Stability of Viscous Shock Waves Associated with Non-Convex Modes

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1 Introduction

In this paper we prove the time asymptotic stability of viscous shock profiles under perturbations of non-vanishing mass, where the profile is associated with a non-convex mode (i.e. with a not necessarily monotone eigenvalue λ). We thus extend the result of [SX] to the case of non-convex modes and the result of [F1] to the case of non-zero mass perturbations.

We consider a system of viscous hyperbolic conservation laws

$$u_t + f(u)_x = \mu u_{xx}. \tag{1}$$

Let u_* be a fixed reference state and let $\phi : \mathbb{R} \to \mathbb{R}^n$ denote the profile of a Laxian shock wave associated with a simple eigenvalue λ of f', i.e., $\phi(x - st)$ solves (1) and thus

$$\mu\phi' = h(\phi) := f(\phi) - s\phi - q \quad , \quad \phi(\pm\infty) = u_\pm \ ,$$

where u_-, u_+, s, q satisfy $f(u_-) - su_- = f(u_+) - su_+ = q$, $\lambda(u_-) > s > \lambda(u_+)$ and u_-, u_+ are close to u_*.

We will prove the following

Theorem 1
There is a positive constant ϵ_0 such that if u_-, u_+ satisfy $|u_\pm - u_*| < \epsilon_0$, then there exists $\beta_0 > 0$ (depending on f, u_-, u_+ and μ) such that whenever the perturbation $u_0 - \phi \in \mathrm{H}^1(\mathbb{R})$ satisfies

$$\int_{-\infty}^{\infty} |u_0(x) - \phi(x - \delta_0)| \ dx \ + \int_{-\infty}^{\infty} (1 + (x - \delta_0)^2)|u_0(x) - \phi(x - \delta_0)|^2 \ dx \ \leq \ \beta_0$$

for some $\delta_0 \in \mathbb{R}$, then the solution $u(x,t)$ to (1) with data $u(\cdot, 0) = u_0$ exists for all times $t > 0$ and has

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \phi(x - st - \delta)| = 0 \tag{2}$$

with a uniquely determined $\delta \in \mathbb{R}$.

Note that situations to which this theorem applies are numerous. A typical case is characterized by the presence of a hypersurface $\mathcal{H} \subset \mathbb{R}^n$ at whose points $r\nabla\lambda = 0$ and $(r\nabla)^2\lambda \neq 0$ (where $\mathbb{R}r = \ker(f' - \lambda I)$), i.e., non-degenerate inflection ("cubic" behavior)! This situation occurs frequently in applications, examples provided by elasticity, nonlinear electromagnetism etc. Particular interest in such situations refers to shock waves for which u_- and u_+ are located on different sides of $\mathcal{H}$ [L1, MP].

The present article continues the paper [F1], where Theorem 1 was proved under the additional assumption that $\int_{-\infty}^{\infty} u_0(x) - \phi(x - \delta_0)dx = 0$, i.e., without diffusion waves. Our proof follows [SX]. At a first level, we combine their method with the ideas introduced in [F1, F2]. However, in the case of non-convex modes *and* non-zero mass perturbation new difficulties arise in the energy estimates (see Section 5) as well as in particular in the pointwise estimates of the so called coupled diffusion wave η (see Section 6), which are derived using (approximate) Green's functions ψ (*dual waves*).

Our proof of the required pointwise estimates for η differs from the proof available in [SX] for the convex case. In the convex case, dual waves in the p-family, going backward in time, expand uniformly, allowing for easier vertical L_1 estimates.

We overcome the difficulty in the general non-convex case by introducing a shape function g that essentially captures the dependence of the "vertical mass" $\int_{-\infty}^{T-1} |\psi(x,t)|dt$ on x (see Lemma 6). The function g, satisfying the simple relation $-(\mu g_x + \lambda_p(\phi)g) = \mathrm{sign}(x - x_g)$ for some $x_g \in \mathbb{R}$, describes how a possible compression in the case of a non-convex flux (i.e. $\exists\ a < b$ s.t. $\lambda_p(\phi(a)) < \lambda_p(\phi(b))$) is compensated by the diffusion introduced by the viscosity. This is in analogy with the role played by the principal weight ω, satisfying $-(\mu\omega_x + \lambda_p(\phi)\omega) = 2\int_{x_0}^{x} |\phi_x(\xi)|d\xi$, used in the energy estimate for the integrated error wave.

For a description of related work prior to the above mentioned, we refer the reader to [F1], and restrict ourselves here to just mentioning [G1, KM, MN1, MN2, N].

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2 Decomposition of u and the *integrated equation*

2.1 Diagonalization

As f' is $\mathbb{R}$-diagonalizable in a neighborhood of the reference state u_* comprising the values of ϕ, we find smooth matrix-valued functions $L = (l_i)$, $R = (r_j)$ (l_i row, r_j column) on this neighborhood with

$$L(u)h'(u)R(u) = \Lambda(u) = \mathrm{diag}(\lambda_1(u), \ldots, \lambda_n(u)) \ ; \quad L(u)R(u) = I.$$

For the rest of this paper, we assume for simplicity in presentation that $h' = f' - s\mathrm{I}$ has only *simple* eigenvalues, $\lambda_1 < \ldots < \lambda_n$, λ_p is identical with $\lambda - s$.

From now we consider u as a function of $(x - st, t)$ and simply write (x, t) again. I.e. u solves

$$u_t + h(u)_x = \mu u_{xx}. \tag{3}$$

For a given u_0, let δ, $\theta_{i,0} \in \mathbb{R}$ be defined by

$$\int_{-\infty}^{\infty} u_0(x) - \phi(x)\ dx\ =:\ \delta(u_+ - u_-) + \sum_{i\neq p} \theta_{i,0} r_i(u_{i,0})$$

where $u_{i,0} := u_{\mathrm{sign}(i-p)} \in \{u_-, u_+\}$.

In the following we will decompose the solution u as $u(x+\delta,t) = \phi(x) + \theta(x,t) + \eta(x,t) + w(x,t)$, where ϕ is the shock profile, θ and η are diffusion waves (θ carries the mass $\sum \theta_{i,0} r_i(u_{i,0})$) and w is an error term. In Section 3 we will discuss properties of ϕ, θ and η. In Section 5 we will estimate w.

2.2 The decoupled diffusion wave θ

For fixed $x_0 \in \mathbb{R}$ (which we will choose later) we define the decoupled diffusion wave $\theta := \sum\limits_{i\neq p} \theta_i r_i(u_{i,0})$ by

$$\theta_i(x,t) = \frac{\theta_{i,0}}{\sqrt{1+t}} v_i\left(\frac{(x-x_0) - \lambda_i(u_{i_0})(1+t)}{2\sqrt{1+t}}\right) \tag{4}$$

as a solution of

$$\begin{aligned} \theta_{i,t} + (\lambda_i(u_{i_0})\theta_i + \frac{1}{2}\nabla\lambda_i(u_{i_0})\cdot r_i(u_{i,0})\theta_i^2)_x - \mu\theta_{i,xx} &= 0 \quad ,t>0 \\ \int_{-\infty}^{+\infty} \theta_i(x,t)dx &= \theta_{i,0} \end{aligned} \tag{5}$$

Furthermore let $\theta_0 := \sum\limits_{i\neq p} |\theta_{i,0}|$.

2.3 The coupled linear diffusion wave η

The diffusion wave θ describes the transport of the mass $\sum_{i\neq p} \theta_{i,0} r_i(u_{i,0})$ only time-asymptotically correct (note that $r_i(u_{i,0})$ and $\lambda_i(u_{i,0})$ represent the amplitude direction and speed respectively at time $t=\infty$). To apply an energy estimate to the error term we need to represent the effects resulting from the coupling of diffusion waves and define the *coupled linear diffusion wave* $\eta : \mathbb{R}\times[0,\infty) \mapsto \mathbb{R}^n$ by

$$\eta_t + (h'(\phi)\eta)_x + \sum_{i\neq p} (h''(\phi)[\theta_i r_i(u_{i,0}), \tilde{V}_i r_i(u_{i,0})])_x - \mu\eta_{xx} = E_{1,x} \tag{6}$$

$$\eta(x,0) \equiv 0 \tag{7}$$

where

$$\begin{aligned} (\tilde{V}_1, \ldots, \tilde{V}_n)^{\mathsf{T}} &:= \left(L(\phi)\int_{-\infty}^{x} \eta(\xi,t)\ d\xi\right)_x \\ E_1 &:= -\frac{1}{2}\sum_{i\neq j, j\neq p} l_i(\phi)\cdot h''(\phi)[\theta_j r_j(u_{j,0}), \theta_j r_j(u_{j,0})] r_i(\phi) \end{aligned} \tag{8}$$

Remark on the notation: By $\mathcal{O}(1)$ we denote a suitable large positive constant that depends only on the reference state u_*, f and μ, but is independent of u_-, u_+ and θ_0 as long as $\epsilon := \max\{|u_+ - u_*|, |u_- - u_*|\}$, $\theta_0 < 1$. With $C(\epsilon)$ we will denote a suitable large constant that may depend on u_- and u_+, but does not depend on θ_0. E.g. after numerous estimates we will arrive at expressions like $(\epsilon + \theta_0 C(\epsilon))\mathcal{O}(1)$ which can be made small by choosing ϵ and (depending on ϵ) θ_0 sufficiently small.

2.4 The integrated equation

Our goal is $\lim\limits_{t\mapsto\infty} \sup\limits_{x\in\mathbb{R}} |w(x,t)| = 0$, which together with decay properties of θ and η implies (2). We will prove this using a weighted energy method.

Subtracting $(\phi+\theta+\eta)_t + (h(\phi+\theta+\eta))_x - \mu(\phi+\theta+\eta)_{xx}$ from $u_t + (h(u))_x - \mu u_{xx}$ and decomposing $h(\phi+\theta+\eta) - h(\phi)$ we get by the definition of the θ and η wave that

$$w_t + (h(\phi+\theta+\eta+w) - h(\phi+\theta+\eta))_x - \mu w_{xx} = -E_x$$

where

$$E = \sum_{i\neq j,p} h''(\phi)[\theta_i r_i(u_{i,0}), \tilde{V}_j r_j(u_{j,0})] + h''(\phi)[\theta, R_x d] + E_2 \quad \text{and} \tag{9}$$

$$|E_2| \leq \mathcal{O}(1)\left(|\theta|^3 + |\eta|^2 + |\eta|^3 + \sum_{i\neq p} |(\phi - u_{i,0})\theta_i r_i(u_{i,0})| + \sum_{j\neq k} |\theta_j \theta_k|\right) \tag{10}$$

(the term E_2 results from the decomposition of the flux – see [SX, F3]). Introducing the integrated variable

$$U(x,t) := \int_{-\infty}^{x} w(\xi,t)\; d\xi$$

we obtain the *integrated equation*

$$U_t + h'(\phi+\theta)U_x - \mu U_{xx} = -E - Q \tag{11}$$

where

$$Q := \int_0^1 h'(\phi+\theta+\eta+\tau U_x) - h'(\phi+\theta) d\tau U_x \tag{12}$$

The integrated equation (11) will be estimated in Section 5 using a weighted energy estimate similar to that in [F1].

3 Properties of the shock and diffusion waves

Lemma 1 (Properties of the shock wave ϕ)
For the shock profile ϕ defined above we have:

$$
\begin{aligned}
|\phi_x| &\leq \mathcal{O}(1)\epsilon && \text{(13a)}\\
|\mu\phi_{xx} - \lambda_p(\phi)\phi_x| &\leq \mathcal{O}(1)|\phi_x|^2 && \text{(13b)}\\
|\phi_{xx}| &\leq \epsilon\mathcal{O}(1)|\phi_x| && \text{(13c)}\\
|\phi_x(x)\exp(-\int_0^x \lambda_p(\xi)\ d\xi)| &\leq \mathcal{O}(1) && \text{(13d)}\\
\int_{\mathbb{R}\setminus\mathcal{P}(T)} |\phi_x(x)|\ dx &\leq \frac{C(\epsilon)}{(1+T)^N} && \text{(13e)}
\end{aligned}
$$

where $\mathcal{P}(T) := \{x\ :\ |x-x_0| \leq \frac{2N}{|\lambda_p(\mathrm{sign}(x-x_0)\infty)|}\log(1+T)\}$ is the T shock region. x_0 will be chosen later (see Lemma 4).

Proof : (13a) follows directly from $\mu\phi_x = h(\phi)$. The proof of (13b) is the same as that of Lemma 2 of [F1] and (13c) and (13d) are consequences of (13b). (13e) follows from (13d) and the definition of the shock region. □

Lemma 2 (Properties of the decoupled diffusion wave θ)
For the diffusion wave θ_i defined above we have:

$$
\begin{aligned}
\left|(\frac{\partial}{\partial x})^k\theta_i(x,t)\right| &\leq \frac{\mathcal{O}(1)\theta_0}{(1+t)^{(k+1)/2}} \qquad k=0,1,2 && \text{(14a)}\\
\left|(\frac{\partial}{\partial t} + \lambda_q(u_{q,0})\frac{\partial}{\partial x})\theta_q\right| &\leq \frac{\mathcal{O}(1)\theta_0}{(1+t)^{3/2}} && \text{(14b)}\\
\left|\int_{-\infty}^x |\theta_{q,x}(\xi,t)|d\xi\right| &\leq \frac{\mathcal{O}(1)\theta_0}{(1+t)^{(1/2)}} && \text{(14c)}\\
\left|(\frac{\partial}{\partial t} + \lambda_q(u_{q,0})\frac{\partial}{\partial x})\int_{-\infty}^x |\theta_{q,x}(\xi,t)|d\xi\right| &\leq \frac{\mathcal{O}(1)\theta_0}{(1+t)^{3/2}} && \text{(14d)}\\
|\theta_i\theta_k| &\leq \mathcal{O}(1)\theta_0^2\exp(-\mathcal{O}(1)(t+|x-x_0|)) \qquad \textit{for } k\neq i && \text{(14e)}\\
|(\phi-u_{i,0})\theta_i| &\leq \mathcal{O}(1)\theta_0\exp(-\frac{1}{C(\epsilon)}(t+|x-x_0|)) && \text{(14f)}\\
|(\phi-u_{i,0})\theta_{i,x}| &\leq \frac{\mathcal{O}(1)\theta_0}{(1+t)^{3/2}} && \text{(14g)}\\
|\theta||\phi_x| &\leq \mathcal{O}(1)\theta_0\exp(-\frac{1}{C(\epsilon)}(t+|x-x_0|)) && \text{(14h)}\\
\int_{\mathbb{R}\setminus\mathcal{K}^i(T)} |\theta_i(x,T)|\ dx &\leq \frac{\mathcal{O}(1)}{(1+T)^N} && \text{(14i)}
\end{aligned}
$$

where $\mathcal{K}^i(T) := \{x\ :\ |x-x_0-\lambda_i(u_{i,0})T| \leq N\log(1+T)\sqrt{T}\}$ is the $t=T$-section of the i diffusion wave region.

Proof : (14a) – (14h) are direct consequences of (4), (5) and Lemma 1. (14i) follows from the definition of the i diffusion wave region. $\square$

For the coupled diffusion wave η the following estimates hold:

Lemma 3 (Properties of the η wave)
Let

$$d := L(\phi)\bar{\eta} = L(\phi) \int_{-\infty} \eta(\xi, \cdot)\ d\xi, \qquad \tilde{V} := d_x = L(\phi)_x R(\phi) d + L(\phi)\eta\ .$$

If ϵ and (depending on ϵ) θ_0 are sufficiently small then we find a sufficiently large constant $K_{d\tilde{V}}$ (depending on ϵ) such that

$$|\eta(x,t)| \ \leq\ \theta_0 K_{d\tilde{V}} \frac{\log(1+t)}{\sqrt{1+t}} \tag{15a}$$

$$||\eta(\cdot,t)||_{\mathrm{L}_2} \ \leq\ \theta_0 C(\epsilon) \frac{(1+\log(1+t))^3}{\sqrt{1+t}} \tag{15b}$$

$$||\theta_j(\cdot,t)\tilde{V}_i(\cdot,t)||_{\mathrm{L}_2} \ \leq\ \theta_0 C(\epsilon) \frac{(1+\log(1+t))^{3/2}}{(1+t)^{5/4}} \qquad \text{for } j \neq i. \tag{15c}$$

We present the proof of Lemma 3 in Section 6, where we use a modified version of the method of [SX].

4 Auxiliary theorems and frame of proof

We consider the integrated equation (11). We have:

Theorem 2 (short-time existence)
Given $\beta_0^* > 0$. If $U_0 \in \mathrm{H}^2(\mathbb{R})$, $||U_0||_{\mathrm{H}^2} \leq c^*$ for some c^* and $||U_0||_{\mathrm{H}^1}$ is sufficiently small there exists $T^* = T^*(\beta_0^*, c^*) > 0$, such that (11) possesses a solution $U \in L^2([0,T^*], \mathrm{H}^2(\mathbb{R}))$ with $U(\cdot,0) = U_0$ and $||U(\cdot,t)||_{\mathrm{H}^1} \leq \beta_0^*$ for $0 \leq t \leq T^*$.

