Structural Stability and Regularity of Entropy Solutions to Hyperbolic Systems of Conservation Laws

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Abstract. The paper is concerned with the qualitative structure of entropy solutions to a strictly hyperbolic, genuinely nonlinear system of conservation laws. We first give an accurate description of the local and global wave-front structure of a BV solution, generated by a front tracking algorithm. We then consider a sequence of exact or approximate solutions u_ν, converging to a solution u in $\mathbf{L}^1$. The convergence of the wave-fronts of u_ν to the corresponding fronts of u is studied, proving a structural stability result in a neighborhood of each point in the t-x plane.

This research was supported in part by TMR project HCL # ERBFMRXCT960033.

1 - Introduction.

This paper is concerned with the qualitative structure of entropy solutions to a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + \big[f(u)\big]_x = 0, \qquad u(t,x) \in I\!R^n, \quad t \geq 0,\, x \in I\!R. \tag{1.1}$$

We consider solutions having small total variation and assume that every characteristic field of (1.1) is genuinely nonlinear [18].

The recent analysis in [5, 6, 7, 10] shows that solutions constructed by wave-front tracking [3, 4] or by the Glimm scheme [16] are unique and depend continuously on the initial data, in the $\mathbf{L}^1$ norm. The main question addressed in this paper is whether this continuous dependence carries over to the wave-front structure of the solutions. In particular, consider a sequence of (exact) entropy solutions u_ν converging to u in $\mathbf{L}^1_{loc}$ as $\nu \to \infty$. Can one establish a one-to-one correspondence between the shock fronts of u and those of u_ν or, else, is it possible for an arbitrarily small perturbation (in the $\mathbf{L}^1$ norm) to destroy the wave-front configuration of the reference solution u? An answer will be provided by introducing a suitable structural stability condition on the function u.

A related question is whether the total amount of i-waves in u_ν, regarded as a positive Radon measure, converges weakly to the amount of i-waves in u. In general, this is not the case.

Example 1. For the Burgers' equation $u_t + (u^2/2)_x = 0$, consider a sequence of (saw-tooth shaped) entropy solutions, defined by

$$u_\nu(t,x) \doteq \frac{\psi(\nu x)}{\nu(t+1)}, \qquad t \geq 0,$$
$$\psi(x) \doteq x - p \qquad \text{if } x \in \,]p - 1/2,\ p + 1/2], \quad p \in Z\!\!\!Z.$$

Clearly, $u_\nu \to 0$ in $\mathbf{L}^1_{loc}$ as $\nu \to \infty$. For each $t \geq 0$, introducing the distributional derivatives $\mu_\nu(t) \doteq D_x u_\nu(t,\cdot)$, we thus have $\mu_\nu(t) \rightharpoonup 0$ in the sense of weak convergence of measures. On the other hand, the corresponding measures of total variation satisfy $|\mu_\nu(t)| \rightharpoonup \mu_\infty(t) \neq 0$. Indeed, in this case, $\mu_\infty(t)$ is the measure with constant density $2/(t+1)$ w.r.t. the Lebesgue measure.

Our results will show that, if $u_\nu \to u$, for the corresponding measures of total variation one may have $|\mu_\nu| \rightharpoonup \mu_\infty \neq |\mu|$, in general. However, the atomic parts of the measures $|\mu|$ and μ_∞ do coincide. Genuinely nonlinear systems of conservation laws thus exhibit a somewhat intermediate behavior between parabolic systems, where convergence of solutions implies convergence of their derivatives, and linear hyperbolic systems, where convergence of solutions has no similar implications concerning derivatives.

Another issue addressed in this paper is the regularity of an entropy solution $u(t,x)$ of (1.1). We recall that, for solutions of 2×2 systems constructed by the Glimm scheme, a detailed qualitative description was obtained by DiPerna [13]. In particular, there exists a countable number of shock curves $x = y_m(t)$, $t\in[a_m, b_m]$, such that :

(i) Each function y_m is Lipschitz continuous. At all (except countably many) times $t\in[a_m,b_m]$, the right and left limits $u^\pm = u\big(t, y_m(t)\pm\big)$ satisfy the Rankine-Hugoniot relations

$$f(u^+) - f(u^-) = \dot y_m\big(u^+ - u^-\big). \tag{1.2}$$

(ii) The function u is continuous outside the set $\big\{(t, y_m(t));\quad t\in[a_m,b_m],\quad m\geq 1\big\}$.

These results were extended by Liu [22] to $n\times n$ systems that need not be genuinely nonlinear. For scalar equations, the structure of solutions was analyzed by Dafermos [12] and Schaeffer [25].

To address the questions of structural stability, we first need to refine the regularity results in [13, 22]. By studying the piecewise constant approximations generated by the front tracking algorithm and then taking the limit, we shall derive various qualitative properties of the entropy solutions of (1.1). We recall that solutions constructed by wave front tracking coincide with those obtained by the Glimm scheme [5]. The uniqueness also holds in the larger class of entropy weak solutions which satisfy the Tame Variation Assumption introduced in [11]. For further uniqueness results see also [14, 19, 20, 21].

Some basic notations are now recalled. Let $A(u) = Df(u)$ be the Jacobian matrix of the flux function $f : I\!R^n \mapsto I\!R^n$ at u. Smooth solutions of (1.1) satisfy the equivalent quasilinear system

$$u_t + A(u)u_x = 0. \tag{1.3}$$

Call $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of $A(u)$ and choose right and left eigenvectors $r_i(u)$ and $l_i(u)$, $i = 1,\ldots,n$, normalized so that

$$|r_i|\equiv 1, \qquad \langle l_i,\ r_j\rangle \equiv \begin{cases} 1 & \text{if}\quad i=j,\\ 0 & \text{if}\quad i\neq j.\end{cases} \tag{1.4}$$

The assumption of genuine nonlinearity provides the existence of a constant $c_0>0$ such that, for all u,

$$\nabla\lambda_i(u)\cdot r_i(u) \ \doteq\ \lim_{\varepsilon\to 0+}\frac{\lambda_i\big(u+\varepsilon\, r_i(u)\big)-\lambda_i(u)}{\varepsilon}\ \geq\ c_0>0, \qquad i=1,\ldots,n. \tag{1.5}$$

By $\sigma\mapsto S_i(\sigma)(u)$ and $\sigma\mapsto R_i(\sigma)(u)$ we denote respectively the i-shock and the i-rarefaction curves through the state u, parametrized by arc-length. We write

$$\Psi_i(\sigma)(u) \doteq \begin{cases} R_i(\sigma)(u) & \text{if}\qquad \sigma\geq 0,\\ S_i(\sigma)(u) & \text{if}\qquad \sigma<0\end{cases} \tag{1.6}$$

for the i-wave curve through u. For simplicity we assume that all of the wave speeds are < 1 in absolute value. This is not restrictive because it can always be achieved by a linear rescaling of the time variable in (1.1).

Following [2, 8, 9, 26], the definition of the Glimm functionals can be extended to general BV functions. Let $u : \mathbb{R} \mapsto \mathbb{R}^n$ be a function of bounded variation. Then $\mu \doteq D_x u$ is a vector-valued measure, which can be decomposed into a continuous and an atomic part : $\mu = \mu_c + \mu_a$. For $i = 1, \ldots, n$, define the signed measure $\mu^i = \mu^i_c + \mu^i_a$ as follows. The continuous part of μ^i is the Radon measure such that

$$\int_{\mathbb{R}} \phi \, d\mu^i_c = \int_{\mathbb{R}} \phi \, l_i(u) \cdot d\mu_c \tag{1.7}$$

for every scalar continuous function ϕ having compact support. The atomic part of μ^i is the measure concentrated on the countable set $\{x_\alpha;\ \alpha = 1, 2, \ldots\}$ where u has a jump, and $\mu^i_a\big(\{x_\alpha\}\big)$ is the strength of the i-th wave in the solution of the Riemann problem with data $u(x_\alpha -)$ and $u(x_\alpha +)$. According to the above definition, one thus has a representation of the form

$$\int \phi \, d\mu^i = \int \phi(x) \tilde{l}_i(x) \cdot du(x), \tag{1.8}$$

where $x \mapsto \tilde{l}_i(x)$ is a BV function (defined for every x) satisfying

$$\Big| \tilde{l}_i(x) - l_i\big(u(x-)\big) \Big| \leq C_1 \big| u(x+) - u(x-) \big| \tag{1.9}$$

for some constant C_1. In particular, $\tilde{l}_i(x) = l_i\big(u(x)\big)$ at points where u is continuous. Note that the right hand side of (1.8) is defined as a Borel-Stieltjes integral.

Call μ^{i+} and μ^{i-} the positive and negative parts of the signed measure μ^i, respectively, so that

$$\mu^i = \mu^{i+} - \mu^{i-}, \qquad |\mu^i| = \mu^{i+} + \mu^{i-}, \qquad \mu^{i\pm} \geq 0. \tag{1.10}$$

The *total strength of waves* in u is defined as

$$V(u) \doteq \sum_{i=1}^n V^i(u), \qquad V^i(u) \doteq |\mu^i|(\mathbb{R}), \tag{1.11}$$

while the *interaction potential* of waves in u is

$$Q(u) \doteq \sum_{1 \leq i < j \leq n} \big(|\mu^j| \otimes |\mu^i|\big)\Big(\big\{(x,y);\ \ x < y\big\}\Big) + \sum_{i=1}^n \big(\mu^{i-} \otimes |\mu^i|\big)\Big(\big\{(x,y);\ \ x \neq y\big\}\Big). \tag{1.12}$$

More generally, the total strength of waves in u inside a Borel set $J \subset \mathbb{R}$ is written

$$V(u;J) \doteq \sum_{i=1}^n V^i(u;J), \qquad V^i(u;J) = |\mu^i|(J). \tag{1.13}$$

We denote by $Q(u;J)$ the interaction potential of u restricted to pairs of waves both contained inside J (i.e., in the right hand side of (1.12), the sets of integration are restricted to $x, y \in J$). Observe that $J \mapsto V(u;J)$ defines a measure on $I\!R$, equivalent to the measure of total variation of u, and that $Q(u;J) \leq \big[V(u;J)\big]^2$.

For solutions with small total variation, obtained as limit of wave-front tracking approximations, the Glimm interaction estimates are well known. In particular, they imply

Proposition 1. *There exist constants $C_0, \delta_0 > 0$ such that, for every solution $u = u(t,x)$ of* (1.1) *satisfying*

$$\Upsilon\big(u(t)\big) \doteq V\big(u(t)\big) + C_0 Q\big(u(t)\big) \leq \delta_0 \qquad t \geq 0, \tag{1.14}$$

the following holds. Let $t = \Lambda_k(x)$, $k = 1, 2$, be the equations of two space-like curves in the t-x plane, with $\Lambda_1 \leq \Lambda_2$. Calling $u_k(x) = u\big(\Lambda_k(x), x\big)$, one then has

$$\begin{aligned} Q(u_2) &\leq Q(u_1), \\ V(u_2) + C_0 Q(u_2) &\leq V(u_1) + C_0 Q(u_1), \\ V^i(u_2) + C_0 Q(u_2) &\leq V^i(u_1) + C_0 Q(u_1), \qquad i = 1, \ldots, n. \end{aligned} \tag{1.15}$$

Throughout this paper we deal with solutions of (1.1) satisfying the uniform bounds (1.14) for all $t \geq 0$ and suitable constants $C_0 \geq 1$ and δ_0. All constants introduced below will solely depend upon the flux f and upon δ_0, but are independent of the solution under consideration. For each $t \geq 0$, $u(t, \cdot)$ is a BV function, in principle defined up to $\mathbf{L}^1$ equivalence. For sake of definiteness, we shall always work with right-continuous representatives, so that $u(t,x) = u(t, x+)$ for all t, x.

