Relaxation Schemes for Curvature-Dependent Front Propagation

by

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ABSTRACT. In this paper we study analytically and numerically a novel relaxation approximation for front evolution according to a curvature dependent local law. In the Chapman-Enskog expansion, this relaxation approximation leads to the level set equation for transport dominated front propagation, that includes the mean curvature as the next order term. This approach yields a new and possibly attractive way of calculating numerically the propagation of curvature dependent fronts. Since the relaxation system is a symmetrizable, semilinear, and linearly convective hyperbolic system without singularities, the relaxation scheme captures the curvature-dependent front propagation, without discretizing directly the complicated, yet singular, mean curvature term.

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1. Introduction.

Front propagation models with curvature-dependent normal velocities arise in a variety of physical phenomena such as flame propagation, solidification, and phase transition problems. Tracking the motion of such fronts globally in time, past possible singularities and changes of topological type, has been the focus of extensive studies in the past few years, see for instance [OS], [CGG], [ESp], [ESS], [BSS], [NPV], and the references therein. In this paper we discuss and validate analytically, asymptotically and numerically a new approach introduced in an earlier note [JK], for studying such surface motion problems, which is based on the relaxation approximations to general Hamilton-Jacobi type equations, developed in [JX2]. These new relaxation schemes approximate the equation of front evolution by a symmetrizable linear-convective hyperbolic system with a stiff relaxation term. Unlike the standard level set method for the same problem, which solves a degenerate, second order parabolic equation with singularity, the relaxation scheme captures the curvature dependence without discretizing directly the complicated, yet singular curvature term.

We first briefly recall some background on relaxation schemes. The relaxation approximation to systems of conservation laws in any space dimension was first introduced systematically by Jin and Xin in [JX1], as a simple and efficient numerical tool for solving multidimensional systems of conservation laws. Some analytical aspects of relaxation systems and their ensuing numerical schemes, mostly for the approximation of a scalar equation, were explored in many recent publications [Nat], [KT], [Nat1], [WX], [Ten], [ADN], [KKM], [CP], [LWY], [TW], [CR]. The main idea is, for a given system of conservation laws, to construct a symmetrizable, linear-convective hyperbolic system with a relaxation term, chosen so that its solution converges to the entropy solution of the given system of conservation laws in the zero relaxation limit. This gives rise to a class of relaxation schemes that are total-variation-diminishing and free of Riemann solvers and characteristic decompositions [JX1].

The Cauchy problem for a Hamilton-Jacobi equation

$$\begin{aligned}&\partial_t u + H(\nabla u) = 0\,, \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0,\infty)\\ &u(\mathbf{x}, 0) = u_0(\mathbf{x})\,, \quad \mathbf{x} \in \mathbf{R}^n\,,\end{aligned} \tag{1.1}$$

can be formulated as a system of conservation laws, simply by considering the equations satisfied by the gradient $\mathbf{p}(x,t) = \nabla u(x,t)$ of the solution of (1.1),

$$\begin{aligned}&\partial_t \mathbf{p} + \nabla H(\mathbf{p}) = 0\,, \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0,\infty)\\ &\mathbf{p}(\mathbf{x}, 0) = \mathbf{p}_0(\mathbf{x}) = \nabla u_0(\mathbf{x})\,, \quad \mathbf{x} \in \mathbf{R}^n\,.\end{aligned} \tag{1.2}$$

A relaxation approximation for a general Hamilton-Jacobi equation was introduced in [JX2] using the special structure of the system (1.2); as in relaxation schemes for conservation laws, a suitable sub-characteristic condition was derived, guaranteeing the dissipative nature of the relaxation approximation.

An important class of Hamilton-Jacobi equations, describing evolving fronts as the zero level set of their solution, arises for positively homogeneous Hamiltonians of degree one, i.e. when $H(\lambda\mathbf{p}) = \lambda H(\mathbf{p})$ for all $\lambda \geq 0$, $\mathbf{p} \in \mathbf{R}^n$. In this case the normal velocity of the front is $V = -H(\mathbf{n})$, where $\mathbf{n}$ is the exterior normal vector to the hypersurface. In [JK] it is pointed out that for a Hamilton-Jacobi equation with Hamiltonian,

$$H(\mathbf{p}) = c|\mathbf{p}|\,, \qquad \mathbf{p} \in \mathbf{R}^n\,, \tag{1.3}$$

describing a front propagating with constant normal velocity c, the relaxation approximation constructed in [JX2],

$$\begin{aligned} \partial_t \mathbf{p} + c\nabla w &= 0\,, \\ \partial_t w + c\nabla\cdot\mathbf{p} &= -\frac{c}{\epsilon}(w - |\mathbf{p}|)\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,, \\ \partial_t u + cw &= 0\,, \end{aligned} \tag{1.4}$$

not only yields the normal velocity c of the front at the leading term, but also captures –in the Chapman-Enskog expansion– the higher order mean curvature correction,

$$V_\epsilon = -c - \epsilon\kappa\,, \tag{1.5}$$

where κ denotes the mean curvature of the front, when the convective speeds of the relaxation approximation satisfy (degenerately) the subcharacteristic condition (2.5) as an equality (i.e., $a = c$). In fact, applying the Chapman-Enskog expansion to the relaxation approximation (1.4) one obtains the level set formulation of the law (1.5), after ignoring the $O(\epsilon^2)$ term:

$$\partial_t u + c|\nabla u| = \epsilon\,\mathrm{tr}\left[\left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\nabla^2 u\right]\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,. \tag{1.6}$$

A related relaxation approximation for anisotropic front propagation problems, where both the transport velocity and the curvature term may be direction-dependent, was also constructed in [JK]. Here, we will only consider isotropic evolutions in order to illustrate the relaxation method in the simplest possible framework. We plan to return to more complex and physically interesting problems in future work.

In this paper we propose to use the relaxation approximation (1.4) to analytically study and numerically compute the front propagation problems (1.6) or (1.5).

This new approach has several advantages. Since the relaxation system (1.4) is a symmetrizable, linearly-convective, hyperbolic system with a special stiff relaxation term without singularities, many important properties, such as finite speed of propagation and well-posedness of the relaxation problem, follow easily. In particular, the asymptotic equivalence between the level set equation (1.6) and the relaxation system (1.4) can be rigorously justified in many physically interesting cases (see Section 3). More importantly, besides its theoretical interest, such an approach provides a robust relaxation scheme to compute numerically propagating fronts problem (1.5) or related problems, by avoiding to discretize directly the mean curvature term in (1.6). Thus, on one hand we do not have a large number of numerical operations (especially in a 3D calculation), while on the other, we circumvent the singularities in the curvature term whenever the gradient of the solution of (1.6) is zero, so we do not need to adjust our grid or to adapt our algorithm in order to avoid related difficulties.

Currently there are two well developed numerical approaches for curvature dependent front propagation problems, which are topology independent and both of which are quite successful in a variety of applications. One is the level set method, developed by Osher and Sethian [OS], that models the front as the zero level set of a higher dimensional function that solves a Hamilton-Jacobi type equation. The second uses the parabolic phase field models such as the Allen-Cahn equation, see [AC, NPV, Ell], and references theirin. Our approach, based on a semilinear hyperbolic system, clearly has several distinct features from these two standard methods.

This paper is organized as follows. In Section 2, we review the relaxation approximation to general Hamilton-Jacobi equations [JX2], and its asymptotic connection with the level set equation of motion of a front propagating with a curvature-dependent normal velocity [JK]. In Section 3, this asymptotic equivalence is further demonstrated for some physically relevant examples such as evolving spheres and corner singularities, corresponding to shock profiles in the conservation law formulation (1.2). We also prove that when the interfaces are graphs in two space dimensions, then the relaxation approximation converges to the solution of the H-J equation as $\epsilon \to 0$. Furthermore, we show that, as long as the latter has a smooth solution, the error between the relaxation approximation (1.4) and the level set equation (1.6) is of order $O(\epsilon^2)$. We note that this error approximation result also holds for general Hamilton-Jacobi equations and is valid in arbitrary space dimensions. In Section 4, we recall the relaxation schemes introduced in [JX2] for general Hamilton-Jacobi equations. Variants of these schemes are then applied to a number of problems involving curvature-dependent front propagation, and the numerical results are reported in Section 5. We conclude the paper in Section 6.

2. Relaxation approximation for front propagation.

In this section we review and make some new observations on the relaxation approximation for general Hamilton-Jacobi equations, and its connection with the level set equation for interfaces propagating according to a curvature dependent local law.

We begin with the Cauchy problem for a Hamilton-Jacobi equation in $\mathbf{R}^n$ is

$$\begin{aligned} &\partial_t u + cH(\nabla u) = 0\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,, \\ &u(\mathbf{x},0) = u_0(\mathbf{x})\,, \quad \mathbf{x} \in \mathbf{R}^n\,. \end{aligned} \tag{2.1}$$

It is well-known that such equations in general do not have classical solutions. Suitable weak solutions, namely the *viscosity* solutions, were introduced by Crandall and Lions in [CL]. In this class of generalized solutions it was shown, under very general conditions, that viscosity solutions to Hamilton-Jacobi equations exist and are unique. For an overview to the theory of viscosity solutions for first and second order equations, we refer to [CIL].

Equation (2.1) has a corresponding system of conservation laws obtained formally by differentiating (2.1),

$$\begin{aligned} &\partial_t \mathbf{p} + c\nabla H(\mathbf{p}) = 0\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,, \\ &\mathbf{p}(\mathbf{x},0) = \mathbf{p}_0(\mathbf{x}) = \nabla u_0(\mathbf{x})\,, \mathbf{x} \in \mathbf{R}^n\,, \end{aligned} \tag{2.2}$$

where

$$\mathbf{p}(x,t) = \nabla u(\mathbf{x},t)\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,. \tag{2.3}$$

The equivalence of the weak solutions obtained in the vanishing viscosity limit of (2.1) and (2.2) for a convex Hamiltonian H, is a classical result in one space dimension [Kr1]. This result was extended in any space dimensions and without the convexity assumption, in [Lio].

In [JX2] the authors introduced a relaxation approximation to (2.2) and subsequently to (2.1). This relaxation approximation is

$$\begin{aligned} \partial_t \mathbf{p} + c\nabla w &= 0\,, \\ \partial_t w + a\nabla\cdot\mathbf{p} &= -\frac{c}{\epsilon}(w - H(\mathbf{p}))\,, \quad (\mathbf{x},t) \in \mathbf{R}^n \times (0,\infty)\,, \\ \partial_t u + cw &= 0\,, \end{aligned} \tag{2.4}$$

where $\epsilon > 0$ is the relaxation time, a is a positive constant satisfying the subcharacteristic stability condition

$$a > c|\nabla_{\mathbf{p}} H|^2\,. \tag{2.5}$$

One can apply a Chapman-Enskog type expansion to system (2.4) to show that its leading order asymptotic solution is governed by (see [JX2])

$$\partial_t u + cH(\nabla u) = \epsilon\left[a\Delta u - c(\nabla_{\mathbf{p}} H(\mathbf{p}))^T \nabla\mathbf{p}\nabla_{\mathbf{p}} H(\mathbf{p})\right], \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)\,. \tag{2.6}$$

It is easy to check that the sub-characteristic condition (2.5) guarantees that the $O(\epsilon)$ correction to the Hamilton-Jacobi equation in (2.1) is dissipative. It should be remarked that though the asymptotic equivalence between (2.4) and (2.6) for small ϵ, is derived formally, the rigorous justification of which can be carried out in many interesting cases. In particular, this asymptotic equivalence can be justified before the formation of singularity on the solution of the Hamilton-Jacobi equation, see Section 3 below.