Theorem 3 (a-priori estimate for U)
If ϵ and (depending on ϵ) θ_0 are chosen sufficiently small, then there exists a constant C, such that the following holds for any solution U in Theorem 2. If $\beta := \sup_{0 \leq t \leq T^*} ||U(\cdot,t)||_{\mathrm{H}^1}$ is sufficiently small, we have for $0 < T < T^*$

$$||U(\cdot,T)||^2_{\mathrm{L}_2} + \int_0^T ||U_x(\cdot,t)||^2_{\mathrm{L}_2}\ dt \leq C(||U(\cdot,0)||^2_{\mathrm{L}_2} + \theta_0).$$

Theorem 4 (a-priori estimate for U_x and U_{xx})
Let U be a solution as in Theorem 2. If ϵ, θ_0 and $\beta := \sup\limits_{0\le t\le T^*} ||U(\cdot,t)||_{\mathrm{H}^1}$ are sufficiently small, then

$$\|U_x(\cdot,T)\|^2_{\mathrm{L}_2} + \int_{t_0}^{T} \|U_{xx}(\cdot,t)\|^2_{\mathrm{L}_2} dt \le C\Big(\|U_x(\cdot,t_0)\|^2_{\mathrm{L}_2} + \int_{t_0}^{T} \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} dt + \frac{\theta_0}{(1+t_0)^{1/4}}\Big) \tag{16}$$

$$\|U_{xx}(\cdot,T)\|^2_{\mathrm{L}_2} \le C(\|U_{xx}(\cdot,0)\|^2_{\mathrm{L}_2} + \int_0^T \|U_x(\cdot,t)\|^2_{\mathrm{L}_2} + \|U_{xx}(\cdot,t)\|^2_{\mathrm{L}_2} dt + \theta_0) \tag{17}$$

for $0 \le t_0 < T \le T^*$.

The proof of Theorem 2 is standard. The proof of Theorem 4 is the same as in [SX] (using Lemma 3). The proof of Theorem 3 fills the next section of this paper.

Proof of Theorem 1: Because of (14a) and (15a) it is sufficient to show $\lim_{t\mapsto\infty}\sup_{x\in\mathbb{R}}|w(x,t)| = 0$. As in [SX] we have $\|U_0\|^2_{\mathrm{H}^1} \le \mathcal{O}(1)(\beta_0+\beta_0^2)$. If $\|U_0\|_{\mathrm{H}^1}$ is sufficiently small we have by Theorem 2 the existence of a solution $U(x,t)$ for $0\le t\le T^*$ with $||U(\cdot,t)||_{\mathrm{H}^1} < \beta_0^*$. If ϵ, θ_0 (by choice of β_0) and β_0^* (and hence β) are sufficiently small, we have by Theorems 3 and 4 (using $t_0 = 0$ in (16)):

$$\begin{aligned} \|U(\cdot,T)\|^2_{\mathrm{H}^1} &\le C(\|U(\cdot,0)\|^2_{\mathrm{H}^1} + \theta_0), \\ \|U(\cdot,T)\|^2_{\mathrm{H}^2} &\le C(\|U(\cdot,0)\|^2_{\mathrm{H}^2} + \theta_0)\ . \end{aligned}$$

Thus, if $||U(\cdot,0)||_{\mathrm{H}^1}$ and θ_0 are sufficiently small, we can extend Theorem 2 to $T^* = \infty$. Theorems 3 and 4 then apply to $T^* = \infty$ as well and from Theorem 3 we get the existence of a sequence (t_n) with $t_n \to \infty$ and

$$||U_x(\cdot,t_n)||^2_{L^2} \to 0 \qquad , \qquad \int_{t_n}^{\infty} ||U_x(\cdot,t)||^2_{L_2}\ dt \to 0$$

From (16) (with $t_0 = t_n$) we get

$$\begin{aligned} &\lim_{t\to\infty}\sup_x |w(x,t)|^2 = \lim_{t\to\infty}\sup_x |U_x(x,t)|^2 \\ &\le \lim_{t\to\infty} ||U_{xx}(\cdot,t)||^2_{L^2}\cdot||U_x(\cdot,t)||^2_{L^2} \le \lim_{t\to\infty} C||U_x(\cdot,t)||^2_{L^2} \\ &\overset{(16)}{\le} C\lim_{t\to\infty}\Big(||U_x(\cdot,t_n)||^2_{L^2} + \int_{t_n}^{t} ||U_x(\cdot,\tau)||^2_{L^2} d\tau + \frac{\theta_0}{(1+t_n)^{1/4}}\Big) \longrightarrow 0 \end{aligned}$$

(as $t_n \to \infty$) which completes the proof. □

5 Energy estimate

5.1 Grouping of terms

We introduce characteristic variables and substitute $U =: R(\phi+\theta)V$ (i.e. $V := L(\phi+\theta)U$), multiply by $V^{\mathsf{T}}WL(\phi+\theta)$, and integrate $\int_{-\infty}^{\infty} dx$ (we drop the dependence on $\phi+\theta$ for L, R and Λ and write $L(x,t)$, $R(x,t)$ and $\Lambda(x,t)$):

$$\int_{-\infty}^{\infty} V^{\mathsf{T}}WV_t + V^{\mathsf{T}}W\Lambda V_x + V^{\mathsf{T}}W(LR_t + \Lambda LR_x)V$$
$$+ \mu(V^{\mathsf{T}}WL)_x(RV)_x dx + \int_{-\infty}^{\infty} V^{\mathsf{T}}WL(E+Q)\ dx \ = \ 0$$

Using integration by parts we obtain

$$\int_{-\infty}^{\infty} \frac{1}{2}(V^{\mathsf{T}}WV)_t - \frac{1}{2}V^{\mathsf{T}}[W_t + (W\Lambda + \mu W_x)_x]V + \mu V_x^{\mathsf{T}}WV_x$$
$$+ V^{\mathsf{T}}[W(LR_t + \Lambda LR_x) + \mu W_x LR_x]V + \mu[V_x^{\mathsf{T}}WLR_xV - V^{\mathsf{T}}WLR_xV_x] +$$
$$+ \mu V^{\mathsf{T}}WL_xR_xV + V^{\mathsf{T}}WLE + V^{\mathsf{T}}WLQ\ dx$$
$$= \int_{-\infty}^{\infty} (T1) + (T2) + (T3) + (T4) + (T5) + (T6) + (T7) + (T8)\ dx \ = \ 0$$

By appropriate choice of the weight matrix W (Section 5.2) (T2) and (T3) become positive definite. (T4) – (T8) will be estimated (Section 5.3).

5.2 Choice of weight functions

We first define a weight function ω which will handle the difficulties caused by the absence of convexity. A motivation for the specific choice of ω can be found in [F2].

Lemma 4
For ϵ sufficiently small there exists $w : \mathbb{R} \to \mathbb{R}$ with $\inf_{x\in\mathbb{R}} w(x)$, $\inf_{x\in\mathbb{R}} (1/w(x)) > 0$ such that

$$\begin{aligned}
-(\omega\lambda_p + \mu\omega_x) \ &= \ 2\int_{x_0}^{x} |\phi_x(\xi)| d\xi \qquad \text{for some } x_0 \in \mathbb{R} && \text{(18a)}\\
|\omega\lambda_p + \mu\omega_x| \ &\le \ 4|u_+ - u_-| && \text{(18b)}\\
|\mu(\omega\phi_x)_x| \ &\le \ 4|u_+ - u_-| \cdot |\phi_x| && \text{(18c)}\\
|\mu\omega\phi_x| \ &\le \ 8|u_+ - u_-|^2 && \text{(18d)}\\
|\omega_x| \ &\le \ M_1|\phi_x| && \text{(18e)}
\end{aligned}$$

where M_1 is a constant depending on the profile ϕ (and thus on f, u_-, u_+), but not on x.

Proof : Apart from (18e) the proof of Lemma 4 can be found in [F1]. Introducing the variable $\tilde{u} = \tilde{p}(x) = \int_0^x \exp(\int_0^\xi \frac{\lambda_p(\eta)}{\mu} d\eta) d\xi$ and using the integral representation of ω resulting from (18a) (see [F1]) it is easy to show that ω can be represented as a function of $\tilde{u}$ where the derivative with respect to $\tilde{u}$ is bounded. (18e) then follows from (13d). $\square$

Remark : The (unique) value of x_0 in (18a) is the one we already referred to (as "to be chosen later") in our definitions of diffusion waves and diffusion wave and shock wave regions. The parameter x_0 serves only to identify a certain state on the profile curve; one could readily make $x_0 = 0$ by switching to a different parameterization of this curve.

We define the diagonal matrix W as

$$W := \mathrm{diag}(\tilde{w}_1, \dots, \tilde{w}_n) = \mathrm{diag}(w_1, \dots, \omega w_p, \dots, w_n)$$

where

$$\begin{aligned} w_i(x,t) &:= \exp\Big(-\textstyle\sum_{q\neq i,p} \int_{-\infty}^x \frac{k_1(1+t)^{\frac{1}{4}}}{\lambda_i(u_{q,0})-\lambda_q(u_{q,0})} |\theta_{q,x}|\ d\xi - k_2 \int_{-\infty}^x |\phi_x|\ d\xi \,\mathrm{sign}(\lambda_i) \\ &\qquad - \textstyle\int_{-\infty}^x \frac{1}{2\mu} \nabla\lambda_i(u_{i,0})\cdot r_i(u_{i,0})\theta_i(y,t)\ dy\Big), \quad i \neq p \end{aligned} \tag{19}$$

$$w_p(x,t) := \exp\Big(-\textstyle\sum_{q\neq p} \int_{-\infty}^x \frac{k_1(t+1)^{\frac{1}{4}}}{\lambda_i(u_{q,0})-\lambda_q(u_{q,0})} |\theta_{q,x}|\ d\xi\Big). \tag{20}$$

Furthermore we define the following constants:

$$\begin{aligned} M_\omega &:= \sup_{x\in\mathbb{R}} \omega(x) & m_\omega &:= \inf_{x\in\mathbb{R}} \omega(x) \\ M_0 &:= \max_{1\le i\le n} \sup_{(x,t)} w_i(x,t) & m_0 &:= \min_{1\le i\le n} \inf_{(x,t)} w_i(x,t). \end{aligned}$$

For the above weight functions we have:

Lemma 5
For given constants c_1, $c_2 > 0$ there is a suitable choice of k_1, k_2 in (19), (20) such that if ϵ and θ_0 are sufficiently small we have for $1 \le i \le n$

$$\begin{aligned} &- \tilde{w}_{i,t} - (\lambda_i(\phi+\theta)\tilde{w}_i + \mu\tilde{w}_{i,x})_x \\ &\ge c_1 \tilde{w}_i (1+t)^{\frac{1}{4}} \sum_{q\neq i,p} |\theta_{q,x}| + \left\{ \begin{array}{ll} c_2 w_i |\phi_x| & \mathit{if}\ i \neq p \\ 2 w_p |\phi_x| & \mathit{if}\ i = p \end{array} \right\} - \theta_0 M_\omega M_0 \frac{\mathcal{O}(1)}{(1+t)^{5/4}}. \end{aligned}$$

Proof : We consider the case $i = p$. The case $i \neq p$ is proved as in [SX].

$$\begin{aligned} &- \tilde{w}_{p,t} - (\lambda_p(\phi+\theta)\tilde{w}_p + \mu\tilde{w}_{p,x})_x = -\omega w_{p,t} - (\lambda_p(\phi+\theta)\omega w_p + \mu(\omega w_p)_x)_x \\ &= -\omega w_p \left(\tfrac{\partial}{\partial t} + \lambda_p(\phi+\theta)\tfrac{\partial}{\partial x}\right) \Big[\textstyle\sum_{q\neq p} \int_{-\infty}^x -\frac{k_1(1+t)^{1/4}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})} |\theta_{q,x}|\ d\xi\Big] \\ &\qquad\qquad - w_p(\omega\lambda_p(\phi+\theta))_x - \mu\omega w_{p,xx} - 2\mu\omega_x w_{p,x} - \mu\omega_{xx} w_p \\ &= \omega w_p \sum_{q\neq p} \left(\tfrac{\partial}{\partial t} + \lambda_q(u_{q,0})\tfrac{\partial}{\partial x}\right) \Big[\textstyle\int_{-\infty}^x \frac{k_1(1+t)^{1/4}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})} |\theta_{q,x}|\ d\xi\Big] \\ &\qquad + \omega w_p \sum_{q\neq p} \tfrac{\lambda_p(\phi+\theta)-\lambda_q(u_{q,0})}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})} k_1(1+t)^{1/4} |\theta_{q,x}| - w_p(\lambda_p(\phi)\omega + \mu\omega_x)_x \\ &\qquad\qquad - w_p((\lambda_p(\phi+\theta) - \lambda_p(\phi))\omega)_x - \mu\omega w_{p,xx} - 2\mu\omega_x w_{p,x} \end{aligned}$$

$$\stackrel{(18a)}{=} \omega w_p \sum_{q\neq p} \tfrac{\lambda_p(\phi+\theta)-\lambda_q(u_{q,0})}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})} k_1(1+t)^{1/4}|\theta_{q,x}| + 2w_p|\phi_x|$$
$$+ \omega w_p \sum_{q\neq p} \left(\tfrac{\partial}{\partial t}+\lambda_q(u_{q,0})\tfrac{\partial}{\partial x}\right)\left[\int_{-\infty}^{x} \tfrac{k_1(1+t)^{1/4}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}|\theta_{q,x}|\ d\xi\right]$$
$$- w_p((\lambda_p(\phi+\theta)-\lambda_p(\phi))\omega)_x - \mu\omega w_{p,xx} - 2\mu\omega_x w_{p,x}$$

The last four terms are estimated as follows:

a) For $q \neq p$ we have

$$\omega w_p\left|\left(\tfrac{\partial}{\partial t}+\lambda_q(u_{q,0})\tfrac{\partial}{\partial x}\right)\left[\int_{-\infty}^{x} \tfrac{k_1(1+t)^{1/4}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}|\theta_{q,x}|\ d\xi\right]\right|$$
$$\leq M_\omega M_0\left|(1+t)^{-3/4}\right|\left|\tfrac{k_1}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}\right|\int_{-\infty}^{x}|\theta_{q,x}|d\xi$$
$$+ M_\omega M_0\left|(1+t)^{1/4}\right|\left|\tfrac{k_1}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}\right|\left|\tfrac{\partial}{\partial t}+\lambda_q(u_{q,0})\tfrac{\partial}{\partial x}\int_{-\infty}^{x}|\theta_{q,x}|\,d\xi\right|$$
$$\stackrel{(14c),(14d)}{\leq} M_\omega M_0\mathcal{O}(1)k_1\theta_0(1+t)^{-5/4}$$

b) $|w_p(\lambda_p(\phi+\theta)-\lambda(\phi))\omega)_x|$

$$\leq w_p|\lambda_p(\phi+\theta)-\lambda_p(\phi)||\omega_x| + w_p|\nabla\lambda_p(\phi+\theta)(\phi_x+\theta_x) - \nabla\lambda_p(\phi)\phi_x||\omega|$$
$$\leq w_p\mathcal{O}(1)\left(|\theta||\omega_x| + \omega|\theta||\phi_x| + \omega|\theta_x|\right)$$
$$\leq w_p\mathcal{O}(1)[M_1+M_\omega]|\theta_0||\phi_x| + \mathcal{O}(1)\tilde{w}_p|\theta_x|$$

c) $|\mu\omega w_{p,xx}|$

$$\leq \mu|\omega w_p|\left[\left(\textstyle\sum_{q\neq p} \tfrac{k_1(1+t)^{(1/4)}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}\theta_{q,x}\right)^2 + \sum_{q\neq p}\left|\tfrac{k_1(1+t)^{(1/4)}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}\tfrac{\partial}{\partial x}|\theta_{q,x}|\right|\right]$$
$$\stackrel{(14a)}{\leq} \mu\omega w_p(\mathcal{O}(1)k_1^2\theta_0^2(1+t)^{-3/2} + \mathcal{O}(1)k_1|\theta_0|(1+t)^{-5/4})$$
$$\leq \theta_0\mathcal{O}(1)M_\omega(1+k_1)^2(1+t)^{-5/4}$$

d) $|\mu\omega_x w_{p,x}| \leq \mu|\omega_x|w_p|\sum_{q\neq p} \tfrac{k_1(1+t)^{(1/4)}}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})}\theta_{q,x}|$

$$\stackrel{(14a)}{\leq} \mu|\omega_x|w_p\mathcal{O}(1)k_1|\theta_0|(1+t)^{-5/4} \leq w_p\mathcal{O}(1)k_1M_1|\theta_0||\phi_x|$$

Altogether we get in the case of $i = p$

$$- \tilde{w}_{p,t} - (\lambda_p(\phi+\theta)\tilde{w}_p + \mu\tilde{w}_{p,x})_x$$
$$\geq \tilde{w}_p \sum_{q\neq p} \tfrac{\lambda_p(\phi+\theta)-\lambda_q(u_{q,0})}{\lambda_p(u_{q,0})-\lambda_q(u_{q,0})} k_1(1+t)^{1/4}|\theta_{q,x}| + w_p|\phi_x|$$
$$- \tilde{w}_p\mathcal{O}(1)|\theta_x| - w_p\mathcal{O}(1)((1+k_1)M_1+M_\omega)|\theta_0||\phi_x|$$
$$- \mathcal{O}(1)\theta_0(1+M_\omega M_0)(1+k_1)^2(1+t)^{-5/4}$$

Thus for given c_1 we can choose k_1 sufficiently large and then θ_0 sufficiently small to complete the proof for the case $i = p$. $\square$