Our first results are concerned with the structure of these solutions in the neighborhood of a point, and along shock curves.

Theorem 2 (Local structure of solutions). *Let $u = u(t,x)$ be a solution of* (1.1). *Fix a point (τ, ξ) and consider the rescaled functions*

$$u^\eta(t,x) \doteq u\big(\tau + \eta t,\ \xi + \eta x\big), \qquad \eta > 0. \tag{1.16}$$

Then, as $\eta \to 0+$, the functions u^η converge in $\mathbf{L}^1_{loc}$ for all $t \in I\!R$ to a self-similar solution $\tilde u : I\!R \times I\!R \mapsto I\!R^n$ of (1.1). *On the upper half plane $\big\{t \geq 0\big\}$, the function $\tilde u$ coincides with the standard solution of the Riemann problem with data $u^- \doteq u(\tau, \xi-)$ and $u^+ \doteq u(\tau, \xi+)$. On the lower half plane $\big\{t < 0\big\}$, $\tilde u$ contains only negative waves, i.e. shock waves and centered compression waves.*

For every t, the measures $\mu^{i\pm}(u^\eta, t)$ of positive and negative i-waves in $u^\eta(t,\cdot)$ converge in the weak topology of measures:

$$\mu^{i+}(u^\eta, t) \rightharpoonup \mu^{i+}(\tilde u, t), \qquad \mu^{i-}(u^\eta, t) \rightharpoonup \mu^{i-}(\tilde u, t), \qquad \text{as } \eta \to 0+\,. \tag{1.17}$$

All of the properties stated in Theorem 2 and below are particular to entropy solutions and are not shared by arbitrary BV functions (for the latter, see [15, 28]). In particular, for an arbitrary BV function $u(t,x)$, the whole family of rescaled functions u^η need not be convergent, but may have several subsequences converging to distinct limits.

Theorem 3 (Structure of shock curves). *Let $u = u(t,x)$ be a solution of (1.1) and let $\varepsilon > 0$ be given. Then there exists a finite number of Lipschitz continuous curves, $x = y_m(t)$ for $t \in [a_m, b_m]$, $m = 1, \ldots, \bar{m}$, such that the following holds.*

For each m and all (except countably many) times $t \in [a_m, b_m]$, the derivative $\dot{y}_m(t)$ and the right and left limits

$$u^- \doteq \lim_{\substack{(s,y)\to(t,y_m(t))\\ y<y_m(s)}} u(s,y), \qquad\qquad u^+ \doteq \lim_{\substack{(s,y)\to(t,y_m(t))\\ y>y_m(s)}} u(s,y) \tag{1.18}$$

exist. The states u^- and u^+ determine a shock wave of some characteristic family, with size $\sigma_m(t) \leq -\varepsilon/2$, and the Rankine-Hugoniot relations (1.2) hold. The total variation of the maps $t \mapsto \sigma_m(t)$ and $t \mapsto \dot{y}_m(t)$ is bounded.

At each point (τ, ξ) outside the set $\Gamma_\varepsilon \doteq \big\{(t, y_m(t));\ \ t \in [a_m, b_m],\ \ m = 1, \ldots, \bar{m}\big\}$ and outside a finite set $\Theta_\varepsilon \doteq \big\{(\tau_\ell, \xi_\ell);$ the function u has small oscillation:

$$\limsup_{(t,x)\to(\tau,\xi)} \big|u(t,x) - u(\tau,\xi)\big| \leq 5\,\varepsilon. \tag{1.19}$$

By applying Theorem 3 in connection with a sequence $\varepsilon_k \to 0$ we reach the following:

Corollary 4 (Regularity of solutions). *Let u be a solution of (1.1). Then there exists a countable set $\Theta \doteq \big\{(t_\ell, x_\ell);\ \ell \geq 1\big\}$ of interaction points and a countable family of Lipschitz continuous shock curves, $\Gamma \doteq \big\{x = y_m(t);\ t \in [a_m, b_m],\ m \geq 1\big\}$ such that the following holds. For each m and each $t \in [a_m, b_m]$ such that $\big(t, y_m(t)\big) \notin \Theta$, the left and right limits (1.18) of u at $\big(t, y_m(t)\big)$ and the shock speed $\dot{y}_m(t)$ exist and satisfy the Rankine-Hugoniot relations (1.2). Moreover, u is continuous outside the set $\Theta \cup \Gamma$.*

The next result shows that the shock curves are preserved under small $\mathbf{L}^1$ perturbations.

Theorem 5 (Stability of shock curves). *Consider a sequence of exact solutions (or wave-front tracking approximations) u_ν converging to u in $\mathbf{L}^1_{loc}$.*

(i) For $\nu \geq 1$, let $y_\nu : [t_\nu^-, t_\nu^+] \mapsto I\!R$ be a shock curve of u_ν of uniformly large strength: $\big|\sigma_\nu(t)\big| > \varepsilon$ for a.e. $t \in [t_\nu^-, t_\nu^+]$ and for some fixed $\varepsilon > 0$. Assume $t_\nu^- \to t^-$, $t_\nu^+ \to t^+$, and $y_\nu(t) \to y(t)$ for every $t \in]t^-, t^+[$. Then $y(\cdot)$ is a shock curve of the limiting solution u. That is, for all but countably many times, the derivative $\dot{y}$ exists together with distinct right and left limits $u^\pm$, satisfying the Rankine-Hugoniot relations.

(ii) Viceversa, let $y : [t^-, t^+] \mapsto I\!R$ *be a shock curve of* u *with strength* $|\sigma(t)| > \varepsilon$ *for a.e.* $t \in [t^-, t^+]$ *and for some fixed* $\varepsilon > 0$. *Then there exists a sequence of shock curves* $y_\nu : I_\nu \mapsto I\!R$ *of* u_ν *such that*

$$\limsup_{\nu\to\infty} I_\nu \supseteq [t^-, t^+], \qquad |\sigma_\nu(t)| \geq c_0\varepsilon, \qquad \lim_{\nu\to\infty} y_\nu(t) = y(t) \tag{1.20}$$

for some constant $c_0 > 0$ and a.e. $t \in [t^-, t^+]$.

In the second part of this paper, relying on Theorem 2, we study the structural stability of an entropy solution u of (1.1). Our aim is to investigate whether the wave-front structure of u in a neighborhood of a given point (τ, ξ) is preserved by nearby solutions, or else can be destroyed by arbitrarily small perturbations in the $\mathbf{L}^1$ norm.

Example 2 (Structural instability). Consider a solution containing three shocks interacting at the same point (τ, ξ). This configuration (figure 1) is structurally unstable, since an arbitrarily small perturbation can make the shocks interact only two at a time (figure 2). Another unstable configuration is provided by a solution containing a centered compression wave.

In order to achieve the local stability of the wave-front configuration of u on a whole neighborhood of a given point, we need to exclude the two cases illustrated by the above example.

Definition. *A solution* u *of* (1.1) *is said to be* **structurally stable** *at the point* (τ, ξ) *if, on the lower half plane* $\{t < 0\}$, *the self-similar solution* $\tilde{u} = \lim_{\eta\to 0+} u^\eta$ *constructed in Theorem 2 satisfies one of the following:*

Case 1 : $\tilde{u}$ *is a constant function.*

Case 2 : $\tilde{u}$ *contains one incoming shock and no other wave.*

Case 3 : $\tilde{u}$ *contains two incoming shocks and no other wave.*

In other words, structural stability does not hold iff $\tilde{u}$ contains three or more incoming shocks, or a centered compression wave.

We recall that, on the set of Radon measures on the open interval $]-M,\ M[$, the Kantorovich-Rubinstein distance d_M is defined as follows: for any two measures μ and μ',

$$d_M(\mu, \mu') \doteq \sup_\phi \langle \phi,\ \mu' - \mu \rangle = \sup_\phi \left| \int_{]-M,M[} \phi\ d\mu' - \int_{]-M,M[} \phi\ d\mu \right|, \tag{1.21}$$

where the supremum is taken over all Lipschitz continuous functions ϕ with $\mathrm{Lip}(\phi) \leq 1$, $\|\phi\|_{\mathbf{L}^\infty} \leq 1$. Similarly the distance $d(\mu, \mu')$ is defined as in (1.21) replacing $]-M, M[$ with $]-\infty, \infty[$.

We first prove a general stability result. Roughly speaking, it says that the $\mathbf{L}^1$ convergence of a sequence of solutions $u_\nu \to u$ implies the convergence of the local wave-front structure in a backward and in a forward neighborhood of each given point (τ, ξ). To state this more precisely,

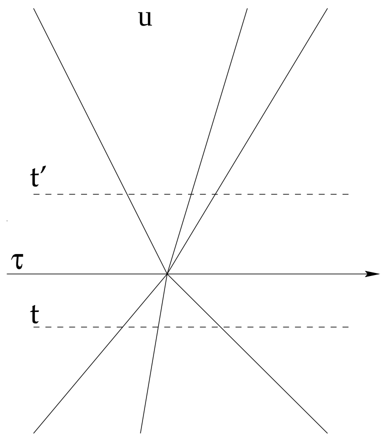

figure 1

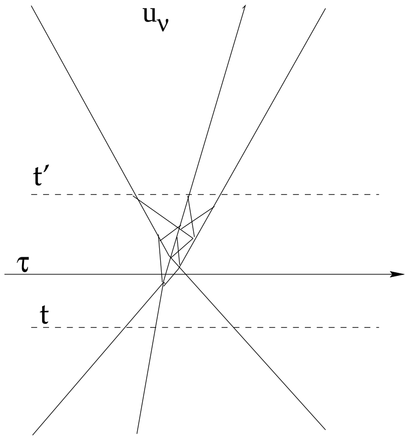

figure 2

consider the measures $\mu^{i\pm}_{\nu,t}$ of positive and negative i-waves in $u_\nu(t,\cdot)$ and define the rescaled measures $\widehat{\mu}^{i\pm}_{\nu,t}$ by

$$\int \phi(x)\, d\widehat{\mu}^{i\pm}_{\nu,t} = \int \phi\left(\frac{x-\xi}{|t-\tau|}\right) d\mu^{i\pm}_{\nu,t} \tag{1.22}$$

for each continuous ϕ having compact support. In connection with the solution u, let $\tilde u \doteq \lim_{\eta\to 0+} u^\eta$ be the corresponding self-similar solution at a point (τ,ξ), constructed in Theorem 2. Call $\tilde\mu^{i+}_- \equiv 0$ and $\tilde\mu^{i-}_- \doteq \mu^{i-}(\tilde u,-1)$, respectively, the measures of positive and negative i-waves in $\tilde u(-1,\cdot)$. Similarly, call $\tilde\mu^{i+}_+$ and $\tilde\mu^{i-}_+$ the measures of positive and negative i-waves in $\tilde u(1,\cdot)$. With the above notations, one has:

Theorem 6 (Local backward and forward structural stability). *Let u_ν be a sequence of solutions, converging to u in L^1_{loc}. Fix $\varepsilon > 0$, $M > 1$ and a point (τ,ξ). Then there exists $\rho > 0$ such that*

$$\limsup_{\nu\to\infty} d_M\big(\widehat{\mu}^{i\pm}_{\nu,t},\ \tilde\mu^{i\pm}_-\big) < \varepsilon \qquad t \in [\tau-\rho,\ \tau[\,, \tag{1.23}$$

$$\limsup_{\nu\to\infty} d_M\big(\widehat{\mu}^{i\pm}_{\nu,t},\ \tilde\mu^{i\pm}_+\big) < \varepsilon \qquad t \in\,]\tau,\ \tau+\rho]. \tag{1.24}$$

The condition $M > 1$ is related to the assumption that all wave speeds are < 1 in absolute value. Theorem 6 can be supplemented with a result on the convergence of the atomic parts of the measures $\widehat{\mu}^{i-}_{\nu,t}$.

