A Note on the Spreading of Characteristics for Nonconvex Conservation Laws

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Abstract

We study the spreading of characteristics for a class of one-dimensional scalar conservation laws for which the flux function has exactly one point of inflection. It is well-known that the characteristic speed satisfies a one-sided Lipschitz estimate in the convex case. Using Dafermos' theory of generalized characteristics [3] we show that the characteristic speed in the nonconvex case satisfies a Hölder estimate at each fixed time.

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1 Introduction

We consider scalar conservation laws of the form

$$\partial_t u(x,t) + \partial_x f(u(x,t)) = 0, \qquad u(x,t=0) = u_0(x), \qquad (x,t) \in \mathbb{R} \times \mathbb{R}^+. \tag{1}$$

The flux function f is assumed to be (at least) of class C^2. It is well–known that this equation has a unique classical solution only locally in time. In general discontinuities develop in finite time no matter how smooth the initial data are. One is thus forced to extend the notion of a solution and consider weak solutions. However, this extension also entails a loss of uniqueness. To obtain uniqueness in the larger class of generalized solutions one introduces entropy conditions which single out the "physical" solution. The theory of existence and uniqueness of entropy solutions to (1) is more or less complete, see for example [8]. However, except for the case where the flux $f(u)$ is convex, the qualitative properties of these solutions are not correspondingly well understood.

For strictly convex equations where the function $f''(u)$ is nonnegative and not identically zero on any interval, the following inequality holds,

$$f'(u(y,t)) - f'(u(x,t)) \leq \frac{y-x}{t}, \quad x \leq y. \tag{2}$$

When the flux f is strictly convex with $f''(u) \geq c > 0$ for all u with $|u| \leq \sup_{x\in\mathbb{R}} |u_0(x)|$, this yields the following stronger estimate,

$$u(y,t) - u(x,t) \leq \frac{y-x}{ct}, \quad x \leq y. \tag{3}$$

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The estimate (3) was first proved by Hopf [6] for Burgers' equation (i.e., when $f(u) = u^2/2$) and later established by Oleinik [12] in the general case. For a proof of (2) see [1]. These estimates contain much information about the structure of the solution. In particular (2) shows that $f'(u(\cdot, t))$ is of locally bounded variation. The inequality also contains the entropy condition that $f'(u(\cdot, t))$ can only decrease across shocks, that is, the characteristics go *into* the shock as time increases. Finally it reveals exactly how rarefaction waves spread out as time increases. For other ways of formulating the entropy condition in the convex case, see [5], [14].

For convex fluxes one has detailed information on the asymptotic behavior of the solution. For periodic data it is known that the solution tends uniformly to the mean value of the initial data with a rate $1/t$. In the case of compactly supported data the solution tends uniformly to zero at a rate $1/\sqrt{t}$, while it tends to an N-wave at the same rate with respect to the L^1–norm. Furthermore, the effect of source terms for convex equations has been studied in several papers, see [9, 10, 11]. In particular, [7] contains a detailed analysis of the asymptotic profile near a blowup point in the case of a superlinear source term. For nonconvex equations much fewer results are known. However, see [13] and references therein.

A key property in the convex case is that no rarefaction waves are generated at positive times. Any collision of shocks always produce a single outgoing shock wave. For general nonconvex equations the situation is complicated by the fact that the solution can contain contact discontinuities from which characteristics radiate as time increase. Moreover, centered rarefaction waves may be generated at positive times. In this note we will consider equations with the former type of behavior. We assume the following normalization:

$$f(0) = f'(0) = f''(0) = 0, \quad uf''(u) < 0 \quad \text{for} \quad u \neq 0. \tag{4}$$

A class of model equations is given by the conservation laws

$$\partial_t u(x,t) - \partial_x[|u(x,t)|^{m-1}u(x,t)] = 0, \tag{5}$$

where $m > 1$. As observed in [3] the assumptions (4) guarantee that no centered rarefaction waves are generated for $t > 0$. However, the solution will in general contain left contact discontinuities along which characteristics are emitted. The theory of generalized characteristics [1, 2] gives a detailed description of the structure of the solution including generic regularity and large time behavior. The analysis in [3] shows that there are certain "convex" regions in which the solution behaves as in the convex case. For these regions we thus know the generic regularity as well as how characteristics spread out in time. Dafermos [3] provides a detailed analysis of the remaining "nonconvex" regions and describes the generic regularity of the solution, see Theorem 2.5 below. In addition the asymptotic behavior of the solution under various assumptions on the initial data was considered. In particular, for the case of periodic initial data it was proved that the characteristic speed decays like $1/t$, while in the case of data with compact support it was showed that the characteristic speed satisfies the estimate

$$f'(u(x,t)) = \frac{x}{t} + O(t^{-\alpha}) \quad \text{as } t \to \infty.$$

Here α is a positive constant depending on the flatness of f at the point of inflection.