5.3 Estimate for (T4) - (T8)

5.3.1 Estimate for (T4)

It is

$$\begin{aligned}
&|V^{\mathsf{T}}(W(LR_t + \Lambda LR_x) + \mu W_x LR_x)V| \\
&= |V^{\mathsf{T}}(W(L_tR + \Lambda L_xR) + \mu W_x L_xR)V| \\
&= \sum_{i,j=1}^{n} |(\tilde{w}_i \frac{\partial}{\partial t} + (\tilde{w}_i\lambda_i(\phi+\theta) + \mu\tilde{w}_{i,x})\frac{\partial}{\partial x})[l_i^{\mathsf{T}}(\phi+\theta)]r_j(\phi+\theta)||V_i||V_j| \\
&\leq \mathcal{O}(1)\sum_{i,j=1}^{n} |\tilde{w}_i\lambda_i(\phi+\theta) + \mu\tilde{w}_{i,x}||\phi_x||V_i||V_j| \\
&\qquad + \mathcal{O}(1)\sum_{i,j=1}^{n} (|\tilde{w}_i(\frac{\partial}{\partial t} + \lambda_i(\phi+\theta)\frac{\partial}{\partial x})\theta| + |\tilde{w}_{i,x}\theta_x|)|V_i||V_j|
\end{aligned} \tag{21}$$

For $i = p$ we have:

$$\begin{aligned}
&|\tilde{w}_i\lambda_i(\phi+\theta) + \mu\tilde{w}_{i,x}| \leq w_p|\omega\lambda_p(\phi+\theta) + \mu\omega_x| + \mu|\omega w_{p,x}| \\
&= w_p(|\omega\lambda_p(\phi) + \mu\omega_x| + w_p\omega\mathcal{O}(1)|\theta| + \mu\omega\sum_{q\neq p}\frac{k_1(1+t)^{\frac{1}{4}}}{\lambda_i(u_{q,0}) - \lambda_q(u_{q,0})}|\theta_{q,x}|) \\
&\leq \mathcal{O}(1)M_0(\epsilon + M_\omega k_1|\theta_0|)
\end{aligned} \tag{22}$$

For $i \neq p$ we have by definition of $\tilde{w}_i$:

$$\begin{aligned}
&|\tilde{w}_i\lambda_i + \mu\tilde{w}_{i,x}| = |w_i\lambda_i + \mu w_{i,x}| \\
&\leq w_i(|\lambda_i| + \mu\mathcal{O}(1)(k_1\theta_0 + k_2|\phi_x| + |\theta_{i,0}|)) \\
&\leq \mathcal{O}(1)M_0(|\lambda_i| + ((1+k_1)\theta_0 + k_2\epsilon))
\end{aligned} \tag{23}$$

For $|(\frac{\partial}{\partial t} + \lambda_i(\phi+\theta)\frac{\partial}{\partial x})\theta|$ we have

$$\begin{aligned}
&|(\frac{\partial}{\partial t} + \lambda_i(\phi+\theta)\frac{\partial}{\partial x})\theta| = |\sum_{q\neq p}(\frac{\partial}{\partial t} + \lambda_i(\phi+\theta)\frac{\partial}{\partial x})\theta_q r_q(u_{q,0})| \\
&\leq \sum_{q\neq p}|(\frac{\partial}{\partial t} + (\lambda_q(u_{q,0}) + \nabla\lambda_q(u_{q,0})r_q(u_{q,0})\theta_q)\frac{\partial}{\partial x})\theta_q r_q(u_{q,0})| \\
&+ \sum_{q\neq p}|(\lambda_i(u_{q,0}) - \lambda_q(u_{q,0}))\theta_{q,x}| + \mathcal{O}(1)\sum_{q\neq p}|\phi - u_{q,0}||\theta_{q,x}| \\
&\qquad + \mathcal{O}(1)\sum_{k,q\neq p}|\theta_k\theta_{q,x}| \\
&\leq \mathcal{O}(1)\left[\sum_{q\neq p}\mu|\theta_{q,xx}| + \sum_{q\neq p,i}|\theta_{q,x}| + \sum_{q\neq p}|\phi - u_{q,0}||\theta_{q,x}| + \sum_{k,q\neq p}|\theta_k\theta_{q,x}|\right] \\
&\overset{(14a),(14g)}{\leq} \mathcal{O}(1)(\sum_{q\neq p,i}|\theta_{q,x}| + \frac{\theta_0}{(1+t)^{3/2}})
\end{aligned} \tag{24}$$

For $|\tilde{w}_{i,x}\theta_x|$ we have:

$$|\tilde{w}_{i,x}\theta_x| \leq \mathcal{O}(1)M_0(1+M_\omega)(\frac{\theta_0}{(1+t)^{5/4}} + \theta_0|\phi_x|). \tag{25}$$

Plugging (22) and (23) together with (24) and (25) into (21), we obtain:

$$\begin{aligned}
&|V^{\mathsf{T}}(W(LR_t + \Lambda LR_x) + \mu W_x LR_x)V| \\
&\leq \sum_{j=1}^{n} \mathcal{O}(1)M_0(\epsilon + M_\omega k_1\theta_0)|\phi_x||V_p||V_j| \\
&\qquad + \sum_{\substack{i,j=1\\ i\neq p}}^{n} \mathcal{O}(1)M_0(|\lambda_i| + ((1+k_1)\theta_0 + k_2\epsilon))|\phi_x||V_i||V_j| \\
&\qquad + \sum_{i,j=1}^{n} \mathcal{O}(1)M_0(1+M_\omega)(\sum_{q\neq p,i}|\theta_{q,x}| + \frac{\theta_0}{(1+t)^{5/4}})|V_i||V_j| \\
&\leq (\frac{1}{8} + \mathcal{O}(1)\frac{M_0}{m_0}(\epsilon + M_\omega k_1\theta_0))w_p|\phi_x||V_p|^2 \\
&\qquad + \sum_{k\neq p} \mathcal{O}(1)\big[\frac{M_0}{m_0}(1 + (1+k_1)\theta_0 + k_2\epsilon\big]^2 w_k|\phi_x||V_k|^2 \\
&\qquad\qquad + \sum_{k=1}^{n} \mathcal{O}(1)M_0(1+M_\omega)(\sum_{q\neq p,i}|\theta_{q,x}| + \frac{\theta_0}{(1+t)^{5/4}})|V_k|^2
\end{aligned}$$

5.3.2 Estimate for (T5)

It is

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mu V_x^{\mathsf{T}} WLR_x V - \mu V^{\mathsf{T}} WLR_x V_x \; dx \\
&= \mu \sum_{i\neq j} \int_{-\infty}^{\infty} \left(\tilde{w}_i l_i \nabla r_j \phi_x - \tilde{w}_j l_j \nabla r_i \phi_x\right) V_j (V_i)_x \; dx \\
&\quad + \mu \sum_{i\neq j} \int_{-\infty}^{\infty} \left(\tilde{w}_i l_i \nabla r_j \theta_x - \tilde{w}_j l_j \nabla r_i \theta_x\right) V_j (V_i)_x \; dx
\end{aligned}$$

The estimate of $\mu\sum_{i\neq j}\left(\tilde{w}_i l_i \nabla r_j \phi_x - \tilde{w}_j l_j \nabla r_i \phi_x\right) V_j (V_i)_x$ is done as in [F1] (see the estimate of (C3) there). Using (13c), (18c), and (18d), we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} \mu \sum_{i\neq j} \left(\tilde{w}_i l_i \nabla r_j \phi_x - \tilde{w}_j l_j \nabla r_i \phi_x\right) V_j (V_i)_x \; dx &\geq -\int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{\mu}{8}\tilde{w}_k |V_{k,x}|^2 \; dx \\
&\quad - \int_{-\infty}^{\infty} \sum_{k=1}^{n} \mathcal{O}(1)\left((1 + k_1 + \frac{M_0}{m_0}M_\omega)\theta_0 + (1 + k_2 + \frac{M_0^2}{m_0^2})\epsilon\right) w_k|\phi_x||V_k|^2.
\end{aligned}$$

We complete the estimate of (T5) by

$$
\begin{aligned}
&\mu \sum_{i\neq j} \left(\tilde{w}_i l_i \nabla r_j \theta_x - \tilde{w}_j l_j \nabla r_i \theta_x\right) V_j (V_i)_x \\
&\geq - \sum_{j=1}^{n} \frac{\mu}{8} \tilde{w}_j |V_{j,x}|^2 - \sum_{i=1}^{n} \mathcal{O}(1)(\tilde{w}_i + \frac{\tilde{w}_i^2}{\tilde{w}_j}) |\theta_x|^2 |V_i|^2 \\
&\geq - \sum_{j=1}^{n} \frac{\mu}{8} \tilde{w}_j |V_{j,x}|^2 - \sum_{i=1}^{n} \mathcal{O}(1)(\tilde{w}_i + \frac{\tilde{w}_i^2}{\tilde{w}_j}) \frac{|\theta_0|^2}{(1+t)^2} |V_i|^2
\end{aligned}
$$

and obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \mu V_x^{\mathsf{T}} W L R_x V - V^{\mathsf{T}} W L R_x V_x \; dx \\
&\geq - \int_{-\infty}^{\infty} \sum_{k=1}^{n} \mathcal{O}(1) \left((1 + k_1 + \frac{M_0}{m_0} M_\omega) \theta_0 + (1 + k_2 + \frac{M_0^2}{m_0^2}) \epsilon \right) w_k |\phi_x| |V_k|^2 \\
&\quad - \sum_{i=1}^{n} \mathcal{O}(1) \frac{M_0^2}{m_0^2} \frac{\theta_0^2}{(1+t)^2} |V_i|^2 - \sum_{k=1}^{n} \frac{\mu}{4} \tilde{w}_k |V_{k,x}|^2 \; dx
\end{aligned}
$$

5.3.3 Estimate for (T6)

$$
\begin{aligned}
&\mu V^{\mathsf{T}} W L_x R_x V \leq \mathcal{O}(1) \sum_{i,j} \tilde{w}_i (\phi_x + \theta_x)^2 |V_i| |V_j| \\
&\leq \mathcal{O}(1) \sum_{i,k} \tilde{w}_i (|\phi_x|^2 + \frac{\theta_{i,0}}{(1+t)^2}) |V_k|^2 \\
&\overset{(18c)}{\leq} \sum_{k=1}^{n} \mathcal{O}(1) \frac{M_0}{m_0} \epsilon w_k |\phi_x| |V_k|^2 + \sum_{k=1}^{n} \mathcal{O}(1) M_\omega M_0 \frac{\theta_0}{(1+t)^2} |V_k|^2
\end{aligned}
$$

5.3.4 Estimate for (T7)

$$
\begin{aligned}
&\left| \int_{-\infty}^{\infty} V^{\mathsf{T}} W L E \; dx \right| \\
&\overset{(9)}{\leq} M_\omega M_0 \mathcal{O}(1) \int_{-\infty}^{\infty} \sum_{\substack{i\neq j \\ j\neq p}} |\theta_j| |\tilde{V}_i| |V| + |\phi_x| |\theta| |d| |V| + |E_2| |V| \; dx \\
&\overset{(10)}{\leq} M_\omega M_0 \mathcal{O}(1) \int_{-\infty}^{\infty} \sum_{\substack{i\neq j \\ j\neq p}} |\theta_j| |\tilde{V}_i| |V| + |\phi_x| |\theta| |d| |V| + |\theta|^3 |V| \\
&\quad + |\eta|^2 |V| + \sum_{i\neq p} |(\phi - u_{i,0}) \theta_i r_i(u_{i,0})| |V| + \sum_{j\neq k} |\theta_j \theta_k| |V| \; dx
\end{aligned}
$$

$$\le M_\omega M_0 \mathcal{O}(1)(\sum_{\substack{i\neq j\\ j\neq p}} \|\theta_j \tilde{V}_i\|_{\mathrm{L}_2}\|V\|_{\mathrm{L}_2} + \|\eta^2 V\|_{\mathrm{L}_1} + \frac{\theta_0}{(1+t)^{5/4}}\|V\|_{\mathrm{L}_2})$$

$$\overset{(15c)}{\le} M_\omega M_0 \mathcal{O}(1)(\|\eta^2 V\|_{\mathrm{L}_1} + \theta_0 \frac{(1+\log(1+t))^{3/2}}{(1+t)^{5/4}}\|V\|_{\mathrm{L}_2})$$

For $\|\eta^2 V\|_{\mathrm{L}_1}$ we find

$$\begin{aligned}
\int_{-\infty}^{\infty} |\eta|^2|V|dx \ &\le\ \|V\|_{\mathrm{L}_\infty}\|\eta\|^2_{\mathrm{L}_2} \ \le\ \|V\|^{1/2}_{\mathrm{L}_2}\|V_x\|^{1/2}_{\mathrm{L}_2}\|\eta\|^2_{\mathrm{L}_2} \\
&\le \frac{1}{4}\beta\|V_x\|^2_{\mathrm{L}_2} + \frac{3}{4}\beta^{-1/3}\|V\|^{2/3}_{\mathrm{L}_2}\|\eta\|^{8/3}_{\mathrm{L}_2} \\
&\overset{(15b)}{\le} \frac{1}{4}\beta\|V_x\|^2_{\mathrm{L}_2} + \frac{3}{4}\beta^{-1/3}\theta_0\mathcal{O}(1)\frac{(1+\log(1+t))^4}{(1+t)^{4/3}}\|V\|^{2/3}_{\mathrm{L}_2}
\end{aligned}$$

Thus

$$\begin{aligned}
&|\int_{-\infty}^{\infty} V^\mathsf{T} WLE\ dx| \\
&\le \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{\mu}{8}\tilde{w}_k|V_k|^2 + \mathcal{O}(1)M_\omega M_0[\beta+\beta^{2/3}]\theta_0\frac{(1+\log(1+t))^4}{(1+t)^{5/4}}
\end{aligned}$$

5.3.5 Estimate for (T8)

Because of (12) we see that $|Q| \le \mathcal{O}(1)(|\eta||U_x| + |U_x|^2)$ and thus we have for (T8)

$$\begin{aligned}
&\Big|\int_{-\infty}^{\infty} V^\mathsf{T} WLQ\ dx\Big| \\
&\le \int_{-\infty}^{\infty} \mathcal{O}(1)(1+\omega)M_0|V|[(|\phi_x|^2+|\theta_x|^2)|V|^2 + |V_x|^2 \\
&\qquad\qquad + |\eta|((|\phi_x|+|\theta_x|)|V| + |V_x|)]\ dx \\
&\overset{(18d)}{\le} \int_{-\infty}^{\infty} \beta M_0\mathcal{O}(1)\epsilon|\phi_x||V|^2\ dx + \int_{-\infty}^{\infty} \beta^3 M_0(1+M_\omega)\frac{\mathcal{O}(1)\theta_0^2}{(1+t)^{3/2}}\ dx \\
&\quad + \int_{-\infty}^{\infty} \beta M_0(1+M_\omega)\mathcal{O}(1)|V_x|^2\ dx + \int_{-\infty}^{\infty} M_0(1+M_\omega)C(\epsilon)\theta_0|\phi_x||V|^2\ dx \\
&\quad + \int_{-\infty}^{\infty} \beta^2 M_0(1+M_\omega)\frac{\mathcal{O}(1)\theta_0}{(1+t)^{5/4}}\ dx + \int_{-\infty}^{\infty} M_0(1+M_\omega)\mathcal{O}(1)|\eta||V||V_x|\ dx
\end{aligned}$$

where $\int_{-\infty}^{\infty}|\eta||V||V_x|\ dx \le \int_{-\infty}^{\infty}|\eta|^2|V|dx + \int_{-\infty}^{\infty}\beta|V_x|^2dx$ and the first integral can be estimated as in (T7).

5.4 Subsumption

Putting together the result of Lemma 5 (positivity for (T2)) and the estimates above, we obtain:

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}\sum_{k=1}^{n} \tilde{w}_k V_k^2 + \sum_{k=1}^{n} \frac{\mu}{2}\tilde{w}_k |V_{k,x}|^2$$
$$+ \sum_{k=1}^{n} \Big[c_1 - \mathcal{O}(1)\frac{M_0}{m_0}(1+M_\omega)\Big]\tilde{w}_k (1+t)^{1/4}\big(\sum_{q\neq i,p}|\theta_{q,x}|\big)V_k^2$$
$$+ \sum_{k\neq p}\Big[c_2 - \mathcal{O}(1)\big(\frac{M_0}{m_0}(1+k_1 M_\omega\theta_0 + k_2\frac{M_0}{m_0}\epsilon)\big)^2\Big] w_k|\phi_x|V_k^2$$
$$+ \Big[1 - \frac{1}{8} - \mathcal{O}(1)\frac{M_0}{m_0}(\epsilon + k_1 M_\omega \theta_0)\Big] w_p|\phi_x|V_k^2 - \theta_0\beta^2\frac{M_0^2}{m_0^2}M_\omega^2\frac{\mathcal{O}(1)}{(1+t)^{5/4}} \;\le\; 0$$

Now we can choose c_1 sufficiently large (by the choice of k_1), then ϵ sufficiently small, then c_2 sufficiently large, and finally (depending on the choice of ϵ) θ_0 and β sufficiently small such that we find $[\ldots] > \frac{1}{2}$ for all the coefficients $[\ldots]$ above. Thus we obtain

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}\sum_{k=1}^{n}\tilde{w}_k V_k^2 + \sum_{k=1}^{n}\frac{\mu}{2}\tilde{w}_k|V_{k,x}|^2 + \sum_{k=1}^{n}\frac{1}{2}\tilde{w}_k(1+t)^{1/4}\big(\sum_{q\neq i,p}|\theta_{q,x}|\big)V_k^2$$
$$+ \sum_{k\neq p}\frac{1}{2}w_k|\phi_x|V_k^2 + \frac{1}{2}w_p|\phi_x|V_k^2 - \theta_0\frac{\mathcal{O}(1)C(\epsilon)}{(1+t)^{5/4}} \;\le\; 0$$

and integrating $\int_0^T dt$ and returning to the original variable U we get

$$||U(\cdot,T)||^2_{\mathrm{L}_2} + \int_0^T ||U_x(\cdot,t)||^2_{\mathrm{L}_2}\;dt \le C\Big(||U(\cdot,0)||^2_{\mathrm{L}_2} + \frac{\theta_0}{(1+T)^{1/4}}\Big).$$

This completes the proof of Theorem 3.