Theorem 7. *In addition to the assumptions of Theorem 6, let $\tilde u(-1,\cdot)$ contain an i-shock of size $\tilde\sigma_i < 0$ at the point $\tilde x$, so that $\tilde\mu^{i-}_-\{\tilde x\} = |\tilde\sigma_i|$. Then one can find $\rho > 0$ such that, for each $t \in [\tau-\rho,\tau[\,$, there exists a sequence of points $x_\nu \to \tilde x$ satiafying*

$$\limsup_{\nu\to\infty} \Big|\widehat{\mu}^{i-}_{\nu,t}\{x_\nu\} - |\tilde\sigma_i|\Big| < \varepsilon. \tag{1.25}$$

An illustration of the above theorems is provided by Figures 1-2. Let u be a solution containing exactly three shocks, interacting at a same point (τ, ξ) and producing three outgoing shocks (fig.1). Consider a sequence of solutions $u_\nu \to u$. Then, in a neighborhood of the point ξ, for each time t smaller but sufficiently close to τ, every function $u_\nu(t, \cdot)$ with $\nu \geq N(t)$ large enough will contain three incoming shocks, and almost no other waves. Observe that here the integer $N(t)$ may well approach infinity as $t \to \tau-$.

Similarly, for each time t' larger but sufficiently close to τ, every function $u_\nu(t', \cdot)$ with $\nu \geq N(t')$ will contain three outgoing shocks and almost no other waves. Again, the integer $N(t')$ depends on t' and may well approach infinity as $t' \to \tau+$.

An explanation of this behavior is as follows. In any perturbed solution u_ν, the three incoming shocks may not interact at a single point (fig.2). However, if u_ν is sufficiently close to u, most of the interactions occur within a small neighborhood of the point (τ, ξ). In particular, if we fix a later time $t' > \tau$, for every solution u_ν close enough to u most of the interactions take place before time t'. Hence, in a small neighborhood J of ξ, the local interaction potential $Q\big(u_\nu(t');\ J\big)$ is small. On J, the function $u_\nu(t', \cdot)$ therefore will contain almost the same waves as the solution to the Riemann problem with data $u(\tau, \xi\pm)$.

In cases where u has a specified wave-front structure at (τ, ξ), in particular if u is structurally stable, one can show that every nearby solution w has a similar wave-front structure on a whole neighborhood of the point (τ, ξ). In the following theorem we consider solutions w that are suitably close to u in the $\mathbf{L}^1$ norm. More precisely, we assume that, for some small $\delta, \bar\rho > 0$, the perturbed solution w satisfies

$$\int_{\xi-\bar\rho}^{\xi+\bar\rho} \big|w(t,x) - u(t,x)\big|\, dx < \delta, \qquad t \in [\tau - \bar\rho,\ \tau]. \tag{1.26}$$

Theorem 8 (Local structural stability). *For some constant $C > 0$ the following holds. Let u be a solution of (1.1) and let $\bar\rho, \varepsilon > 0$ and a point (τ, ξ) be given with $\tau \geq \bar\rho > 0$. Let $\tilde u = \lim_{\eta \to 0+} u^\eta$ be as in Theorem 2.*

(1) *Assume that, for $t < 0$, the total amount of waves in $\tilde u$ satisfies $V\big(\tilde u(t)\big) < \varepsilon$. Then there exist $\rho, \delta > 0$ such that, for any other solution w of (1.1), the integral estimate (1.26) implies*

$$V\big(w(t);\ [\xi - 2\rho,\ \xi + 2\rho]\big) < C\,\varepsilon, \qquad t \in [\tau - \rho,\ \tau + \rho]. \tag{1.27}$$

(2) *Assume that, for $t < 0$, the function $\tilde u$ contains a single i-shock of strength $|\tilde\sigma| > C\varepsilon$ plus other waves whose total strength is $< \varepsilon$. Then there exist $\rho, \delta > 0$ such that, for any other solution w of (1.1), the integral estimate (1.26) implies the following. For every $t \in [\tau - \rho,\ \tau + \rho]$, the* *function w also has an i-shock at a point $y(t) \in [\xi - 2\rho,\ \xi + 2\rho]$. The strength of all other waves in $w(t)$ is*

$$V\Big(w(t);\ [\xi - 2\rho,\ \xi + 2\rho] \setminus \big\{y(t)\big\}\Big) < C\,\varepsilon, \qquad t \in [\tau - \rho,\ \tau + \rho]. \tag{1.28}$$

(3) *Assume that, for $t < 0$, the function $\tilde{u}$ contains two large incoming shocks with strengths $|\tilde{\sigma}|, |\tilde{\sigma}'| > C\varepsilon$, plus other waves whose total strength is $< \varepsilon$. Then there exist $\rho, \delta > 0$ such that, for any other solution w of* (1.1)*, the integral estimate* (1.26) *implies the following. The function w also contains two incoming shocks, interacting at some point (τ^*, ξ^*), with $|\tau^* - \tau| + |\xi^* - \xi| < \rho/2$. For $t \in [\tau - \rho, \tau^*[$, these shocks are located at points $y(t), y'(t) \in [\xi - 2\rho,\ \xi + 2\rho]$. The strength of all other waves in $w(t)$ is*

$$V\Big(w(t);\ [\xi - 2\rho,\ \xi + 2\rho] \setminus \big\{y(t), y'(t)\big\}\Big) < C\varepsilon, \qquad t \in [\tau - \rho,\ \tau^*]. \tag{1.29}$$

Observe that, if u is structurally stable at (τ, ξ), then the assumptions of the above theorem hold with an arbitrary choice of $\varepsilon > 0$.

The proofs of our results rely heavily on the decay estimates for positive waves of genuinely nonlinear families, established in [7]. They generalize the classical Oleinik estimates to the case of vector-valued solutions. In analogy with (1.13), we denote by $V^{i+}\big(u(t); J\big)$ the amount of positive i-waves of $u(t, \cdot)$ contained inside the set J. The Lebesgue measure of J is denoted by $\mathrm{meas}(J)$.

Proposition 9 (Decay of positive waves). *There exist constants $C_2, \kappa > 0$ such that every solution u of* (1.1) *satisfies*

$$V^{i+}\big(u(t'); J\big) \leq \frac{meas(J)}{\kappa(t' - t)} + C_2\left(Q(u(t)) - Q(u(t'))\right) \tag{1.30}$$

for every Borel set $J \subset I\!R$, $i = 1, \ldots, n$ and $t' > t \geq 0$.

When J is an interval, inequalities of the form (1.30) also hold for approximate solutions constructed by wave front tracking. We recall that, when restricted to a set of functions with suitably small total variation, the Glimm functionals

$$u \mapsto Q(u), \qquad u \mapsto \Upsilon(u) \doteq V(u) + C_0 Q(u) \tag{1.31}$$

are lower semi-continuous w.r.t. the $\mathbf{L}^1$-convergence. For a proof, see [2].

Proposition 10. *There exist constants $C_0 \geq 1$ and $\delta_0 > 0$ such that the following holds. Let $u, u_\nu : I\!R \mapsto I\!R^n$ be functions with bounded total variation such that $\Upsilon(u), \Upsilon(u_\nu) \leq \delta_0$, and assume that $u_\nu \to u$ in $\mathbf{L}^1$. Then, for every open interval $J \subseteq I\!R$, one has*

$$Q(u; J) \leq \liminf_{\nu \to \infty} Q(u_\nu; J), \tag{1.32}$$

$$\Upsilon(u;J) \leq \liminf_{\nu\to\infty} \Upsilon(u_\nu;J), \tag{1.33}$$

Observe however that one may have

$$V(u) > \limsup_{\nu\to\infty} V(u_\nu). \tag{1.34}$$

Indeed, let $w = w(t,x)$ be a solution of (1.1) containing two shocks that interact at a time τ, and assume that the total amount of waves is increased by the interaction. Choosing $u \doteq w(\tau)$, $u_\nu \doteq w(\tau - 2^{-\nu}, \cdot)$, we then have $u_\nu \to u$ together with (1.34).

Throughout the following, we shall assume that the constants C_0, δ_0 are chosen so that Propositions 1 and 10 hold, and so that the decrease $\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)$ provides a bound on the total amount of interaction and cancellation in the solution u on the interval $]t,t']$, for every $t' > t \geq 0$.

We conclude this section by stating some additional results on the global behavior of waves, for solutions of (1.1). For $i = 1,\ldots,n$, consider the measures $\mu^i(u,t) = \mu^{i+}(u,t) + \mu^{i-}(u,t)$ of i-waves in $u(t,\cdot)$. If $u = u(t,x)$ is a BV solution of (1.1), the maps $t \mapsto \mu^{i\pm}(u,t)$ have bounded variation w.r.t. the Kantorovich-Rubinstein distance (1.21). More precisely, the total variation of these maps is estimated in terms of the variation of $Q(u)$ or $\Upsilon(u)$.

Proposition 11. *For some positive constant C_3 the following holds. Given any solution u of* (1.1) *satisfying* (1.14)*, for $i = 1,\ldots,n$ and all $0 \leq t < t'$ one has*

$$d\big(\mu^i(u,t'),\ \mu^i(u,t)\big) \leq C_3\Big(Q\big(u(t)\big) - Q\big(u(t')\big)\Big) + (t'-t)\delta_0. \tag{1.35}$$

$$d\big(\mu^{i\pm}(u,t'),\ \mu^{i\pm}(u,t)\big) \leq C_3\Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)\Big) + (t'-t)\delta_0. \tag{1.36}$$

In particular, during a time interval $[t_1,\ t_2]$ where no interaction takes place, the maps $t \mapsto \mu^{i\pm}(u,t)$ are Lipschitz continuous with constant δ_0. The validity of Proposition 11 depends on the fact that δ_0 is an upper bound for the total strength of waves, and that all wave speeds are < 1 in absolute value. From Proposition 11 and a compactness argument one obtains:

Proposition 12. *For some positive constant C_4 the following holds. Let $(u_\nu)_{\nu\geq 1}$ be a sequence of solutions to* (1.1) *satisfying* (1.14) *and having uniformly bounded support. By possibly extracting a subsequence, there exist measures $\mu^{i\pm}_\infty(t)$ and functions Υ_∞, Q_∞ such that, for all $t' > t \geq 0$, one has*

$$Q_\infty(t) = \lim_{\nu\to\infty} Q\big(u_\nu(t)\big), \qquad \Upsilon_\infty(t) = \lim_{\nu\to\infty} \Upsilon\big(u_\nu(t)\big), \tag{1.37}$$

$$\mu^{i\pm}(u_\nu,t) \rightharpoonup \mu^{i\pm}_\infty(t), \tag{1.38}$$

$$d\big(\mu^{i\pm}_\infty(t'),\ \mu^{i\pm}_\infty(t)\big) \leq C_4\big(\Upsilon_\infty(t) - \Upsilon_\infty(t')\big) + (t'-t)\delta_0. \tag{1.39}$$

Observe that, for any (right-continuous, by convention) solution u of (1.1), the functions $t \mapsto Q\big(u(t)\big)$, $t \mapsto \Upsilon\big(u(t)\big)$ are always right-continuous. On the other hand, $t \mapsto Q_\infty(t+)$ denotes the right-continuous version of the BV function Q_∞. In the following, we assume that the measures $\mu_\infty^{i\pm}$ and the function Q_∞ are given by (1.37)–(1.39). The next results describe the relations between the measures $\mu_\infty^{i\pm}(t)$, describing the asymptotic amount of positive and negative i-waves in $u_\nu(t,\cdot)$ as $\nu \to \infty$, and the corresponding measures $\mu^{i\pm}(u,t)$, describing the amount of i-waves in the limiting solution $u(t,\cdot)$. To avoid misunderstandings, we remark that, given a weakly convergent sequence of positive measures $\mu_k \rightharpoonup \mu$, the limit of the atomic parts of μ_k may well be different from the atomic part of μ.

