(u_\Delta(x_i,t^m)) - B(u_\Delta(x_i,t^n))| = \mathcal{O}\big(\alpha + \Delta x\big) = \mathcal{O}\big(\sqrt{|m-n|\Delta t} + \Delta x\big).$$

Now the proof of (58) is completed. $\square$

§5. Convergence Results.

Now we will employ the regularity properties established for $\{U_j^n\}$ and $\{B(U_j^n)\}$ in §5 to prove that the approximate solutions generated by (21) in fact converges to the solution of (1) in the sense of Definition 2.1. We start by showing that a subsequence of the family of approximate solutions converges to a function u and that this limit inherits the properties of the approximate solutions (see Lemma 5.2). Finally, using the cell entropy inequality of Lemma 4.5 and the properties of the interpolant we show that this limit satisfies the entropy inequality of Definition 2.1. The arguments needed to prove this turn out to be rather involved due to the double nonlinearity of the problem. In particular, we will see that it is important how the linear interpolant is defined.

Recall that u_Δ denotes the interpolant of the discrete values $\{U_j^n\}$ given by (57). Similary we define w_Δ as the interpolant of the discrete values $\{B(U_j^n)\}$ given by

$$w_\Delta(x,t) = \begin{cases} B(U_j^n) + \frac{B(U_{j+1}^n)-B(U_j^n)}{\Delta x}(x-x_j) + \frac{B(U_{j+1}^{n+1})-B(U_{j+1}^n)}{\Delta t}(t-t^n), & (x,t)\in T_{j,n}^L, \\ B(U_j^n) + \frac{B(U_{j+1}^{n+1})-B(U_j^{n+1})}{\Delta x}(x-x_j) + \frac{B(U_j^{n+1})-B(U_j^n)}{\Delta t}(t-t^n), & (x,t)\in T_{j,n}^U. \end{cases} \tag{63}$$

For later use, observe that the following important relations hold

$$\partial_x u_\Delta = D_+U_j^n, \qquad \partial_x w_\Delta = D_+B(U_j^n) \tag{64}$$

on the parallelogram P_j^n with vertices (x_j,t^{n-1}), (x_j,t^n), (x_{j+1},t^n) and (x_{j+1},t^{n+1}), i.e., $P_j^n = T_{j,n-1}^U \cup T_{j,n}^L$. Similary,

$$\partial_t u_\Delta = \frac{U_j^{n+1}-U_j^n}{\Delta t} \tag{65}$$

on the parallelogram Q_j^n with vertices (x_{j-1},t^n),(x_j,t^n), (x_j,t^{n+1}) and (x_{j+1},t^{n+1}), i.e., $Q_j^n = T_{j-1,n}^L \cup T_{j,n}^U$. Note also that for $(x,t)\in R_j^n$ neither w_Δ nor $B(u_\Delta)$ will introduce new minima or maxima, that is

$$\min\Big(B(U_j^n),B(U_{j+1}^n),B(U_j^{n+1}),B(U_{j+1}^{n+1})\Big) \le w_\Delta, B(u_\Delta) \le \max\Big(B(U_j^n),B(U_{j+1}^n),B(U_j^{n+1}),B(U_{j+1}^{n+1})\Big). \tag{66}$$

This follows from the definition of u_Δ, w_Δ and the fact that $B(s)$ is monotone. The next technical lemma deals with the interpolation error associated with the linear interpolant (63) of Hölder continuous functions.

Lemma 5.1. *Assume that $G(x,t)\in C^{1,\frac12}(Q_T)$ and let $\Pi_\Delta G(x,t)$ denote the interpolant given by*

$$\Pi_\Delta G(x,t) = \begin{cases} G(x_j,t^n) + \frac{G(x_{j+1},t^n)-G(x_j,t^n)}{\Delta x}(x-x_j) + \frac{G(x_{j+1},t^{n+1})-G(x_{j+1},t^n)}{\Delta t}(t-t^n), & (x,t)\in T_{j,n}^L, \\ G(x_j,t^n) + \frac{G(x_{j+1},t^{n+1})-G(x_j,t^{n+1})}{\Delta x}(x-x_j) + \frac{G(x_j,t^{n+1})-G(x_j,t^n)}{\Delta t}(t-t^n), & (x,t)\in T_{j,n}^U. \end{cases}$$