6 Estimate for the η wave – Proof of Lemma 3

In this section we proof the estimate of the η wave, i.e. Lemma 3. Introducing the integrated variable $\bar{\eta} := \int_{-\infty}^{x}\eta(\xi,t)\;d\xi$ we obtain from (6)

$$\bar{\eta}_t + h'(\phi)\bar{\eta}_x + \sum_{j\neq p} h''(\phi)[\theta_j r_j(u_{j,0}), \tilde{V}_j r_j(u_{j,0})] - \mu\bar{\eta}_{xx} = E_1.$$

Changing to *characteristic Variables* d, $\tilde{V} = d_x$ with $\bar{\eta} =: R(\phi)d$ we have

$$d_t + \Lambda(\phi)d_x - \mu d_{xx} = \tilde{E} \tag{26}$$

where $d(\cdot,0) = 0$ and by differentiating (26)

$$\tilde{V}_t + (\Lambda(\phi)\tilde{V})_x - \mu\tilde{V}_{xx} = \tilde{E}_x \tag{27}$$

where $\tilde{V}(\cdot, 0) = 0$ and

$$\begin{aligned} \tilde{E} \; &= L(\phi)E_1 - L(\phi)\sum_{j\neq p} h''(\phi)[\theta_j r_j(u_{j,0}), \tilde{V}_j r_j(u_{j,0})] \\ &\quad - \Lambda L R_x d + 2\mu L R_x d_x + \mu L R_{xx} d \\ E_1 \; &:= -\frac{1}{2}\sum_{i\neq j, j\neq p} l_i(\phi)\cdot h''(\phi)[\theta_j r_j(u_{j,0}), \theta_j r_j(u_{j,0})] r_i(\phi). \end{aligned} \tag{28}$$

Note that we diagonalize with $L(\phi)$, $R(\phi)$ (in contrast to Section 5). We drop the dependence on ϕ and write $L(x) = L(\phi(x))$ etc.

6.1 The dual waves

To derive pointwise estimates for d and $\tilde{V}$ (and thus for $\eta = R_x d + R\tilde{V}$) we define approximate Green functions, also known as *dual waves.*

The dual operator of $\mathrm{I}\frac{\partial}{\partial t} + \Lambda\frac{\partial}{\partial x} - \mu\mathrm{I}\frac{\partial^2}{\partial x^2}$ is given by $-\mathrm{I}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\Lambda - \mu\mathrm{I}\frac{\partial^2}{\partial x^2}$. We approximate this by $-\mathrm{I}\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\bar{\Lambda} - \mu\mathrm{I}\frac{\partial^2}{\partial x^2}$ and define for fixed $(x', T) \in \mathbb{R}\times[0,\infty)$ the (approximate) dual wave of d_i by

$$\begin{aligned} -\psi_{i,t} - (\bar{\lambda}_i\psi_i)_x - \mu\psi_{i,xx} \;&=\; 0\;, \quad 0 < t < T \\ \psi_i(x,T) \;&=\; \delta(x - x') \end{aligned}$$

where the approximated speeds $\bar{\Lambda}(x) := \mathrm{diag}(\bar{\lambda}_1(x), \ldots, \bar{\lambda}_n(x))$ will be defined later as $\lambda_i(\pm\infty)$ or $\lambda_i(x)$. Multiplying (26) by $\mathrm{diag}(\psi_1, \ldots, \psi_n)$ and integrating $\int_{-\infty}^{\infty} dx$, $\int_0^T dt$ we obtain

$$d_i(x', T) = \int_0^T \int_{-\infty}^{\infty} \psi_i(x,t)[\tilde{E}(x,t) + (\Lambda(x) - \bar{\Lambda}(x))d_x(x,t)]_i \; dxdt. \tag{29}$$

In the same way we obtain a representation for $\tilde{V}_i(x', T)$. Defining the (approximated) dual wave $\tilde{\psi}_i$ of $\tilde{V}_i$ by

$$\begin{aligned} -\tilde{\psi}_{i,t} - \bar{\lambda}_i\tilde{\psi}_{i,x} - \mu\tilde{\psi}_{i,xx} \;&=\; 0\;, \quad 0 < t < T \\ \tilde{\psi}_i(x,T) \;&=\; \delta(x - x') \end{aligned}$$

we obtain

$$\tilde{V}_i(x', T) = \int_0^T \int_{-\infty}^{\infty} \tilde{\psi}_{i,x}(x,t)[-\tilde{E}(x,t) + (\Lambda(x) - \bar{\Lambda}(x))d_x(x,t)]_i \; dxdt. \tag{30}$$

Using the representation (29) and (30) we prove pointwise estimates for d and $\tilde{V}$ (and by this Lemma 3) by estimating the integral on the right hand side. To do this we need several properties of the dual wave: L_1 and L_∞ estimates in the space variable (see Lemma 8), L_1 estimates in the time variable (vertical estimates) (see Lemma 6) and localization in space-time (see Lemma 8).

Since we think of the dual waves propagating backward in time we set $s := T - t$ in the following (the s here has nothing to do with the shock speed).

We complete the definition of the dual waves: Let $1 =: s_1 \le s_2^i \le s_3^i \le T$ and

$$\bar\lambda_i(\cdot,t) := \begin{cases} \lambda'_{i,0} := \lambda_i(\phi(\mathrm{sign}(x'-x_0)\infty)) & \text{for } T-t \le s_2^i \\ \lambda_i & \text{for } T-t \in (s_2^i, s_3^i) \\ \lambda'''_{i,0} := \lambda_i(u_{i,0}) & \text{for } T-t \ge s_3^i \text{ and } i \ne p \end{cases}$$

(for $T-t \ge s_3^i$ and $i = p$ we define the dual wave differently – see below). Thus we have $\psi_i(x,t) = \tilde\psi_i(x,t) = D(x,s,x',\lambda'_{i,0})$ for $s = T-t \le s_2^i$, where

$$D(x,s,\xi,\lambda) := \frac{1}{\sqrt{4\pi s}} \exp\big(-\frac{(x-\xi-\lambda s)^2}{4s}\big).$$

For $s = T-t \ge s_3^i$ we define

$$\begin{aligned} \psi_i(x,t) &= D(\cdot, s-s_3^i, x, \lambda'''_{i,0}) * \psi_i(\cdot, s_3^i) && \text{for } i \ne p \\ \tilde\psi_i(x,t) &= D(\cdot, s-s_3^i, x, \lambda'''_{i,0}) * \tilde\psi_i(\cdot, s_3^i) && \text{for } i \ne p \end{aligned}$$

and

$$\begin{aligned} \psi_p(x,t) &= D(\cdot, s-s_3^p, x, \lambda_p(u_-)) * (\psi_p(\cdot, s_3^p)\chi_{(-\infty,x_0]}) \\ &\quad + D(\cdot, s-s_3^p, x, \lambda_p(u_+)) * (\psi_p(\cdot, s_3^p)\chi_{(x_0,\infty)}) \\ &= \psi_p^-(x,t) + \psi_p^+(x,t) \\ \tilde\psi_{p,x}(x,t) &= D(\cdot, s-s_3^p, x, \lambda_p(u_-)) * (\tilde\psi_{p,x}(\cdot, s_3^p)\chi_{(-\infty,x_0]}) \\ &\quad + D(\cdot, s-s_3^p, x, \lambda_p(u_+)) * (\tilde\psi_{p,x}(\cdot, s_3^p)\chi_{(x_0,\infty)}) \\ &= \tilde\psi_{p,x}^-(x,t) + \tilde\psi_{p,x}^+(x,t). \end{aligned}$$

The times s_2^i, s_3^i are defined by

$$\begin{aligned} s_2^i &:= \sup\{s_2 : \forall\ s_1 \le s < s_2 \le T : \int_{\mathcal{P}(T)} \psi_i(x, T-s)dx \le (1+T)^{-N}\} \\ s_3^i &:= \inf\{s_3 : \forall\ s_2^i \le s_3 < s \le T : \int_{\mathcal{P}(T)} \psi_i(x, T-s)dx \le (1+T)^{-N}\}. \end{aligned}$$

This choice ensures that we use the exact speeds when the dual wave interacts with the shock wave (i.e. $\int_{\mathcal{P}(T)} \psi_i(x,T-s)dx$ is large). We will see later that $s_3^i - s_2^i \le \mathcal{O}(1)\log(1+T)\sqrt{1+T}$ (see Section 6.2.2).

With this definition the stated representation formulas (29), (30) hold for $i \ne p$. In the case $i = p$ the representation formulas have a slightly different error term caused by the approximated speeds $\lambda_p(\pm\infty)$. In this case we have (see [SX])

$$\begin{aligned} d_p(x',T) &= \int_0^{T-s_3^p} \int_{-\infty}^{\infty} (\psi_p^-(\lambda_p - \lambda_p(-\infty)) + \psi_p^+(\lambda_p - \lambda_p(+\infty)))d_{p,x}\ dxdt \\ &+ \int_{T-s_3^p}^{T} \int_{-\infty}^{\infty} \psi_p(x,t)(\lambda_p(x) - \bar\lambda_p(x))d_{p,x}(x,t)\ dxdt + \int_0^T \int_{-\infty}^{\infty} \psi_p(x,t)[\tilde E(x,t)]_p dxdt \end{aligned} \tag{29$_p$}$$

$$\tilde V_p(x',T) = \int_0^{T-s_3^p} \int_{-\infty}^{\infty} (\tilde\psi^-_{p,x}(\lambda_p - \lambda_p(-\infty)) + \tilde\psi^+_{p,x}(\lambda_p - \lambda_p(+\infty)))d_{p,x}\ dxdt$$

$$+ \int_{T-s_3^p}^{T} \int_{-\infty}^{\infty} \tilde\psi_{p,x}(x,t)(\lambda_p - \bar\lambda_p)d_{p,x}(x,t)\ dxdt - \int_0^T \int_{-\infty}^{\infty} \tilde\psi_{p,x}(x,t)[\tilde E(x,t)]_p dxdt. \tag{30_p}$$

6.2 Properties of the dual waves

Since we only need properties of ψ_i and $\tilde\psi_{i,x}$ (see (29), (30)) and both solve the same equation they can be treated simultaneously. We begin by deriving the vertical estimates.

6.2.1 Vertical estimates

Lemma 6 (Vertical estimates)
Let $\psi : \mathbb{R} \times [s_1, \infty) \to \mathbb{R}$ be a smooth solution of

$$\psi_s - (\lambda_i \psi)_x - \mu\psi_{xx} = 0 \qquad\qquad i = 1, \ldots, n. \tag{31}$$

with $\psi(\cdot, s_1) \in \mathrm{L}_1(\mathbb{R}) \cap \mathrm{H}^1(\mathbb{R})$ and let $g(x) := \min\{\pm \int_x^{\pm\infty} \mathrm{e}^{\int_x^\xi \frac{\lambda_i(\eta)}{\mu} d\eta} d\xi\}$. Then

$$\int_{s_1}^{\infty} |\psi(x,s)| ds \ \leq\ \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})}\ g(x)/\mu \tag{32a}$$

$$\int_{s_1}^{\infty} |\lambda_i \psi(x,s) + \mu\psi_x(x,s)| ds \ \leq\ \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})} \tag{32b}$$

$$\int_{s_1}^{\infty} |\psi(x,s)| ds \ \leq\ \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})} \Big/ \min_x |\lambda_i(x)| \qquad \textit{for } i \neq p \tag{32c}$$

$$\int_{s_1}^{\infty} |\psi(x,s)| ds\ |\phi_x(x)| \ \leq\ \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})} \mathcal{O}(1) \cdot \epsilon \tag{32d}$$

$$\int_{s_1}^{\infty} |(\psi(x,s)\phi_x(x))_x| ds \ \leq\ \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})} \mathcal{O}(1)|\phi_x|/\mu \tag{32e}$$

Remark : In the corresponding Lemma in [SX] only (32c) was mentioned. The observation that there exists an L_1 vertical estimate even for the case $i = p$ would have slightly simplified the proof there. In our case of a non-convex flux function this vertical estimate (i.e. (32d), (32e)) is essential for the forthcoming estimate of the η wave. In addition we use a grouping of terms which was not used in [SX] and which enables certain cancellations (e.g. (32b)).

Note the similarity between our weight function ω of Lemma 4 and the function g: g satisfies $g\lambda_p + \mu g_x = -\mathrm{sign}(x - x_g)$ (for some x_g), while ω satisfies $\omega\lambda_p + \mu\omega_x = -2\int_{x_0} |\phi_x(\xi)| d\xi$.

Proof : Let $\mathrm{sign}_\delta(x) := \mathrm{sign}(x)\cdot\min\{|x|/\delta\,,1\}$ and $\rho \in \mathrm{C}_0^\infty(\mathbb{R}),\ \rho \geq 0$. Multiply (31) by $\mathrm{sign}_\delta(\psi)\rho$ and integrate by parts to obtain

$$\int_{-\infty}^{\infty} \rho\,\mathrm{sign}_\delta(\psi)\psi_s - \rho\,\mathrm{sign}_\delta(\psi)(\lambda_i\psi)_x + \mu\rho_x\mathrm{sign}_\delta(\psi)\psi_x\ dx + \mu \int_{|\psi|<\delta} \frac{\rho}{\delta}\psi_x^2\ dx = 0.$$

For $\delta \to 0$ we get

$$\int_{-\infty}^{\infty} \rho|\psi|_s - \rho(\lambda_i|\psi| + \mu|\psi|_x)_x\ dx \leq 0$$

and integration $\int_{s_1}^\infty\ ds$ gives with $v(x) := \int_{s1}^\infty |\psi(x,s)|ds$

$$\int_{-\infty}^{\infty} -\rho(\lambda_i v + \mu v_x)_x\ dx \leq \int_{-\infty}^{\infty} \rho\psi(x,s_1)\ dx$$

Considering the limit $\rho(y) \to \rho^\infty(y) = \begin{cases} 1 & y \leq x \\ 0 & y > x \end{cases}$ we obtain

$$-(\lambda_i(x)v(x) + \mu v_x(x)) \leq \int_{-\infty}^{x} |\psi(\xi,s_1)|d\xi \leq \|\psi(\cdot,s_1)\|_{\mathrm{L}_1(\mathbb{R})} \tag{33}$$

and considering the limit $\rho(y) \to \rho^\infty(y) = \begin{cases} 0 & y \leq x \\ 1 & y > x \end{cases}$ we obtain

$$(\lambda_i(x)v(x) + \mu v_x(x)) \leq \int_{x}^{\infty} |\psi(\xi,s_1)|d\xi \leq \|\psi(\cdot,s_1)\|_{\mathrm{L}_1(\mathbb{R})} \tag{34}$$

Thus we have for $\lambda_i(\infty) < 0$

$$\begin{aligned} v(x) &= v(x)\mathrm{e}^{\int_0^x \frac{\lambda_i(\eta)}{\mu}d\eta}\mathrm{e}^{-\int_0^x \frac{\lambda_i(\eta)}{\mu}d\eta} \\ &= \int_x^\infty -(v(\xi)\mathrm{e}^{\int_0^\xi \frac{\lambda_i(\eta)}{\mu}d\eta})_\xi\ d\xi\mathrm{e}^{-\int_0^x \frac{\lambda_i(\eta)}{\mu}d\eta} \\ &= \frac{1}{\mu}\int_x^\infty -(\lambda_i v + \mu v_x)(\xi)\mathrm{e}^{\int_x^\xi \frac{\lambda_i(\eta)}{\mu}d\eta}d\xi \leq \|\psi(\cdot,s_1)\|_{\mathrm{L}_1(\mathbb{R})}\frac{1}{\mu}\int_x^\infty \mathrm{e}^{\int_x^\xi \frac{\lambda_i(\eta)}{\mu}d\eta}\ d\xi \end{aligned}$$

and for $\lambda_i(-\infty) > 0$ by using the integration $\int_{-\infty}^x$ instead of $\int_x^\infty$

$$v(x) \leq \|\psi(\cdot,s_1)\|_{\mathrm{L}_1(\mathbb{R})}\frac{1}{\mu}\int_{-\infty}^x \mathrm{e}^{\int_x^\xi \frac{\lambda_i(\eta)}{\mu}d\eta}\ d\xi$$

Since for $i \neq p$ either $\lambda_i(\infty) < 0$ or $\lambda_i(-\infty) > 0$ holds and for $i = p$ both cases apply we arrive at (32a). (In the case of $i \neq p$ one of the two integrals in the definition of g is equal to ∞ while the other one remains finite).