The aim of this note is to supplement the analysis in [3] by establishing pointwise estimates describing the spreading of characteristics in the nonconvex regions of the solution. In particular we are interested in the regularity of $f'(u(\cdot, t))$ close to a left contact discontinuity.

The estimate (2) shows that $f'(u(\cdot,t))$ satisfies a one-sided Lipschitz condition in the convex case. In the present case the characteristic speed may have infinite gradients. However, provided the flux f is not too flat near the point of inflection we show that $f'(u(\cdot,t))$ satisfies a Hölder estimate at each fixed time.

After recalling the definition and key properties of generalized characteristics, we state the main regularity results from [3]. We then use these results to give a detailed analysis of $f'(u(\cdot,t))$ when $u(x,t)$ solves (1) and $f(u)$ satisfies (4).

Apart from their intrinsic interest the results presented below were motivated by the work of Tadmor [14] on local error estimates for convex equations. The results may be viewed as a first step toward a corresponding analysis of nonconvex equations. Recently Tegnander [15] obtained similar results based on a front tracking argument.

2 Preliminaries

We define an *admissible BV solution* of (1) to be a bounded measurable function defined on $\mathbb{R}\times\mathbb{R}^+$, with distributional derivatives $\partial_t u$, $\partial_x u$ which are locally finite Borel measures, satisfying the entropy inequality

$$\partial_t\eta(u(x,t))+\partial_x q(u(x,t))\geq 0, \tag{6}$$

for every entropy entropy–flux pair $(\eta(u),q(u))$. For initial data u_0 in $\mathrm{L}^\infty\cap\mathrm{BV}_{loc}$, it is well–known that (1) has a unique admissible BV solution, see [8]. We normalize the solution to be continuous from the left.

A *generalized characteristic* associated with an admissible BV solution $u(x,t)$ of (1) is a Lipschitz curve $\xi:[t_0,t_1]\to\mathbb{R}$, $0\leq t_0<t_1<\infty$, which satisfies the differential inequality

$$\xi'(t)\in[f'(u(\xi(t)+,t)),f'(u(\xi(t),t))], \tag{7}$$

for almost all $t\in[t_0,t_1]$. From the theory of differential equations with discontinuous right-hand side (see [4]), it follows that through any point of $\mathbb{R}\times(0,\infty)$ there is a funnel of generalized characteristics passing through this point. These lie between a minimal one and a maximal one. The backward maximal (minimal) generalized characteristic through a given point $(\bar{x},\bar{t})$ is the rightmost (leftmost) characteristic passing through $(\bar{x},\bar{t})$ traveling with minimal (maximal) speed. The forward maximal (minimal) through $(\bar{x},\bar{t})$ is the rightmost (leftmost) characteristic passing through $(\bar{x},\bar{t})$ traveling with maximal (minimal) speed.

As in the case of a convex conservation law, classical characteristics and shocks are examples of generalized characteristics. Another possibility is the situation of a generalized characteristic traveling with a speed which coincides with the characteristic speed to the left (right) of itself. We call these *left (right) contacts.* We note that the normalization (4) implies that all right contacts are also left contacts, while the converse is not true in general. In the convex case contacts are necessarily classical characteristics whereas in the present case a left contact can be a line of discontinuity. In the following we refer to generalized characteristics as characteristics. By a *genuine* shock we mean a curve of discontinuity $x=\chi(t)$, $t\in[a,b]$, for which

$$f'(u(\chi(t),t))<\dot{\chi}(t)<f'(u(\chi(t),t)),\qquad \text{for almost all } t\in[a,b].$$

2.1 Extremal Characteristics

The following results give a detailed description of the extremal backward characteristics of the solution of (1). A basic fact is that these are contacts.

Proposition 2.1 *The minimal (maximal) backward characteristic through a point* $(x,t) \in \mathbb{R} \times (0,\infty)$ *is a left (right) contact on* $[0,t]$.

Proof. This is Lemma 2.3 in [3]. ∎

The following proposition describes the solution along a left contact. This result will be used below when we consider the behavior of $f'(u(\cdot,t))$ close to a left contact discontinuity.