The relaxation approximation (2.4) has several advantages, such as finite speed of propagation and linear convection. Moreover, it might provide an alternative way to construct the weak solution of (2.1). The convergence of the solutions to relaxation systems such as (2.4) to the entropy weak solutions of the associated inviscid conservation laws has been the focus of many recent studies, see [CLL, KT, Nat, WX, CR]. We also note that the relaxation approximation (2.4) is considerably simpler than those for general conservation laws [JX1] since only one rate equation is introduced in (2.4) for any space dimensions due to the special structure of the system (2.2).

A particularly important of application of the Hamilton-Jacobi equations lies in the description of an evolving interface [Whi, OJK, OS]. In the simplest such example, a front Γ_t propagating with constant outward normal velocity $V = -c$ can be described by the zero level set of an auxiliary function u solving the Hamilton-Jacobi equation (2.1), where $H(\nabla u) = |\nabla u|$:

$$\partial_t u + c|\nabla u| = 0\,, \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)\,. \tag{2.7}$$

As noted in [JK], if one chooses the critical value $a = c$ in the subcharacteristic condition (2.5), then (2.4) becomes

$$\begin{aligned} \partial_t \mathbf{p} + c\nabla w &= 0\,, \\ \partial_t w + c\nabla\cdot\mathbf{p} &= -\frac{c}{\epsilon}(w - |\mathbf{p}|)\,, \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)\,, \\ \partial_t u + cw &= 0\,, \end{aligned} \tag{2.8}$$

and the Chapman-Enskog expansion, which leads to (2.6) from (2.4) gives

$$\partial_t u + c|\nabla u| = \epsilon\,\mathrm{tr}\left[\left(I - \frac{\nabla u \otimes \nabla u}{|\nabla u|^2}\right)\nabla^2 u\right]\,, \quad (\mathbf{x}, t) \in \mathbf{R}^n \times (0, \infty)\,. \tag{2.9}$$

This equation is exactly the *level set equation* for a front Γ_t propagating with normal velocity

$$V_\epsilon = -c - \epsilon\kappa\,, \tag{2.10}$$

where κ is the mean curvature of the hypersurface Γ_t, and u is negative inside Γ_t. Note that the mean curvature is given by the formula

$$\kappa = \nabla\cdot\mathbf{n} = -\frac{1}{|\nabla u|}\mathrm{tr}\left[\left(I - \frac{\nabla u\otimes\nabla u}{|\nabla u|^2}\right)\nabla^2 u\right]\,, \tag{2.11}$$

where $\mathbf{n} = \frac{\nabla u}{|\nabla u|}$ is the outnormal of the front and the velocity is $V = \frac{u_t}{|\nabla u|}$.

The level set equation (2.9) that involves the curvature term is a second order, singular and degenerate parabolic equation. Similar to general Hamilton-Jacobi equations, such an equation in general has only weak–viscosity–solutions as shown by Chen, Giga and Goto [CGG] and Evans and Spruck [ESp]. In this context, the front is recovered as the level set $\Gamma_t = \{x : u(x,t) = 0\}$ of the viscosity solution u of (2.9), so the level set formulation allows for the interpretation of (2.10) in a weak sense past all possible singularities and changes of topological type of the front.

An important feature of the level set formulation for (2.9) or any other such local law, is the *geometric* property of the corresponding PDE (2.9): the zero level set of the solution u remains invariant under the transformation $u \mapsto \phi(u)$, for a smooth increasing function ϕ with $\phi(0) = 0$. This property guarantees that the evolution of the zero level set Γ_t depends only on the zero level set Γ_0 of the initial datum and its sign inside and outside Γ_0, and not on the particular choice of the initial condition of (2.9). The geometric property of the level set equation (2.9) is manifested in the relaxation system (2.8) as follows: assume that $(u, \mathbf{p}, w)$ is a solution to (2.8) with initial data

$$(u, \mathbf{p}, w)(x, t = 0) = (u_0(x), \nabla u_0(x), |\nabla u_0(x)|)) \tag{2.12}$$

Let ϕ be a smooth, non-decreasing function with the property that $\phi(0) = 0$. Set

$$(\bar u, \bar{\mathbf{p}}, \bar w) = (\phi(u), \phi'(u)\mathbf{p}, \phi'(u)w)\,. \tag{2.13}$$

It follows from (2.8) that

$$\begin{aligned} \partial_t\bar{\mathbf{p}} + c\nabla\bar w &= 0\\ \partial_t\bar w + c\nabla\cdot\bar{\mathbf{p}} &= -\frac{c}{\epsilon}(\bar w - |\bar{\mathbf{p}}|) + c\phi''(u)(|\mathbf{p}|^2 - w^2)\\ \partial_t\bar u + c\bar w &= 0, \end{aligned} \tag{2.14}$$

where one has used the assumption that $\phi'(u) \geq 0$ and the fact $\mathbf{p} = \nabla u$ which follows from (2.8a), (2.8c) and (2.12). Note that the last term in (2.16b) is of order $O(\epsilon)$, due to (2.8b). Thus, we may conclude that the geometrical property of the level set formulation is kept approximately up to order $O(\epsilon)$.

Going back to (2.8) we underline the observation that the Chapman-Enskog expansion leading to (2.9) (for the "degenerate" choice $a = c$) indicates that the relaxation approximation yields a front propagating with normal velocity $V_\epsilon = -c - \epsilon\kappa$, including the mean curvature as the next order correction to the front velocity $V = -c$. Curvature corrections of this type arise in a variety of physical phenomena, one of the most striking being in phase transition problems, for instance the Allen-Cahn theory [AC]. Furthermore, in problems where anisotropy plays a significant role, the mean curvature is typically replaced by a weighted scalar depending on the curvature tensor [AG]. A relaxation model similar to (2.8)) was developed for anisotropic front propagation in [JK].

Compared to the second order, singular and degenerate level set equation (2.9) the relaxation system (2.8) is a first order semilinear hyperbolic system without singularity. While it keeps the advantages of the level set equation such as the ability to handle geometric singularity and changes of topology, it provides some additional advantages. First, it is always well-posed without any singularity and degeneracy (see the discussion in the next paragraph). Secondly, it allows one to capture the curvature dependency automatically without directly solving the complicated, yet singular, curvature term numerically. Being semilinear allows one to solve it numerically in a strikingly simple way by avoiding the Riemann solvers.

Finally we point out an important theoretical feature of the relaxation approximation here, i.e., the relaxation system (2.8)is a symmetrizable semilinear hyperbolic system with linear convection. This simple fact insures the well-posed of smooth solutions to the relaxation system for any fixed $\epsilon > 0$. To show the symmetrizability, one may rewrite the system (2.8) in terms of variable $(u, \mathbf{p}, w)^t$ as follows

$$\partial_t \begin{pmatrix} u \\ \mathbf{p} \\ w \end{pmatrix} + \sum_{i=1}^n A_i \partial_{x_i} \begin{pmatrix} u \\ \mathbf{p} \\ w \end{pmatrix} = \begin{pmatrix} -w \\ 0 \\ -\frac{c}{\epsilon}(w - H(\mathbf{p})) \end{pmatrix} \tag{2.15}$$

where A_i $(i = 1, 2, \ldots, n)$ are constant $(n+2) \times (n+2)$ matrices with all entries being zero except the $(i+1, n+2)$-th entry being one and the $(n+2, i+1)$-th entry being c. System (2.15) can be symmetrized as follows. Let C be a $(n+2) \times (n+2)$ constant matrix given by

$$C = \operatorname{diag}\{1, \sqrt{c}, \sqrt{c}, \ldots, \sqrt{c}, 1\}. \tag{2.16}$$

Set

$$\begin{pmatrix} u \\ \mathbf{q} \\ w \end{pmatrix} = C \begin{pmatrix} u \\ \mathbf{p} \\ w \end{pmatrix} . \tag{2.17}$$

In terms of the new variable $(u, \mathbf{q}, w)^t$, the system (2.15) is transformed into

$$\partial_t \begin{pmatrix} u \\ \mathbf{q} \\ w \end{pmatrix} + \sum_{i=1}^{n} B_i \partial_{x_i} \begin{pmatrix} u \\ \mathbf{q} \\ w \end{pmatrix} = \begin{pmatrix} -w \\ 0 \\ -\frac{1}{\epsilon}\left(w - H\left(\frac{\mathbf{q}}{\sqrt{c}}\right)\right) \end{pmatrix} \tag{2.18}$$

where $B_i = CA_iC^{-1}$. Direct computation shows that all the entries of B_i are zero except that the $(i+1, n+2)$-th and $(n+2, i+1)$-th entries are $\sqrt{c}$. Thus B_i $(i = 1, 2, \ldots, n)$ are symmetric matrices. And so (2.18) is a symmetric hyperbolic system.

3. Rigorous asymptotics of the relaxation systems (2.4) and (2.8)

We now turn to the rigorous justification of the asymptotic equivalence, as ϵ becomes small, of the relaxation system (2.8) to the curvature-dependent Hamilton-Jacobi equation (2.9). Without loss of generality we take $c = 1$. The goal is to show that the deviations of solutions to (2.8) from those of (2.9) are at most of order $O(\epsilon^2)$. Although there seems no doubt about this asymptotic equivalence, yet its rigorous justification poses a challenging mathematical problem due to the possible development of singularities for the solutions to (2.1) and the highly singular nature of the relaxation limit. However, in the following, we will demonstrate this asymptotic equivalence for some special (but physically relevant) flows in two and three dimensions. We will start with the weakly nonlinear limit of the two systems (2.8) and (2.9), in two space dimensions.

3.1 Evolution of a sphere

We first simplify the relaxation system (2.8) for interfaces with rotational symmetry. Assume first that a solution (u, p, w) of (2.8) satisfies

$$u(\mathbf{x}, t) = u(|\mathbf{x}|, t)\,, \quad \mathbf{p}(x, t) = \frac{\mathbf{x}}{|\mathbf{x}|} u_r(|\mathbf{x}|, t)\,, \quad w(\mathbf{x}, t) = w(|\mathbf{x}|, t)\,,$$

where we set $r = |\mathbf{x}|$. Then (2.8) yields

$$\begin{aligned} u_t + w &= 0\,, \\ p_t + w_r &= 0\,, \\ w_t + p_r + \frac{n-1}{r} p &= -\frac{1}{\epsilon}(w - |p|)\,. \end{aligned}$$

The form of this system indicates that the relaxation system (2.8) retains for all times the rotational symmetry of initial data with the same property.