To prove (32b) we conclude from (33) and (34)

$$|(\lambda_i v + \mu v_x)(x)| \leq \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})}$$

and thus

$$\int_{s_1}^{\infty} |\lambda_i \psi(x,s) + \mu \psi_x(x,s)| \; ds \leq \|\psi(\cdot, s_1)\|_{\mathrm{L}_1(\mathbb{R})}$$

(32c) follows from (32a) and

$$\text{i) for } \lambda_i > 0\text{:} \quad \int_{-\infty}^{x} \mathrm{e}^{\int_x^\xi \frac{\lambda_k(\eta)}{\mu} d\eta} \; d\xi \leq \int_{-\infty}^{x} \mathrm{e}^{-\frac{\min|\lambda_i|}{\mu}(x-\xi)} \; d\xi = \mu \Big/ \min_\xi |\lambda_i(\xi)|$$

$$\text{ii) for } \lambda_i < 0\text{:} \quad \int_{x}^{\infty} \mathrm{e}^{\int_x^\xi \frac{\lambda_k(\eta)}{\mu} d\eta} \; d\xi \leq \int_{x}^{\infty} \mathrm{e}^{-\frac{\min|\lambda_i|}{\mu}(\xi-x)} \; d\xi = \mu \Big/ \min_\xi |\lambda_i(\xi)|$$

To prove (32d) it is sufficient to examine $g(x)|\phi_x(x)|$.

Because of $\lambda_p(-\infty) > 0 > \lambda_p(+\infty)$ there exist x_-^*, x_+^* such that $\lambda_p(x) > \frac{\lambda_p(-\infty)}{2}$ for $x \in (-\infty, x_-^*)$ and $\lambda_p(x) < \frac{\lambda_p(+\infty)}{2}$ for $x \in (x_+^*, \infty)$. Thus

$$\text{for } x < x_-^* : \int_{-\infty}^{x} \mathrm{e}^{\int_x^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} \; d\xi \leq \int_{-\infty}^{x} \mathrm{e}^{-\frac{\lambda_p(-\infty)}{2\mu}(x-\xi)} \; d\xi = \frac{2\mu}{\lambda_p(-\infty)},$$

$$\text{for } x > x_+^* : \int_{x}^{\infty} \mathrm{e}^{\int_x^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} \; d\xi \leq \int_{x}^{\infty} \mathrm{e}^{\frac{\lambda_p(\infty)}{2\mu}(\xi-x)} \; d\xi = -\frac{2\mu}{\lambda_p(\infty)},$$

i.e. g is bounded and we have $g(x)|\phi_x(x)| \to 0$ if $x \to \pm\infty$. From this we see that $g|\phi_x|$ assumes its maximum at some $x_M \in \mathbb{R}$. We assume that $g(x_M)|\phi_x(x_M)| = \int_{-\infty}^{x_M} \mathrm{e}^{\int_{x_M}^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} d\xi |\phi_x(x_M)|$ (the other case can be treated in the same way). Then

$$\begin{aligned}
M := \max_x(g(x)|\phi_x|) = g(x_M)|\phi_x(x_M)| &= \int_{-\infty}^{x_M} \mathrm{e}^{\int_{x_M}^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} d\xi |\phi_x(x_M)| \\
&= \Big| \int_{-\infty}^{x_M} \Big[\int_{-\infty}^{x} \mathrm{e}^{\int_x^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} d\xi \phi_x(x) \Big]_x \; dx \Big| \\
&\leq \int_{-\infty}^{x_M} |\phi_x + [\phi_{xx} - \frac{\lambda_p}{\mu}\phi_x] \int_{-\infty}^{x} \mathrm{e}^{\int_x^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} d\xi | \; dx \\
&\overset{(13b)}{\leq} \int_{-\infty}^{\infty} |\phi_x| \; dx + \mathcal{O}(1) \int_{-\infty}^{\infty} |\phi_x|^2 \int_{-\infty}^{x} \mathrm{e}^{\int_x^\xi \frac{\lambda_p(\eta)}{\mu} d\eta} d\xi \; dx \\
&\leq 2\epsilon + \mathcal{O}(1) \int_{-\infty}^{\infty} |\phi_x| \; dx M \leq 2\epsilon + \mathcal{O}(1)\epsilon M.
\end{aligned}$$

and for ϵ sufficiently small we obtain (32d).

Finally (32e) follows directly from

$$(\psi\phi_x)_x \;=\; \psi_x\phi_x + \psi\phi_{xx} \;=\; \frac{1}{\mu}(\mu\psi_x + \lambda_p\psi)\phi_x \;+\; \frac{1}{\mu}(\mu\phi_{xx} - \lambda_p\phi_x)\psi$$

and (32b), (13b) and (32d). □

We now prove a time decay result for the p dual wave that will be used to prove localization for the same wave. The result is similar to Lemma 6.2 of [SX] except that we need to use the weight ω due to the absence of convexity.

Lemma 7

Let ω be the weight function defined in Lemma 4, i.e. we have $\mu\omega_x + \lambda_p(\phi)\omega_x = -2\int_{x_0}^x |\phi_x| d\xi$, $\omega(0) = \omega_0$. If $\psi : \mathbb{R} \times [s_1, \infty) \to \mathbb{R}$ is a smooth solution of

$$\psi_s - (\lambda_p\psi)_x - \mu\psi_{xx} = 0 \tag{35}$$

where $\psi(\cdot, s_1) \in \mathrm{L}_1(\mathbb{R}) \cap \mathrm{H}^1(\mathbb{R})$ then we have for any

$$0 \;<\; c_1 \;\leq\; \frac{1}{2(1+\mu)} \inf_{|x-x_0|<1} |\phi_x(x)| \Big/ (1 + \sup_x \omega(x))$$

with $\rho(x) := \exp(-c_1\sqrt{1 + (x-x_0)^2})$ and $c_2 = \mu \min\{2c_1, \frac{1}{4}c_1^2\}$ (independent of s) that

$$\int_{-\infty}^{\infty} \frac{\rho(x)}{\omega(x)} \psi^2(x,s)\, dx \leq \mathrm{e}^{-2c_2(s-s_1)} \int_{-\infty}^{\infty} \frac{\rho(x)}{\omega(x)} \psi^2(x,s_1)\, dx\ . \tag{36}$$

Proof : We substitute $\psi(x,s) =: \omega(x)v(x,s)$ in (35) and obtain for v

$$\omega v_s - (\lambda_p\omega v)_x - \mu(\omega v)_{xx} = 0.$$

Multiplying by ρv and integrating $\int_{-\infty}^{\infty} dx$ we obtain

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial s}\rho\omega v^2 - \rho v(\lambda_p\omega v)_x - \mu\rho v(\omega v)_{xx}\, dx = 0$$

and by direct calculation we have

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial s}\rho\omega v^2 + \left(-\frac{1}{2}\rho(\lambda_p\omega + \mu\omega_x)_x + \frac{1}{2}\rho_x(\lambda_p\omega + \mu\omega_x) - \mu\frac{\rho_x^2}{\rho}\omega\right) v^2\, dx \leq 0$$

Considering the two cases $x \in [x_0 - 1, x_0 + 1]$ and $x \notin [x_0 - 1, x_0 + 1]$ and applying the choice of ρ, c_1 and c_2 we see that

$$-\frac{1}{2}\rho(\lambda_p\omega + \mu\omega_x)_x + \frac{1}{2}\rho_x(\lambda_p\omega + \mu\omega_x) - \mu\frac{\rho_x^2}{\rho}\omega \geq \rho\omega c_2$$

and thus we have

$$\int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial s}\rho\omega v^2 + c_2\rho\omega v^2\, dx \leq 0 \quad \text{i.e.} \quad \frac{\partial}{\partial s}\int_{-\infty}^{\infty} \rho\omega v^2\, dx \leq -2c_2 \int_{-\infty}^{\infty} \rho\omega v^2\, dx.$$

Resubstituting $v = \frac{\psi(x,s)}{\omega(x)}$ and integrating $\int_{s_1}^{s} ds$ we obtain (36). □

6.2.2 L_1 and L_∞ spatial estimates

To prove the localization result we define for fixed i, x' and $T > 0$ regions of the strip $\mathbb{R} \times [0, T]$. We first subdivide $[0, T]$ into

$$\mathcal{S}_1^i := [0, s_1], \qquad \mathcal{S}_2^i := (s_1, s_2^i], \qquad \mathcal{S}_3^i := (s_2^i, s_3^i], \qquad \mathcal{S}_4^i := (s_3^i, T].$$

Depending on the time interval where the dual wave $\psi(\cdot, t)$ or $\tilde{\psi}(\cdot, t)$ is located we define a time-slice of the i *dual wave region* as follows ($t = T - s$):

$$\begin{aligned}
\mathcal{W}_s^i &:= \{x : |x - x' + \lambda'_{i,0} s| \le N \log(1+T)\sqrt{s}\} \ \text{ for } s \in \mathcal{S}_1^i \cup \mathcal{S}_2^i \\
\mathcal{W}_s^i &:= \left\{x : |x - x_i'' + \lambda_i(\phi(x))(s - s_2^i)| \le N \log(1+T)\sqrt{s - s_2^i} + \frac{1}{2}k_i'' + c_{\mathcal{W}}\epsilon(s - s_2^i)\right\} \\
&\qquad \text{for } s \in \mathcal{S}_3^i \text{ and } i \neq p \\
\mathcal{W}_s^p &:= \left\{x : |x - x_p'' + \lambda_p(\phi(x))(s - s_2^p)| \le N \log(1+T)\sqrt{s - s_2^p} + \frac{1}{2}k_p'' + c_{\mathcal{W}}\epsilon(s - s_2^p)\right\} \\
&\qquad \cap \left\{x : |x - x_0| > -\frac{2N}{c_1}\log(1+T) + \frac{c_2}{c_1}s\right\} \ \text{ for } s \in \mathcal{S}_3^p \\
\mathcal{W}_s^i &:= \left\{x : |x - x_i''' + \lambda_{i,0}'''(s - s_3^i)| \le N \log(1+T)\sqrt{s - s_3^i} + \frac{1}{2}k_i'''\right\} \\
&\qquad \text{for } s \in \mathcal{S}_4^i \text{ and } i \neq p \\
\mathcal{W}_s^p &:= \left\{x : |x - x_p^- + \lambda_p(u_-)(s - s_3^p)| \le N \log(1+T)\sqrt{s - s_3^p} + \frac{1}{2}k_p^-\right\} \\
&\qquad \cup \left\{x : |x - x_p^+ + \lambda_p(u_+)(s - s_3^p)| \le N \log(1+T)\sqrt{s - s_3^p} + \frac{1}{2}k_p^+\right\}
\end{aligned}$$

where

> x_i'' is the center and k_i'' is the width of $\mathcal{W}_s^i$ at time $s = s_2^i$,
>
> x_i''' is the center and k_i''' is the width of $\mathcal{W}_s^i$ at time $s = s_3^i$ (for $i \neq p$) and
>
> $x_p^\pm$, $k_p^\pm$ are the center and the width of $\mathcal{W}_s^p \cap [x_0, \pm\infty)$ at time $s = s_3^i$ respectively.

Lemma 8 (Properties of the dual waves)
For a suitable choice of the constant $c_{\mathcal{W}}$ (depending only on $\max \nabla\lambda_i$ in the definition of $\mathcal{W}_s^i$ we have ($\mathcal{W}^i(t) := \mathcal{W}_{T-t}^i$):

Localization:

$$\|\psi_i(\cdot, t)\|_{\mathrm{L}_1(\mathbb{R}\setminus\mathcal{W}^i(t))} \le \frac{C(\epsilon)}{(1+T)^N} \qquad \forall\, 0 \le t \le T \tag{37}$$

$$\|\tilde{\psi}_{i,x}(\cdot, t)\|_{\mathrm{L}_1(\mathbb{R}\setminus\mathcal{W}^i(t))} \le \frac{C(\epsilon)}{(1+T)^N} \qquad \forall\, 0 \le t \le T \tag{38}$$

L_1*-norm estimates:*

$$\|\psi_i(\cdot,\tau_1)\|_{\mathrm{L}_1(\mathbb{R})} \le \|\psi_i(\cdot,\tau_2)\|_{\mathrm{L}_1(\mathbb{R})} \qquad \forall\ 0 \le \tau_1 < \tau_2 \le T \tag{39}$$

$$\|\tilde{\psi}_{i,x}(\cdot,\tau_1)\|_{\mathrm{L}_1(\mathbb{R})} \le \|\tilde{\psi}_{i,x}(\cdot,\tau_2)\|_{\mathrm{L}_1(\mathbb{R})} \qquad \forall\ 0 \le \tau_1 < \tau_2 \le T \tag{40}$$

For $(i \neq p)$ or $(i = p$ and $x' \notin \mathcal{P}(T))$:

$$\|\tilde{\psi}_{i,x}(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} \le \begin{cases} \frac{1}{\sqrt{T-t}} & \forall\ T - s_2^i < t < T \\ \frac{\mathcal{O}(1)}{\sqrt{s_2^i+(t_3-t)}} & \forall\ 0 < t < t_3 := T - s_3^i \end{cases} \tag{41}$$

L_∞*-norm estimates:*

For $(i \neq p)$ or $(i = p$ and $x' \notin \mathcal{P}(T))$:

$$\|\psi_i(\cdot,\tau_1)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \mathcal{O}(1)\|\psi_i(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} \tag{42}$$

$$\|\tilde{\psi}_{i,x}(\cdot,\tau_1)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \mathcal{O}(1)\|\tilde{\psi}_{i,x}(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} \tag{43}$$

$$\|\psi_i(\cdot,t)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \begin{cases} \frac{1}{\sqrt{T-t}} & \forall\ T - s_2^i < t < T \\ \frac{\mathcal{O}(1)}{\sqrt{s_2^i+(t_3-t)}} & \forall\ 0 < t < t_3 := T - s_3^i \end{cases} \tag{44}$$

$$\|\tilde{\psi}_{i,x}(\cdot,t)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \begin{cases} \frac{1}{T-t} & \forall\ T - s_2^i < t < T \\ \frac{\mathcal{O}(1)}{s_2^i+(t_3-t)} & \forall\ 0 < t < t_3 := T - s_3^i \end{cases} \tag{45}$$

with a constant $C(\epsilon)$ depending on u_+, u_- only (i.e. on the specific profile ϕ).

Remark : The localization result implies that the interaction time $[T - s_2^i, T - s_3^i]$ is precisely the time strip where the dual wave region $\mathcal{W}^i$ intersects the shock region $\mathcal{P}(T)$ (this holds for $T \ge T_0(\epsilon)$ sufficiently large). Because the width of $\mathcal{P}(T)$ is $C(\epsilon)N\log(1+T)$ we see that for $i \neq p$ $s_3^i - s_2^i \le \mathcal{O}(1)(N\log(1+T)\sqrt{T} + C(\epsilon)N\log(1+T))$ which can be bounded by $\mathcal{O}(1)N\log(1+T)\sqrt{T}$ for $T \ge T_0(\epsilon)$ sufficiently large. The same holds for $i = p$. Indeed, we have $s_3^p \le C(\epsilon)N\log(1+T) \le \mathcal{O}(1)N\log(1+T)\sqrt{T}$ for $T \ge T_0(\epsilon)$ sufficiently large. Thus the maximal width of $\mathcal{W}_s^i$ (or of the components of $\mathcal{W}_s^p$) is $2N\log(1+T)\sqrt{T} + c_{\mathcal{W}}\epsilon\mathcal{O}(1)N\log(1+T)\sqrt{T} \le \mathcal{O}(1)N\log(1+T)\sqrt{T}$.

Proof : (41), (44) and (45) are direct consequences of the explicit definition of ψ_i and $\tilde{\psi}_i$ outside the time strip $[T - s_2^i, T - s_3^i]$ (i.e. before and after interacting with the shock region) and the other estimates. Note that for $x' \notin PSR(T)$ the p dual wave does not interact with the shock region.

Since ψ_i and $\tilde{\psi}_{i,x}$ satisfy the same equation, i.e.

$$-\psi_t - (\lambda\psi)_x - \mu\psi_{xx} = 0 \tag{46}$$

with $\lambda := \bar{\lambda}_i$, we obtain (39) and (40) as well as (42) and (43) simultaneously by letting $\psi := \psi_i$ and $\psi := \tilde{\psi}_{i,x}$ respectively. The proof of (37) and (38) will be presented at the end.

As in the proof of Lemma 6 we obtain by multiplication of (46) by $\mathrm{sign}_\delta(\psi)\rho$, with $\rho \in \mathrm{C}_0^\infty(\mathbb{R})$, $\rho \ge 0$, integration $\int_{-\infty}^{\infty} dx$ and going to the limit $\delta \to 0$

$$\int_{-\infty}^{\infty} -\rho(x)|\psi(x,t)|_t - \rho(x)(\lambda(x)|\psi| + \mu|\psi|_x)_x \ dx \ \le \ 0.$$

and with $\rho \to 1$ we get $-\frac{\partial}{\partial t}\|\psi(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} \leq 0$. Integration $\int_{\tau_1}^{\tau_2} dt$ gives (39) and (40) respectively.