Proposition 2.2 *Let* χ *be a left contact characteristic on some interval* (a,b). *Then, if* $u(\chi(t),t) \geq 0$ *(or* ≤ 0*) for almost all* $t \in (a,b)$, *then* $t \mapsto u(\chi(t),t)$ *is nonincreasing (or nondecreasing, respectively) on* (a,b). *In particular* χ *is a convex curve.*

Proof. This is Lemma 2.5 in [3]. ∎

The next theorem describes the maximal backward characteristics. These are convex polygons with vertices at left contacts and thus have a particularly simple structure.

Theorem 2.3 *Given* $(\bar{x},\bar{t}) \in \mathbb{R}\times(0,\infty)$, *let* ξ *be the maximal backward characteristic through* $(\bar{x},\bar{t})$. *If* $u(\bar{x},\bar{t}) \neq 0$ *or* $u(\bar{x}+,\bar{t}) \neq 0$, *then there is a finite mesh* $0 = t_0 < t_1 < \cdots < t_{n+1} = \bar{t}$ *such that* ξ *is a convex polygonal with vertices at the points* $(\xi(t_j),t_j)$, $j = 0,\ldots,n+1$. *Furthermore,*

$$u(\xi(t),t) = u(\xi(t)+,t) = u(\xi(t_{j+1}),t_{j+1}), \quad t_j < t < t_{j+1}, \quad j = 0,\ldots,n,$$

$$u(\xi(t_j),t_j) = u(\xi(t_{j+1}),t_{j+1}) \quad j = 1,\ldots,n,$$

$$\begin{cases} u_0(\xi(0)) \geq u(\xi(t_1)+,t_1) & \text{if } u(\xi(t_1)+,t_1) > 0, \\ u_0(\xi(0)) \leq u(\xi(t_1)+,t_1) & \text{if } u(\xi(t_1)+,t_1) < 0, \end{cases}$$

$$\dot{\xi}(t) = f'(u(\xi(t_{j+1})+,t_{j+1})), \quad t_j < t < t_{j+1}, \quad j = 0,\ldots,n,$$

$$f'(u(\xi(t_j),t_j)) = \frac{f(u(\xi(t_j)+,t_j)) - f(u(\xi(t_j),t_j))}{u(\xi(t_j)+,t_j) - u(\xi(t_j),t_j)}, \quad j = 0,\ldots,n.$$

Proof. This is Theorem 2.1 in [3]. ∎

We include the following result for maximal forward characteristics which shows that once a discontinuity has developed it never disappears.

Theorem 2.4 *Given a point* $(\bar{x},\bar{t}) \in \mathbb{R}\times(0,\infty)$ *of discontinuity of an admissible solution* u *and let* χ *denote the maximal forward characteristic passing through* $(\bar{x},\bar{t})$. *Then* χ *is a shock on* $[\bar{t},\infty)$. *Also, the functions* $u(\chi(t),t)$, $u(\chi(t)+,t)$ *are continuous from the right on* $[\bar{t},\infty)$, χ *is differentiable from the right on* $[\bar{t},\infty)$, *and*

$$\frac{d^+}{dt}\chi(t) = \frac{f(u(\chi(t)+,t)) - f(u(\chi(t),t))}{u(\chi(t)+,t) - u(\chi(t),t)}, \quad t \in [\bar{t},\infty).$$

Proof. This is Theorem 2.3 in [3]. ∎

These results are used to study the generic regularity of solutions.

2.2 Generic Regularity

We now assume that the flux f is of class C^∞ and that the initial data satisfies the following assumptions:

(i) u_0 is of class C^∞;

(ii) u_0 coincides outside a compact interval with a function v_0 which satisfies

$$\frac{d}{d\,x} f'(v_0(x)) > 0.$$

(iii) u_0 changes sign across a finite number of points $-\infty < y_1 < \cdots < y_l < \infty$.

The set of initial data u_0 which satisfy (i)–(iii) forms an open dense subset of C^∞ when equipped with the standard topology of uniform convergence of all derivatives. This justifies the term *generic*.