Proposition 13. *In the same setting as in Proposition 12, let $t \mapsto \mu_\infty^{i\pm}(t+)$ be the right continuous version of the BV map $t \mapsto \mu_\infty^{i\pm}(t)$. Then the atomic parts of the measures $\mu^{i\pm}(u,t)$ and $\mu_\infty^{i\pm}(t+)$ coincide for all $t \geq 0$.*

The function $t \mapsto Q\big(u(t)\big)$ is continuous at a time $\tau > 0$ if and only if the function Q_∞ is continuous at τ. In the positive case, the atomic parts of the measures $\mu^i(u,\tau)$ and $\mu_\infty^i(\tau)$ are purely negative and coincide.

Propositions 11–13 also hold for a sequence of approximate solutions constructed by wave-front tracking.

2 - Preliminary estimates.

First of all we recall the algorithm of wave-front tracking [3, 4]. One starts with a piecewise constant function $\bar{u}$ at time $t = 0$. At each point of jump, the corresponding Riemann problem is approximately solved within the class of piecewise constant functions. This yields an approximate solution defined up to the first time $t_1 > 0$ where an interaction between two wave fronts takes place. The new Riemann problem is then solved again within the class of piecewise constant functions, extending the solution up to some time $t_2 > t_1$ where the second interaction takes place, etc.

In order to keep finite the total number of wave-fronts, we shall use two distinct procedures for solving a Riemann problem: an accurate method, which possibly introduces several new fronts, and a simplified method, which avoids the introduction of new wave-fronts. For given Riemann data u^-, u^+, these two resolution methods are described below. The algorithm involves two (strictly positive) parameters: δ, bounding the maximum size of rarefaction fronts, and ρ, determining which Riemann problems will be accurately solved. By $[\![s]\!]$ we denote the integer part of s.

Accurate Riemann solver: Let $\omega_0 = u^-$, $\omega_1, \ldots, \omega_n = u^+$ be the constant states present in the solution of the Riemann problem. The piecewise constant approximation u is obtained by replacing each centered rarefaction wave with several propagating discontinuities. If the states ω_{i-1}, ω_i are connected by an i-rarefaction wave of size $\sigma_i > 0$, we divide this jump into $p_i \doteq 1 + [\![\sigma_i/\delta]\!]$ equal parts of size $< \delta$ and insert the intermediate states

$$\omega_{i,\ell} \doteq R_i(\ell\sigma_i/p_i)(\omega_{i-1}), \qquad \ell = 0, 1, \ldots, p_i.$$

Here $\sigma \mapsto R_i(\sigma)(\omega)$ denotes the i-rarefaction curve through the state ω. Each small jump $(\omega_{i,\ell-1}, \omega_{i,\ell})$ is then propagated with speed $\lambda_i(\omega_{i,\ell})$, i.e. with the characteristic speed of its right state.

Simplified Riemann Solver: Assume that the Riemann problem is determined by the interaction of two waves of distinct families, say $i < j$, with sizes σ_i, σ_j. Call u^l, u^m, u^r the left, middle and right states before the interaction. With the notation (1.6), we thus have $u^m = \Psi_j(\sigma_j)(u^l)$, $u^r = \Psi_i(\sigma_i)(u^m)$. The Riemann problem is then solved in terms of two outgoing wave-fronts of the same families, still with sizes σ_i, σ_j. The solution involves a new middle state $\tilde{u}^m \doteq \Psi_i(\sigma_i)(u^l)$ and a new right state $\tilde{u}^r \doteq \Psi_j(\sigma_j)(\tilde{u}^m)$. In general, $\tilde{u}^r \neq u^r$. The jump $(\tilde{u}^r, u^r)$ is then propagated along a *non-physical* wave-front, travelling with a fixed speed $\hat{\lambda}$, larger than all characteristic speeds.
In the case that both incoming wave-fronts belong to the same i-th family and have sizes σ_i, σ_i', the Riemann problem is solved by a single outgoing i-wave of size $\sigma_i + \sigma_i'$, together with a non-physical wave-front connecting the states $\tilde{u}^r \doteq \Psi_i(\sigma_i + \sigma_i')(u^l)$ and u^r, travelling with speed $\hat{\lambda}$.
Finally, in the case where a non-physical front interacts with an i-wave of size σ_i, the Riemann problem is solved in terms of an outgoing i-wave of the same size σ_i, and a non-physical front travelling with speed $\hat{\lambda}$.

To complete the description of the algorithm, it remains to specify which Riemann solver is used at any given interaction. The choice is made in connection with a threshold parameter $\rho > 0$:

- The accurate method is used at time $t = 0$, and at every interaction where the product of the strengths of the incoming waves is $\geq \rho$.
- The simplified method is used at every interaction involving a non-physical wave-front, and also at interactions where the product of the strengths of the incoming waves is $< \rho$.

In the above, we tacitly assumed that only two wave-fronts interact at any given point. This can always be achieved by an arbitrarily small change in the speed of one of the interacting fronts. We shall also adopt the provision that, in the Accurate Riemann Solver, rarefaction fronts of the same family of one of the incoming fronts are never partitioned (even if their strength is $> \delta$). This guarantees that every wave-front can be uniquely continued forward in time, unless it gets completely cancelled by interacting with another front of the same family and opposite sign.

Given any initial data $\bar{u}$ having sufficiently small total variation and using the above algorithm under natural conditions on ρ_ν and δ_ν as $\nu \to \infty$, we obtain a sequence $(u_\nu)_{\nu \geq 1}$ of piecewise constant approximate solutions such that $u_\nu(0) \to \bar{u}$ in $\mathbf{L}^1$. Each u_ν is defined for all $t \in [0, \infty)$ and has a finite number of lines of discontinuity in the t-x plane. The estimates in [3, 4] show that:

(i) The total variation of $u_\nu(t)$ remains uniformly small.

(ii) The maximum size of the rarefaction fronts in u_ν approaches zero.

(iii) The total strength of all non-physical waves approaches zero.

Furthermore the interaction estimates in Proposition 1 hold for the functions u_ν, uniformly in ν. In view of (i), Helly's theorem guarantees the existence of a subsequence, strongly convergent in $\mathbf{L}^1_{loc}$. By (ii) and (iii), this limit is indeed an entropy weak solution to (1.1).

We shall adopt the following conventions, introduced in [4]: All non-physical wave-fronts belong to a (fictitious) $(n+1)$-th characteristic family. A non-physical front located at x has strength $\sigma \doteq \big|u(t, x(t+)) - u(t, x(t)-)\big|$ and approaches all physical fronts located at points $y > x$. Two non-physical fronts never approach each other. With these conventions, all estimates (1.14)-(1.15) still hold.

We recall a useful technical property of front-tracking approximations [9]: the interaction potential for u_ν is small in some forward neighborhood of each given point, uniformly with respect to ν.

Lemma 14. *Let $(u_\nu)_{\nu \geq 1}$ be a sequence of approximate solutions of (1.1), generated by wave-front tracking. Let a point (τ, ξ) and $\varepsilon > 0$ be given. Then there exists $\rho > 0$ such that, by setting*

$$J(t) \doteq \big[\xi - \rho + (t - \tau), \quad \xi + \rho - (t - \tau)\big]$$

and possibly extracting a subsequence, one has

$$\limsup_{\nu \to \infty} \; Q\big(u_\nu(t);\; J(t)\big) < \varepsilon, \qquad\qquad t \in \,]\tau,\; \tau + \rho]. \tag{2.1}$$

Toward the proof of Theorem 2, we first describe two situations where the $\mathbf{L}^1$ convergence $u_\nu \to u$ implies the weak convergence $\mu_\nu^{i\pm} \rightharpoonup \mu^{i\pm}$ of the measures of i-waves associated with a sequence of functions $u_\nu = u_\nu(x)$. It is assumed that either the positive waves and the local interaction potential of u_ν approach zero, or else the global interaction potential of the u_ν approaches zero. This result will be used later with a sequence of solutions or approximate solutions $u_\nu = u_\nu(t, x)$ and combined with Lemma 14. For notations, we refer to (1.7)–(1.13).

Lemma 15. *Let $u_\nu : I\!\!R \mapsto I\!\!R^n$ be a uniformly bounded sequence of BV functions, converging to u pointwise and hence in $\mathbf{L}^1_{loc}$. Call $\mu_\nu^{i\pm}$, $\mu^{i\pm}$ the measures of positive and negative i-waves in u_ν, u, respectively.*

(1) *For all $i = 1, \ldots, n$, assume the weak convergence $\mu_\nu^{i+} \rightharpoonup 0$. Moreover, assume that for each x and every $\varepsilon > 0$ there exists $\rho > 0$ such that*

$$\limsup_{\nu\to\infty} Q\big(u_\nu;\ [x-\rho,\ x+\rho]\big) < \varepsilon. \tag{2.2}$$

Then one has the weak convergence

$$\mu_\nu^{i-} \rightharpoonup \mu^{i-}, \qquad \mu_\nu^{i+} \rightharpoonup \mu^{i+}. \tag{2.3}$$

(2) *In addition, if $\mu^{i-}\big(\{\bar x\}\big) = |\bar\sigma| > 0$, then there exists a sequence of points $x_\nu \to \bar x$ such that $\mu_\nu^{i-}\big(\{x_\nu\}\big) \doteq |\sigma_\nu| \to |\bar\sigma|$.*

(3) *The above results remain true if the conditions $\mu_\nu^{i+} \rightharpoonup 0$ and (2.2) are replaced by the single assumption*

$$\lim_{\nu\to\infty} Q\big(u_\nu;\ [a,b]\big) = 0 \qquad \textit{for all }\ a<b. \tag{2.4}$$

The remainder of this section is devoted to the proof of Lemma 15, given in several steps. The main idea is to decompose the domain $I\!\!R$ into a set J, where the convergence $u_\nu \to u$ is nearly uniform, and a finite collection of small intervals I_j around jump points, such that the interaction potential $Q(u_\nu;\ I_j)$ is very small. On each interval I_j the weak convergence $\mu_\nu^{i\pm} \rightharpoonup \mu^{i\pm}$ follows from a stability property of the Riemann problem.

1. As a preliminary we observe that, by the stability of solutions to the Riemann problem, there exists an increasing continuous function $\varphi = \varphi(\delta)$ with $\varphi(0) = 0$ such that the following holds. For any left and right states u^-, u^+, let $\sigma_1, \ldots, \sigma_n$ be the sizes of the waves present in the solution of the corresponding Riemann problem. Let $v : [a,b] \mapsto I\!\!R^n$ be any BV function which satisfies

$$\big|v(a) - u^-\big| < \delta, \qquad \big|v(b) - u^+\big| < \delta, \qquad Q\big(v;\ [a,b]\big) < \delta. \tag{2.5}$$

Then, for $i = 1, \ldots, n$, the measures $\mu^{i\pm}\big(v\big)$ of i-waves in v satisfy

$$\Big|\mu^{i\pm}\big(v; [a,b]\big) - \sigma_i^\pm\Big| < \varphi(\delta), \tag{2.6}$$

where $\sigma_i^+ = \max(\sigma_i, 0)$ and $\sigma_i^- = \max(-\sigma_i, 0)$.