The L_∞ estimates (42), (43) are obtained via duality. For ψ we define the dual wave ψ^* by

$$\psi_t^* + \lambda\psi_x^* - \mu\psi_{xx}^* = 0 \tag{47}$$

and prove an L_1 estimate for ψ^*:

$$\|\psi^*(\cdot,\tau_2)\|_{\mathrm{L}_1(\mathbb{R})} \leq \mathcal{O}(1)\|\psi^*(\cdot,\tau_1)\|_{\mathrm{L}_1(\mathbb{R})} \tag{48}$$

If (48) holds we can conclude from

$$\begin{aligned}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,t)dx &= \int_{-\infty}^{\infty} \psi^*(x,t)\psi_t(x,t) + \psi_t^*(x,t)\psi_(x,t)dx \\ &= \int_{-\infty}^{\infty} \psi^*(x,t)(\psi_t(x,t) + (\lambda(x)\psi(x,t))_x + \psi_{xx}(x,t)\ dx = 0,\end{aligned}$$

thus

$$\int_{-\infty}^{\infty} \psi^*(x,\tau_1)\psi(x,\tau_1)dx = \int_{-\infty}^{\infty} \psi^*(x,\tau_2)\psi(x,\tau_2)dx,$$

that

$$\|\psi(\cdot,\tau_1)\|_{\mathrm{L}_\infty(\mathbb{R})} = \sup \frac{\int_{-\infty}^{\infty} \psi^*(x,\tau_1)\psi(x,\tau_1)dx}{\|\psi^*(\cdot,\tau_1)\|_{\mathrm{L}_1(\mathbb{R})}} \overset{(48)}{\leq} \mathcal{O}(1)\|\psi(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})}.$$

It remains to prove (48). Let $\sigma(x) := \exp(k_2\int_{-\infty}^{x}|\phi_x(\xi)|\ d\xi \mathrm{sign}(\lambda))$ (with the same constant k_2 as in (19)). As in the proof of Lemma 6 we obtain by multiplication of (47) by $\mathrm{sign}_\delta(\psi^*)\sigma$, integration $\int_{-\infty}^{\infty} dx$ and going to the limit $\delta \to 0$

$$\int_{-\infty}^{\infty} |\psi^*(x,t)|_t\sigma(x)\ dx - (\lambda(x)\sigma(x) + \mu\sigma_x)_x|\psi^*(x,t)|\ dx \\ + \limsup_{\delta\to 0}\frac{1}{\delta}\int_{|\psi^*|<\delta} \sigma(x)|\psi^*(x,t)|^2\ dx = 0$$

Choosing k_2 sufficiently large and ϵ sufficiently small we get $\frac{\partial}{\partial t}\|\psi^*(\cdot,t)\sigma\|_{\mathrm{L}_1(\mathbb{R})} \leq 0$ and after integration $\int_{\tau_1}^{\tau_2} dt$ (48).

We now turn to the localization result (37) and (38). Because $\bar{\lambda}_i$ is constant if the dual wave is outside the strip $\mathbb{R}\times\mathcal{S}_3^i$, the only non-trivial part is the proof for $s \in \mathcal{S}_3^i$. We consider the case $i = p$ and prove for $\psi(\cdot,s) := \psi_p(\cdot,T-s)$ or $\psi(\cdot,s) := \tilde{\psi}_{p,x}(\cdot,T-s)$ and $\bar{\mathcal{W}} := \{|x-x_0| \leq -\frac{2N}{c_1}\log(1+T) + \frac{c_2}{c_1}s\}$ that

$$\|\psi(\cdot,s)\|_{\mathrm{L}_1(\bar{\mathcal{W}})} \leq \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^N}.$$

ψ fulfills the assumption of Lemma 7. With ρ, ω defined as in Lemma 7 we have

$$
\begin{aligned}
&\|\psi(\cdot,s)\|_{\mathrm{L}_1(\bar{\mathcal{W}})} \le \|(\rho/\omega)^{-1/2}\|_{\mathrm{L}_2(\bar{\mathcal{W}})}\|(\rho/\omega)^{1/2}\psi(\cdot,s)\|_{\mathrm{L}_2(\mathbb{R})} \\
&\overset{(36)}{\le} \|(\rho/\omega)^{-1/2}\|_{\mathrm{L}_2(\bar{\mathcal{W}})}\mathrm{e}^{-\frac{c_2}{2}(s-s_1)}\|(\rho/\omega)^{1/2}\psi(\cdot,s_1)\|_{\mathrm{L}_2(\mathbb{R})} \\
&\le \sqrt{M_\omega}\Big(\int\limits_{\bar{\mathcal{W}}} \exp(c_1\sqrt{1+(x-x_0)^2})\ dx\Big)^{1/2}\mathrm{e}^{-\frac{c_2}{2}(s-s_1)}\|(\rho/\omega)^{1/2}\psi(\cdot,s_1)\|_{\mathrm{L}_2(\mathbb{R})} \\
&\le \sqrt{\frac{2M_\omega}{c_1}}\mathrm{e}^{\frac{c_2}{2}(s_1)}\frac{1}{(1+T)^N}\|(\rho/\omega)^{1/2}\psi(\cdot,s_1)\|_{\mathrm{L}_2(\mathbb{R})} \le \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^N}
\end{aligned}
$$

To complete the proof of the localization result we have to consider

$$
\|\psi(\cdot,s)\|_{\mathrm{L}_1(\left\{x:|x-x_i''+\lambda_i(\phi(x))(s-s_2^i)|\le N\log(1+T)\sqrt{s-s_2^i}+\frac{1}{2}k_i''+c_{\mathcal{W}}\epsilon(s-s_2^i)\right\})}
$$

for $\psi(\cdot,s):=\psi_i(\cdot,T-s)$ or $\psi(\cdot,s):=\tilde{\psi}_{i,x}(\cdot,T-s)$ $(i=1,\dots,n)$. As shown in [SX] this can be estimated by $\frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^N}$ too. □

6.3 Estimate of the η wave

We are now prepared to start the proof of Lemma 3. We begin by proving a preliminary version of Lemma 3 using induction in time, namely

Lemma 9
If $N>3$, ϵ and (depending on ϵ) θ are sufficiently small then we find a sufficiently large constant $K_{d\tilde{V}}$ (depending on ϵ, but not on θ) such that

$$
|d(x,t)| \ \le\ \theta_0 K_{d\tilde{V}}\begin{cases}\frac{1+\log(1+t)}{(1+t)^{1/2}} & (x,t)\in\mathcal{P}\\ \frac{(1+\log(1+t))^{3/2}}{(1+t)^{1/4}} & (x,t)\notin\mathcal{P}\end{cases} \tag{49a}
$$

$$
|\tilde{V}_i(x,t)| \ \le\ \theta_0 K_{d\tilde{V}}\begin{cases}\frac{(1+\log(1+t))^{3/2}}{(1+t)^{3/4}} & (x,t)\in\cup_{j\ne p}\mathcal{K}^j\\ \frac{1+\log(1+t)}{(1+t)^{1/2}} & \textit{otherwise}\end{cases} \tag{49b}
$$

After we have completed the proof of Lemma 9 we will refine our argumentation (using this Lemma) to complete the proof of Lemma 3. Considering the representation (29) we see that

$$
\begin{aligned}
d_j(x',T) = \int\limits_0^T\int\limits_{-\infty}^{\infty} &\psi_j E_{1,j} - \psi_j l_j\cdot\sum_{i\ne p} f''(\phi)[\theta_i r_i(u_{i,0}),\tilde{V}_i r_i(u_{i,0})] \\
&+\psi_j(\lambda_j-\bar{\lambda}_j)d_{j,x} - \psi_j\lambda_j\sum_{k=1}^n l_j r_{k,x}d_k + \mu 2\psi_j\sum_{k=1}^n l_j r_{k,x}d_{k,x} \\
&+\mu\psi_j\sum_{k=1}^n l_j r_{k,xx}d_k\ dxdt,
\end{aligned}
$$

where $E_{1,j}=-\frac{1}{2}\sum_{j\ne i,p} l_j(\phi)\cdot h''(\phi)[\theta_i r_i(u_{i,0}),\theta_i r_i(u_{i,0})]$, and from (30) an analogous representation for $\tilde{V}_j$ where ψ_j is replaced by $\tilde{\psi}_{j,x}$ and $(E_1)_j$ by $-(E_1)_j$ (considering

(29_p) or (30_p) gives similar representations). Now let $2 \le T_0 \le T_1 \le T \le T_2 := T_1 + 1$ and define

the T_1 *interaction regions*:

$$\mathcal{E}^{p,j}(t) := \mathcal{P}(T_1) \cap \mathcal{W}^j(t) \qquad \mathcal{E}^{p,j} := \cup_{0\le t\le T_1} \mathcal{E}^{p,j}(t)$$
$$\mathcal{E}^{i,j}(t) := \mathcal{K}^i(t) \cap \mathcal{W}^j(t) \qquad \mathcal{E}^{i,j} := \cup_{0\le t\le T_1} \mathcal{E}^{i,j}(t)$$

the T_1 *induction assumptions*:

$$\|d(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{P}(t))} \le \theta_0 K_{d\tilde{V}} \frac{1+\log(1+t)}{(1+t)^{1/2}} \qquad \forall\ 0 \le t \le T_1 \qquad (\mathrm{IV1}[T_1])$$
$$\|\tilde{V}(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{P}(t))} \le \theta_0 K_{d\tilde{V}} \frac{1+\log(1+t)}{(1+t)^{1/2}} \qquad \forall\ 0 \le t \le T_1 \qquad (\mathrm{IV2}[T_1])$$
$$\|\tilde{V}_i(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{K}^i(t))} \le \theta_0 K_{d\tilde{V}} \frac{(1+\log(1+t))^{3/2}}{(1+t)^{3/4}} \qquad \forall\ 0 \le t \le T_1 \qquad (\mathrm{IV3}[T_1])$$
$$\|d(\cdot,t)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \theta_0 K_{d\tilde{V}} \qquad \forall\ 0 \le t \le T_1 \qquad (\mathrm{IV4}[T_1])$$
$$\|\tilde{V}(\cdot,t)\|_{\mathrm{L}_\infty(\mathbb{R})} \le \theta_0 K_{d\tilde{V}} \qquad \forall\ 0 \le t \le T_1 \qquad (\mathrm{IV5}[T_1])$$

the T_1 *ϕ-ψ_j interaction*:

$$\begin{aligned} d^{T_1}_{\phi,j}(x',T) := \int_0^{T_1}\int_{-\infty}^{\infty} & -\psi_j\lambda_j \sum_{k=1}^n l_j r_{k,x} d_k \\ & + 2\mu\psi_j \sum_{k=1}^n l_j r_{k,x} d_{k,x} + \mu\psi_j \sum_{k=1}^n l_j r_{k,xx} d_k \ dxdt \\ \tilde{V}^{T_1}_{\phi,j}(x',T) := \int_0^{T_1}\int_{-\infty}^{\infty} & -\tilde{\psi}_{x,j}\lambda_j \sum_{k=1}^n l_j r_{k,x} d_k \\ & + 2\mu\tilde{\psi}_{x,j} \sum_{k=1}^n l_j r_{k,x} d_{k,x} + \tilde{\psi}_{x,j} \sum_{k=1}^n l_j r_{k,xx} d_k \ dxdt \end{aligned}$$

the T_1 *θ-ψ_j interaction*:

$$d^{T_1}_{\theta,j}(x',T) := \int_0^{T_1}\int_{-\infty}^{\infty} \psi_j(E_1)_j - \psi_j l_j \cdot \sum_{i\ne p} f''(\phi)[\theta_i r_i(u_{i,0}), \tilde{V}_i r_i(u_{i,0})] \ dxdt$$
$$\tilde{V}^{T_1}_{\theta,j}(x',T) := \int_0^{T_1}\int_{-\infty}^{\infty} -\tilde{\psi}_{x,j}(E_1)_j - \tilde{\psi}_{x,j} l_j \cdot \sum_{i\ne p} f''(\phi)[\theta_i r_i(u_{i,0}), \tilde{V}_i r_i(u_{i,0})] \ dxdt$$

6.3.1 Induction basis

By a standard short time estimate we see that for any fixed $T_0 > 0$ there exists a sufficiently large constant $K_{d\tilde{V}}$ such that (IV1[T_1]) - (IV5[T_1]) hold for $T_1 = T_0$.

6.3.2 Induction step

In the integral defining $d^{T_1}_{\phi,j}$ and $d^{T_1}_{\theta,j}$ we see that $d(x,t)$ only enters for $t \le T_1$. Thus, using the induction assumption (IV1[T_1])-(IV5[T_1]), we can prove

Lemma 10 (induction step - part 1)
Let (IV1[T_1]) - (IV5[T_1]) *hold,* $0 < T_0 \le T_1 \le T \le T_1 + 1$. *If* T_0 *is sufficiently large we have*

$$|d^{T_1}_{\phi,j}(x',T)| \le \epsilon \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \begin{cases} \frac{1+\log(1+T)}{(1+T)^{1/2}} & x' \in \mathcal{P}(T) \\ \frac{(1+\log(1+T))^{3/2}}{(1+T)^{1/4}} & x' \notin \mathcal{P}(T) \end{cases} \tag{50}$$

$$|d^{T_1}_{\theta,j}(x',T)| \le C(\epsilon)\theta_0 \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \begin{cases} \frac{1+\log(1+T)}{(1+T)^{1/2}} & x' \in \mathcal{P}(T) \\ \frac{(1+\log(1+T))^{3/2}}{(1+T)^{1/4}} & x' \notin \mathcal{P}(T) \end{cases} \tag{51}$$

$$|\tilde{V}^{T_1}_{\phi,j}(x',T)| \le \epsilon \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \begin{cases} \frac{(1+\log(1+T))^{3/2}}{(1+T)^{3/4}} & (x',T) \in \cup_{i\neq p}\mathcal{K}^i \\ \frac{1+\log(1+T)}{(1+T)^{1/2}} & \textit{otherwise.} \end{cases} \tag{52}$$

$$|\tilde{V}^{T_1}_{\theta,j}(x',T)| \le C(\epsilon)\theta_0 \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \begin{cases} \frac{(1+\log(1+T))^(3/2)}{(1+T)^{3/4}} & (x',T) \in \cup_{i\neq p}\mathcal{K}^i \\ \frac{1+\log(1+T)}{(1+T)^{1/2}} & \textit{otherwise.} \end{cases} \tag{53}$$

We present the proof of this lemma at the end. We now complete the induction step (and thus the proof of Lemma 9) by showing that under the assumption (IV1[T_1])-(IV5[T_1]) we can prove (IV1[T])-(IV5[T]) for any $T_1 \le T \le T_2 := T_1 + 1$ if only ϵ and θ are sufficiently small. Let

$$\mathcal{M}_d := \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathbb{R}}} |d(x',T)| \qquad \mathcal{M}_{d,\mathcal{P}} := \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathcal{P}(T)}} |d(x',T)|$$

$$\mathcal{M}_{\tilde{V}} := \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathbb{R}}} |\tilde{V}(x',T)| \qquad \mathcal{M}_{\tilde{V},\mathcal{P}} := \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathcal{P}(T)}} |\tilde{V}(x',T)|$$

$$\mathcal{M}_{\tilde{V},\mathcal{K}^i} := \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathcal{K}^i(T)}} |\tilde{V}(x',T)|$$

Then we have for any $x' \in \mathbb{R}$ and $T_1 \le T \le T_2$ (considering (29))

$$|d_j(x',T)| \le |d^{T_1}_{\phi,j}(x',T)| + |d^{T_1}_{\theta,j}(x',T)| + \int_0^T \int_{-\infty}^{\infty} |\psi_j||\lambda_j - \bar{\lambda}_j||d_{j,x}| \, dxdt$$
$$+ \int_{T_1}^T \int_{-\infty}^{\infty} \psi_j(E_1)_j - \psi_j l_j \cdot \sum_{i\neq p} f''(\phi)[\theta_i r_i(u_{i,0}), \tilde{V}_i r_i(u_{i,0})]$$

$$- \psi_j \lambda_j \sum_{k=1}^n l_j r_{k,x} d_k + 2\mu\psi_j \sum_{k=1}^n l_j r_{k,x} d_{k,x} + \mu\psi_j \sum_{k=1}^n l_j r_{k,xx} d_k \; dxdt.$$

For the last integral we find

$$\Big| \int_{T_1}^{T} \int_{-\infty}^{\infty} \dots \; dxdt \Big| \le \mathcal{O}(1) \int_{T_1}^{T} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} dt \cdot \Big[\frac{\theta_0^2}{1+T} + \theta_0 (\sum_{i=1}^n \mathcal{M}_{\tilde V, \mathcal{K}^i} + \frac{\mathcal{M}_{\tilde V}}{(1+T)^N}) + \epsilon(\mathcal{M}_{d,\mathcal{P}} + \frac{C(\epsilon)\mathcal{M}_d}{(1+T)^N}) + \epsilon(\mathcal{M}_{\tilde V,\mathcal{P}} + \frac{C(\epsilon)\mathcal{M}_{\tilde V}}{(1+T)^N}) \Big]$$

and for the second last integral

$$\begin{aligned} \int_0^T \int_{-\infty}^{\infty} |\psi_j||\lambda_j - \bar\lambda_j||d_{j,x}| \; dxdt &\le \int_{T-s_1}^{T} \int_{-\infty}^{\infty} |\psi_j||\lambda_j - \bar\lambda_j||d_{j,x}| \; dxdt + \frac{C(\epsilon)\mathcal{M}_{\tilde V}}{(1+T)^{N-1}} \\ &\le \int_{T-s_1}^{T} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} \; dt \epsilon \mathcal{M}_{\tilde V,\mathcal{P}} + \frac{C(\epsilon)\mathcal{M}_{\tilde V}}{(1+T)^{N-1}}. \end{aligned}$$