We let ϕ_i denote the maximal forward characteristic starting at $(y_i, 0)$. If ϕ_{i-1} and ϕ_i collide, we let $s_i^* > 0$ denote the time of collision. In that case we have $\phi_{i-1}(t) = \phi_i(t)$ for all $t \in [s_i^*, \infty)$. The characteristics ϕ_i partition the upper half-plane in $l+1$ regions $\mathcal{R}_1, \ldots, \mathcal{R}_{l+1}$ given by

$$\mathcal{R}_i = \{(x,t) \mid 0 \leq t < s_i^*,\ \phi_{i-1}(t) < x < \phi_i(t)\} \quad i = 1, \ldots, l+1,$$

where we let

$$\phi_0 \equiv -\infty, \quad \phi_{l+1} \equiv \infty, \quad \text{and } s_i^* = \infty \text{ whenever } \phi_{i-1}(t) < \phi_i(t) \text{ for all } t \in [0, \infty).$$

It follows that any backward characteristic from a point in $\mathcal{R}_i$ remains within $\mathcal{R}_i$ and that the sign of $u(x,t)$ is constant on $\mathcal{R}_i$.

We now single out two special types of characteristics. A classical characteristic is said to *graze* upon ϕ_i at time $t = \tau$ if it is tangent to ϕ_i at $(\phi_i(\tau), \tau)$. Also, a *weak wave* of order m, $m \geq 1$, is a classical characteristic $\mathcal{L}$ such that $u(x,t)$ is C^{m-1} on an open neighborhood $\mathcal{U}$ of $\mathcal{L}$, it is C^∞ on $\mathcal{U} \setminus \mathcal{L}$, and its partial derivatives of order m has jump discontinuities across $\mathcal{L}$. We can now state the main regularity result from [3].

Theorem 2.5 *Assume that the flux is C^∞ smooth and satisfies the assumptions (4), and assume that the data satisfy (i)–(iii) above. Then the domain $\mathbb{R} \times [0,\infty)$ of an admissible solution of (1) generically consists of four disjoint subsets $\mathcal{D}$, $\mathcal{S}$, $\mathcal{W}$, $\mathcal{I}$, with the following properties:*

(a) $\mathcal{D}$ is a dense set on which $u(x,t)$ is C^∞.

(b) $\mathcal{S}$ is a finite union of C^∞ arcs which are genuine shocks or left contact discontinuities.

(c) $\mathcal{W}$ is a finite union of classical characteristics which are weak waves. All weak waves emanate tangentially out of left contact discontinuities and terminate by colliding with discontinuities. A weak wave of order 1 is generated whenever a characteristic graze upon a left contact discontinuity, while waves of order $m \geq 2$ are generated when a weak wave of order $m-1$ hits a left contact discontinuity from the right.

(d) $\mathcal{I}$ is the finite set of shock generation points, shock interaction points and points of interaction between shocks and weak waves.

2.3 Convex and Nonconvex Regions

In order to describe the solution in more detail we now divide each region $\mathcal{R}_i$ further into two subregions. The idea is to identify where the solution behaves as in the case of a convex flux and where the behavior is influenced by the presence of the inflection point of f.

We let σ_i denote the minimal forward characteristic starting from $x = y_i$ at time zero. Note that the maximal backward characteristic through points (x,t) with $x \in [\phi_{i-1}(t), \sigma_i(t))$ lies strictly to the left of ϕ_i on $[0,t]$, whereas if $x \in [\sigma_i(t), \phi_i(t))$, then the maximal backward characteristic meets ϕ_i at some positive time. We now define the following subregions,

$$\begin{aligned} \mathcal{G}_i &= \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ \mid \phi_{i-1}(t) < x < \sigma_i(t)\}, && (8)\\ \mathcal{H}_i &= \{(x,t) \in \mathbb{R}\times\mathbb{R}^+ \mid \sigma_i(t) < x < \phi_i(t)\}, && (9) \end{aligned}$$

where $\phi_{l+1} \equiv \infty$. We refer to $\mathcal{G}_i$, $\mathcal{H}_i$, $i = 1,\ldots,l+1$, as the *convex* and *nonconvex* regions of the solution, respectively.

The behavior in the region $\mathcal{G}_i$ is easily analyzed. The domain of dependence of $\mathcal{G}_i$ is the interval (y_{i-1}, y_i) where u_0 does not cross the inflection point of $f(u)$. It follows that the behavior of $u(x,t)$ on $\mathcal{G}_i$ is governed by the same mechanism as in the case of convex equations. We thus have the following result for the convex region.