Then the following error estimate holds

$$\|\Pi_\Delta G - G\|_{L^\infty(Q_T)} \le C\big(\Delta x + \sqrt{\Delta t}\big).$$

Proof. To see this, let (x,t) be an arbitrary point in Q_T. Then (x,t) is contained in some rectangle R_j^n and we have

$$|\Pi_\Delta G(x,t) - G(x,t)| \le |\Pi_\Delta G(x,t) - G(x_j,t^n)| + |G(x_j,t^n) - G(x,t)| \tag{67}$$

For the first term on the right hand side of (67) we have

$$|\Pi_\Delta G(x,t) - G(x_j,t^n)| \le \begin{cases} |G(x_{j+1},t^n)-G(x_j,t^n)| + |G(x_{j+1},t^{n+1})-G(x_{j+1},t^n)|, & (x,t)\in T_{j,n}^L, \\ |G(x_{j+1},t^{n+1})-G(x_j,t^{n+1})| + |G(x_j,t^{n+1})-G(x_j,t^n)|, & (x,t)\in T_{j,n}^U. \end{cases}$$

Therefore, since $G(x,t)\in C^{1,\frac12}(Q_T)$, it follows that the first and the second term on the right hand side of (67) is of order $\Delta x + \sqrt{\Delta t}$. $\square$

Now we show that the following compactness and convergence results hold.

Lemma 5.2. *There exists a function $u \in L^\infty(Q_T) \cap BV(Q_T)$, with $B(u) \in C^{1,\frac{1}{2}}(Q_T)$, such that*

$$
(68) \qquad \begin{cases} (a) & u_\Delta(x,t) \to u(x,t), & \text{in } L^1_{\text{loc}}(Q_T) \text{ and pointwise a.e. in } Q_T. \\ (b) & w_\Delta(x,t) \to B(u(x,t)), & \text{uniformly on compact sets in } Q_T. \\ (c) & \partial_x w_\Delta \overset{*}{\rightharpoonup} \partial_x B(u), & \text{in } L^\infty_{\text{loc}}(Q_T). \\ (d) & A(\partial_x w_\Delta) \overset{*}{\rightharpoonup} A(\partial_x B(u)), & \text{in } L^\infty_{\text{loc}}(Q_T). \end{cases}
$$

Proof. The functions $u_\Delta(x,t)$ and $w_\Delta(x,t)$ satisfy the following estimates:

$$
(69) \qquad ||u_\Delta||_{L^\infty(Q_T)} \le \mathrm{C}, \qquad |u_\Delta|_{BV(Q_T)} \le \mathrm{C},
$$

and

$$
(70) \qquad |w_\Delta(y,s) - w_\Delta(x,t)| \le \mathrm{C}\big(|x-y| + \sqrt{|t-s|} + \Delta x + \sqrt{\Delta t}\big), \qquad \forall x,y,s,t.
$$

The first estimate of (69) follows immediately from the definition of the linear interpolant u_Δ and Lemma 4.2. The second estimate of (69) is a consequence of the following two estimates:

$$
\iint_{Q_T} |\partial_x u_\Delta| \, dt\, dx = \sum_{j,n} \iint_{P_j^n} |\partial_x u_\Delta| \, dt\, dx = \Delta x \Delta t \sum_{j,n} |D_+ U_j^n| \le T|u_0|_{BV}.
$$

Here we have used (64) and Lemma 4.2. Similary, by using (65) and Lemma 4.4 we obtain the estimate

$$
\iint_{Q_T} |\partial_t u_\Delta| \, dt\, dx = \sum_{j,n} \iint_{Q_j^n} |\partial_t u_\Delta| \, dt\, dx = \Delta x \Delta t \sum_{j,n} \frac{|U_j^{n+1} - U_j^n|}{\Delta t} \le C_0 \cdot T,
$$

where $C_0 = |h(U_j^0, U_{j+1}^0) - A(D_+ B(U_j^0))|_{BV(\mathbb{Z})}$. The estimate (70) requires argument. Let (x,t) and (y,s) be some arbitrary given points and choose two rectangles $R_{x,t}$ and $R_{y,s}$ such that $(x,t) \in R_{x,t}$ and $(y,s) \in R_{y,s}$ (they may coinside). Moreover, let (x_i, t^m) and (x_j, t^n) denote vertices of $R_{x,t}$ and $R_{y,s}$ respectively, such that