We now compare the evolution of a sphere according to the level set formulation (2.9) and the relaxation system (2.8). We construct explicit solutions of the two models by separating variables, assuming first the ansatz for the solution of (2.8),

$$u(\mathbf{x},t) = u(|\mathbf{x}|) - t\,, \quad \mathbf{p}(\mathbf{x},t) = \frac{\mathbf{x}}{|\mathbf{x}|}u_r(|\mathbf{x}|)\,, \quad w(\mathbf{x},t) = 1\,,$$

where as before $r = |\mathbf{x}|$; similarly, in (2.9) we assume

$$u(\mathbf{x},t) = v(|\mathbf{x}|) - t\,.$$

The corresponding evolutions are given by

$$\Gamma_t^{RS} = \{\mathbf{x} : u(|\mathbf{x}|) = t\}\,, \quad \Gamma_t^{LS} = \{\mathbf{x} : v(|\mathbf{x}|) = t\}\,,$$

and as common initial data we consider the sphere,

$$\Gamma_0 = \{\mathbf{x} : |\mathbf{x}| = R_0\} = \{\mathbf{x} : u(|\mathbf{x}|) = 0\} = \{\mathbf{x} : v(|\mathbf{x}|) = 0\} \tag{3.1}$$

Then setting

$$p(r) = u_r(r)\,, \quad q(r) = v_r(r)\,,$$

we obtain from (2.8c) and (2.9) respectively, that

$$p_r + \frac{n-1}{r}p = -\frac{1}{\epsilon}(1 - |p|)\,, \tag{3.2}$$

and

$$\frac{n-1}{r}q = -\frac{1}{\epsilon}(1 - |q|)\,. \tag{3.3}$$

In dimension $n = 3$ we easily obtain a solution of (3.2):

$$p(r) = u_r(r) = -1 + \frac{2\epsilon}{r} - \frac{2\epsilon^2}{r^2}\,.$$

In addition, (3.3) yields

$$q(r) = v_r(r) = -1 + \frac{2\epsilon}{2\epsilon + r}\,.$$

By employing the initial condition (3.1) we find

$$u(r) = R_0 - 2\epsilon \log R_0 - \frac{2\epsilon^2}{R_0} - r + 2\epsilon \log r + \frac{2\epsilon^2}{r}\,, \tag{3.4}$$

and

$$v(r) = R_0 - 2\epsilon \log(R_0 + \epsilon) - r + 2\epsilon \log(r + \epsilon)\,. \tag{3.5}$$

It is clear that away from $r = 0$,

$$u(r) = v(r) + \Big(\frac{1}{R_0^2} - \frac{1}{r^2}\Big)\epsilon^3 + O(\epsilon^4)\,.$$

Furthermore we can estimate the error between the interfaces Γ_t^{RS} and Γ_t^{LS}: let $R(t)$ and $\rho(t)$ be the radii of the circles obtained by the evolutions of the relaxation system and the level set formulation respectively. Then,

$$u(R(t)) = t \quad \text{and} \quad v(\rho(t)) = t\,.$$

Expanding (3.4) and (3.5) we obtain,

$$R(t) = \rho(t) + \Big(\frac{1}{R_0 - t} + \frac{1}{(R_0 - t)^2} - \frac{1}{R_0^2}\Big)O(\epsilon^3) + O(\epsilon^4)\,. \tag{3.6}$$

Notice that the estimate deteriorates at the two spheres shrink and blows up in the vicinity of the vanishing time.

3.2 Shock profile

Next, we compare the structures of the viscous shock profile and relaxation shock profile in two space dimensions, assuming that the front is the graph of a function $v_0 : \mathbf{R} \mapsto \mathbf{R}$. In this case the propagating front with normal velocity $V_\epsilon = -1 - \epsilon\kappa$ is governed by

$$v_t + \sqrt{1 + v_x^2} = \epsilon \frac{v_{xx}}{1 + v_x^2}\,, \tag{3.7}$$

where the front $\Gamma_t^G = \{(x, y) \in \mathbf{R}^2 : y = u(x, t)\}$. If we set $q(x, t) = v_x(x, t)$, then we obtain the conservation law

$$q_t + (\sqrt{1 + q^2})_x = \epsilon\Big(\frac{q_x}{1 + q^2}\Big)_x\,. \tag{3.8}$$

Similarly, for the relaxation system (2.10)), we may define the front $\Gamma_t^{RS} = \{(x, y) \in \mathbf{R}^2 : y = u(x, t)\}$, where $p(x, t) = u_x(x, t)$ and

$$\begin{aligned} u_t + w &= 0\,, \\ p_t + w_x &= 0\,, \\ w_t + p_x &= -\frac{1}{\epsilon}(w - \sqrt{1 + p^2})\,. \end{aligned} \tag{3.9}$$

The viscous shock profiles are smooth traveling wave solutions to (3.8) and (3.9) respectively, connecting the same states u_l, u_r. The ensuing interfaces are given by the graphs of the integrated variables and correspond to a smoothed (due to the curvature term) corner singularity, defined by straight lines with slopes u_l, u_r.

A traveling wave solution $q(x,t) = Q(\frac{x-st}{\epsilon})$ of (3.8), where $s = s(u_r, u_l)$, is given by the Rankine-Hugoniot condition

$$s = \frac{\sqrt{1+u_r^2} - \sqrt{1+u_l^2}}{u_r - u_l}\,, \tag{3.10}$$

satisfies

$$Q' = B_1(Q)\,, \tag{3.11}$$

where,

$$B_1(Q) := \left(1+Q^2\right)\left[\,-s(Q-u_l) + \sqrt{1+Q^2} - \sqrt{1+u_l^2}\,\right],$$

and

$$Q(-\infty) = u_l\,,\;\; Q(\infty) = u_r\,. \tag{3.12}$$

The solution to (3.11)-(3.12) exists uniquely up to a phase shift if $u_l > u_r$. Similarly, a traveling wave solution $\big(p(x,t), w(x,t)\big) = \big(P(\frac{x-st}{\epsilon}), W(\frac{x-st}{\epsilon})\big)$ of (3.9), where $s = s(u_r, u_l)$ is given by (3.10), solves

$$\begin{aligned} &-sP' + W' = 0\,,\\ &-sW' + P' = \sqrt{1+P^2} - W\,, \end{aligned}$$

with

$$P(-\infty) = u_l\,,\;\; W(-\infty) = \sqrt{1+u_l^2}\,,\;\; P(\infty) = u_r\,,\;\; W(\infty) = \sqrt{1+u_r^2}\,. \tag{3.13}$$

Consequently, we obtain that $W(\xi) = \sqrt{1+u_l^2} + s[P(\xi) - u_l]$, and

$$P' = B_2(P)\,, \tag{3.14}$$

where

$$B_2(P) := \frac{1}{1-s^2}\left[\,-s(P-u_l) + \sqrt{1+P^2} - \sqrt{1+u_l^2}\,\right].$$

Note that the traveling waves P and Q are monotone and translation invariant, i.e., $\overline{P}(\xi) = P(\xi+\xi_0)$ and $\overline{Q}(\xi) = Q(\xi+\xi_0)$ are solutions of (3.14) and (3.11) respectively, for any constant ξ_0. In the next Theorem we compare the shock P and Q, selecting first the parameter $\xi_0 = 0$:

Theorem 3.1. *Let (u_l, u_r), $u_l > u_r$ be constant states and $s = s(u_r, u_l)$ be the shock speed satisfying the Rankine-Hugoniot condition* (3.10)*. If the states (u_l, u_r) are connected by the solutions of* (3.14) *and* (3.11)*, P and Q respectively, then, provided we fix both profiles so that for $u_0 \in (u_r, u_l)$,*

$$P(0) = Q(0) = u_0\,, \quad \textit{where} \;\; s = \frac{u_0}{\sqrt{1+u_0^2}}\,,$$

the following hold:
(a) The profiles agree to first order at the point $\xi = 0$, i.e.

$$P'(0) = Q'(0)\,.$$

(b) There is a positive constant C such that

$$||P - Q||_{L^\infty(\mathbf{R})} \leq C(u_l - u_r)^2\,,$$

for $u_l - u_r$ sufficiently small.

A similar result to Part (b) of the previous Theorem was obtained by Caflisch in [Caf] for the shock profiles of the Broadwell system and the corresponding Navier-Stokes equations. We point out that the variation of P and Q is of order $O(u_l - u_r)$, so a priori it is only known that $||P - Q||_{L^\infty} = O(u_l - u_r)$. Another aspect of the agreement of the two profiles is exhibited in Part (a) of the Theorem, which essentially says that the two profiles are identical to first order at the "center" of the shock $\xi = 0$. Consequently the corresponding fronts agree to second order at the tip of the corner defined by the slopes u_l, u_r, which is also the point where a singularity occurs, as $\epsilon \to 0$.

Proof. (a). Since the speed s satisfies (3.10), there is $u_0 \in (u_r, u_l)$ such that $s = \frac{u_0}{\sqrt{1+u_0^2}}$. Furthermore, due to the monotonicity of both shock profiles, we may fix the profiles so that $P(0) = Q(0) = u_0$. Evaluating (3.11), (3.14) at the point $\xi = 0$, we have

$$(1 - s^2)P'(0) = \frac{1}{1+u_0^2}Q'(0)\,,$$

which in view of the value of s yields that $P'(0) = Q'(0)$.

Before we proceed with the proof of Part (b) we recall a Lemma in [Caf]:

Lemma 3.2. *Let $c, \alpha, \delta, u_1, u_2 \in \mathbf{R}$, with $c > 0\,, \delta = u_2 - u_1 > 0$. Let $B \in C^3([u_1, u_2])$. Suppose that*

$$B(u_1) = B(u_2) = 0\,, \tag{3.15}$$

$$B'(u_1) < 0 < B'(u_2)\,, \tag{3.16}$$

and for all $u \in [u_1, u_2]$,

$$B(u) < 0 \ \ \textit{if}\ \ u \neq u_1\,, u_2\,, \tag{3.17}$$

$$1/c < B''(u) < c\,, \tag{3.18}$$

$$|\alpha - \frac{1}{2}B''(u)| < c\delta\,. \tag{3.19}$$

Define the functions u, v *by*

$$u' = B(u)\,, \quad u(0) = u_0\,,$$

$$v' = \alpha(v - u_1)(v - u_2)\,, \quad v(0) = u_0\,, \tag{3.20}$$

where $u_0 \in (u_r, u_l)$. *Then if* $\delta = u_2 - u_1$ *is sufficiently small compared to c, we have*

$$||u - v||_{L^\infty(\mathbf{R})} \leq D\delta^2\,,$$

where D *depends only on c.*

Proof of Part (b). It suffices to show that the functions B_1, B_2 in (3.11), (3.14) respectively, satisfy the assumptions of Lemma 3.2 for some suitable constants α, c and $u_1 = u_r\,, u_2 = u_l$. Then the result follows by the successive comparison of the profiles P and Q with the solution of (3.20).

Clearly (3.15) is satisfied due to (3.10). Since $H(p) = \sqrt{1+p^2}$ is convex and $u_l > u_r$ –which is equivalent to an entropy condition–, we have that (3.16) also holds for both B_1, B_2. Moreover the convexity of H implies (3.17) for B_1 and therefore for B_2. Again by convexity, we have that B_1 satisfies (3.18), if we pick c so that

$$1 < c < \big(\min_{r \in [u_r, u_l]} B_1''(r)\big)^{-1}\,. \tag{3.21}$$

Evaluating the second derivative of B_2 in $[u_r, u_l]$, we readily see that (3.18) holds for δ small enough (compared to c above). Finally, (3.19) implies (3.21) if we pick an α such that $c/2 - c\delta < \alpha < c\delta + 1/2c$. This is possible, if in addition to (3.21), we impose the condition

$$1 < c < \frac{1}{\sqrt{1 - 4\delta}}\,.$$

3.3 Error Estimates

In this section we will prove the asymptotic equivalence of the relaxation approximation to both the inviscid Hamilton-Jacobi equation and its regularization (2.6), including the case where the higher order term is the mean curvature, (2.9). We will show that not only do the solutions to the relaxation approximation converge uniformly to the solutions to the Hamilton-Jacobi equations, but also the errors between the solutions to the relaxation approximation (2.8) and the ones to the curvature regularization (2.9) are of second order in ϵ, when the interfaces are graphs. We note that this error approximation result also holds for general Hamilton-Jacobi equations and is valid in arbitrary space dimensions. For simplicity in the presentation, we will only state this result for the 2-dimensional case.