Proposition 2.6 *The solution is generically of class C^∞ in $\mathcal{G}_i$ except from a finite set of smooth shock curves and a finite set of points where shocks are generated or collide. For $(x,t),(y,t) \in \mathcal{G}_i$ with $x \le y$ the solution $u(x,t)$ satisfies the Lipschitz estimate*

$$\mathrm{sign}[f''(u(x,t))](f'(u(y,t)) - f'(u(x,t))) \le \frac{y-x}{t}. \qquad (10)$$

Proof. See [1] and [2]. ∎

The study of $u(x,t)$ in the nonconvex regions is more involved. We first describe the boundary curves of $\mathcal{H}_i$ in more detail, while the spreading of characteristics in $\mathcal{H}_i$ will be treated in the next section.

At time zero ϕ_i starts as a classical characteristic with zero speed. At some later time $t = t_i^*$ it may collide with a shock after which it propagates either as a genuine shock, with characteristics impinging upon it from both sides, or as a left contact discontinuity with characteristics emitted to the left. Note that these parts of the ϕ_i are the only left contacts present in the solution which are not at the same time right contacts. We denote by J_i the set of times $t \in (t_i^*, s_i^*)$ at which ϕ_i is a left contact discontinuity. The interior of J_i is denoted J_i^o. It was proved in [3] that ϕ_i generically is of class C^∞ on J_i^o except at a finite number of points across which some derivative $\phi_i^{(m)}$, $m \ge 1$, experiences a jump discontinuity.

The characteristics σ_i starts out at time $t = 0$ from the point $x = y_i$ and stay together with ϕ_i until the first time at which a classical characteristic originating at the x–axis grazes upon ϕ_i. At that point the two curves separate and σ_i follows the grazing characteristic for some time. The solution is continuous but experiences a jump in the first derivative across σ_i, that is, σ_i is now a weak wave of order 1. As time increases σ_i may:

- collide with ϕ_{i-1}; or
- collide with a shock in $\mathcal{G}_i$; or

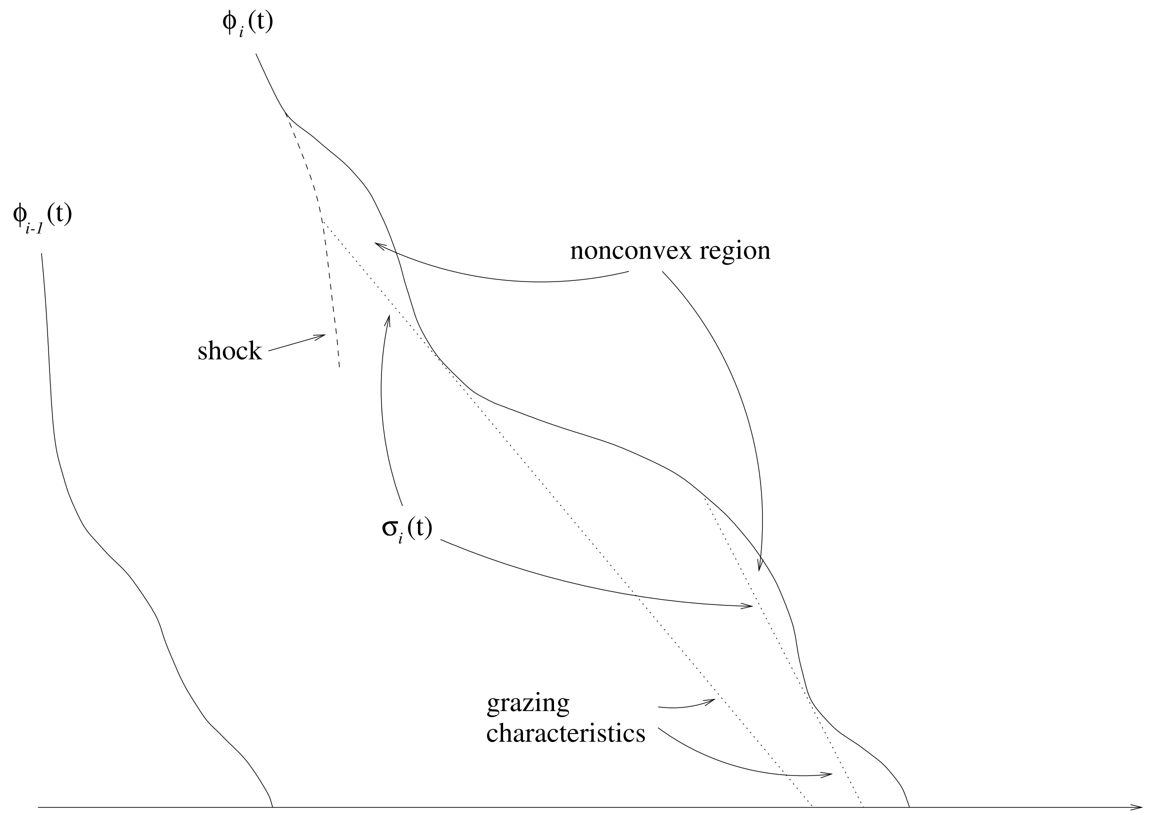

Figure 1: The characteristics bounding a nonconvex region

- pass through a shock generation point and develop into a shock; or
- collide with ϕ_i and develops into a shock.