$$
|x_j - x_i| + \sqrt{|t^n - t^m|} \le |x-y| + \sqrt{|t-s|}.
$$

Then, we have

$$
\begin{aligned}
|w_\Delta(y,s) - w_\Delta(x,t)| &\le |w_\Delta(y,s) - w_\Delta(x_i,t^m)| + |w_\Delta(x_i,t^m) - w_\Delta(x_j,t^n)| + |w_\Delta(x_j,t^n) - w_\Delta(x,t)| \\
&=: E_1 + E_2 + E_3
\end{aligned}
$$

Clearly, by (58)

$$
\begin{aligned}
E_2 = |B(U_i^m) - B(U_j^n)| &\le \mathrm{C}\big(|x_i - x_j| + \sqrt{|m-n|\Delta t} + \Delta x\big) \\
&\le \mathrm{C}\big(|x-y| + \sqrt{|t-s|} + \Delta x\big).
\end{aligned}
$$

Now estimate (70) follows since we have, in view of (66), that

$$
E_1, E_3 \le \mathrm{C}\big(\Delta x + \sqrt{\Delta t}\big).
$$

By virtue of estimates (69), $\{u_\Delta\}$ is bounded in $W^{1,1}(\mathcal{K}) \subset BV(\mathcal{K})$ for each compact set $\mathcal{K}$. Using that $BV(\mathcal{K})$ is compactly imbedded in $L^1(\mathcal{K})$ it is not difficult to show that $\{u_\Delta\}$, passing if necessary to a subsequence, converges in $L^1_{loc}(Q_T)$ and pointwise almost everywhere in Q_T to a function u,

$$
u \in L^\infty(Q_T) \cap BV(Q_T).
$$

Next we discuss convergence properties of the sequence $\{w_\Delta\}$. By estimate (70) we can repeat the proof of the Ascoli-Arzela theorem to conclude that there is a subsequence of $\{w_\Delta\}$ and a limit w,

$$
w \in C^{1,\frac{1}{2}}(Q_T)
$$

such that

$$
w_\Delta \to w, \qquad \text{uniformly on compact sets and pointwise in } Q_T.
$$

By the continuity of w and the pointwise convergence, we conclude that $w = B(u)$. To see this, let (x,t) be an arbitrary point such that $u_\Delta(x,t) \to u(x,t)$, i.e. $B(u_\Delta(x,t)) \to B(u(x,t))$. We have

$$|B(u(x,t)) - w(x,t)| \le |B(u(x,t)) - B(u_\Delta(x,t))| + |B(u_\Delta(x,t)) - w_\Delta(x,t)| + |w_\Delta(x,t) - w(x,t)|.$$

Since $w_\Delta(x,t) \to w(x,t)$, we only have to check that $|B(u_\Delta(x,t)) - w_\Delta(x,t)|$ must tend to zero. For this purpose, assume that (x,t) is contained in a rectangle $R_{x,t}$. Then, in view of (66) we have

$$|B(u_\Delta(x,t)) - w_\Delta(x,t)| \le |B(u_\Delta(x_j,t^n)) - w_\Delta(x_i,t^m)| = |B(U_j^n) - B(U_i^m)| \le C\big(\Delta x + \sqrt{\Delta t}\big),$$

where (x_j,t^n) and (x_i,t^m) are appropiate chosen vertices of the rectangle $R_{x,t}$. Hence $w = B(u)$ almost everywhere in Q_T. By the continuity of w, this must hold for all points in Q_T.

Now we continue showing the convergence result (c) of (68). From (55) and (64) it follows that

$$\|\partial_x w_\Delta\|_{L^\infty(Q_T)} \le C.$$

Hence there is a limit function W such that $\|W\|_{L^\infty(Q_T)} \le C$ and, passing if necessary to a subsequence

$$\partial_x w_\Delta \overset{*}{\rightharpoonup} W, \qquad \text{in } L^\infty(Q_T).$$