The same estimate can be derived considering (29_p). Thus we have for all $j = 1, \dots, n$:

$$\begin{aligned} |d_j(x',T)| \le & |d^{T_1}_{\phi,j}(x',T)| + |d^{T_1}_{\theta,j}(x',T)| \\ & + \mathcal{O}(1)(\epsilon + \theta_0)(\mathcal{M}_{d,\mathcal{P}} + \mathcal{M}_{\tilde V,\mathcal{P}} + \sum_{i=1}^n \mathcal{M}_{\tilde V,\mathcal{K}^i}) \\ & + \mathcal{O}(1)\Big(\frac{C(\epsilon)\mathcal{M}_d}{(1+T)^N} + \frac{C(\epsilon)\mathcal{M}_{\tilde V}}{(1+T)^{N-1}} + \frac{\theta_0^2}{1+T} \Big). \end{aligned}$$

In the same way (replacing ψ_j by $\tilde\psi_{j,x}$) we obtain

$$\begin{aligned} |\tilde V_j(x',T)| \le & |\tilde V^{T_1}_{\phi,j}(x',T)| + |\tilde V^{T_1}_{\theta,j}(x',T)| \\ & + \mathcal{O}(1)(\epsilon + \theta_0)(\mathcal{M}_{d,\mathcal{P}} + \mathcal{M}_{\tilde V,\mathcal{P}} + \sum_{i=1}^n \mathcal{M}_{\tilde V,\mathcal{K}^i}) \\ & + \mathcal{O}(1)\Big(\frac{C(\epsilon)\mathcal{M}_d}{(1+T)^N} + \frac{C(\epsilon)\mathcal{M}_{\tilde V}}{(1+T)^{N-1}} + \frac{\theta_0^2}{1+T} \Big). \end{aligned}$$

From this we obtain by $\mathcal{M}_{d,\mathcal{P}} \le \mathcal{M}_d$ etc. that for ϵ and θ_0 sufficiently small and T_0 sufficiently large

$$\begin{aligned} \mathcal{M}_d &\le \mathcal{O}(1) \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathbb{R}}} \left(|d^{T_1}_{\phi,j}(x',T)| + |d^{T_1}_{\theta,j}(x',T)| + \frac{\mathcal{O}(1)\theta_0^2}{1+T} \right) \\ \mathcal{M}_{\tilde V} &\le \mathcal{O}(1) \sup_{\substack{T_1 \le T \le T_2 \\ x' \in \mathbb{R}}} \left(|\tilde V^{T_1}_{\phi,j}(x',T)| + |\tilde V^{T_1}_{\theta,j}(x',T)| + \frac{\mathcal{O}(1)\theta_0^2}{1+T} \right) \end{aligned}$$

and thus that

$$\mathcal{M}_{d,\mathcal{P}} \leq \mathcal{O}(1) \sup_{\substack{T_1 \leq T \leq T_2 \\ x' \in \mathcal{P}(T)}} \left(|d^{T_1}_{\phi,j}(x',T)| + |d^{T_1}_{\theta,j}(x',T)| + \frac{\mathcal{O}(1)\theta_0^2}{1+T} \right)$$

$$\mathcal{M}_{\tilde{V},\mathcal{P}} \leq \mathcal{O}(1) \sup_{\substack{T_1 \leq T \leq T_2 \\ x' \in \mathcal{P}(T)}} \left(|\tilde{V}^{T_1}_{\phi,j}(x',T)| + |\tilde{V}^{T_1}_{\theta,j}(x',T)| + \frac{\mathcal{O}(1)\theta_0^2}{1+T} \right)$$

$$\mathcal{M}_{\tilde{V},\mathcal{K}^i} \leq \mathcal{O}(1) \sup_{\substack{T_1 \leq T \leq T_2 \\ x' \in \mathcal{K}^i(T)}} \left(|\tilde{V}^{T_1}_{\phi,j}(x',T)| + |\tilde{V}^{T_1}_{\theta,j}(x',T)| + \frac{\mathcal{O}(1)\theta_0^2}{1+T} \right)$$

and by Lemma 10 (using $T_1 \geq T_2/2$ to resolve the $\sup_{T_1 \leq T_2}$) we see that (IV1[T_1]) - (IV5[T_1]) hold if ϵ and (depending on ϵ) θ_0 are chosen suitably small and T_0 sufficiently large. Thus we see that (IV1[T_1]) - (IV5[T_1]) holds for all T_1. In particular we obtain (49a), (49b) for $t = T_1$.

To complete the proof it remains to show Lemma 10. We prove Lemma 10 by proving two auxiliary Lemmas that estimate the coupling of the dual waves and the shock and diffusion waves respectively:

Lemma 11 (Estimate for the ψ_j-ϕ_x-coupling)
Let $0 \leq \tau_1 < \tau_2 \leq T_1$ and the induction assumptions (IV1[T_1]) - (IV5[T_1]) *hold. Then we have for $\mathcal{E}^{p,j} \subset \mathbb{R} \times [\tau_1, \tau_2]$ and T_0 sufficiently large:*

$$|d^{T_1}_{\phi,j}(x',T)| \leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\psi_j(\cdot,\tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \Big[\epsilon \frac{1+\log(1+T)}{(1+\tau_1)^{1/2}} + \frac{C(\epsilon)}{(1+\tau_1)^N}\Big] \tag{54}$$

$$|d^{T_1}_{\phi,j}(x',T)| \leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} (1+\log(1+T)) \cdot \Big[\epsilon\sqrt{\tau_2-\tau_1} + \frac{C(\epsilon)}{(1+\tau_1)^{N-1}}\Big] \quad (j \neq p) \tag{55}$$

$$|\tilde{V}^{T_1}_{\phi,j}(x',T)| \leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\tilde{\psi}_{j,x}(\cdot,\tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \Big[\epsilon \frac{1+\log(1+T)}{(1+\tau_1)^{1/2}} + \frac{C(\epsilon)}{(1+\tau_1)^N}\Big] \tag{56}$$

$$|\tilde{V}^{T_1}_{\phi,j}(x',T)| \leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\tilde{\psi}_{j,x}(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} (1+\log(1+T)) \cdot \Big[\epsilon\sqrt{\tau_2-\tau_1} + \frac{C(\epsilon)}{(1+\tau_1)^{N-1}}\Big] \quad (j \neq p) \tag{57}$$

where $\tau_2^* = \min\{\tau_2, T - s_1\}$.

Proof : Using integration by parts we have

$$\begin{aligned}
&|d^{T_1}_{\phi,j}(x',T)| \\
&= \Big| \sum_{k=1}^{n} \int_0^{T_1} \int_{-\infty}^{\infty} -(\psi_j\lambda_j + \mu\psi_{j,x}) l_j r_{k,x} d_k - \mu\psi_j l_{j,x} r_{k,x} d_k + \mu\psi_j l_j r_{k,x} d_{k,x} \; dxdt \Big| \\
&\leq \Big| \sum_{k=1}^{n} \int_{\tau_1}^{\tau_2^*} \int_{-\infty}^{\infty} -(\psi_j\lambda_j + \mu\psi_{j,x}) l_j r_{k,x} d_k - \mu\psi_j l_{j,x} r_{k,x} d_k + \mu\psi_j l_j r_{k,x} d_{k,x} \; dxdt \Big| \\
&\quad + \mathcal{O}(1) \int_{\tau_2^*}^{T_1} \int_{-\infty}^{\infty} |\phi_x(x)|(|d(x,t)| + |\tilde{V}(x,t))||\psi_j(x,t)| dxdt + \frac{\theta_0 K_{d\tilde{V}} \mathcal{O}(1)}{(1+T)^{N-1}}
\end{aligned}$$

For $\tau_2 < T - s_1$ the last integral can be estimated by $\frac{\theta_0 K_{d\tilde{V}} \mathcal{O}(1)}{(1+T)^{N-1}}$. In the other case we use the induction assumption (IV1$[T_1]$), (IV2$[T_1]$) to estimate it by $\epsilon \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \frac{1+\log(1+T)}{(1+T)^{1/2}} \|\psi_j(\cdot, \tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} + \frac{\theta_0 K_{d\tilde{V}} \mathcal{O}(1)}{(1+T)^{N-1}}$.

For the three summands of the first integral we have

$$
\begin{aligned}
&\Big| \int_{\tau_1}^{\tau_2^*} \int_{-\infty}^{\infty} (\lambda_j \psi_j + \mu \psi_{j,x}) l_j r_{k,x} d_k dx dt \Big| \\
&\leq \mathcal{O}(1) \int_{\tau_1}^{\tau_2^*} \int_{\mathcal{P}(t)} |\lambda_j \psi_j + \mu \psi_{j,x}| |\phi_x| |d_k| dx + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\lambda_j \psi_j + \mu \psi_{j,x}| |\phi_x| |d_k| dx dt \\
&\overset{\substack{(\mathrm{IV1}[T_1]) \\ (\mathrm{IV4}[T_1])}}{\leq} \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \int_{\tau_1}^{\tau_2^*} \int_{\mathcal{P}(t)} |\lambda_j \psi_j + \mu \psi_{j,x}| |\phi_x| dx \frac{1+\log(1+t)}{(1+t)^{1/2}} \\
&\qquad + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\lambda_j \psi_j + \mu \psi_{j,x}| |\phi_x| dx dt \\
&\overset{(32b)}{\leq} \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \|\psi_j(\cdot, \tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \sup_{\tau_1 \leq t \leq \tau_2^*} \Big[\epsilon \frac{1+\log(1+t)}{(1+t)^{1/2}} + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\phi_x| dx \Big] \\
&\overset{(13e)}{\leq} \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \|\psi_j(\cdot, \tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \Big[\epsilon \frac{1+\log(1+T)}{(1+\tau_1)^{1/2}} + \frac{C(\epsilon)}{(1+\tau_1)^N} \Big]
\end{aligned}
$$

$$
\begin{aligned}
&\Big| \int_{\tau_1}^{\tau_2^*} \int_{-\infty}^{\infty} \mu \psi_j \sum_{k=1}^{n} l_{j,x} r_{k,x} d_k \ dx \ dt \Big| \\
&\leq \mathcal{O}(1) \int_{\tau_1}^{\tau_2^*} \int_{\mathcal{P}(t)} |\psi_j| |\phi_x| |\phi_x| |d_k| dx + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\psi_j| |\phi_x| |\phi_x| |d_k| dx dt \\
&\overset{\substack{(\mathrm{IV1}[T_1]) \\ (\mathrm{IV4}[T_1])}}{\leq} \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \int_{\tau_1}^{\tau_2^*} \int_{\mathcal{P}(t)} |\psi_j| |\phi_x| |\phi_x| dx \frac{1+\log(1+t)}{(1+t)^{1/2}} \\
&\qquad + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\psi_j| |\phi_x| |\phi_x| dx dt \\
&\leq \|d\|_{\mathrm{L}_\infty(\mathcal{E}^{p,j})} \mathcal{O}(1) \int_{\tau_1}^{\tau_2^*} \int_{-\infty}^{\infty} |\psi_j| |\phi_x| |\phi_x| \ dx \ dt \ + \ \frac{\theta_0 K_{d\tilde{V}} \mathcal{O}(1)}{(1+T)^{N-1}} \\
&\overset{(32d)}{\leq} \mathcal{O}(1) \theta_0 K_{d\tilde{V}} \|\psi_j(\cdot, \tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \sup_{\tau_1 \leq t \leq \tau_2^*} \Big[\epsilon \frac{1+\log(1+t)}{(1+t)^{1/2}} + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\phi_x| dx \Big]
\end{aligned}
$$

$$\stackrel{(13e)}{\leq} \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\psi_j(\cdot,\tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})}\Big[\epsilon\frac{1+\log(1+T)}{(1+\tau_1)^{1/2}} + \frac{C(\epsilon)}{(1+\tau_1)^N}\Big]$$

$$\Big|\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \psi_j \sum_{k=1}^{n} \mu l_j r_{k,x} d_{k,x}\ dx\ dt\Big| = \Big|\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \sum_{k=1}^{n} \mu l_j \nabla r_k[\psi_j\phi_x] d_{k,x}\ dx\ dt\Big|$$

$$\leq \Big|\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \sum_{k=1}^{n} \mu [l_j\nabla r_k]_x(\psi_j\phi_x) d_k\ dx\ dt\Big| + \Big|\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \sum_{k=1}^{n} \mu [l_j\nabla r_k](\psi_j\phi_x)_x d_k\ dx\ dt\Big|$$

The first integral can be estimated as above. Thus we consider

$$\Big|\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \mu \sum_{k=1}^{n} [l_j\nabla r_k](\psi_j\phi_x)_x d_k\ dx\ dt\Big| \leq \mathcal{O}(1)\int_{\tau_1}^{\tau_2^*}\int_{-\infty}^{\infty} \sum_{k=1}^{n} |\psi_j\phi_x)_x||d_k|\ dx\ dt$$

$$\stackrel{(32e)}{\leq} \mathcal{O}(1)\|\psi_j(\cdot,\tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})} \int_{-\infty}^{\infty} |\phi_x| \sup_{\tau_1\leq t\leq\tau_2} |d(x,t)| dx$$

$$\stackrel{\substack{(\mathrm{IV1}[T_1])\\(\mathrm{IV4}[T_1])}}{\leq} \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\psi_j(\cdot,\tau_2^*)\|_{\mathrm{L}_1(\mathbb{R})}\Big[\epsilon\frac{1+\log(1+T)}{(1+\tau_1)^{1/2}} + \frac{C(\epsilon)}{(1+\tau_1)^N}\Big]$$

In sum we conclude (54). Next we prove (55):

$$|d_{\phi,j}^{T_1}(x',T)| \leq \mathcal{O}(1)\int_0^{T_1}\int_{-\infty}^{\infty} |\psi_j||\phi_x|(|d|+|V|)\ dxdt$$

$$\leq \mathcal{O}(1)\int_{\tau_1}^{\tau_2}\int_{-\infty}^{\infty} |\psi_j||\phi_x|(|d|+|V|)\ dxdt\ +\ \frac{\theta_0 K_{d\tilde{V}}\mathcal{O}(1)}{(1+T)^{N-1}}$$

and

$$\int_{\tau_1}^{\tau_2}\int_{-\infty}^{\infty} |\psi_j||\phi_x|(|d|+|V|)\ dxdt$$

$$\leq \mathcal{O}(1)\int_{\tau_1}^{\tau_2}\int_{\mathcal{P}(t)} |\psi_j||\phi_x|(|d|+|V|)\ dx + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\psi_j||\phi_x|(|d|+|V|)\ dxdt$$

$$\stackrel{(42)}{\leq} \mathcal{O}(1)\|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})}\int_{\tau_1}^{\tau_2}\int_{\mathcal{P}(t)} |\phi_x|(|d|+|V|)\ dx + \int_{\mathbb{R}\setminus\mathcal{P}(t)} |\phi_x|(|d|+|V|)\ dxdt$$

$$\leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}}\|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})}\int_{\tau_1}^{\tau_2} \epsilon\frac{1+\log(1+t)}{(1+t)^{1/2}} + \frac{C(\epsilon)}{(1+t)^N}dt$$

$$\leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}}\|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})}(1+\log(1+T))\Big[\epsilon\sqrt{\tau_2-\tau_1} + \frac{C(\epsilon)}{(1+\tau_1)^{N-1}}\Big]$$

The proofs of (56) and (57) are obtained by replacing ψ_j with $\tilde{\psi}_{j,x}$. $\square$

Lemma 12 (Estimate for the ψ_j-θ_i-coupling)
Let $0 \leq \tau_1 < \tau_2 \leq T_1$ and the induction assumptions (IV1[T_1]) - (IV5[T_1]) *hold. Then we have for $\mathcal{E}^{i,j}_{\tau_1,\tau_2} := \mathcal{E}^{i,j} \cap \mathbb{R} \times [\tau_1, \tau_2]$ and T_0 sufficiently large:*

$$\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i|^2 \, dxdt \;\leq\; \theta_0^2 \mathcal{O}(1) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_1(\mathbb{R})} \int\limits_{\tau_1}^{\tau_2} \frac{1}{1+t} dt \tag{58}$$

$$\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i|^2 \, dxdt \;\leq\; \theta_0^2 \mathcal{O}(1) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} \sqrt{\tau_2 - \tau_1} \quad \text{for } j \neq p \tag{59}$$

$$\begin{aligned} &\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i||\tilde{V}_i| \, dxdt \\ &\leq\; \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} (1 + \log(1+T)) \int\limits_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} \frac{1}{(1+t)^{5/4}} \, dt \end{aligned} \tag{60}$$

$$\begin{aligned} &\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i||\tilde{V}_i| \, dxdt \\ &\leq\; \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} (1 + \log(1+T)) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} (\tau_2 - \tau_1)^{1/4} \quad \text{for } j \neq p \end{aligned} \tag{61}$$

The same holds when ψ_j is replaced by $\tilde{\psi}_{j,x}$.