2. Fix $\varepsilon > 0$. Call $\mu_\nu^\# \doteq \sum_i (\mu_\nu^{i+} + \mu_\nu^{i-})$ the total measure of waves in u_ν. By extracting a subsequence, we can assume the pointwise convergence $u_\nu(x) \to u(x)$ for each $x \in I\!\!R$, together with the weak convergence $\mu_\nu^\# \rightharpoonup \mu^\#$ for some positive Radon measure $\mu^\#$. Let $\mathcal{F} \doteq \{x_1, \ldots, x_N\}$ be the points such that $\mu^{\#}\big(\{x_j\}\big) \geq \varepsilon/4$. Then every other point $x \notin \mathcal{F}$ admits an open neighborhood I_x such that

$$\limsup_{\nu\to\infty} \sup_{y,y'\in I_x} \big|u_\nu(y) - u_\nu(y')\big| < \varepsilon/3. \tag{2.7}$$

Recalling the representation $d\mu^i_\nu = \tilde l^\nu_i \cdot du_\nu$ in (1.8)-(1.9), this implies

$$\limsup_{\nu\to\infty} \sup_{y,y'\in I_x} \big|\tilde l^\nu_i(y) - \tilde l^\nu_i(y')\big| < C_1\varepsilon/3. \tag{2.8}$$

From (2.7) and the pointwise convergence $u_\nu \to u$, it also follows

$$\sup_{y,y'\in I_x} \big|u(y) - u(y')\big| < \varepsilon/2, \qquad \sup_{y,y'\in I_x} \big|\tilde l_i(y) - \tilde l_i(y')\big| < C_1\varepsilon/2. \tag{2.9}$$

3. Choose $\delta^* > 0$ such that $\varphi(\delta^*) < \varepsilon/N$. By assumption (2.2), we can find disjoint intervals $I_j \doteq [x_j - \rho_j,\ x_j + \rho_j]$ such that, for each $j = 1, \ldots, N$,

$$\limsup_{\nu\to\infty} Q\big(u_\nu;\ I_j\big) < \delta^*. \tag{2.10}$$

$$\mu^{\#}\big(I_j \setminus \{x_j\}\big) < \frac{\varepsilon}{N}. \tag{2.11}$$

Let ϕ be a continuous scalar function with compact support. Using a continuous partition of unity, we can construct continuous functions $\tilde\phi, \phi_j$ such that

$$\tilde\phi \equiv 0 \qquad \text{in a neighborhood of each point } x_j, \tag{2.12}$$

$$\mathrm{supp}(\phi_j) \subset I_j, \tag{2.13}$$

$$\phi(x) = \tilde\phi(x) + \sum_{j=1}^N \phi_j(x), \qquad \big|\phi(x)\big| = \big|\tilde\phi(x)\big| + \sum_{j=1}^N \big|\phi_j(x)\big|, \qquad x \in I\!R. \tag{2.14}$$

4. For $j = 1, \ldots, N$, choose $\rho'_j \in\,]0, \rho_j[$ small enough so that

$$\tilde\phi(x) = 0, \qquad \big|\phi_j(x) - \phi_j(x_j)\big| < \varepsilon/N \qquad \text{for all } \ x \in I'_j \doteq [x_j - \rho'_j,\ x_j + \rho'_j]. \tag{2.15}$$

On the open set

$$J \doteq I\!R \setminus \Big(\bigcup_{j=1}^N I'_j\Big),$$

define the multifunction

$$\Lambda_i(x) \doteq \Big\{z \in I\!R^n;\ \ \max\big\{|z - \tilde l_i(y-)|, |z - \tilde l_i(y+)|, |z - \tilde l_i(y)|\big\} \leq C_1\varepsilon \ \text{ for all } \ y \ \text{ with } \ |y - x| < \rho\Big\}$$

for $i = 1, \cdots, n$ and for some $\rho > 0$ to be determined. Observe that Λ_i has closed convex values and is lower semicontinuous [1], i.e. for every x and each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\Lambda_i(x) \subset B\big(\Lambda_i(y), \varepsilon\big) \qquad \text{whenever} \ \ |y - x| < \delta.$$

Here $B\big(\Lambda_i(y),\varepsilon\big)$ denotes the ε-neighborhood around the set $\Lambda_i(y)$. Moreover, by (2.9), choosing $\rho>0$ small enough the values of Λ_i are non-empty. A selection theorem of Michael [1, 23] thus provides the existence of a continuous function $\hat l_i$ such that

$$\hat l_i(x)\in\Lambda_i(x)\qquad\text{for all}\ \ x\in J.\tag{2.16}$$

By construction, for all ν sufficiently large, this selection satisfies

$$\big|\hat l_i(x)-\tilde l_i^\nu(x)\big|<C_1\varepsilon,\quad \big|\hat l_i(x)-\tilde l_i(x)\big|<C_1\varepsilon\qquad\text{for all}\ \ x\in J.\tag{2.17}$$

5. Using (1.8)-(1.9), we now estimate the limit as $\nu\to\infty$ of

$$\int\phi\,(d\mu^i_\nu-d\mu^i)=\int\tilde\phi\,(d\mu^i_\nu-d\mu^i)+\sum_{j=1}^N\int\phi_j\,(d\mu^i_\nu-d\mu^i).\tag{2.18}$$

Concerning the first term on the right hand side of (2.18) we have

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left|\int\tilde\phi\,\big(\tilde l_i^\nu\cdot du_\nu-\tilde l_i\cdot du\big)\right|\\
&\le\limsup_{\nu\to\infty}\left\{\left|\int\tilde\phi\,(\tilde l_i^\nu-\hat l_i)\cdot du_\nu\right|+\left|\int\tilde\phi\,\hat l_i\cdot(du_\nu-du)\right|+\left|\int\tilde\phi\,(\hat l_i-\tilde l_i)\cdot du\right|\right\}\\
&\le\sup_x\big|\tilde\phi(x)\big|\cdot\limsup_{\nu\to\infty}\left(\sup_{x\in J}\big|\tilde l_i^\nu(x)-\hat l_i(x)\big|\right)\left(\limsup_{\nu\to\infty}|du_\nu|(I\!R)+|du|(I\!R)\right)\\
&\le C\,\varepsilon\,\sup_x\big|\phi(x)\big|
\end{aligned}\tag{2.19}$$

in view of (2.17), where $|du|$ is the measure of total variation of u.

6. Concerning the second term on the right hand side of (2.18), for each j we have

$$\begin{aligned}
&\limsup_{\nu\to\infty}\left|\int\phi_j(d\mu^i_\nu-d\mu^i)\right|\\
&\le\big|\phi_j(x_j)\big|\cdot\limsup_{\nu\to\infty}\big|\mu^i_\nu(I'_j)-\mu^i(I'_j)\big|+\sup_{x\in I'_j}\big|\phi_j(x)-\phi_j(x_j)\big|\cdot\limsup_{\nu\to\infty}\Big[|\mu^i_\nu|(I'_j)+|\mu^i|(I'_j)\Big]\\
&+\sup_x\big|\phi_j(x)\big|\cdot\limsup_{\nu\to\infty}\Big(|\mu^i_\nu|+|\mu^i|\Big)\Big(]x_j-\rho_j,\ x_j-\rho'_j]\cup[x_j+\rho'_j,\ x_j+\rho_j[\Big)\\
&\le\sup_x\big|\phi(x)\big|\cdot\left\{\frac{\varepsilon}{N}+\frac{\varepsilon}{N}\cdot2\mu^\#(I'_j)+2\frac{\varepsilon}{N}\right\}.
\end{aligned}\tag{2.20}$$

Indeed, the estimate on the first term follows from (2.10), (2.6) and the choice of δ^*. The second term is estimated by (2.15) and the third by (2.11).

7. The previous arguments show that, from every subsequence $u_{\nu'}$ one can extract a further subsequence, say $u_{\nu''}$, for which the following holds. For every continuous function ϕ with compact support, one has

$$\limsup_{\nu\to\infty}\left|\int\phi(d\mu^i_{\nu''}-d\mu^i)\right|\le C\,\varepsilon\,\sup_x\big|\phi(x)\big|\tag{2.21}$$

for some constant C independent of ϕ and ε. Since $\varepsilon > 0$ in (2.21) is arbitrary, this proves the weak convergence

$$\mu^i_\nu \rightharpoonup \mu^i. \tag{2.22}$$

8. On one hand, assuming that $\mu^{i+}_\nu \rightharpoonup 0$, for every $\phi \geq 0$ we have

$$\int \phi \, d\mu^i = \lim_{\nu\to\infty} \int \phi \, d\mu^i_\nu \leq 0.$$

Therefore $\mu^{i+} \equiv 0$. By (2.22), it is thus clear that $\mu^{i-}_\nu \rightharpoonup \mu^{i-} = \mu^i$. On the other hand if, instead, (2.4) holds, we have in particular, for each i,

$$\limsup_{\nu\to\infty} \ \mu^{i+}_\nu(\mathbb{R}) \cdot \mu^{i-}_\nu(\mathbb{R}) = 0.$$

Hence (by possibly taking a subsequence) we either have $\mu^{i+}_\nu \rightharpoonup 0$ or $\mu^{i-}_\nu \rightharpoonup 0$. In both cases, from (2.22), we deduce $\mu^{i\pm}_\nu \rightharpoonup \mu^{i\pm}$.

9. Finally, assume that $\mu^{i-}\big(\{\bar x\}\big) = |\bar\sigma| > 0$. By definition of μ^{i-}, this means that u has a jump at $\bar x$ and that the solution to the Riemann problem with data $u(\bar x-)$, $u(\bar x+)$ contains an i-shock of size $\bar\sigma < 0$. By the pointwise convergence $u_\nu \to u$ and the assumption (2.2) or (2.4), there exists a sequence $\rho_\nu \to 0$ such that

$$u_\nu(x-\rho_\nu) \to u(\bar x-), \qquad\qquad u_\nu(x+\rho_\nu) \to u(\bar x+),$$

$$Q\big(u_\nu;\ [\bar x-\rho_\nu,\ \bar x+\rho_\nu]\big) \to 0.$$

By the stability of the solution of the Riemann problem, stated in (2.5)-(2.6), u_ν thus contains a shock of strength $\sigma_\nu \to \bar\sigma$ at some point $x_\nu \in [\bar x-\rho_\nu,\ \bar x+\rho_\nu]$. This establishes part (2), completing the proof of Lemma 15.

3 - Local structure of solutions.

This section contains a proof of Theorem 2, concerning the local asymptotic structure of a solution $u = u(t,x)$ of (1.1), in the neighborhood of a given point (τ,ξ) with $\tau > 0$. By assumption, u is the limit of a sequence of front-tracking approximations u_ν. As in [13], the main technique consists in letting $\nu\to\infty$ much faster compared with the limit $\eta \to 0+$ of the rescaling parameter. In this way, as $\eta\to 0+$ and $\nu = \nu(\eta) \to \infty$, the total amount of interaction and cancellation in the rescaled approximations

$$u^\eta_\nu(t,x) \doteq u_\nu(\tau+\eta t,\ \xi+\eta x)$$

converges to zero on every set of the form

$$\Gamma_{r_2} \setminus \Gamma_{r_1} \qquad\qquad 0 < r_1 < r_2,$$

where Γ_r is the square $\{|t| \leq r, \quad |x| \leq r\}$. The proof is given in several steps.

1. Let $(\eta_k)_{k\geq 1}$ be a sequence of strictly positive numbers, decreasing to zero. Consider the rescaled functions

$$u^k(t,x) \doteq u(\tau + \eta_k t,\ \xi + \eta_k x) \tag{3.1}$$

and call $\mu_t^{k,i\pm}$ the measures of positive and negative i-waves in $u^k(t,\cdot)$. Since $u(t,\cdot)$ has bounded variation and varies Lipschitz continuously in time (w.r.t. the $\mathbf{L}^1$ distance), the same property holds for all functions $u^k(t,\cdot)$. Moreover, the measures $\mu_t^{k,i\pm}$ have uniformly bounded total mass. Extracting a subsequence, we can assume that there exist a measurable function $\tilde u = \tilde u(t,x)$ and bounded measures $\tilde\mu_t^{i\pm}$ such that

$$u^k \to \tilde u \qquad \text{in } \mathbf{L}^1_{loc}\big(\mathbb{R}_+ \times \mathbb{R};\ \mathbb{R}^n\big), \tag{3.2}$$

$$u^k(t,\cdot) \to \tilde u(t,\cdot) \qquad \text{in } \mathbf{L}^1_{loc}(\mathbb{R};\ \mathbb{R}^n) \ \text{ for each } t \in \mathbb{Q}. \tag{3.3}$$

In the following steps, we will show that the limit $\tilde u$ is self-similar and does not depend on the choice of the subsequence η_k.