Since

$$\iint\limits_{Q_T} w_\Delta \partial_x \phi \, dt\, dx \to \iint\limits_{Q_T} B(u)\partial_x\phi\, dt\, dx, \qquad \phi \in C_0^\infty(Q_T),$$

it is obvious that $W = \partial_x B(u)$ and (c) follows. Finally we show why (d) is satisfied. Due to the fact that

$$\|A(\partial_x w_\Delta)\|_{L^\infty(Q_T)} \le \tilde{C},$$

(see (54)) we know there is a function $\overline{A}(x,t)$ in $L^\infty(Q_T)$ such that, again passing if necessary to a subsequence,

$$A(\partial_x w_\Delta) \overset{*}{\rightharpoonup} \overline{A}, \qquad \text{in } L^\infty(Q_T).$$

We now show that $\overline{A} = A(\partial_x B(u))$ by using a discrete version of the arguments used by Yin [18]. Let $\Pi_\Delta B(u)$ be the interpolant of the discrete values $B(u(x_j,t^n))$ defined as in Lemma 5.1. For the moment, assume that $\mathcal{K}$ is a compact subset of Q_T of the form $\mathcal{K} = \cup_{j,n} P_j^n$ where $(j,n) \in \{J_1,\dots,J_2\} \times \{N_1,\dots,N_2\}$. We then have

$$\begin{aligned}
&\iint\limits_{\mathcal{K}} A(\partial_x w_\Delta)\,(\partial_x w_\Delta - \partial_x B(u))\; dt\, dx \\
&= \iint\limits_{\mathcal{K}} A(\partial_x w_\Delta)\,(\partial_x w_\Delta - \partial_x \Pi_\Delta B(u))\; dt\, dx + \iint\limits_{\mathcal{K}} A(\partial_x w_\Delta)\,(\partial_x \Pi_\Delta B(u) - \partial_x B(u))\; dt\, dx \\
&=: E_1 + E_2.
\end{aligned} \tag{71}$$

First, we estimate E_1 as follows (recall (64)).

$$\begin{aligned}
E_1 &= \iint\limits_{\mathcal{K}} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x \Pi_\Delta B(u)\big)\, dt\, dx \\
&= \sum_{j,n} \iint\limits_{P_j^n} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x \Pi_\Delta B(u)\big)\, dt\, dx \\
&= \Delta x \Delta t \sum_{j,n} A(D_+ B(U_j^n))\big[D_+ B(U_j^n) - D_+ B(u(x_j,t^n))\big] \\
&= -\Delta x\Delta t \sum_{j,n} D_- A(D_+ B(U_j^n))\,\big[B(U_j^n) - B(u(x_j,t^n))\big] \\
&\quad + \Delta x \Delta t \sum_n \left(A(D_+ B(U_{J_2}^n))\,\big[B(U_{J_2}^n) - B(u(x_{J_2},t^n))\big] - A(D_+B(U_{J_1}^n))\,\big[B(U_{J_1}^n) - B(u(x_{J_1},t^n))\big]\right) \\
&= -\Delta x \Delta t \sum_{j,n} \Big(\frac{U_j^n - U_j^{n-1}}{\Delta t} + D_- h(U_j^n, U_{j+1}^n)\Big)\,\big[w_\Delta(x_j,t^n) - B(u(x_j,t^n))\big] \\
&\quad + \Delta x \Delta t \sum_n \left(A(D_+ B(U_{J_2}^n))\,\big[B(U_{J_2}^n) - B(u(x_{J_2},t^n))\big] - A(D_+B(U_{J_1}^n))\,\big[B(U_{J_1}^n) - B(u(x_{J_1},t^n))\big]\right),
\end{aligned}$$

where we have used the finite difference scheme (21) for the last equality. Hence, by Lemmas 4.2, 4.4 and (54)

$$|E_1| \le (C_0T + T(\max|\partial_u h| + \max|\partial_v h|)|u_0|_{\mathrm{BV}}) \cdot \|w_\Delta - B(u)\|_{L^\infty(\mathcal{K})} + C\Delta x. \tag{72}$$