Subsequently σ_i follows the shock and may again collide with ϕ_i, stay together for some time, branch out along a grazing characteristic which started at time zero, etc. It was proved in [3] that generically the process of branching out along grazing characteristics takes place only a finite number of times. Figure 1 shows a typical situation.

3 Spreading of Characteristics in Nonconvex Regions

Our aim in this section is to establish estimates on the characteristic speed $f'(u)$ in the regions $\mathcal{H}_i$. We assume throughout that the solution behaves as in the generic case described by Theorem 2.5.

Consider first a general situation where characteristics at some time s spread out until some later time t. That is, defining

$$\theta_s(x) = f'(u(x,s)),$$

we assume that $\theta'_s(x)$ is positive on some interval. Since the solution is constant along classical characteristics we have the relation

$$\theta_t(x) = \theta_s(x - (t-s)\theta_t(x)).$$

Differentiation with respect to x gives

$$\theta_t'(x) = \frac{\theta_s'(\xi(x))}{1 + (t-s)\theta_t'(\xi(x))},$$

where $\xi(x) = x - (t-s)\theta_t(x)$. We conclude that

$$\theta_t'(x) \leq \frac{1}{t-s}. \tag{11}$$

We now choose one of the nonconvex regions $\mathcal{H}_i$ and simplify notation by dropping the subscript i. Thus,

$$\sigma(t) \equiv \sigma_i(t), \quad \phi(t) \equiv \phi_i(t), \quad J \equiv J_i, \quad \text{etc.}$$

We fix a time t for which $\sigma(t) < x < \phi(t)$, and define

$$\theta(x) = f'(u(x,t)), \qquad \sigma(t) < x < \phi(t).$$

By the definition of the nonconvex regions we have that all characteristics intersecting the interval $(\sigma(t), \phi(t))$ at time t are straight line characteristics which were emitted tangentially from ϕ at earlier times. The function θ is negative and increasing on $(\sigma(t), \phi(t))$. Due to the presence of at most finitely many weak waves the derivative of θ is defined almost everywhere. It follows from the previous estimate that $\theta'(x)$ (when defined) is bounded on any compact subset of $(\sigma(t), \phi(t))$ and that

$$\lim_{x \downarrow \sigma(t)} \theta'(x) < \infty.$$

Furthermore, if $t \in [(t^*, s^*) \setminus J]^o$, then

$$\lim_{x \uparrow \phi(t)} \theta'(x) < \infty.$$

However, for $t \in clJ$ this last limit may be infinite. In particular this shows that the characteristic speed $f'(u)$ does not satisfy a Lipschitz estimate of the form (10) in the nonconvex region.

In order to obtain estimates on θ near a left contact discontinuity we need to assume that the graph of the flux is not too flat at the point of inflection. From the assumptions (4) it follows that for each $u \neq 0$ there exists a unique $u^* \neq u$ with the property that

$$f'(u) = \frac{f(u^*) - f(u)}{u^* - u}. \tag{12}$$

This is exactly the relation satisfied across a left contact discontinuity with left state u and right state u^*. We now assume that there exists a differentiable function $\rho(u)$ with the following properties:

(a) $0 < \rho(u) < 1$ for $u \neq 0$;

(b) $\rho(u) f'(u^*) = f'(u)$, where u and u^* satisfy (12);

(c) $\text{sign}(u)\rho'(u) \leq 0$, for $u \neq 0$.