Proof :

$$\begin{aligned} \iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i|^2 \, dxdt \;&\leq\; \int\limits_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathcal{E}^{i,j}(t))} \|\theta_i^2\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \, dt \\ &\overset{(39)}{\leq}\; \theta_0^2 \mathcal{O}(1) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_1(\mathcal{E}^{i,j}(t))} \int\limits_{\tau_1}^{\tau_2} \frac{1}{1+t} \, dt \end{aligned}$$

$$\begin{aligned} \iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i|^2 \, dxdt \;&\leq\; \int\limits_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \|\theta_i^2\|_{\mathrm{L}_1(\mathcal{E}^{i,j}(t))} \, dt \\ &\overset{\substack{(42)\\ j\neq p}}{\leq}\; \theta_0^2 \mathcal{O}(1) \; \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} \int\limits_{\tau_1}^{\tau_2} \frac{1}{\sqrt{1+t}} \, dt \\ &\leq\; \theta_0^2 \mathcal{O}(1) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} \sqrt{\tau_2 - \tau_1} \end{aligned}$$

$$
\begin{aligned}
&\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i||\tilde{V}_i| \; dxdt \\
&\quad \le \int_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathcal{E}^{i,j}(t))} \|\theta_i(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \|\tilde{V}_i(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \; dt \\
&\quad \le \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} (1+\log(1+T)) \int_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})} \frac{1}{(1+t)^{5/4}} \; dt
\end{aligned}
$$

$$
\begin{aligned}
&\iint\limits_{\mathcal{E}^{i,j}_{\tau_1,\tau_2}} |\psi_j||\theta_i||\tilde{V}_i| \; dxdt \\
&\quad \le \int_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \|\theta_i(\cdot,t)\|_{\mathrm{L}_1(\mathcal{E}^{i,j}(t))} \|\tilde{V}_i(\cdot,t)\|_{\mathrm{L}_\infty(\mathcal{E}^{i,j}(t))} \; dt \\
&\quad \le \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} \int_{\tau_1}^{\tau_2} \|\psi_j(\cdot,t)\|_{\mathrm{L}_\infty(\mathbb{R})} \frac{(1+\log(1+t))}{(1+t)^{3/4}} \; dt \\
&\quad \overset{\substack{(42)\\ j\neq p}}{\le} \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} (1+\log(1+T)) \|\psi_j(\cdot,\tau_2)\|_{\mathrm{L}_\infty(\mathbb{R})} (\tau_2-\tau_1)^{1/4}
\end{aligned}
$$

□

Using Lemma 11 and 12 we prove Lemma 10 and thus complete the induction argument above. In the following proofs of Lemma 10 and Lemma 3 we will focus on the interaction between the shock and the dual waves, because the interaction between dual waves and diffusion waves can be treated as in the case of a convex mode – see [SX].

Proof of Lemma 10: We first consider a j dual wave interacting with the shock region:

Due to the width of the shock-region we have for $j = p$ we have

$$\mathcal{E}^{p,j} = \mathcal{E}^{p,p} \subset \mathbb{R} \times [T - C(\epsilon)\mathcal{O}(1)\log(1+T), T] \subset \mathbb{R} \times [\frac{1}{4}T, T].$$

for $T_0 = T_0(\epsilon)$ sufficiently large (this is clear for $x' \in \mathcal{P}(T)$ and in the other case we have $\mathcal{E}^{p,j} = \emptyset$). Due to the width of the dual wave region ($\approx \mathcal{O}(1)\log(1+T)\sqrt{T}$ for $T \ge T_0(\epsilon)$ sufficiently large) we see that for $j \neq p$

$$\mathcal{E}^{p,j} \subset \mathbb{R} \times [\tau_0, \tau_0 + \mathcal{O}(1)\log(1+T)\sqrt{T}]$$

(for some τ_0). Thus we have for $T_0 = T_0(\epsilon)$ sufficiently large that for $j \neq p$

$$\text{either} \quad \mathcal{E}^{p,j} \subset \mathbb{R} \times [\frac{1}{4}T, T] \quad \text{or} \quad \mathcal{E}^{p,j} \subset \mathbb{R} \times [0, \frac{3}{4}T].$$

In the first case we use (54) and obtain for $T \geq T_0$ and T_0 sufficiently large

$$|d_{\phi,j}^{T_1}(x',T)| \leq \epsilon\mathcal{O}(1)\theta_0 K_{d\tilde{V}} \frac{1+\log(1+T)}{(1+T)^{1/2}}.$$

In the second case we use (55) to obtain

$$\begin{aligned}
&|d_{\phi,j}^{T_1}(x',T)| \\
&\leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \|\psi_j(\cdot,\frac{3}{4}T)\|_{\mathrm{L}_\infty(\mathbb{R})}(1+\log(1+T))[\epsilon\sqrt{\sqrt{T}} + \frac{C(\epsilon)}{(1+\tau_1)^{N-1}}] \\
&\leq \mathcal{O}(1)\theta_0 K_{d\tilde{V}} \frac{1+\log(1+T)}{\sqrt{1+T}}[\epsilon\sqrt{\sqrt{T}} + C(\epsilon)] \ \leq \ \epsilon\mathcal{O}(1)\theta_0 K_{d\tilde{V}} \frac{1+\log(1+T)}{(1+T)^{1/4}}.
\end{aligned}$$

Now (50) follows immediately from the fact that

$$x' \in \mathcal{P}(T) \ \Rightarrow \ \mathcal{E}^{p,j} \subset \mathbb{R} \times [\frac{1}{4}T, T] \qquad\qquad \forall\, j = 1, \ldots, n.$$

(52) follows from the observation that we have for $\mathcal{E}^{p,j} \subset \mathbb{R} \times [0, \gamma_2 T]$ ($\gamma_2 < 1$ fixed)

$$\begin{aligned}
&|\tilde{V}_{\phi,j}^{T_1}(x',T)| \leq \epsilon\mathcal{O}(1)\theta_0 K_{d\tilde{V}}(1+\log(1+T))\sqrt{\sqrt{T}}\|\tilde{\psi}_{j,x}(\cdot,\gamma_2 T)\|_{\mathrm{L}_\infty(\mathbb{R})} \\
&\overset{(56)}{\leq} \epsilon\mathcal{O}(1)K_{d\tilde{V}}(1+\log(1+T))\sqrt{\sqrt{T}}\frac{1}{1+T} \leq \epsilon\mathcal{O}(1)\theta_0 K_{d\tilde{V}} \frac{1+\log(1+T)}{(1+T)^{3/4}}
\end{aligned}$$

and

$$x' \in \mathcal{K}^i(T) \ \Rightarrow \ \mathcal{E}^{p,j} \subset \mathbb{R} \times [0, \gamma_2 T] \qquad\qquad \forall\, i,j \neq p$$

with γ_2 independent of ϵ.

We now turn to the interaction of the j dual wave with the i diffusion wave. Note that we have for $i \neq j$

$$\mathcal{E}^{i,j} \subset \mathbb{R} \times [\tau_0, \tau_0 + C(\epsilon)\mathcal{O}(1)\log(1+T)\sqrt{T}]$$

(for some τ_0) and thus either $\mathcal{E}^{i,j} \subset \mathbb{R} \times [\gamma(\epsilon)T, T]$ or $\mathcal{E}^{i,j} \subset \mathbb{R} \times [0, \frac{3}{4}T]$, where $\gamma(\epsilon) > 0$. In the first case we have

$$\begin{aligned}
&\Big|\int_0^{T_1}\int_{-\infty}^{\infty} \psi_j(E_1)_j \ dxdt\Big| \\
&\leq \ \mathcal{O}(1)\sum_{i\neq j}\int_0^{T_1}\int_{-\infty}^{\infty} |\psi_j||\theta_i|^2 \ dxdt \ \leq \ \mathcal{O}(1)\sum_{i\neq j}\iint_{\mathcal{E}^{i,j}} |\psi_j||\theta_i|^2 \ dxdt + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\overset{(58)}{\leq} \sum_j \theta_0^2\mathcal{O}(1)C(\epsilon)\|\psi_j(\cdot,\tau^*)\|_{\mathrm{L}_1(\mathbb{R})}\frac{\log(1+T)\sqrt{T}}{1+T} \\
&\qquad + \mathcal{O}(1)\int_{T-s_1}^{T} \|\psi_j(\cdot,t)\|_{\mathrm{L}_1(\mathbb{R})}dt\frac{\theta_0^2}{1+T} + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\leq \theta_0^2 C(\epsilon)\mathcal{O}(1)\frac{\log(1+T)}{\sqrt{1+T}}
\end{aligned}$$

where $\tau^* = \min\{\tau_2, T - s_1\}$ for τ_2 such that $\mathcal{E}^{i,j} \subset \mathbb{R} \times [0, \tau_2]$. The second case can only occur for $j \neq p$ and thus we have

$$\begin{aligned}
\Big|\int_0^{T_1}\int_{-\infty}^{\infty} \psi_j (E_1)_j \ dxdt\Big| \ &\leq\ \mathcal{O}(1) \sum_{i\neq j} \iint_{\mathcal{E}^{i,j}} |\psi_j||\theta_i|^2 \ dxdt + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\overset{(59)}{\leq}\ \sum_j \theta_0^2 \mathcal{O}(1) C(\epsilon) \|\psi_j(\cdot, \frac{3}{4}T)\|_{\mathrm{L}_\infty(\mathbb{R})} \sqrt{\sqrt{T}} + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\leq\ \theta_0^2 C(\epsilon)\mathcal{O}(1) \frac{\log(1+T)}{(1+T)^{1/4}}
\end{aligned}$$

Furthermore we have for any $i, j \neq p$

$$\begin{aligned}
\Big|\int_0^{T_1}\int_{-\infty}^{\infty} \psi_j \theta_i \tilde{V}_i \ dx\ dt\Big| &\leq \iint_{\mathcal{E}^{i,j}} |\psi_j||\theta_i||\tilde{V}_i| \ dxdt + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\leq \int_0^{\frac{1}{2}T}\int_{\mathcal{E}^{i,j}(t)} |\psi_j||\theta_i||\tilde{V}_i| \ dxdt + \int_{\frac{1}{2}T}^{T_1}\int_{\mathcal{E}^{i,j}(t)} |\psi_j||\theta_i||\tilde{V}_i| \ dxdt + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}} \\
&\overset{(60),(61)}{\leq} \theta_0^2 \mathcal{O}(1) K_{d\tilde{V}} \frac{\log(1+T)}{(1+T)^{1/4}} + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}}
\end{aligned}$$

and for any $i \neq j$ and $\mathcal{E}^{i,j} \subset \mathbb{R} \times [\gamma T, T]$ (from (60))

$$\Big|\int_0^{T_1}\int_{-\infty}^{\infty} \psi_j \theta_i \tilde{V}_i \ dx\ dt\Big| \leq \theta_0 \mathcal{O}(1)^2 K_{d\tilde{V}} \frac{\log(1+T)}{(1+T)^{3/4}} + \frac{C(\epsilon)\mathcal{O}(1)}{(1+T)^{N-1}}$$

(51) now follows from

$$x' \in \mathcal{P}(T) \ \Rightarrow\ \mathcal{E}^{i,j} \subset \mathbb{R} \times [\gamma T, T] \text{ for all } i \neq p \text{ and } \mathcal{E}^{i,i} = \emptyset$$

with $\gamma = \gamma(\epsilon)$. The estimate (53) follows similar to that noting that we gain a factor of $1/\sqrt{(1+T)}$ when replacing ψ_j by $\tilde{\psi}_{j,x}$ as in the proof of (51) (or see [SX]). □

Using Lemma 9 we can proof a first part of Lemma 3:

$$|\eta(x,t)| \ \leq\ \mathcal{O}(1)(|\phi_x||d| + |\tilde{V}|) \ \leq\ \theta_0 C(\epsilon)\mathcal{O}(1) \frac{1 + \log(1+t)}{(1+t)^{1/2}}.$$

We complete the proof of Lemma 3 by proving the L_2 estimates. Now that the above induction argument is compete, we can use Lemma 11 and 12 for $T_1 = T$ and $\forall\ T > 0$. Noting that

$$\tilde{V}_j(x', T) = \tilde{V}^T_{\phi,j}(x', T) + \tilde{V}^T_{\theta,j}(x', T) + \int_0^T\int_{-\infty}^{\infty} \tilde{\psi}_{j,x}(\lambda_j - \bar{\lambda}_j) d_{j,x} \ dxdt$$

we have

$$|\tilde{V}_j(x',T)| \le |\tilde{V}^T_{\phi,j}(x',T)| + \mathcal{O}(1)\sum_{i\neq p}|\tilde{V}^T_{\theta_i,j}(x',T)| + \mathcal{O}(1)\sum_{i=1}^{n}|\tilde{V}^T_{\tilde{V}_i,j}(x',T)| + |\int_0^T\int_{-\infty}^{\infty}\tilde{\psi}_{j,x}(\lambda_j-\bar{\lambda_j})d_{j,x}\ dxdt|,$$

where $\tilde{V}^T_{\theta_i,j}(x',T) = \int_{-\infty}^{\infty}\int_0^T|\tilde{\psi}_{j,x}||\theta_i|^2\ dxdt$ and $\tilde{V}^T_{\tilde{V}_i,j}(x',T) = \int_{-\infty}^{\infty}\int_0^T|\tilde{\psi}_{j,x}||\theta_i||\tilde{V}_i|\ dxdt$. Furthermore

$$\|\tilde{V}_j(\cdot,T)\|^2_{\mathrm{L}_2} \le \mathcal{O}(1)\big(\|\tilde{V}^T_{\phi,j}(\cdot,T)\|^2_{\mathrm{L}_2} + \sum_{i\neq p}\|\tilde{V}^T_{\theta_i,j}(\cdot,T)\|^2_{\mathrm{L}_2} + \sum_{i=1}^{n}\|\tilde{V}^T_{\tilde{V}_i,j}(\cdot,T)\|^2_{\mathrm{L}_2} + \frac{(1+\log(1+T))^2}{(1+T)}\big).$$

For $\mathcal{E}^{p,j}\subset\mathbb{R}\times[\tau_1,\tau_2]$ we have

$$|\tilde{V}^T_{\phi,j}(x',T)| \overset{(56)}{\le} \epsilon\mathcal{O}(1)\theta_0 K_{d\tilde{V}}\frac{1+\log(1+T)}{(1+\tau_1)^{1/2}(1+T-\tau_2)^{1/2}}.$$

and thus for $j\neq p$

$$|\tilde{V}^T_{\phi,j}(x',T)| \le \theta_0 C(\epsilon)(1+\log(1+T))$$
$$\big((\max\{1,T-\frac{1}{\lambda'_{j,0}}(x'-x_0)-c\sqrt{T}\log(1+T)\})^{1/2}(\max\{1,\frac{1}{\lambda'_{j,0}}(x'-x_0)\})^{1/2}\big)^{-1}$$

for $x'\in[x_0+\lambda'_{j,0}T+\operatorname{sign}(p-j)c\sqrt{T}\log(1+T), x_0-\operatorname{sign}(p-j)C(\epsilon)\log(1+T)]$ (note that for $j=p$ the case is even simpler because we have $\tau_1>\frac{1}{4}T$ if T_0 is sufficiently large). For $x'\notin[x_0-2\max|\lambda_k|T, x_0+2\max|\lambda_k|T]$ there is no interaction at all ($\mathcal{E}^{i,j}=\emptyset$) and since the dual waves are explicit we see that $|\tilde{V}_j(x',T)|\le\theta_0\mathcal{O}(1)\exp(-c(\epsilon)(|x-x_0|+T))$.

Combining these two observations with Lemma 9 we get

$$\|\tilde{V}^T_{\phi,j}(\cdot,T)\|^2_{\mathrm{L}_2(\mathbb{R})} \le \theta_0 C(\epsilon)\frac{(1+\log(1+T))^3}{1+T}.$$

In the same way one can proceed with $\tilde{V}^T_{\theta_i,j}(x',T)$ and $\tilde{V}^T_{\tilde{V}_i,j}(x',T)$ leading to

$$\|\tilde{V}_j(\cdot,t)\|^2_{\mathrm{L}_2(\mathbb{R})} \le \theta_0 C(\epsilon)\frac{(1+\log(1+t))^3}{1+t}.$$

Thus we have

$$\|\eta(\cdot,t)\|^2_{\mathrm{L}_2(\mathbb{R})} \le \|\phi_x d(\cdot,t)\|^2_{\mathrm{L}_2(\mathbb{R})} + \|\tilde{V}_j(\cdot,t)\|^2_{\mathrm{L}_2(\mathbb{R})} \le \theta_0 C(\epsilon)\frac{(1+\log(1+t))^3}{1+t}.$$

Furthermore we have for $j \neq i$

$$\|\theta_j(\cdot,t)\tilde{V}_i(\cdot,t)\|^2_{L_2} \leq \mathcal{O}(1) \int\limits_{\mathcal{K}_j(t)} |\theta_j(x,t)|^2 |\tilde{V}_i(x,t)|^2 dx + \theta_0^2 \mathcal{O}(1) \frac{(1+\log(1+t))^2}{(1+t)^{N+1}}$$

$$\leq \int\limits_{\mathcal{K}_j(t)} \mathcal{O}(1)\theta_0^2 \frac{(1+\log(1+t))^2}{(1+t)|x-x_0-\lambda_i t|} dx + \theta_0^2 \frac{(1+\log(1+t))^2}{(1+t)^{N+1}}$$

$$\leq \theta_0^2 \mathcal{O}(1) \frac{(1+\log(1+t))^3}{(1+t)^{5/2}}$$

completing the proof of Lemma 3.

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