In order to estimate E_2 let $\omega_\delta(x)$ denote a standard mollifier in the x variable with support in $[-\delta,\delta]$. Let $A^\delta(\cdot,t) = \omega_\delta(\cdot) * A(\partial_x w_\Delta(\cdot,t))$. For E_2 we then have

$$\begin{aligned}
E_2 &= \iint_{\mathcal{K}} A(\partial_x w_\Delta)\,(\partial_x \Pi_\Delta B(u) - \partial_x B(u))\; dt\, dx \\
&= \iint_{\mathcal{K}} A^\delta(x,t)\,(\partial_x \Pi_\Delta B(u) - \partial_x B(u))\; dt\, dx + \iint_{\mathcal{K}} \big(A(\partial_x w_\Delta(x,t)) - A^\delta(x,t)\big)\,(\partial_x \Pi_\Delta B(u) - \partial_x B(u))\; dt\, dx \\
&= -\iint_{\mathcal{K}} \partial_x A^\delta(x,t)\,(\Pi_\Delta B(u) - B(u))\; dt\, dx + \iint_{\mathcal{K}} \big(A(\partial_x w_\Delta(x,t)) - A^\delta(x,t)\big)\,(\partial_x \Pi_\Delta B(u) - \partial_x B(u))\; dt\, dx \\
&=: E_{2,1} + E_{2,2}.
\end{aligned}$$

Clearly, in view of Lemma 5.1,

$$|E_{2,1}| \le \int_0^T |A^\delta(\cdot,t)|_{BV(\mathbb{R})}\, dt \cdot \|\Pi_\Delta B(u) - B(u)\|_{L^\infty(\mathcal{K})} = \mathcal{O}(\Delta x + \sqrt{\Delta t}), \tag{73}$$

due to the Hölder continuity of $B(u)$ and the fact that

$$|A^\delta(\cdot,t)|_{BV(\mathbb{R})} \le |A(\partial_x w_\Delta(\cdot,t))|_{BV(\mathbb{R})} = |A(D_+ B(U_j^n))|_{BV(\mathbb{Z})} \le \mathrm{C} \qquad \text{(for some appropiate } n),$$

which is true because of (35). Moreover, we have

$$|E_{2,2}| \le \|\partial_x \Pi_\Delta B(u) - \partial_x B(u)\|_{L^\infty(\mathcal{K})} \cdot \left\|A^\delta(x,t) - A(\partial_x w(x,t))\right\|_{L^1(\mathcal{K})} = \mathcal{O}(\delta), \tag{74}$$

since $\|\partial_x B(u)\|_{L^\infty(Q_T)} \le C$. From (71),(72), (68)b, (73) and (74) it follows that

$$\lim_{\Delta\to 0} \iint_{\mathcal{K}} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x B(u)\big)\; dt\, dx = 0. \tag{75}$$

Note that for a general compact set $\mathcal{K}$ we can split $\mathcal{K}$ into two sets $\mathcal{K}_P$ and $\Delta\mathcal{K}$ such that

$$\mathcal{K} = \mathcal{K}_P \cup \Delta\mathcal{K}, \qquad \mathcal{K}_P = \cup_{j,n} P_j^n, \qquad \mathrm{meas}(\Delta\mathcal{K}) = \mathcal{O}(\Delta x + \Delta t).$$

Hence

$$\begin{aligned}
&\iint_{\mathcal{K}} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x B(u)\big)\; dt\, dx \\
&\qquad = \iint_{\mathcal{K}_P} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x B(u)\big)\; dt\, dx + \iint_{\Delta\mathcal{K}} A(\partial_x w_\Delta)\big(\partial_x w_\Delta - \partial_x B(u)\big)\; dt\, dx.
\end{aligned}$$

In light of the analysis above, the first term tends to zero. Because the integrand of the last integral is uniformly bounded, it follows that this term is of order $\Delta x + \Delta t$ and thus tends to zero. Hence (75) holds for all compact $\mathcal{K} \subset Q_T$. On the other hand, since $A(\partial_x B(u))$ is in $L^\infty(Q_T)$ we have by (68)c

$$\lim_{\Delta\to 0} \iint_{\mathcal{K}} A\,(\partial_x B(u))\,(\partial_x w_\Delta - \partial_x B(u))\; dt\, dx = 0. \tag{76}$$

From (75) and (76) it follows that

$$\lim_{\Delta\to 0} \iint_{\mathcal{K}} \overline{a}_\Delta \big(\partial_x w_\Delta - \partial_x B(u)\big)^2\; dt\, dx = \lim_{\Delta\to 0} \iint_{\mathcal{K}} \big(A(\partial_x w_\Delta) - A(\partial_x B(u))\big)\big(\partial_x w_\Delta - \partial_x B(u)\big)\; dt\, dx = 0, \tag{77}$$

where

$$\overline{a}_\Delta = \overline{a}_\Delta(x,t) = \int_0^1 a\left(\xi\partial_x w_\Delta + (1-\xi)\partial_x B(u)\right)\, d\xi = \frac{A(\partial_x w_\Delta) - A(\partial_x B(u))}{\partial_x w_\Delta - \partial_x B(u)}. \tag{78}$$