Theorem 3.3. *Assume that the Hamiltonian H in (1.1) is smooth. Let $u^{(0)} \in C^{\ell}([0,T]; H^{\ell+3}(\mathbf{R}^2))$ ($\ell > 5$) be the solution to (1.1). Then $\exists\ \epsilon_0 > 0$ such that if $0 < \epsilon \leq \epsilon_0$, there exist $u^\epsilon, \widehat{u^\epsilon} \in C([0,T]; H^3(\mathbf{R}^2))$, which solve*

$$\begin{cases} \partial_t u^\epsilon + H(\nabla u^\epsilon) = \epsilon(a\Delta u^\epsilon - \partial_t^2 u^\epsilon) \\ u^\epsilon(x,0) = u_0(x) \\ \partial_t u^\epsilon(x,0) = -H(\nabla u_0(x)) \end{cases} \tag{3.22}$$

and

$$\begin{cases} \partial_t \widehat{u^\epsilon} + H(\nabla \widehat{u^\epsilon}) = \epsilon[a\Delta \widehat{u^\epsilon} - (\nabla_p H(\nabla \widehat{u^\epsilon}))^t \nabla^2 \widehat{u^\epsilon}(\nabla_p H(\nabla \widehat{u^\epsilon})), \\ \widehat{u^\epsilon}(x,0) = u_0(x) \end{cases} \tag{3.23}$$

respectively. Furthermore, there exists a constant C depending only on $u^{(0)}$ (but independent of ϵ) such that

$$\sup_{0 \leq t \leq T} \|u^\epsilon - \widehat{u^\epsilon}\|_{L^\infty(\mathbf{R}^2)} \leq C\epsilon^2 . \tag{3.24}$$

Remark 1: It will be clear from our analysis that a similar conclusion holds true for arbitrary space dimensions.

Remark 2: We note that Theorem 3.3 includes the evolution of interfaces according to (1.5) in arbitrary space dimensions, in the case they always remain graphs, since then the hamiltonian $H(\mathbf{p}) = \sqrt{1 + |\mathbf{p}|^2}$, is smooth and the conditions of Theorem 3.3 are fulfilled.

Remark 3: It should be also pointed out that (3.22) is not a regular perturbation problem of (1.1) due to the inconsistency of the initial velocity $\partial_t u(x,0)$ and $\partial_t u^\epsilon(x,0)$. Thus a weak initial layer will appear which is needed for the analysis.

Proof of Theorem 3.3

Step 1: Asymptotic ansatz. We first construct an approximate solution to (3.22). Set $\tau = \frac{t}{\epsilon}$, and

$$U_a = (u^0(x,t) + \epsilon u^{(1)}(x,t) + \epsilon^2 u^{(2)}(x,t)) + (\epsilon^2 U^{(2)}(x,\tau) + \epsilon^3 U^{(3)}(x,\tau)) \tag{3.25}$$

where $u^{(0)}(x,t)$ is the solution to the Cauchy problem (1.1) assumed in the theorem, $u^{(1)}(x,t)$ and $u^{(2)}(x,t)$ solve the following initial value problems respectively.

$$\begin{cases} \partial_t u^{(1)} + \nabla_p H(\nabla u^{(0)})\nabla u^{(1)} = a\Delta u^{(0)} - \partial_t^2 u^{(0)} \\ u^{(1)}(x,0) \equiv u_0^{(1)}(x) = 0, \end{cases} \tag{3.26}$$

$$\begin{cases} \partial_t u^{(2)} + \nabla_p H(\nabla u^{(0)})\nabla u^{(2)} \\ \qquad = a\Delta u^{(1)} - \partial_t^2 u^{(1)} - \frac{1}{2}\nabla_p^2 H(\nabla u^{(0)})(\nabla u^{(1)}, \nabla u^{(1)}) + \nabla_p H(\nabla u^{(0)})\nabla u^{(1)}(x,0) \\ u^{(2)}(x,t=0) \equiv u_0^{(2)}(x) = 0, \end{cases} \tag{3.27}$$

while $U^{(2)}(x,\tau)$ and $U^{(3)}(x,\tau)$ are given respectively by

$$U^{(2)}(x,\tau) = (-\partial_t u^{(1)}(x,0)) + (e^{-\tau}\partial_t u^{(1)}(x,0)), \tag{3.28}$$

$$\begin{aligned} U^{(3)}(x,\tau) = {} & (-\partial_t u^{(2)}(x,0) - \nabla_p H(\nabla u^{(0)}(x,0))\partial_t \nabla u^{(1)}(x,0)) \\ & + e^{-\tau}[\partial_t u^{(2)}(x,0) + (1-\tau)\nabla_p H(\nabla u^{(0)}(x,0))\partial_t \nabla u^{(1)}(x,0)]. \end{aligned} \tag{3.29}$$

First, it should be noted that for $u^{(0)}(x,t)$ sufficiently smooth there exist unique smooth solutions $u^{(1)}$ and $u^{(2)}$ to (3.26) and (3.27) respectively, so that $u^{(1)} \in C^{\ell-1}([0,T]; H^{\ell+1}(\mathbf{R}^2))$ and $u^{(2)} \in C^{\ell-2}([0,T]; H^{\ell-1}(\mathbf{R}^2))$. Consequently, $U^{(2)}(x,\tau)$ and $U^{(3)}(x,\tau)$, given by (3.28)–(3.29) respectively, are well-defined, smooth functions, i.e.,

$$U^{(2)}(\cdot, \frac{\cdot}{\epsilon}) \in C^{\ell-2}([0,T]; H^{\ell+1}(\mathbf{R}^2)), \qquad U^{(3)}(\cdot, \frac{\cdot}{\epsilon}) \in C^{\ell-4}([0,T]; H^{\ell}(\mathbf{R}^2)) \,.$$

Next, we claim that U_a is a desired approximate solution to the problem (3.22). Indeed, it follows from (1.1), (3.26)–(3.29), and direct calculation that

$$\begin{cases} \partial_t U_a + H(\nabla U_a) - \epsilon(a\Delta U_a - \partial_t^2 U_a) = \epsilon^3 \mathcal{E}_a, \\ U_a(x,0) = u_0(x), \\ \partial_t U_a(x,0) = -H(\nabla u_0(x)), \end{cases} \tag{3.30}$$

where the error term $\epsilon^3\mathcal{E}_a$ is given by

$$
\begin{aligned}
\epsilon^3\mathcal{E}_a(x,t) = &\left\{ H\left(\sum_{i=0}^{2}\epsilon^i\nabla u^{(i)}\right) - H(\nabla u^{(0)}) - \epsilon\nabla_p H(\nabla u^{(0)})\nabla u^{(1)} \right.\\
&- \epsilon^2\left[\nabla_p H(\nabla u^{(0)})\nabla u^{(2)} + \frac{1}{2}\nabla_p^2 H(\nabla u^{(0)})(\nabla u^{(1)}, \nabla u^{(2)})\right]\\
&\left. + \epsilon^3(\partial_t^2 u^{(2)} - a\Delta u^{(2)})\right\} + \left\{ -a\sum_{i=2}^{3}\epsilon^{i+1}\Delta U^{(i)} + \right.\\
&H\left(\sum_{i=0}^{2}\epsilon^i\nabla u^{(i)} + \sum_{i=2}^{3}\epsilon^i\nabla U^{(i)}\right)\\
&- H\left(\sum_{i=0}^{2}\epsilon^i\nabla u^{(i)}\right) + \epsilon^2\nabla_p H(\nabla u^{(0)})\partial_t\nabla u^{(1)}(x,0)\\
&\left. - \epsilon^2\nabla_p H\left(\nabla\sum_{i=0}^{2}\epsilon^2 u^i(x,0)\right)e^{-\tau}\partial_t\nabla u^{(1)}(x,0)\right\}
\end{aligned}
\tag{3.31}
$$

It follows from the construction of the asymptotic ansatz that

$$
\begin{cases}
\|\epsilon^3\mathcal{E}_a\|_{L^\infty} \le \epsilon^3 c,\\
\sup\limits_{0\le t\le T}\|\epsilon^3\mathcal{E}_a(\cdot,t)\|_{H^3(\mathbf{R}^2)} \le \epsilon^3 c,
\end{cases}
\tag{3.32}
$$

where, and from now on in this subsection, c will denote a generic positive constant independent of ϵ.

Step 2: Perturbation analysis. With the uniform asymptotic solution U_a constructed in the previous step, we can look for solutions to (3.22) of the form

$$
u^\epsilon(x,t) = U_a(x,t) + \epsilon^2 v(x,t)
\tag{3.33}
$$

where the new unknown function $v(x,t)$ solves

$$
\begin{aligned}
&\partial_t v + \nabla_p H(\nabla U_a)\nabla v - \epsilon(a\Delta v - \partial_t^2 v) = \mathcal{E} \equiv \mathcal{E}_1 + \mathcal{E}_2\\
&v(x,0) = 0\\
&\partial_t v(x,0) = 0
\end{aligned}
\tag{3.34}
$$

with $\mathcal{E}_1 = \epsilon\mathcal{E}_a$ given in (3.31), and

$$
\mathcal{E}_2 = -\epsilon^{-2}(H(\nabla U_a + \epsilon^2\nabla v) - H(\nabla U_a) - \epsilon^2\nabla_p H(\nabla U_a)\nabla v).
\tag{3.35}
$$

Thus the analysis of the problem (3.22) is reduced to estimate the solution to the error problem (3.34). The problem (3.34) is an initial valued problem for a quasi-linear second order hyperbolic equation for each $\epsilon > 0$, so the local (in time) existence and regularity of the solutions in space $C^1([0,t];H^3(\mathbf{R}^2))$ are standard. Hence, to prove the global (up to T) existence of the solution and derive the desired estimate claimed in Theorem 3.3, we need the following a priori estimate.

Proposition 3.4. *There exists uniform constants C_1 and C_2 such that if*

$$\epsilon \sup_{0\le t\le T} \|v(t,\cdot)|_{H^3(\mathbf{R}^2)} \le C_1, \tag{3.36}$$

then

$$\sup_{0\le t\le T} \|v(t,\cdot)\|_{H^3(\mathbf{R}^2)} \le C_2\epsilon \tag{3.37}$$

Assume Proposition 3.4 holds for the moment. Let us continue our argument. Combining the standard continuity argument with Proposition 3.3 shows that there exists a positive constant ϵ_0 such that for all $0 < \epsilon \le \epsilon_0$, (3.34) has a unique solution $v \in C^0([0,T], H^3(\mathbf{R}^2)) \cap C^1([0,T], L^2(\mathbf{R}^2))$. Furthermore, the uniform estimate (3.37) holds. Consequently, there exists a unique solution, $u^\epsilon(x,t)$, to (3.22) with the property that

$$\|u^\epsilon(\cdot,\cdot) - U_a(\cdot,\cdot)\|_{L^\infty([0,T]\times\mathbf{R}^2)} \le C\epsilon^3 \tag{3.38}$$

which justify the asymptotic ansatz U_a rigorously.

It remains to prove Proposition 3.4.