2. Consider a sequence of wave-front tracking approximations u_ν converging to u. For each t, call $\mu_{\nu,t}^{i\pm}$ the measures of positive and negative i-waves in $u_\nu(t,\cdot)$, and let

$$\mu_{\nu,t}^{\#} \doteq \sum_{i=1}^{n} \big(\mu_{\nu,t}^{i+} + \mu_{\nu,t}^{i-}\big)$$

be the total measure of waves in $u_\nu(t)$. To each u_ν, we also associate a purely atomic positive measure μ_ν^{IC} in both variables $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, keeping track of the total amount of Interaction and Cancellation. More precisely, let (t,x) be a point where two wave-fronts in u_ν interact. Suppose that the incoming fronts belong to the families i, i' and have sizes σ, σ'. We then set

$$\mu_\nu^{IC}\big\{(t,x)\big\} \doteq |\sigma\sigma'| + \begin{cases} |\sigma| + |\sigma'| - |\sigma + \sigma'| & \text{if} \quad i = i', \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Taking a subsequence, for some positive measures $\bar\mu_\tau^{i\pm}$ on the real line and $\bar\mu^{IC}$ on $\mathbb{R}^2$, we can assume the weak convergence

$$\mu_{\nu,\tau}^{i\pm} \rightharpoonup \bar\mu_\tau^{i\pm}, \qquad \mu_\nu^{IC} \rightharpoonup \bar\mu^{IC}. \tag{3.5}$$

For any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\bar\mu_\tau^{i\pm}\big([\xi - \delta,\ \xi + \delta] \setminus \{\xi\}\big) < \varepsilon, \tag{3.6}$$

$$\bar\mu^{IC}\big(\Gamma_\delta \setminus \{(\tau,\xi)\}\big) < \varepsilon, \tag{3.7}$$

where we defined the square

$$\Gamma_\delta \doteq [\tau - \delta,\ \tau + \delta] \times [\xi - \delta,\ \xi + \delta]. \tag{3.8}$$

By (3.5)–(3.7), there exist decreasing sequences $\varepsilon_\nu, \delta_\nu, \rho_\nu \to 0$, with $\delta_\nu > 2\rho_\nu > 0$, such that

$$\mu^{i\pm}_{\tau,\nu'}\big([\xi - \delta_\nu,\ \xi + \delta_\nu] \setminus [\xi - \rho_{\nu'},\ \xi + \rho_{\nu'}]\big) < \varepsilon_\nu \qquad \nu \le \nu'. \tag{3.9}$$

$$\mu^{IC}_{\nu'}\big(\Gamma_{\delta_\nu} \setminus \Gamma_{\rho_{\nu'}}\big) < \varepsilon_\nu \qquad \nu \le \nu'. \tag{3.10}$$

3. We shall be concerned first with the region where $t < \tau$. Using the approximate conservation laws for positive and negative i-waves:

$$\Big|[\text{Strength of incoming waves}] - [\text{Strength of outgoing waves}]\Big| \le C \cdot [\text{Interaction} + \text{Cancellation}] \tag{3.11}$$

for $u_{\nu'}$ on the region

$$\big\{(t,x);\quad t \in [\tau - \rho_{\nu'},\ \tau],\quad \rho_{\nu'} + (\tau - t) \le |x - \xi| \le \rho_\nu - (\tau - t)\big\},$$

we obtain the estimate

$$\mu^{i\pm}_{\nu',t}\big([\xi - \delta_\nu/2,\ \xi + \delta_\nu/2] \setminus [\xi - 2\rho_{\nu'},\ \xi + 2\rho_{\nu'}]\big) \le C\varepsilon_\nu, \tag{3.12}$$

for all $\nu \le \nu'$ and all $t \in [\tau - \rho_{\nu'},\ \tau]$. Indeed, the strength of outgoing waves is here bounded by (3.9) while the amount of interaction and cancellation is bounded by (3.10).

Our next goal is to derive a relation between the position of wave-fronts in $u_{\nu'}$ and their speed. Fix any

$$t \in J_{\nu,\nu'} \doteq [\tau - \delta_\nu/4,\ \tau - \rho_{\nu'}].$$

Consider the wave-fronts of $u_{\nu'}(t,\cdot)$ contained inside the interval $[\xi - \delta_\nu/4,\ \xi + \delta_\nu/4]$. Suppose that these fronts are located at points $x_\alpha(t)$ and have sizes $\sigma_\alpha(t)$, for $\alpha \in \mathcal{A}$. We split the set $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ as follows:

$\alpha \in \mathcal{A}_1$ if, during the time interval $J_{\nu,\nu'}$, the front x_α loses at least half of its strength, due to interactions and cancellations.

$\alpha \in \mathcal{A}_2$ if $\alpha \notin \mathcal{A}_1$ and, at time $\tau - \rho_{\nu'}$, the front x_α lies outside the interval $[\xi - 2\rho_{\nu'},\ \xi + 2\rho_{\nu'}]$.

$\alpha \in \mathcal{A}_3$ if $\alpha \notin \mathcal{A}_1$ and, at time $\tau - \rho_{\nu'}$, the front x_α lies inside the interval $[\xi - 2\rho_{\nu'},\ \xi + 2\rho_{\nu'}]$.

Observe that $\mathcal{A}_1$ includes all the fronts that are not defined in the whole time interval $J_{\nu,\nu'}$.

Recalling (3.10) and (3.12), one obtains the estimate

$$\sum_{\alpha \in \mathcal{A}_1 \cup \mathcal{A}_2} \big|\sigma_\alpha(t)\big| \le C'\,\varepsilon_\nu. \tag{3.13}$$

If $\alpha \in \mathcal{A}_3$, then by definition

$$\left|x_\alpha(\tau - \rho_{\nu'}) - \xi\right| \leq 2\rho_{\nu'}.$$

Hence, by the mean value theorem, at some intermediate time $t' \in [t,\ \tau - \rho_{\nu'}]$ there holds

$$\begin{aligned}\left|\dot{x}_\alpha(t') - \frac{\xi - x_\alpha(t)}{\tau - \rho_{\nu'} - t}\right| &\leq \left|\dot{x}_\alpha(t') - \frac{x_\alpha(\tau - \rho_{\nu'}) - x_\alpha(t)}{\tau - \rho_{\nu'} - t}\right| + \left|\frac{x_\alpha(\tau - \rho_{\nu'}) - \xi}{\tau - \rho_{\nu'} - t}\right| \\ &\leq \frac{2\,\rho_{\nu'}}{\tau - \rho_{\nu'} - t}.\end{aligned} \tag{3.14}$$

Observe that, at each interaction, the change in the quantity

$$\Psi(t) \doteq \sum_\alpha \left|\sigma_\alpha(t)\right|\dot{x}_\alpha(t)$$

is controlled by the amount of interaction plus cancellation. From (3.10) and (3.14) it thus follows

$$\sum_{\alpha \in \mathcal{A}_3} \left|\sigma_\alpha(t)\right| \left|\dot{x}_\alpha(t) - \frac{\xi - x_\alpha(t)}{\tau - \rho_{\nu'} - t}\right| \leq 2\,\rho_{\nu'} \sum_{\alpha \in \mathcal{A}_3} \frac{\left|\sigma_\alpha(t)\right|}{\tau - \rho_{\nu'} - t} + C\,\varepsilon_\nu.$$

Keeping ν fixed, and letting $\nu' \to \infty$ for the wave-fronts $\mathcal{W}_{\nu'}$ of $u_{\nu'}(t, \cdot)$ contained inside the interval $[\xi - \delta_\nu/4,\ \xi + \delta_\nu/4]$, we obtain the estimate

$$\limsup_{\nu' \to \infty} \sum_{\alpha \in \mathcal{W}_{\nu'}} \left|\sigma_\alpha(t)\right| \left|\dot{x}_\alpha(t) - \frac{\xi - x_\alpha(t)}{\tau - t}\right| \leq C\varepsilon_\nu, \qquad t \in [\tau - \delta_\nu/4,\ \tau[. \tag{3.15}$$

4. As in (3.1), consider the rescaled functions

$$u_\nu^k(t, x) \doteq u_\nu(\tau + \eta_k t,\ \xi + \eta_k x). \tag{3.16}$$

By (3.9), (2.1), (3.10) and (3.15) we can choose a sequence $\nu(k) \to \infty$ increasing fast enough so that, as $k \to \infty$, the following holds.

$$\frac{\rho_{\nu(k)}}{\eta_k} \to 0, \qquad u^k_{\nu(k)} \to \tilde{u} \qquad \text{in } \mathbf{L}^1_{loc}, \tag{3.17}$$

$$\mu^{IC}_{\nu(k)} \rightharpoonup 0 \qquad \text{on } I\!R^2 \setminus \{(0,0)\}, \tag{3.18}$$

$$\lim_{k \to \infty} Q\big(u^k_{\nu(k)}(t, \cdot);\ [a, b]\big) = 0 \qquad \text{for every } t > 0,\ a < b, \tag{3.19}$$

$$\lim_{k \to \infty} \sum_{\alpha \in \mathcal{W}^{[a,b]}_{\nu(k)}} \left|\sigma^{\nu(k)}_\alpha(t)\right| \left|\dot{x}^{\nu(k)}_\alpha(t) - \frac{x^{\nu(k)}_\alpha(t)}{t}\right| = 0 \qquad \text{for every } t < 0,\ a < b, \tag{3.20}$$

$$\lim_{k \to \infty} \Upsilon\big(u^k_{\nu(k)}(t)\big) - \Upsilon\big(u^k_{\nu(k)}(t')\big) = 0 \qquad \text{for every } \ t < t' < 0. \tag{3.21}$$

In (3.18), $\mu^{IC}_{\nu(k)}$ denotes the measure of interaction and cancellation of the rescaled piecewise constant function $u^k_{\nu(k)}$, while the summation in (3.20) extends to the family $\mathcal{W}^{[a,b]}_{\nu(k)}$, of all wave-fronts of $u^k_{\nu(k)}(t,\cdot)$ contained in the interval $[a,b]$. Recall that Υ was defined at (1.14).