Using (78) and Hölder's inequality we now deduce

$$\begin{aligned}\left|\iint_{Q_T}\left(A\left(\partial_x w_\Delta\right) - A(\partial_x B(u))\right)\phi\, dt\, dx\right| &\le \|\phi\|_{L^\infty(Q_T)}\iint_{\operatorname{supp}\phi}\sqrt{\overline{a}_\Delta}\cdot\left|\sqrt{\overline{a}_\Delta}\left(\partial_x w_\Delta - \partial_x B(u)\right)\right| dt\, dx\\ &\le \|\phi\|_{L^\infty(Q_T)}\Big(\iint_{\operatorname{supp}\phi}\overline{a}_\Delta\, dt\, dx\Big)^{\frac12}\cdot\Big(\iint_{\operatorname{supp}\phi}\overline{a}_\Delta\left(\partial_x w_\Delta - \partial_x B(u)\right)^2 dt\, dx\Big)^{\frac12}.\end{aligned}$$

Since $\overline{a}_\Delta \le a_\infty < \infty$ it follows that

$$\lim_{\Delta\to 0}\iint_{Q_T}\left(A(\partial_x w_\Delta) - A(\partial_x B(u))\right)\phi\, dt\, dx = 0, \qquad \phi \in C_0^\infty(Q_T).$$

This concludes the proof of (d) and thus the lemma. $\square$

The next two technical lemmas will be used in the sequel.

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^2$ and $g_j(x) \to g(x)$ a.e. in Ω. Then there exists a set F, which is at most countable, such that for any $c \in \mathbb{R}\backslash F$,*

$$\operatorname{sign}(g_j(x) - c) \to \operatorname{sign}(g(x) - c), \qquad a.e.\ in\ \Omega.$$

The proof is elementary and is omitted.

Lemma 5.4. *Let $\tilde u_\Delta$ be a piecewise constant interpolant of the discrete data points $\{U_j^n\}$ defined such that $\tilde u_\Delta\big|_{P_j^n} = U_j^n$. Then, passing if necessary to a subsequence, $\tilde u_\Delta \to u$ pointwise a.e. in Q_T, where u denotes the limit function obtained in Lemma 5.2.*

Proof. Clearly we have

$$\begin{aligned}&\iint_{Q_T}|\tilde u_\Delta - u_\Delta|\, dt\, dx\\ &= \sum_{j,n}\iint_{P_j^n}|\tilde u_\Delta - u_\Delta|\, dt\, dx = \sum_{j,n}\iint_{T^U_{j,n-1}}|\tilde u_\Delta - u_\Delta|\, dt\, dx + \sum_{j,n}\iint_{T^L_{j,n}}|\tilde u_\Delta - u_\Delta|\, dt\, dx\\ &=: S_1 + S_2.\end{aligned}$$

For S_1 we have

$$\begin{aligned}S_1 &= \sum_{j,n}\iint_{T^U_{j,n-1}}|\tilde u_\Delta - u_\Delta|\, dt\, dx\\ &= \sum_{j,n}\iint_{T^U_{j,n-1}}\left|(U_{j+1}^n - U_j^n)\Big(\frac{x-x_j}{\Delta x}\Big) + (U_j^n - U_j^{n-1})\Big(\frac{t-t^{n-1}}{\Delta t} - 1\Big)\right| dt\, dx\\ &\le \frac12\Delta x\Delta t\sum_{j,n}\left(|U_{j+1}^n - U_j^n| + |U_j^n - U_j^{n-1}|\right) \le \frac{T}{2}\left(|u_0|_{\mathrm{BV}}\Delta x + C_0\Delta t\right),\end{aligned}$$

where C_0 is given by (62). Similary, we have $S_2 \le \frac12 T\left(|u_0|_{\mathrm{BV}}\Delta x + C_0\Delta t\right)$. Hence

$$\iint_{Q_T}|\tilde u_\Delta - u_\Delta|\, dt\, dx \le T\left(|u_0|_{\mathrm{BV}}\Delta x + C_0\Delta t\right),$$

from which the lemma follows. $\square$

We continue by showing that the limit u satisfies the integral inequality (10).