Step 3: Stability estimates. For simplicity in the notation, we will use D^α ($\alpha = 0,1,2,3$) to denote any space derivatives of order α. Apply D^α to the equation in (3.34) and rewrite the resulting equation as

$$\begin{cases} \partial_t D^\alpha v + \nabla_p H(\nabla U_a)\cdot\nabla D^\alpha v - \epsilon(a\Delta D^\alpha v - \partial_t^2 D^\alpha v)\,, \\ \qquad = -[D^\alpha, \nabla_p H(\nabla U_a)\cdot\nabla]v + D^\alpha\mathcal{E}\,, \\ D^\alpha v(x,0) = \partial_t D^\alpha v(x,0) = 0 \end{cases} . \tag{3.39}$$

Let λ be a positive constant to be determined. Multiplying (3.39) first by $\lambda D^\alpha v$, $\partial_t D^\alpha v$, summing up the resulting expression, and integrating it over $[0,t]\times\mathbf{R}^2$, one can obtain after some standard manipulations that

$$
\begin{aligned}
&\tfrac{1}{2}\int(\lambda|D^\alpha v|^2+2\lambda\epsilon D^\alpha v\partial_t D^\alpha v+\epsilon|\partial_t D^\alpha v|^2)(x,t)dx\\
&\quad+\frac{a\epsilon}{2}\int|D^\alpha\nabla v(x,t)|^2dx+\int\int(\epsilon\lambda a|\nabla D^\alpha v|^2+(1-\lambda\epsilon)|\partial_t D^\alpha v|^2\\
&\quad+\partial_t D^\alpha v(\nabla_p H(\nabla U_a))\nabla D^\alpha v)(x,t)dxds\\
&=\int\int\frac{\lambda}{2}\mathrm{div}_x\nabla_p H(\nabla U_a)|D^\alpha v|^2dxds-\int\int\lambda D^\alpha v[D^\alpha,\nabla_p H(\nabla U_a)\nabla]vdxds\\
&\quad+\int\int\lambda D^\alpha vD^\alpha\mathcal{E}dxds\\
&\quad-\int\int\partial_t D^\alpha v[D^\alpha,\nabla_p H(\nabla U_a)\nabla]vdxds+\int\int\partial_t D^\alpha vD^\alpha\mathcal{E}dxds\,.
\end{aligned}
\tag{3.40}
$$

We now estimate each term in (3.40) separately. Set $\lambda=\frac{1}{2\epsilon}$. First, there exists an uniform constant C_0 such that

$$
\begin{aligned}
&\frac{1}{2}\int\left(\lambda|D^\alpha v|^2+2\lambda\epsilon D^\alpha v\partial_t D^\alpha v+\epsilon|\partial_t D^\alpha v|^2+a\epsilon|\nabla D^\alpha v|^2\right)(x,t)dx\\
&\qquad\ge c_0\int\left(\frac{1}{\epsilon}|D^\alpha v|^2(x,t)+\epsilon(|\partial_t D^\alpha v|^2+|\nabla D^\alpha v|^2)(x,t)\right)dx\\
&\qquad\equiv c_0E_\alpha(t,\epsilon).
\end{aligned}
\tag{3.41}
$$

Next, it follows from the stability condition ($a\ge|\nabla_p H(\nabla u^{(0)})|^2$) and the structure of U_a that

$$
\begin{aligned}
&\int\int\left(\epsilon\lambda a|\nabla D^\alpha v|^2+(1-\lambda\epsilon)|\partial_t D^\alpha v|^2+\partial_t D^\alpha v(\nabla_p H(\nabla U_a)\right)\nabla D^\alpha v)(x,s)dxds\\
&\qquad\ge -C_1\epsilon\int\int(|\partial_t D^\alpha v|^2+|\nabla D^\alpha v|^2)(x,s)dxds.
\end{aligned}
\tag{3.42}
$$

Using the structure of U_a, one verifies easily that

$$
\begin{aligned}
&\int\int\Bigg(\frac{\lambda}{2}\mathrm{div}_x\nabla_p H(\nabla U_a)|D^\alpha v|^2-\lambda D^\alpha v[D^\alpha,\nabla_p H(\nabla U_a)\nabla]v\\
&\qquad-\partial_t D^\alpha vD^\alpha,\nabla_p H(\nabla U_a)\nabla]v\Bigg)(x,s)dxds\le C_2\sum_{\beta\le\alpha}\int_0^t E_\beta(s,\epsilon)ds.
\end{aligned}
\tag{3.43}
$$

Next, one obtains from the error estimate (3.42) that

$$\begin{aligned}
&\int\int(\lambda D^\alpha v D^\alpha \mathcal{E}_1 + \partial_t D^\alpha v D^\alpha \mathcal{E}_1)(x,s)dxds \\
&\le \int_0^t E_\alpha(s,\epsilon)ds + \epsilon^{-1} C_3 \int_0^t \int |D^\alpha \mathcal{E}_1|^2(x,s)dxds \\
&\le \int_0^t E_\alpha(s,\epsilon)ds + C_3\epsilon \int_0^t \int |D^\alpha \mathcal{E}_a|^2(x,s)dxds \\
&\le \int_0^t E_\alpha(s,\epsilon)ds + C_4\epsilon.
\end{aligned} \tag{3.44}$$

Finally, we estimate the integral involving nonlinear terms. By the definition of $\mathcal{E}_2$ (3.35), and the a priori assumption (3.36), one shows by repeated use of the Gagliardo-Nirenberg and Hölder inequalities that

$$\begin{aligned}
&\lambda \int\int(\lambda D^\alpha v D^\alpha \mathcal{E}_2 + \partial_t D^\alpha v D^\alpha \mathcal{E}_2)(x,s)dxds \\
&\le \int_0^t E_\alpha(s,\epsilon)ds + C_5\epsilon^{-1}\int\int |D^\alpha \mathcal{E}_2|^2(x,s)dxds \le C_6 \sum_{\beta\le\alpha}\int E_\beta(s,\epsilon)ds.
\end{aligned} \tag{3.45}$$

As a consequence of (3.40)–(3.45), one has shown that under the assumptions (3.36), it holds true that

$$E_\alpha(t,\epsilon) \le C_7 \left(\sum_{\beta\le\alpha}\int_0^t E_\beta(s,\epsilon)ds + \epsilon\right) \tag{3.46}$$

Summing up (3.46) for all α (with $\alpha \le 3$) and applying Gronwall's inequality yields that for all $t \in [0,T]$

$$\epsilon^{-1}\|v(\cdot,t)\|^2_{H^3(\mathbf{R}^2)} + \epsilon(\|\partial_t v(\cdot,t)\|^2_{H^3(\mathbf{R}^2)} + \|\nabla v(\cdot,t)\|^2_{H^3(\mathbf{R}^2)}) \le C_8(T)\epsilon \tag{3.47}$$

which gives the desired estimate (3.37) immediately. This completes the proof of Proposition 3.3.

Step 4: Viscous problem. We now analyze the viscous problem (3.23). Note that (3.23) is a regular perturbation problem of (1.1). Thus in the absence of discontinuity in the solution to the inviscid problem (1.1), it becomes standard now to justify the asymptotic form of solution $\widehat{u^\epsilon}(x,t)$ to (3.23) by asymptotic expansions and energy estimates. Indeed, in a similar way as in Steps 1–3 (but much simpler since there is no initial layer), one can show that for suitably small ϵ, there exists a unique solution $\widehat{u^\epsilon}(x,t) \in C([0,T]; H^3(R^2))$ to (3.23) with the property that

$$\widehat{u^\epsilon}(x,t) = u^{(0)}(x,t) + \epsilon u^{(1)}(x,t) + \epsilon^2 \hat{v}(x,t) \tag{3.48}$$

where $u^{(0)}$ and $u^{(1)}$ solves (1.1) and (3.26) respectively, while $\hat{v}(x,t)$ is a solution to the following problem

$$\begin{aligned}&\partial_t \hat{v} + \nabla_p H(\nabla u^{(0)} + \epsilon \nabla u^{(1)})\nabla \hat{v} \\ &\qquad - \epsilon(a\Delta \hat{v} - \nabla_p H(\nabla u^{(0)} + \nabla u^{(1)}\epsilon)\nabla^2 \hat{v} \nabla_p H(\nabla u^{(0)} + \epsilon \nabla u^{(1)})) = \mathcal{E}_3 \\ &\hat{v}(x, t=0) = 0\,,\end{aligned} \tag{3.49}$$

with

$$\begin{aligned}\mathcal{E}_3 = \epsilon^{-2}(&H(\nabla u^{(0)} + \epsilon\nabla u^{(1)} + \epsilon^2 \nabla \hat{v}) - H(\nabla u^{(0)} + \epsilon \nabla u^{(1)}) \\ &- \epsilon^2 \nabla_p H(\nabla u^{(0)} + \epsilon \nabla u^{(1)})\nabla \hat{v} + \epsilon\{\nabla_p H(\nabla \hat{u}^\epsilon)\nabla^2 \hat{u}^\epsilon \nabla_p H(\nabla \hat{u}^\epsilon) \\ &- \nabla_p H(\nabla u^{(0)} + \epsilon \nabla u^{(1)})\nabla^2(u^{(0)} + \epsilon u^{(1)})\nabla_p H(\nabla u^{(0)} + \epsilon\nabla u^{(1)})\} \\ &+ \{H(\nabla u^{(0)} + \epsilon \nabla u^{(1)}) - H(\nabla u^{(0)}) - \epsilon \nabla_p H(\nabla u^{(0)})\nabla u^{(1)}\} \\ &+ \epsilon\{\nabla_p H(\nabla u^{(0)} + \epsilon\nabla u^{(1)})\nabla^2(u^{(0)} + \epsilon u^{(1)})\nabla_p H(\nabla u^{(0)} + \epsilon \nabla u^{(1)}) \\ &- \nabla_p H(\nabla u^{(0)})\nabla u^{(0)} \nabla_p H(\nabla u^{(0)})\}\,.\end{aligned} \tag{3.50}$$

It is not hard to show, by energy estimates, that (3.49) has a unique smooth solution $\hat{v}(x,t)$ such that

$$\|\hat{v}(\cdot,\cdot)\|_{L^\infty(\mathbf{R}^2\times[0,T])} \le C \tag{3.51}$$

Hence the desired error estimate, (3.24), follows form (3.25), (3.33), (3.38), (3.48), and (3.51). Consequently, the proof of Theorem 3.3 is completed.

3.4 Zero relaxation limit

In this section we justify the zero relaxation limit of the system (3.9) to the inviscid Hamilton-Jacobi equation

$$v_t + \sqrt{1 + v_x^2} = 0\,, \quad v(x,0) = v_0(x)\,, \quad (x,t) \in \mathbf{R} \times (0,T)\,. \tag{3.52}$$

Such a limit can be obtained using the techniques developed in [Nat]. First notice that the characteristic of (3.52) is $p/\sqrt{1+p^2}$. With the choice of $a = 1$, the subcharacteristic condition (2.5) is satisfied when the front remains a graph; i.e.,

$$1 - \left(\frac{p}{\sqrt{1+p^2}}\right)^2 = \frac{1}{1+p^2} > 0\,,$$

if $p \in L^\infty(\mathbf{R} \times [0,T])$. This is true provided that the initial data $p_0^\epsilon, w_0^\epsilon$ of (3.9bc) are uniformly bounded in $L^\infty(\mathbf{R})$, for all $\epsilon > 0$ (see [Nat]). Similarly we can obtain the rate of convergence following [KT]. We summarize these results below under the following assumptions on the initial data $(u_0^\epsilon, p_0^\epsilon, w_0^\epsilon)$ of (3.9) and v_0 of (3.52):

(A1) $u_0^\epsilon, v_0 \in W^{1,1}(\mathbf{R}) \cap W^{1,\infty}(\mathbf{R})$, $p_0^\epsilon = \partial_x u_0^\epsilon$, $w_0^\epsilon \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$, and $\|u_0^\epsilon\|_{1,1} \leq C < \infty$, $\|u_0^\epsilon\|_{1,\infty} \leq C < \infty$ $\|w_0^\epsilon\|_1 \leq C < \infty$, $\|w_0^\epsilon\|_\infty \leq C < \infty$.