5. We now show that, for $t<0$, the measures of positive waves in the rescaled functions approach zero on bounded sets. More precisely, fix any $t<0$ and any interval $[a,b]$. Call $\mu^{i+}_{k,t}$ the measure of positive i-waves in $u^k(t,\cdot)$. Let $\varepsilon>0$ be given. Choose $\tau'<\tau$ so that

$$Q\big(u(\tau')\big)-Q\big(u(\tau-)\big)<\varepsilon.$$

Then, for all k so large that

$$\frac{\tau-\tau'}{\eta_k}>\frac{1}{\varepsilon}-t,$$

we can apply the decay estimate (1.30) to u^k along the time interval $\big[(\tau'-\tau)/\eta_k,\ t\big]$ and deduce

$$\begin{aligned}\mu^{i+}_{k,t}\big([a,b]\big)&\leq\frac{b-a}{\kappa\left(t-\frac{\tau'-\tau}{\eta_k}\right)}+C_2\Big(Q\big(u^k((\tau'-\tau)/\eta_k)\big)-Q\big(u^k(t)\big)\Big)\\&\leq\frac{\varepsilon(b-a)}{\kappa}+C_2\,\varepsilon.\end{aligned}$$

Since ε was arbitrary, for each $t<0$ we obtain the weak convergence

$$\mu^{i+}_{k,t}\rightharpoonup 0\qquad\text{as}\ \ k\to\infty. \tag{3.22}$$

Similarly, call $\mu^{i+}_{\nu(k),t}$ the measure of positive i-waves in $u^k_{\nu(k)}(t,\cdot)$. When J is an interval, the decay estimates (1.30) also hold for front tracking approximations [9], hence for every $t<0$ the above argument yields

$$\mu^{i+}_{\nu(k),t}\rightharpoonup 0\qquad\text{as}\ \ k\to\infty. \tag{3.23}$$

By (3.18) every point (t,x) with $t<0$ has a neighborhood, say $\mathcal{N}\doteq[t-1,\ t/2]\times[x-1,\ x+1]$, such that the total amount of interaction and cancellation in u^k, $u^k_{\nu(k)}$ approaches zero as $k\to\infty$. As a consequence, the assumption (2.2) of Lemma 15 must hold.

We can now apply Lemma 15 and deduce the weak convergence

$$\mu^{i\pm}_{k,t}\rightharpoonup\tilde\mu^{i\pm}_t,\qquad\mu^{i\pm}_{\nu(k),t}\rightharpoonup\tilde\mu^{i\pm}_t\qquad\text{as}\ \ k\to\infty \tag{3.24}$$

for all $t<0$. Here $\tilde\mu^{i\pm}_t$ denote the positive and negative measures of i-waves in the limiting solution $\tilde u(t,\cdot)$ introduced at (3.3).

The weak convergence (3.24) also holds for $t>0$. Indeed, this follows again from Lemma 15 because (3.19) yields (2.4).

6. This step will establish the self-similarity of the function $\tilde{u}$, i.e.

$$\tilde{u}(t,x) = \tilde{u}(\lambda t, \lambda x) \qquad\qquad \lambda > 0. \tag{3.25}$$

From the definition of $\tilde{u}$ it is clear that

$$\tilde{u}(0,x) = \begin{cases} u(\tau, \xi+) & \text{if} \quad x > 0, \\ u(\tau, \xi-) & \text{if} \quad x < 0. \end{cases} \tag{3.26}$$

By (3.19)-(3.20), recalling that all wave speeds are < 1 in absolute value, we deduce

$$\begin{aligned} \tilde{u}(t,x) &= u(\tau, \xi-) \qquad \text{if } x \leq -|t|, \\ \tilde{u}(t,x) &= u(\tau, \xi+) \qquad \text{if } x \geq |t|. \end{aligned} \tag{3.27}$$

By (3.18), for $t > 0$ the function $\tilde{u}$ concides with the solution of the Riemann problem with data (3.27), hence (3.25) is clear. On the region where $t < 0$, the identity (3.25) will be proved by showing that

$$\int \phi\Big(\frac{x}{t}\Big)\, d\tilde{\mu}_t^{i-} = \int \phi\Big(\frac{x}{t'}\Big)\, d\tilde{\mu}_{t'}^{i-} \tag{3.28}$$

for every $t < t' < 0$ and every Lipschitz function ϕ with compact support. By (3.27), we can restrict the above integrations to the domains $[t,\ -t]$ and $[t',\ -t']$ respectively.

As a preliminary, fix $t < t' < 0$ and consider any front tracking approximate solution u defined on the strip $[t,t'] \times I\!R$. Call σ_α, x_α, for $\alpha \in \mathcal{W}_*^-(t)$, the sizes and locations of the negative wave-fronts in $u(t,\cdot)$ contained in the interval $[2t,\ -2t]$. We use the similar notation σ_β, x_β, for $\beta \in \mathcal{W}_*^-(t')$, for the fronts in $u(t',\cdot)$. For each $\beta \in \mathcal{W}_*^-(t')$, let $\mathcal{I}_\beta$ be the set of indices α such that the wave-front located at x_α at time t, reaches x_β at time t'. Recalling (1.14), the total amount of interaction and cancellation on the interval $]t,t']$ can be bounded in terms of $\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)$. We now have the estimates

$$\sum_{\beta \in \mathcal{W}_*^-(t')} \left| \sigma_\beta - \sum_{\alpha \in \mathcal{I}_\beta} \sigma_\alpha \right| \leq C \cdot \Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)\Big), \tag{3.29}$$

$$\sum_{\beta \in \mathcal{W}_*^-(t')} \left| |\sigma_\beta| \dot{x}_\beta - \sum_{\alpha \in \mathcal{I}_\beta} |\sigma_\alpha| \dot{x}_\alpha \right| \leq C \cdot \Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)\Big), \tag{3.30}$$

$$\sum_{\beta \in \mathcal{W}_*^-(t')} \left| |\sigma_\beta| x_\beta - \sum_{\alpha \in \mathcal{I}_\beta} |\sigma_\alpha| \big[x_\alpha + (t'-t)\dot{x}_\alpha\big] \right| \leq C \cdot \Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big)\Big). \tag{3.31}$$

Moreover, calling σ_α, $\alpha \in \mathcal{W}_{canc}^-$ the sizes of negative fronts in $u(t,\cdot)$ which cannot be prolonged to the whole interval $[t,t']$ (i.e., those which disappear due to cancellations with positive fronts), one has

$$\sum_{\alpha \in \mathcal{W}_{canc}^-(t)} |\sigma_\alpha| \leq C \cdot \Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big) \tag{3.32}$$

Applying the estimates (3.29)–(3.32) to the waves σ_α^k, σ_β^k in the approximate solutions $u_{\nu(k)}^k$ and using (3.21) we thus obtain

$$\lim_{k\to\infty} \sum_{\beta\in\mathcal{W}_*^-(t')} \left| \sigma_\beta^k - \sum_{\alpha\in\mathcal{I}_\beta} \sigma_\alpha^k \right| = 0, \qquad \lim_{k\to\infty} \sum_{\alpha\in\mathcal{W}_{canc}^-(t)} |\sigma_\alpha^k| = 0, \tag{3.33}$$

$$\lim_{k\to\infty} \sum_{\beta\in\mathcal{W}_*^-(t')} \left| |\sigma_\beta^{(k)}| x_\beta^{(k)} - \sum_{\alpha\in\mathcal{I}_\beta} |\sigma_\alpha^{(k)}| (t'/t) x_\alpha^{(k)} \right| = 0. \tag{3.34}$$

Moreover, by (3.20) and the fact that all wave speeds are < 1 it follows

$$\lim_{k\to\infty} \sum_{2|t|<|x_\alpha|<M} |\sigma_\alpha^k| = 0, \qquad \lim_{k\to\infty} \sum_{2|t'|<|x_\beta|<M} |\sigma_\beta^k| = 0 \tag{3.35}$$

for any large constant M. From (3.33)-(3.35) we deduce

$$\lim_{k\to\infty} \sum_{\alpha\in\mathcal{W}_*^-(t)} \phi\Big(\frac{x_\alpha^k}{t}\Big) \cdot |\sigma_\alpha^k| = \lim_{k\to\infty} \sum_{\beta\in\mathcal{W}_*^-(t')} \phi\Big(\frac{x_\beta^k}{t}\Big) \cdot |\sigma_\beta^k|.$$

By Lemma 15, this yields (3.28).

7. It remains to show that the limit function $\tilde{u}$ is independent of the specific sequence η_k. This will imply, in particular, that the whole family of rescaled functions u^η converges to $\tilde{u}$ as $\eta \to 0$. We only need to consider the case $t < 0$.

Another preliminary estimate is needed. For any front tracking approximation u and any time $t < 0$, define a positive Radon measure γ_t on the interval $[-1, 1]$ as follows:

$$\gamma_t(J) \doteq \sum_{\alpha\in\mathcal{W}_*^-(t),\, \dot{x}_\alpha\in J} |\sigma_\alpha| \tag{3.35}$$

for every Borel set J, where $\mathcal{W}_*^-(t)$ is the set of negative waves in $u(t,\cdot)$ contained in the interval $[2t,\ -2t]$. Consider the Kantorovich-Rubinstein distance $d(\gamma_t, \gamma_{t'})$ introduced at (1.21) (with M replaced by ∞). By standard interaction estimates it now follows

$$d(\gamma_t, \gamma_{t'}) \leq C \cdot \Big(\Upsilon\big(u(t)\big) - \Upsilon\big(u(t')\big) \Big), \tag{3.36}$$

since the right hand side controls the total amount of interaction and cancellation over the interval $[t, t']$.

Now consider two sequences $\eta_k, \eta_k^* \to 0$ and assume that the corresponding rescaled functions u^{η_k}, $u^{\eta_k^*}$ converge to self-similar solutions $\tilde{u}$, u^*, respectively. By a simple relabeling, it is not restrictive to assume $\eta_k^*/\eta_k \to 0$. Define the corresponding Radon measures $\tilde{\gamma}, \gamma^*$, by letting $\tilde{\gamma}(J)$,

$\gamma^*(J)$ be the total amount of (negative) waves in $\tilde{u}(-1,\cdot)$, $u^*(-1,\cdot)$ respectively, contained inside the Borel set $-J \doteq \{x;\ -x \in J\}$.

Construct a sequence of front tracking approximations $u_\nu \to u$. Let $\varepsilon > 0$ be given. Define the measures $\gamma_\nu^{\eta_k}$ as in (3.35), taking $u = u_\nu^{\eta_k}$, $t = -1$. Similarly, define $\gamma_\nu^{\eta_k^*}$ as in (3.35), taking $u = u_\nu^{\eta_k^*}$, $t = -1$.

Since the functions $t \mapsto \Upsilon\big(u_\nu(t)\big)$ are non-increasing and uniformly bounded, by possibly taking a subsequence we can assume that there exists $\tau_\varepsilon < \tau$ such that

$$\limsup_{\nu\to\infty} \Big[\Upsilon\big(u_\nu(\tau_\varepsilon)\big) - \Upsilon\big(u_\nu(t)\big)\Big] < \varepsilon \qquad t \in [\tau_\varepsilon, \tau[\,. \tag{3.37}$$

Choose $\bar{k}, \bar{\nu}$ large enough so that

$$\eta_{\bar{k}}, \eta_{\bar{k}}^* < \tau - \tau_\varepsilon, \tag{3.38}$$

$$d(\gamma_\nu^{\eta_k},\ \tilde{\gamma}) < \varepsilon, \qquad d(\gamma_\nu^{\eta_k^*},\ \gamma^*) < \varepsilon \qquad\qquad k \geq \bar{k},\ \nu \geq \bar{\nu}. \tag{3.39}$$

By (3.37)-(3.38) it follows

$$\limsup_{\nu\to\infty} d(\gamma_\nu^{\eta_k}, \gamma_\nu^{\eta_k^*}) \leq C \cdot \limsup_{\nu\to\infty} \Big[\Upsilon\big(u_\nu(\tau-\eta_k)\big) - \Upsilon\big(u_\nu(\tau-\eta_k^*)\big)\Big] \leq C\varepsilon. \tag{3.40}$$

Together (3.39) and (3.40) imply

$$d(\tilde{\gamma}, \gamma^*) \leq 2\varepsilon + C\varepsilon.$$

Since ε was arbitrary, $\tilde{\gamma} = \gamma^*$. Because of self-similarity, this implies $\tilde{u} = u^*$, completing the proof of Theorem 2.

4 - Structure of shock fronts.

We begin this section with a proof of Theorem 3, given in several steps.