Lemma 5.4. *Let ϕ be a nonnegative test function with compact support on $\mathbb{R} \times [0,T\rangle$ and $c \in \mathbb{R}$. Then the limit function $u(x,t)$ of Lemma 5.2 satisfies the integral inequality (10).*

Proof. Let ϕ be a suitable test function and put $\phi_j^n = \phi(j\Delta x, n\Delta t)$. Multiplying the cell entropy inequality (50) by $\phi_j^n \Delta x$, summing over all j and n and applying summation by parts, we get

$$\begin{aligned}\Delta x \Delta t \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \Big(|U_j^{n+1} - c| \Big[\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} \Big] + \big(h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c)\big) D_+ \phi_j^n \\ - \operatorname{sign}(U_j^{n+1} - c) A(D_+ B(U_j^{n+1})) D_+ \phi_j^n \Big) + \Delta x \sum_j |U_j^0 - c| \phi_j^0 \geq 0.\end{aligned} \tag{79}$$

For the first term we have

$$\Delta x \Delta t \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} |U_j^{n+1} - c| \Big[\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} \Big] = \iint_{Q_T} |\tilde{u}_\Delta - c| \partial_t \phi \, dt\, dx + \mathcal{O}(\Delta x + \Delta t).$$

Using Lemma 4.2 and the fact that h is consistent with f, we can obviously write

$$\begin{aligned}\Delta x \sum_{j\in\mathbb{Z}} \big(h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c)\big) D_+ \phi_j^n \\ = \Delta x \sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j^{n+1} - c) \big(f(U_j^{n+1}) - f(c) \big) D_+ \phi_j^n + \mathcal{O}(\Delta x).\end{aligned}$$

Hence we have for the second term of (79)

$$\begin{aligned}\Delta x \Delta t \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \big(h(U_j^{n+1} \vee c, U_{j+1}^{n+1} \vee c) - h(U_j^{n+1} \wedge c, U_{j+1}^{n+1} \wedge c)\big) D_+ \phi_j^n \\ = \iint_{Q_T} \operatorname{sign}(\tilde{u}_\Delta - c) \left(f(\tilde{u}_\Delta) - f(c) \right) \partial_x \phi \, dt\, dx + \mathcal{O}(\Delta x + \Delta t),\end{aligned}$$

For the discrete diffusion term of (79) we now have

$$\begin{aligned}\Delta x \Delta t \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \operatorname{sign}(U_j^{n+1} - c) A(D_+ B(U_j^{n+1})) D_+ \phi_j^n \\ = \sum_{n=0}^{N-1} \sum_{j\in\mathbb{Z}} \iint_{P_j^{n+1}} \operatorname{sign}(\tilde{u}_\Delta - c) A(\partial_x w_\Delta) \partial_x \phi \, dt\, dx + \mathcal{O}(\Delta x + \Delta t) \\ = \iint_{Q_T} \operatorname{sign}(\tilde{u}_\Delta - c) A(\partial_x w_\Delta) \partial_x \phi \, dt\, dx + \mathcal{O}(\Delta x + \Delta t).\end{aligned}$$

Hence, we can replace (79) by

$$\begin{aligned}\iint_{Q_T} |\tilde{u}_\Delta - c| \partial_t \phi + \operatorname{sign}(\tilde{u}_\Delta - c) \left(f(\tilde{u}_\Delta) - f(c) \right) \partial_x \phi - \operatorname{sign}(\tilde{u}_\Delta - c) A \left(\partial_x w_\Delta \right) \partial_x \phi \, dt\, dx \\ + \int_{\mathbb{R}} |u_0 - c| \phi(x,0)\, dx \geq -\mathrm{C}(\Delta x + \Delta t).\end{aligned}$$

Using (68) and Lemma 5.3 and 5.4 we conclude that u satisfies (10) for almost all $c \in \mathbb{R}$. To complete the proof, note that $A(\partial_x B(u)) = 0$ a.e. in $E_c = \{(x,t) \in Q_T : u(x,t) = c\}$ for any constant c. Therefore, by using an approximate procedure the result holds for all c. $\square$

This completes our discussion when u_0 has compact support and f, A, B are locally C^1. For $u_0 \in \mathcal{B}(f,A,B)$ not necessarily compactly supported and f, A, B merely locally Lipschitz continuous, we approximate u_0 by a compactly supported function u_0^p and f, a, b by a smoother function f^p, a^p, b^p, compute the difference approximation of the resulting problem and then let $p \to \infty$ and $\Delta t, \Delta x \to 0$.