As an example consider the model equation (5). In this case ρ is constant and equal to the unique positive root of the algebraic equation

$$m\rho^{1+1/m} + (m+1)\rho - 1 = 0. \tag{13}$$

Assuming that (a)–(c) are satisfied we now proceed to study the profile of $\theta(x)$ close to $(\phi(t), t)$ for $t \in J^o$. Without loss of generality we assume that ϕ is C^∞ smooth in a neighborhood of t. For $x \in (\sigma(t), \phi(t))$ we let $\tau = \tau(x)$ denote the time at which the characteristic through (x, t) was emitted from ϕ. We choose x close enough to $\phi(t)$ to guarantee that ϕ is C^∞ smooth also at time τ. We have

$$x = \phi(\tau(x)) + \dot{\phi}(\tau(x))(t - \tau(x)). \tag{14}$$

Differentiation with respect to x yields

$$\ddot{\phi}(\tau(x))\tau'(x) = \frac{1}{t - \tau(x)}. \tag{15}$$

Recall that $\theta(x) = f'(u(x,t)) = \dot{\phi}(\tau(x))$, whence

$$\theta'(x) = \frac{1}{t - \tau(x)}. \tag{16}$$

As noted above this shows that the characteristic speed cannot satisfy a Lipschitz condition as in the convex case. Now, differentiating once more with respect to x and using (15) and (16) gives

$$\frac{\theta''(x)}{(\theta'(x))^3} = \frac{1}{\ddot{\phi}(\tau(x))}. \tag{17}$$

To proceed we must estimate the curvature of ϕ at time τ. We let ξ and ζ denote the maximal backward characteristics through (x, t) and $(\phi(t), t)$, respectively. According to Theorem 2.3 these are convex polygonal lines with vertices at the points where they cross left contact discontinuities. Following ζ backward in time from $(\phi(t), t)$ we let $\bar{t}$ be the first time ζ crosses a left contact discontinuity which we denote by ψ. By the smoothness of ϕ at time t and by choosing x sufficiently close to $\phi(t)$ we may assume that ξ crosses ψ at some time $\bar{\tau} = \bar{\tau}(\tau)$ and that ψ is smooth on an open neighborhood of $[\bar{\tau}, \bar{t}]$. Recalling that u is normalized to be continuous from the left we set

$$v(\tau) = u(\phi(\tau), \tau),$$

so that

$$\phi(\tau) = \dot{\psi}(\bar{\tau})(\tau - \bar{\tau}) + \psi(\bar{\tau}) = \frac{\dot{\phi}(\tau)}{\rho(v(\tau))}(\tau - \bar{\tau}) + \psi(\bar{\tau}).$$

Differentiating with respect to τ and using Proposition 2.2 together with the properties of ρ gives

$$\dot{\phi}(\tau) \geq \ddot{\phi}(\tau)(\tau - \bar{\tau}) + \frac{\dot{\phi}(\tau)}{\rho(v(\tau))}.$$

Defining $\rho_0 = \inf\{\rho(v) : |v| \leq \|u_0\|_\infty\}$ and observing that $\tau(x) - \bar{\tau} > \tau(x) - \bar{t} > 0$ for x close enough to $\phi(t)$, we conclude that

$$\frac{1}{\ddot{\phi}(\tau(x))} \geq \left(\frac{\rho_0}{\rho_0 - 1}\right) \frac{\tau(x) - \bar{t}}{\dot{\phi}(\tau(x))}.$$

Substituting this into (17) and using (16) we obtain the estimate

$$\frac{\theta''(x)}{(\theta'(x))^3} \geq \left(\frac{\rho_0}{\rho_0 - 1}\right) \frac{\theta'(x)(t-\bar{t}\,) - 1}{\theta(x)\theta'(x)},$$

which may be written as

$$\frac{d}{dx}\left(\frac{\theta(x)}{\theta'(x)}\right) \geq \left(\frac{\rho_0}{1-\rho_0}\right)\theta'(x)(t-\bar{t}\,) + \left(\frac{1-2\rho_0}{1-\rho_0}\right).$$

Integration from x to $\phi(t)$ finally yields the following differential inequality for $\theta(x)$:

$$\theta'(x) \leq -\frac{(1-\rho_0)\theta(x)}{\rho_0(t-\bar{t}\,)(\dot{\phi}(t) - \theta(x)) + (1-2\rho_0)(\phi(t) - x)}.$$

We let $\mu(x)$ be the solution of the problem

$$\mu'(x) = -\frac{(1-\rho_0)\mu(x)}{\rho_0(t-\bar{t}\,)(\dot{\phi}(t) + \mu(x)) + (1-2\rho_0)(\phi(t)-x)}, \qquad \mu(\phi(t)) = -\dot{\phi}(t), \tag{18}$$

and conclude that $\theta(x) \geq -\mu(x)$ for x close to $\phi(t)$. It is readily verified that the solution of (18) is implicitly given as the solution of the algebraic equation

$$(x-A)\mu(x)^{-a} + (t-\bar{t}\,)\mu(x)^{1-a} = C, \tag{19}$$

where

$$\begin{aligned} A &= \frac{\rho_0}{1-2\rho_0}(t-\bar{t}\,)\dot{\phi}(t) + \phi(t), \\ a &= \frac{1-2\rho_0}{1-\rho_0}, \\ C &= (\phi(t) - A)(-\dot{\phi}(t))^{-a} + (t-\bar{t}\,)(-\dot{\phi}(t))^{1-a}. \end{aligned}$$