(A2) If $p_0^\epsilon = \partial_x u_0^\epsilon$, $p_0 = \partial_x v_0$, assume that $\|p_0^\epsilon - p_0\|_1 = o(1)$ and $\|w_0^\epsilon - \sqrt{1+(p_0^\epsilon)^2}\|_1 = O(\epsilon)$.

(A3) For all $\delta > 0$, there exists an $M > 0$ such that

$$\int_{\mathbf{R}\setminus[-M,M]} |p_0^\epsilon| + |w_0^\epsilon|\, dx \leq \delta \qquad \text{for all} \quad \epsilon > 0\,.$$

Theorem 3.5. *Assume that* $(u_0^\epsilon, p_0^\epsilon, w_0^\epsilon)\,, v_0$ *satisfy* (A1)-(A3). *Then,*
(i) The solution u^ϵ *of* (3.9) *converges locally uniformly in* $\mathbf{R} \times [0,T]$ *to the unique viscosity solution of* (3.7) *as* $\epsilon \to 0$.
(ii) If in addition $p_0^\epsilon, w_0^\epsilon$ *are uniformly bounded in* $BV(\mathbf{R})$ *and* $p_0 = \partial_x v_0 \in BV(\mathbf{R})$, *then we have the following estimate on the rate of convergence*

$$\|u^\epsilon - v\|_{L^\infty(\mathbf{R}\times[0,T])} \leq \|p_0^\epsilon - p_0\|_{L^1(\mathbf{R})} + \|u_0^\epsilon - v_0\|_{L^\infty(\mathbf{R})} + O(\sqrt{\epsilon})\,,$$

where $O(\sqrt{\epsilon})$ *depends on the* BV *bounds of* $p_0, p_0^\epsilon, w_0^\epsilon$ *and the terminal time* T.

Remark: The same techniques also justify the zero relaxation limit for the relaxation approximation constructed in [JX2] for general Hamilton-Jacobi equations in one space dimension under the subcharacteristic condition (2.5). Notice also that Theorem 3.5 is valid for general relaxation approximations to Hamilton-Jacobi equations and conservation laws, thus justifying to first order the formal Chapman-Enskog expansions found in [CLL], [JX1] and also (2.6), in the case of one space dimension.

4. A relaxation scheme.

The scheme we use is the relaxation scheme, introduced by Jin and Xin [JX2] for general Hamilton-Jacobi equation, based on (2.4). This is an upwind discretization for (2.4)) with dimension splitting. In the case of 2D, let $\mathbf{p} = (p, q)$, then (2.4)) reads

$$\partial_t p + c\partial_x w = 0\,, \tag{4.1a}$$

$$\partial_t q + c\partial_y w = 0\,, \tag{4.1b}$$

$$\partial_t w + a\partial_x p + a\partial_y q = -\tfrac{c}{\epsilon}(w - H(p,q))\,, \tag{4.1c}$$

$$\partial_t u + cw = 0\,. \tag{4.1d}$$

Since (4.1a-c) is a semi-linear system of hyperbolic equations, one can apply the upwind scheme dimension-by-dimension easily. Integrating over the grid cell $c_{ij} = [x_{i-1/2}, x_{i+1/2}] \times [y_{j-1/2}, y_{j+1/2}]$, one can define the cell average for a generic variable v over c_{ij} by

$$v_{ij} = \frac{1}{\Delta x \Delta y} \int_{x_{i-1/2}}^{x_{i+1/2}} \int_{y_{j-1/2}}^{y_{j+1/2}} v(x,y)\, dxdy\,. \tag{4.2}$$

Integrating (2.4)a-c) over c_{ij}, we have

$$\partial_t p_{ij} + \frac{c}{\Delta x}(w_{i+1/2,j} - w_{i-1/2,j}) = 0\,, \tag{4.3a}$$

$$\partial_t q_{ij} + \frac{c}{\Delta y}(w_{i,j+1/2} - w_{i,j-1/2}) = 0\,, \tag{4.3b}$$

$$\partial_t w_{ij} + a\frac{1}{\Delta x}(p_{i+1/2,j} - p_{i-1/2,j}) + a\frac{1}{\Delta y}(q_{i,j+1/2} - q_{i,j-1/2}) = -\tfrac{c}{\epsilon}(w_{ij} - H(p_{ij}, q_{ij})) \tag{4.3c}$$

$$\partial_t u_{ij} + c\tfrac{1}{4}(w_{i+1/2,j} + w_{i-1/2,j} + w_{i,j+1/2} + w_{i,j-1/2}) = 0 \tag{4.3d}$$

Special attention needs to be paid to (4.3d). As understood in [JX2] via a traveling wave analysis, if one naively uses $\partial_t u_{i,j} + w_{ij} = 0$ to approximate (4.1d) one does not get the correct viscosity solution for the inviscid Hamilton-Jacobi equation.

To define the numerical fluxes the second order upwind scheme (with slope limiter [vL]) maybe used dimension-by-dimension. In x-direction the Riemann invariants are $p \pm \frac{1}{\sqrt{a}}\, w$, which move with characteristic speeds $\pm\sqrt{a}$, thus the second order upwind discretization to $p \pm \frac{1}{\sqrt{a}}\, w$ is

$$\begin{aligned} (p + \tfrac{1}{\sqrt{a}}\, w)_{i+1/2,j} &= (p + \tfrac{1}{\sqrt{a}}\, w)_{ij} + \tfrac{1}{2} h_j^x \sigma_{ij}^{x,+}\ , \\ (p - \tfrac{1}{\sqrt{a}}\, w)_{i+1/2,j} &= (p - \tfrac{1}{\sqrt{a}}\, w)_{i+1,j} - \tfrac{1}{2} h_{j+1}^x \sigma_{i+1,j}^{x,-}\ , \end{aligned} \tag{4.4}$$

where $\sigma_{ij}^{\pm}$ are the limited slopes of $p \pm \frac{1}{\sqrt{a}}\, w$ defined in (4.7), and h_i^x is the mesh size in the x-direction. This gives

$$\begin{aligned} p_{i+1/2,j} &= \tfrac{1}{2}(p_{ij} + p_{i+1,j}) - \frac{1}{2\sqrt{a}}\,(w_{i+1,j} - w_{ij}) \\ &\quad + \frac{1}{4}\,(h_i^x \sigma_{ij}^{x,+} - h_{i+1}^x \sigma_{i+1,j}^{x,-})\,, \\ w_{i+1/2,j} &= \tfrac{1}{2}(w_{ij} + w_{i+1,j}) - \frac{\sqrt{a}}{2}\,(p_{i+1,j} - p_{ij}) \\ &\quad + \tfrac{\sqrt{a}}{4}\,(h_i^x \sigma_{ij}^{x,+} + h_{i+1}^x \sigma_{i+1,j}^{x,-})\,. \end{aligned} \tag{4.5}$$

Similarly in the y-direction we have the following numerical fluxes:

$$\begin{aligned} q_{i+1/2,j} &= \tfrac{1}{2}(q_{ij} + q_{i,j+1}) - \frac{1}{2\sqrt{a}}\,(w_{i,j+1} - w_{ij}) \\ &\quad + \frac{1}{4}\,(h^y_j \sigma^{y,+}_{ij} - h^y_{j+1}\sigma^{y,-}_{i,j+1})\,, \\ w_{i,j+1/2} &= \tfrac{1}{2}(w_{ij} + w_{i,j+1}) - \frac{\sqrt{a}}{2}\,(q_{i,j+1} - q_{ij}) \\ &\quad + \tfrac{\sqrt{a}}{4}\,(h^y_j \sigma^{y,+}_{ij} + h^y_{j+1}\sigma^{y,-}_{i,j+1})\,, \end{aligned} \tag{4.6}$$

where h^y_j is the mesh size in y-direction. The limited slopes are defined as

$$\begin{aligned} \sigma^{x,\pm}_{ij} &= \frac{1}{h^x_i}\left(p_{i+1,j} \pm \frac{1}{\sqrt{a}}\,w_{i+1,j} - p_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}\right)\,\phi(\theta^{x,\pm}_{ij})\,, \\ \theta^{x,\pm}_{ij} &= \frac{p_{ij} \pm \frac{1}{\sqrt{a}}\,w_{ij} - p_{i-1,j} \mp \frac{1}{\sqrt{a}}\,w_{i-1,j}}{p_{i+1,j} \pm \frac{1}{\sqrt{a}}\,w_{i+1,j} - p_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}}\,; \\ \sigma^{y,\pm}_{ij} &= \frac{1}{h^y_j}\left(q_{i,j+1} \pm \frac{1}{\sqrt{a}}\,w_{i,j+1} - q_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}\right)\,\phi(\theta^{y,\pm}_{ij})\,, \\ \theta^{y,\pm}_{ij} &= \frac{q_{ij} \pm \frac{1}{\sqrt{a}}\,w_{ij} - q_{i,j-1} \mp \frac{1}{\sqrt{a}}\,w_{i,j-1}}{q_{i,j+1} \pm \frac{1}{\sqrt{a}}\,w_{i,j+1} - q_{ij} \mp \frac{1}{\sqrt{a}}\,w_{ij}}\,, \end{aligned} \tag{4.7}$$

with (see [Swe])

$$\phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}\,. \tag{4.8}$$

For the time discretization, denoting (4.3a-c) by

$$\partial_t U + A(U) = B(U) \tag{4.9}$$

where $A(U)$ stands for the convection terms, and $B(U)$ for the relaxation term. When ϵ is not too small, and one wants to study the influence of the curvature term, one has to numerically resolve the small relaxation term by choosing Δt and Δx sufficiently small (compared to ϵ), then one can just use the classical Euler or Runge-Kutta method to solve (4.9). In case one is interested only in the leading term, or the viscosity solution to the Hamilton-Jacobi equation, one can just use a second order (for $\epsilon \ll \Delta t$) Runge-Kutta type splitting [Jin]:

$$U^* = U^n + B(U^*)\;; \tag{4.10a}$$

$$U^{(1)} = U^* - kA(U^*)\;, \tag{4.10b}$$

$$U^{**} = U^{(1)} + B(U^{**})\;; \tag{4.10c}$$

$$U^{(2)} = U^{**} - kA(U^{**})\;, \tag{4.10d}$$

$$U^{n+1} = \tfrac{1}{2}\,(U^n + U^{(2)})\;. \tag{4.10e}$$

Although we use implicit relaxation term, since the relaxation term is not fully-ranked, in (4.10a) and (4.10c) one can always solve for p and q first, using which to update $H(p,q)$, then solve for the unknown w which is linear in both equations. Thus (4.10) can be implemented explicitly. Hence whether to use (4.10) depends on if one wants to resolve the order ϵ behavior. In this paper, we are interested in the $O(\epsilon)$ dependence in curvature, thus we need to resolve ϵ both spatially and temporally, and the second order explicit Runge-Kutta method will be used in Section 5.