We are now ready to state our main result:

Theorem 5.5. *Suppose $u_0 \in \mathcal{B}(f, A, B)$ and the fluxes f, A, B are locally Lipschitz continuous. Assume also that A satisfies the structural condition*

$$A(-\infty) = -\infty, \qquad A(+\infty) = +\infty.$$

Then the entire sequence $\{u_\Delta\}$ defined by (21) and (57) converges in $L^1_{\mathrm{loc}}(Q_T)$ and pointwise a.e. in Q_T to a BV entropy weak solution u of the initial value problem

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x A(b(u)\partial_x u), & (x,t) \in Q_T = \mathbb{R} \times \langle 0, T\rangle, \\ u(x,0) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

where

$$A(s) = \int_0^s a(\xi)\, d\xi, \qquad a(s) \geq 0, \qquad b(s) \geq 0.$$

Remark. *From Lemma 5.2 it is clear that the following results hold without assuming (6): There is a function $u(x,t) \in L^\infty(Q_T) \cap BV(Q_T)$ and a function $\overline{A}(x,t) \in L^1(Q_T)$ such that*

$$\begin{aligned} &u_\Delta(x,t) \to u(x,t), && \text{in } L^1_{\mathrm{loc}}(Q_T) \text{ and pointwise a.e. in } Q_T, \\ &A(\partial_x w_\Delta) \overset{*}{\rightharpoonup} \overline{A}(x,t), && \text{in } L^\infty(Q_T). \end{aligned}$$

Furthermore, in view of Lemma 5.4 it follows that the following integral inequality is satisfied

$$\iint_{Q_T} \Big(|u - c|\partial_t \phi + \operatorname{sign}(u-c)\big(f(u) - f(c) - \overline{A}\big)\partial_x \phi \Big)\, dt\, dx + \int_{\mathbb{R}} |u_0 - c|\, dx \geq 0. \tag{80}$$

We let $C(0,T; L^1(\mathbb{R}))$ denote the usual Bochner space consisting of all continuous functions $u : [0,T] \to L^1(\mathbb{R})$ for which the norm $\|u\|_{C(0,T;L^1(\mathbb{R}))} = \sup_{t\in[0,T]} \|u(t)\|_{L^1(\mathbb{R})}$ is finite. A closer inspection of the arguments leading to Theorem 5.5 will reveal that $\{U_\Delta(t)\}$ converges in $C(0,T; L^1(\mathbb{R}))$ to the unique BV entropy weak solution $u(t)$, with $u(0) = u_0$, of the initial value problem (1). A reexamination of the proofs leading to Theorem 5.5 also shows that we have proved the following result on existence and properties of solutions of (1):

Corollary 5.6. *Let f and A, B be locally Lipschitz continuous. Then for any initial function $u_0 \in \mathcal{B}(f, A, B)$ there exists a BV entropy weak solution $u \in C(0,T; L^1(\mathbb{R}))$ of the initial value problem (1). Denoting this solution by $\mathcal{S}_t u_0$, we have the following properties:*

(1) $t \to \mathcal{S}_t u_0$ *is Lipschitz continuous into* $L^1(\mathbb{R})$ *and* $\|\mathcal{S}_t u_0\|_{BV(\mathbb{R})} \leq \|u_0\|_{BV(\mathbb{R})}$,
(2) $\|\mathcal{S}_t u_0 - \mathcal{S}_t v_0\|_{L^1(\mathbb{R})} \leq \|u_0 - v_0\|_{L^1\mathbb{R})}$,
(3) $u_0 \leq v_0$ *implies* $\mathcal{S}_t u_0 \leq \mathcal{S}_t v_0$,
(4) $m \leq u_0 \leq M$ *implies* $m \leq \mathcal{S}_t u_0 \leq M$.

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