This result immediately yields information on the singularity of θ' at the point $x = \phi(t)$. Let $\tilde{\mu}(x) = \mu(x - \phi(t))$ for small $x < 0$ and consider the inverse function $\nu = \tilde{\mu}^{-1}$. From (19) it follows that ν satisfies the equation

$$\nu(y) = A - (t-\bar{t}\,)y + Cy^a, \qquad y \geq -\dot{\phi}(t).$$

A straightforward computation shows that

$$\nu(-\dot{\phi}(t)) = 0, \quad \nu'(-\dot{\phi}(t)) = 0, \quad \nu''(-\dot{\phi}(t)) = \left(\frac{\rho_0}{1-\rho_0}\right)\frac{t-\bar{t}}{\dot{\phi}(t)} < 0.$$

It follows that μ satisfies a corresponding Hölder estimate

$$\mu(x) = -\dot{\phi}(t) + c\sqrt{\frac{\phi(t)-x}{t-\bar{t}}} + \text{higher order terms}. \tag{20}$$

Since $\theta(x) \geq -\mu(x)$ and $\lim_{x\uparrow\phi(t)}\theta(x) = \infty$ it follows that the characteristic speed is Hölder continuous from the left at $x = \phi(t)$ and satisfies the estimate

$$0 > f'(u(x,t)) \geq f'(u(\phi(t),t)) - c\sqrt{\frac{\phi(t)-x}{t-\bar{t}}}, \tag{21}$$

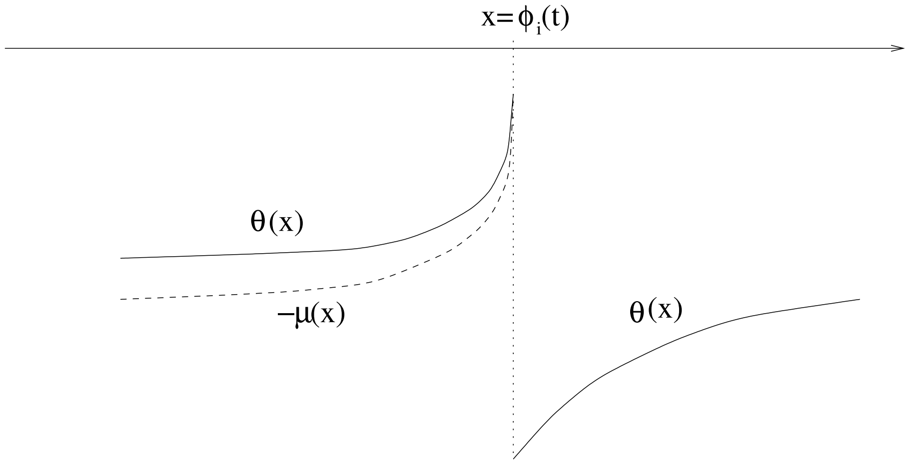

Figure 2: Profile of $\theta(x)$ and $-\mu(x)$

for x close to $\phi(t)$ and where the constant c is given by

$$c = \left(\frac{2(1-\rho_0)|\dot{\phi}(t)|}{\rho_0(t-\bar{t}\,)} \right)^{1/2}.$$

We summarize the results in the following theorem.

Theorem 3.1 *Assume that the flux f is C^∞ smooth and satisfies the conditions (4). Also assume that there exists a function ρ such that (a)–(b) hold and that the initial data satisfy (i)–(iii) above. Then, generically, the gradient of the characteristic speed $f'(u)$ is bounded from either side at every point except those lying on left contact discontinuities. At these points the gradient is bounded from the right while $f'(u)$ satisfies the Hölder estimate (21) from the left.*

Remarks. The result is valid for more general initial data. In particular one may allow for a finite number of jump discontinuities in the initial data.

Of course, the time $\bar{t}$ in the estimate (21) may be close to t. However, if the data crosses the inflection point of f only once , then $\bar{t} = 0$ and the characteristic speed is Hölder continuous from the left with exponent $1/2$ and coefficient $O(t^{-1/2})$.

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