5. Numerical Results.

In this section we give four numerical examples to demonstrate the feasibility of the relaxation schemes as a tool to compute convection dominated front propagation by mean curvature. We will compare with the level set method, and different ϵ's. When we say inward motion we take $H(\nabla u) = -|\nabla u|$, while the outward motion corresponds to $H(\nabla u) = |\nabla u|$. In all examples we use $a = c = 1$, with 100 grid points between -1.5 and 1.5 in both x and y direction, and take $\Delta t = 0.01$ for $\epsilon = 0$ and 0.0025 for other ϵ's.

Example 1. We begin with a unit ball that moves inward with $\epsilon = 0.1$ and 0.05. The initial level set function is given by $u(x,y) = x^2 + y^2 - 1$. The results by the relaxation scheme and the level set method are depicted in Fig. 5.1. Both methods give similar results. For $\epsilon = 0.1$, the relaxation scheme gives a slightly slower speed when the radius gets small. Such a difference becomes inviscible with smaller ϵ. An analytical justification of these observations in three dimensions is provided in Section 3.3.

Example 2. We begin with a square of side length $\sqrt{2}$. The initial level set function is $u(x,y) = |x| + |y| - 1$, while we define $p = sign x$ and $q = sign y$. We use 100 grid points between -1.5 and 1.5 in both x and y direction. First we compare the inward motion with $\epsilon = 0.1, 0.05$ and 0. In all cases both the relaxation scheme and the level set method give almost the same results, as shown in Fig. 5.2a. We then compute the outward motion and display the results in Fig. 5.2b. For $\epsilon = 0.1$ the outward motion near the four corners shows some difference, indicating that the asymptotic expansion is not valid near the corners for this rarefaction case. However such a discrepancy disappears as one reduces ϵ. The results by the relaxation scheme and the level set method are depicted in Fig. 5.1. Both methods give similar results. For $\epsilon = 0.1$, the relaxation scheme gives a slightly slower speed when the radius gets small. Such a difference becomes invisible with smaller ϵ.

Example 3. The initial boundary is a four-leaf rose described by $u(x,y) = (x^2 + y^2)^{3/2} - 2xy$ for $xy > 0$ and $u(x,y) = (x^2+y^2)^{3/2}+2xy$ for $xy < 0$. We compute both inward and outward motion for $\epsilon = 0.05$. For the inward case the pinch-off time near the center of the rose is different, and after the pinch-off both methods give essentially the same result. In the outward case there is slight difference near the four spikes but overall speaking the solutions are very good approximation of each other.

Remark: When the curvature is very large, or becomes singular, the difference between the relaxation scheme and the level set method can be observed numerically. This simply means that when the curvature is not $O(1)$, the Chapman-Enskog expansion breaks down and the relaxation solution thus deviates from the level set solution and, higher order terms in the expansion become significant. This requires a physical definition of the nature of the singularity, thus is out of the scope of this paper.

6. Conclusions.

In this paper we presented the relaxation model (2.8) and relaxation schemes (4.3) for evolving interfaces with curvature-dependent local laws. The interfaces are calculated as the level sets of the solution u of (2.8c) and are related to a standard level set method through the Chapman-Enskog expansion. The relaxation schemes (4.3) do not compute directly the curvature term, so we avoid the singularities in the curvature term whenever the gradient of the solution of (2.9) is zero. As a result, we do not need to adjust our grid or to adapt our algorithm in order to circumvent such difficulties. We have employed both rigorous justification and numerical experiments to validate such an approach.

References:

[AC] **Allen S.M. and Cahn J.W., 1979.** A macroscopic theory for antiphase boundary motion and its application to antiphase domain coarsening, *Acta Metal.* 27, pp. 1085–1095.

[AG] **Angenent S.B. and Gurtin M.E., 1989.** Multiphase thermomechanics with interfacial structure, 2. Evolution of an isothermal interface, *Arch. Rat. Mech. Anal.* 108, pp. 323-391.

[ADN] **Aregba-Driollet D. and Natalini R., 1996.** Convergence of relaxation schemes for conservation laws , *Applic. Anal.*, to appear.

[BSS] **Barles G., Soner H. M. and Souganidis P.E., 1993.** Front Propagation and Phase Field Theory, *SIAM J. Cont. Opt.* 31, pp. 439–469.

[Caf] **Caflisch R.E., 1979.** Navier-Stokes and Boltzmann shock profiles for a model of gas dynamics, *Comm. Pure Appl. Math.* 22, pp.521-554.

[CLL] **Chen G.-Q., Levermore C.D. and Liu T.-P., 1994.** Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure Appl. Math.*, 47, pp. 789-830.

[CGG] **Chen Y.-G., Giga Y., and Goto S., 1991.** Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations, *J. Diff. Geom.* 33, pp. 749–786.

[Cho] **Chopp D. L. 1993.** Computing minimal surfaces via level set curvature flow, *J. Comp. Phys.* 106, pp. 77-91.

[CP] **Coquel F. and Perthame B., 1997.** Relaxation of energy and approximate Riemann solvers for general pressure laws in fluid dynamics, preprint.

[CR] **J.F. Collet and M. Rascle, 1996.** Convergence of the Relaxation to a Scalar Nonlinear Hyperbolic Equation in on Space Dimension, *Z. Angew. Math. Phys.* 47, 400-409.

[CEL] **Crandall M., Evans L.C. and Lions P.L., 1984** Some properties of viscosity solutions to Hamilton-Jacobi equations, *Trans. AMS*, 282, pp. 487-502.

[CIL] **Crandall M.G., Ishii I. and Lions P.L., 1992.** User's guide to viscosity solutions of second order partial differential equations, *Bul. AMS*, 27, pp. 1-67.

[CL] **Crandall M.G. and Lions P.L., 1983.** Viscosity solutions of Hamilton-Jacobi equations, *Trans. AMS*, 277, pp. 1-42.

[Ell] **Elliot, C.M., 1996.** Approximations of Curvature Dependent Interface Motion, preprint.

[ESS] **Evans L.C., Soner H. M. and Souganidis P.E., 1992.** Phase transitions and generalized motion by mean curvature, *Comm. Pure and Appl. Math.*, 35, pp. 1097–1123.

[ESp] **Evans L.C., and Spruck J., 1991.** Motion of level sets by mean curvature I, *J. Diff. Geom.* 33, pp. 635–681.

[Jin] **Jin S., 1995** Runge-Kutta methods for hyperbolic conservation laws with stiff relaxation terms, *J. Comp. Phys.* 122, pp. 51-67.

[JK] **Jin S. and Katsoulakis M., 1997.** Relaxation approximations to front propagation, *J. Diff. Eqn.*, 138, pp. 380-387.

[JX1] **Jin S. and Xin Z.P., 1995.** The relaxation schemes for systems of conservation laws in arbitrary space dimensions, *Comm. Pure Appl. Math.*, 48, pp. 235-277.

[JX2] **Jin S. and Xin Z.P., 1996.** Numerical Passages from Systems of Conservation Laws to Hamilton-Jacobi Equations, *SIAM J. Num. Anal.*, accepted.

[KKM] **Katsoulakis M.A., Kossioris G. and Makridakis Ch., 1997.** Convergence and error estimates of relaxation schemes for multidimensional conservation laws, submitted.

[KT] **Katsoulakis M.A. and Tzavaras A.E., 1997.** Contractive relaxation systems and the scalar multidimensional conservation law, *Comm. PDE*, 22 pp. 195-233.

[Kr1] **Kruzkov S.N., 1975.** Generalized solutions to the Hamilton-Jacobi equations of Eikonal type I, *Math. USSR-Sb.*, 27, pp. 406-466.

[Kr2] **Kruzkov S.N., 1970.** First-order Quasilinear equations with several space variables, *Math. USSR Sb.*, 10, pp. 217-273.

[Lio] **Lions P.L., 1982.** Generalized Solutions of Hamilton-Jacobi Equations, Research Notes in Mathematics 69, Pitman Advanced Publishing Program.

[LWY] **Liu H., Wang J. and Yang T., 1996.** Nonlinear stability and existence of stationary discrete traveling waves for the relaxing schemes, preprint.

[Nat] **Natalini R., 1996.** A discrete kinetic approximation of entropy solutions to multidimensional scalar conservation laws, preprint.

[Nat1] **Natalini R., 1996.** Convergence to equilibrium for the relaxation approximations of conservation laws, *Comm. Pure Appl. Math.* 49, pp. 795-823.

[NPV] **Nochetto R. H., Paolini M. and Verdi C., 1997.** A dynamic mesh algorithm for curvature dependent evolving interfaces, *J. Comp. Phys.*, to appear.

[OJK] **Ohta T., Jasnow D. and Kawasaki K., 1982.** Universal scaling in the motion of random interfaces, *Phys. Rev. Lett.* 49, pp. 1223–1226.

[OS] **Osher S. and Sethian J.A., 1988.** Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations, *J. Comp. Phys.*, 78, pp. 12-49.

[Set] **Sethian J.A., 1996.** Level Set Methods, Cambridge University Press.

[Swe] **P.R. Sweby, 1984.** High resolution schemes using flux limiters for hyperbolic conservation laws, *SIAM J. Num. Anal.* 21, pp. 995-1101.

[TW] **H.Z. Tang and H.M. Wu, 1998.** Analysis of the relaxing schemes for scalar conservation laws, preprint.

[Ten] **Teng, Z.-H., 1997.** First-order L^1-convergence for the relaxation approximation to conservation laws, preprint.

[vL] **B. van Leer, 1979.** Towards the ultimate conservative difference schemes V. A second-order sequel to Godunov's method, *J. Comp. Phys.* 32, pp. 101-136.

[WX] **Wang W. C., Xin Z. P., 1998.** On the asymptotic limit of the initial boundary value problem of 1-d scalar conservation law with relaxation extension, *Comm. Pure Appl. Math.*, 51, pp. 0505-0535.

[Whi] **Whitham G.B.., 1974** Linear and Nonlinear Waves, Wiley, New York.

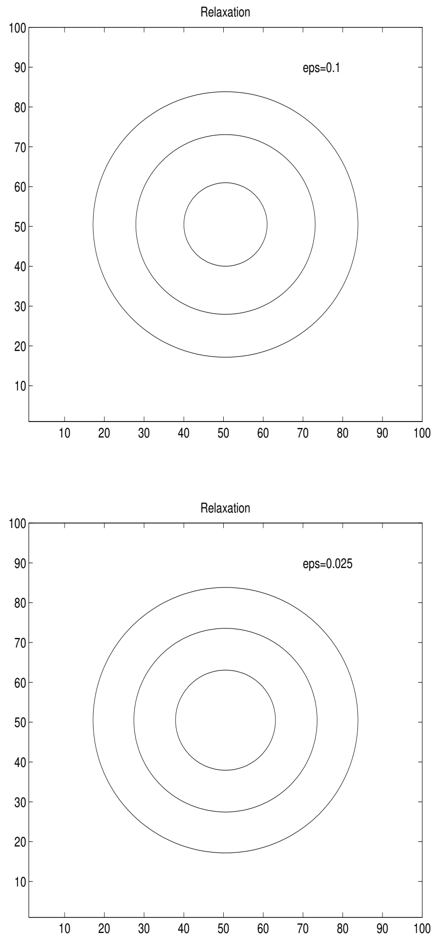

Fig.5.1. A shrinking circle by relaxation method (left) and the level set method (right). Top: $\epsilon = 0.1$. Bottom: $\epsilon = 0.025$. Output time $t = 0, 0.3, 0.6$.

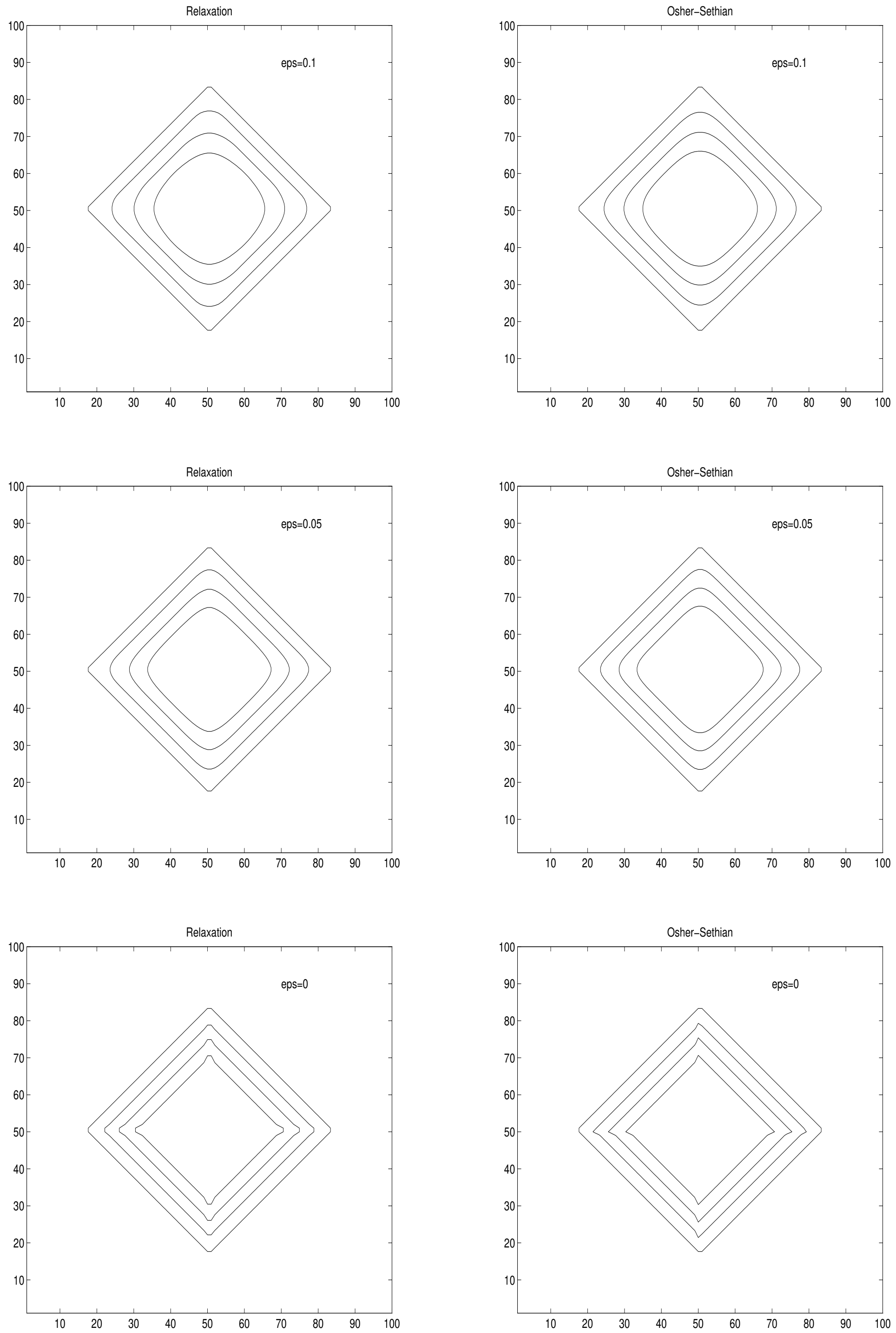

Fig.5.2a. A shrinking square by relaxation method (left) and the level set method (right). Top: $\epsilon = 0.1$. Center: $\epsilon = 0.05$. Bottom: $eps = 0$. Output time $t = 0, 0.1, 0.2, 0.3$.

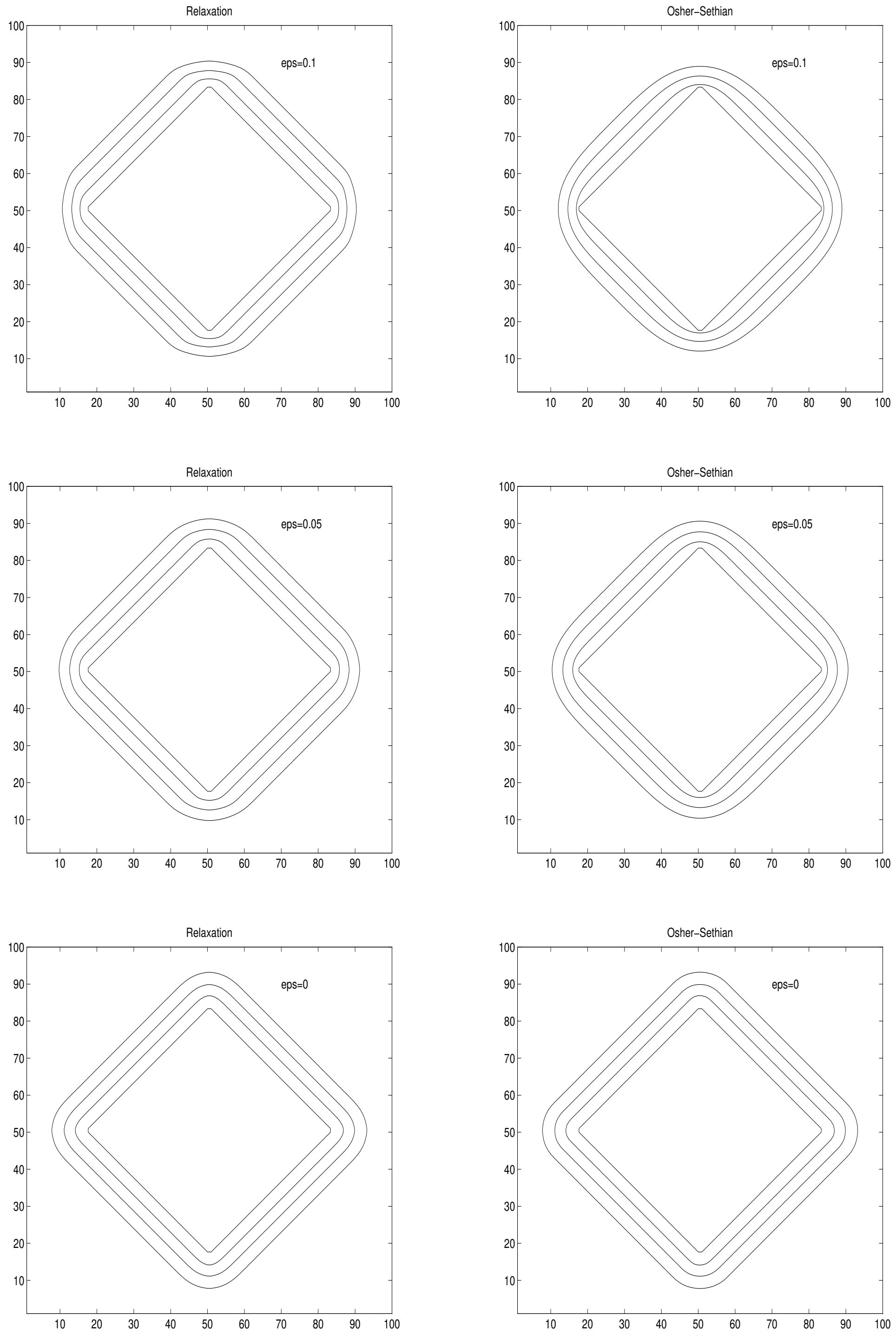

Fig.5.2b. An expanding square by relaxation method (left) and the level set method (right). Top: $\epsilon = 0.1$. Center: $\epsilon = 0.05$. Bottom: $eps = 0$. Output time $t = 0, 0.1, 0.2, 0.3$.

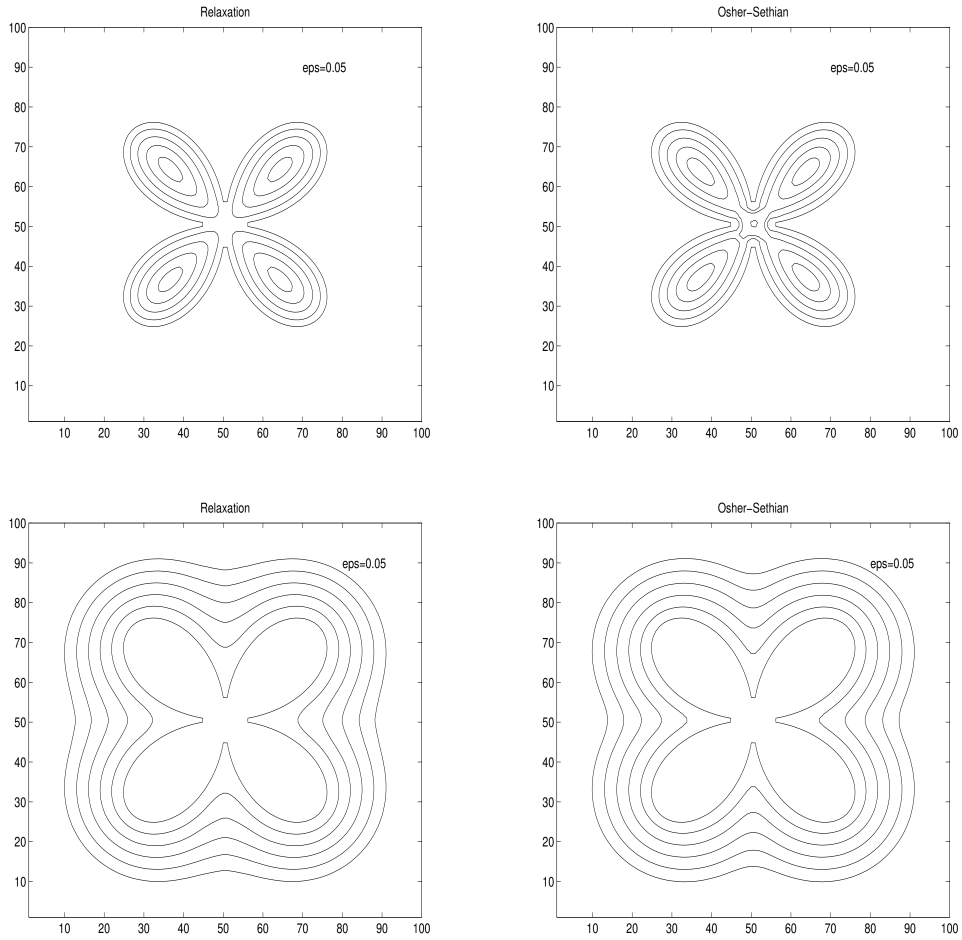

Fig.5.3. Shrinking (top) and expanding (bottom) four-leaf roses for $\epsilon = 0.05$. Left: relaxation. Right: level set. Output time $t = 0, 0.05, 0.1, 0.15, 0.2$ for the top figures and $t = 0, 0.1, 0.2, 0.3, 0.4, 0.5$ for the bottom figures.