1. Let u be a solution of (1.1) and let $\varepsilon > 0$ be given. Consider a sequence $u_\nu \to u$ of approximate solutions constructed by wave front tracking. Following [7, p. 29], by an *ε-shock front* in u_ν of the i-th family we mean a polygonal line in the t-x plane, with nodes (t_0, x_0), $(t_1, x_1), \ldots, (t_N, x_N)$ having the properties:

(i) The points (t_k, x_k) are interaction points, with $0 \leq t_0 < t_1 < \cdots$

(ii) On each segment joining (t_{k-1}, x_{k-1}) with (t_k, x_k), the function u_ν has an i-shock with size $\sigma_k \leq -\varepsilon/2$. Moreover, $\sigma_{\bar{k}} \leq -\varepsilon$ for at least one index $\bar{k} \in \{1, \ldots, N\}$.

(iii) For $k \neq N$, if two incoming i-shocks interact at the node (t_k, x_k), then the shock coming from (t_{k-1}, x_{k-1}) has the larger speed, i.e. is the one coming from the left.

An ε-shock front which is maximal w.r.t. the set theoretical inclusion will be called a *maximal ε-shock front*. As in [7], one can prove that two maximal ε-shock fronts either are disjoint or coincide. Observe that the number of shock fronts in $u_\nu(0,\cdot)$ of size $\sigma < -\varepsilon/2$ is uniformly bounded. Moreover, a new shock of size $< -\varepsilon$ can arise at some time $t > 0$ only if a uniformly large amount of interaction takes place. Therefore (see [7] for details), for fixed $\varepsilon > 0$, the number of maximal ε-shock fronts remains uniformly bounded as $\nu \to \infty$.

2. Call $x_{\nu,m} : [t^-_{\nu,m},\ t^+_{\nu,m}] \mapsto I\!R$, $m = 1, \ldots, N(\nu)$ the set of maximal ε-shock fronts in u_ν. By possibly extracting a subsequence, we can assume the convergence

$$x_{\nu,m}(\cdot) \to y_m(\cdot) \qquad\qquad t^\pm_{m,\nu} \to t^\pm_m, \qquad m \in \{1, \cdots, \bar N\}, \tag{4.1}$$

where $y_m \ : [t^-_m, t^+_m] \mapsto I\!R$ are Lipschitz continuous paths. Observe that two limiting paths may in principle coincide, on some interval at least. For each ν, call μ^{IC}_ν the positive, atomic measure of Interaction and Cancellation in the approximate solution u_ν, according to (3.4). By extracting a further subsequence, we can assume the weak convergence $\mu^{IC}_\nu \rightharpoonup \bar\mu^{IC}$. Call $\Theta \doteq \big\{(\tau_\ell, \xi_\ell);\ \ \ell = 1, 2, \ldots\big\}$ the set of atoms of $\bar\mu^{IC}$.

3. Consider any one of the curves $y_m(\cdot)$ in (4.1). We claim that, for every $t \in]t^-_m, t^+_m[$ such that $\big(t, x_m(t)\big) \notin \Theta$, the derivative $\dot y_m(t)$ exists and the corresponding left and right states defined at (1.18) satisfy the Rankine-Hugoniot equations (1.2). Indeed, since in a front-tracking approximation the change in the speed of a shock of strength $\big|\sigma_m(t)\big| \geq \varepsilon/2$ is controlled in terms of the amount of interaction and cancellation, the assumption $\bar\mu^{IC}\big\{(t, y_m(t))\big\} = 0$ implies the following. For every $\varepsilon' > 0$, there exists $\delta > 0$ such that

$$\limsup_{\nu\to\infty}\ \sup_{|s-t|<\delta} \big|\dot x_{\nu,m}(s) - \dot x_{\nu,m}(t)\big| < \varepsilon'. \tag{4.2}$$

From (4.2) and the uniform convergence $x_{\nu,m} \to y_m$ on a neighborhood of t we deduce

$$\dot y_m(t) = \lim_{\nu\to\infty} \dot x_{\nu,m}(t). \tag{4.3}$$

3. Next we claim that, for each $\varepsilon' > 0$, there exists $\rho > 0$ such that, setting

$$u^\pm_\nu \doteq u_\nu\big(t,\ x_{\nu,m}(t) \pm\big), \qquad \Gamma_\rho \doteq [t-\rho,\ t+\rho] \times \big[y_m(t) - \rho,\ y_m(t) + \rho\big],$$

for all ν large enough there holds

$$\big|u_\nu(s,x) - u^-_\nu\big| < \varepsilon' \qquad \text{if}\ \ (s,x) \in \Gamma_\rho,\ \ x < x_{\nu,m}(s), \tag{4.4}$$

$$\big|u_\nu(s,x) - u^+_\nu\big| < \varepsilon' \qquad \text{if}\ \ (s,x) \in \Gamma_\rho,\ \ x > x_{\nu,m}(s). \tag{4.5}$$

Indeed, in the opposite case, there would exist $\varepsilon' > 0$ and sequences $t_\nu \to t$, $\rho_\nu \to 0$ such that each function $u_\nu(t_\nu, \cdot)$ contains waves of total strength $> \varepsilon'$, disjoint from the large shock at $x_{m,\nu}$, inside an interval of arbitrarily small size, say $\big]x_{\nu,m}(t_\nu),\ x_{\nu,m}(t_\nu) + \rho_\nu\big[$. By strict hyperbolicity and genuine nonlinearity, this would generate a uniformly positive amount of interaction in an arbitrarily small neighborhood of the point $\big(t,\ y_m(t)\big)$, against the assumption. Therefore (4.4)-(4.5) hold. Recalling (4.3), since u_ν^-, u_ν^+ satisfy the Rankine-Hugoniot equations for all ν, the same is true for the states u^-, u^+ in (1.18). Since the total variation of the maps $t \mapsto \dot{x}_{\nu,m}(t)$ and $t \mapsto \big|\sigma_{\nu,m}(t)\big|$ is bounded uniformly w.r.t. ν (in terms of the total amount of interaction and cancellation along the given path), the maps $t \mapsto \dot{y}_m(t)$, $t \mapsto \big|\sigma_m(t)\big|$ are in BV. This establishes the first part of Theorem 3.

4. Let $\Gamma_\varepsilon \doteq \big\{(t, y_m(t));\ \ t \in [t_m^-, t_m^+],\ \ m = 1, \ldots, \bar{N}\big\}$ be the union of the larger shock fronts in u, and call $\Theta_\varepsilon \doteq \big\{(t_\ell, x_\ell);\ \ell = 1, \ldots, \bar{\ell}\big\}$ the set of atoms of $\bar{\mu}^{IC}$ with $\bar{\mu}^{IC}\big\{(t_\ell, x_\ell)\big\} \geq \varepsilon^3$. Consider any point $(\tau, \xi) \notin \Gamma_\varepsilon \cup \Theta_\varepsilon$. Then there exists $\bar{\nu}$ and a neighborhood V of (τ, ξ), say

$$V \doteq \big\{(t, x);\ \ |t - \tau| < \rho,\ \ |x - \xi| < 2\rho\big\}$$

such that

$$\mu_\nu^{IC}(V) < \varepsilon^3 \qquad \text{for all } \ \nu \geq \bar{\nu}. \tag{4.6}$$

We now apply the decay estimates (1.30) to the functions u_ν, with $J \doteq [\xi - \rho',\ \xi + \rho']$, $\rho' \doteq \varepsilon^3 \rho$, on the time interval $[\tau - \rho,\ \tau]$. For all $\nu \geq \bar{\nu}$ sufficiently large, the amount of positive waves in $u_\nu(\tau, \cdot)$ contained in the interval $[\xi - \rho',\ \xi + \rho']$ is thus $\leq C\varepsilon^3$. Moreover, we can assume that all shocks of strength $|\sigma| \geq \varepsilon$ remain outside V since $(\tau, \xi) \notin \Gamma_\varepsilon$.

Now assume that, for some sequence $\rho_\nu \to 0$, each $u_\nu(\tau, \cdot)$ has an amount $> 2\varepsilon$ of waves inside the interval $[\xi - \rho_\nu,\ \xi + \rho_\nu]$. Then, since by assumption there are no shocks of strength $\geq \varepsilon$ inside V, there would be a uniformly positive amount of interaction and cancellation in an arbitrarily small neighborhood of (τ, ξ). This would imply, $\bar{\mu}^{IC}\big\{(\tau, \xi)\big\} > c_0 \varepsilon^2$ for some $c_0 > 0$, in contradiction with (4.6). Therefore, for some fixed $\rho'' > 0$ sufficiently small, the total strength of waves in $u_\nu(\tau, \cdot)$ contained in $[\xi - \rho'',\ \xi + \rho'']$ is $\leq 2\varepsilon$ for all ν large.

5. Consider the neighborhood

$$\mathcal{N} \doteq \big\{(t, x);\ \ |t - \tau| + |x - \xi| < \rho''\big\}$$

of the point (τ, ξ). For each ν, call Λ_ν^τ the total strength of waves in $u_\nu(\tau, \cdot)$ contained in the interval $[\xi - \rho'',\ \xi + \rho'']$. Moreover, for $(t, x) \in \mathcal{N}$, call $\Lambda_\nu^{(t,x)}$ the total strength of waves in u_ν that cross the space-like segment joining the two points $(\tau,\ \xi + \rho'')$ and (t, x). By standard interaction estimates it follows

$$\big|\Lambda_\nu^{(t,x)} - \Lambda_\nu^\tau\big| \leq C\mu_\nu^{IC}(\mathcal{N}). \tag{4.7}$$

Using (4.7) and letting $\nu \to \infty$ we obtain

$$\begin{aligned}
&\limsup_{\nu\to\infty} \sup_{(t,x)\in\mathcal{N}} \big|u_\nu(t,x) - u_\nu(\tau,\xi)\big| \\
&\qquad \le \limsup_{\nu\to\infty} \sup_{(t,x)\in\mathcal{N}} \Big(\big|u_\nu(t,x) - u_\nu(\tau,\xi+\rho'')\big| + \big|u_\nu(\tau,\xi+\rho'') - u_\nu(\tau,\xi)\big|\Big) \\
&\qquad \le 2\limsup_{\nu\to\infty} \Lambda_\nu^\tau + C\cdot \limsup_{\nu\to\infty} \mu_\nu^{IC}(\mathcal{N}) \\
&\qquad \le 4\varepsilon + C\varepsilon^3.
\end{aligned}$$

This establishes (1.19), completing the proof of Theorem 3.

Corollary 4 is clear, taking as Θ the set of all atoms of the measure $\bar\mu^{IC}$.

Concerning Theorem 5, in the case where the u_ν are front tracking approximations of u, the result is already implied by the above proof of Theorem 3. We can take here $c_0 \doteq 1/10$, for instance. Observe that the sets I_ν may be the union of a finite number of intervals, not necessarily one single interval.

To treat the case where the u_ν are exact solutions, one can approximate each u_ν with a sequence of front tracking approximations $u_{\nu,m} \to u_\nu$, and work with a suitable diagonal subsequence $u_{\nu,m(\nu)} \to u$.

5 - Local structural stability.

The aim of this section is to prove Theorems 6–8 on the local structural stability of solutions. We begin by proving a result on the weak convergence of wave measures, derived from Lemma 15. According to (1.13)-(1.14), by $\Upsilon(u;J) \doteq V(u;J) + C_0 Q(u;J)$ we denote the sum of the strength of waves in u and of the interaction potential, restricted to the interval J. The shorter notation $\Upsilon^{Mt}(u) \doteq \Upsilon\big(u;\ [-Mt,\ Mt]\big)$ will also be used. The next lemma shows that, if a solution u remains very close to a self-similar solution $\tilde u$ for a long time, and if u contains almost no interaction and cancellation, then the wave-fronts of u must be close to those of $\tilde u$.

Lemma 17. *Let $\tilde u$ be a self-similar solution defined for $t<0$, and fix $M''>M'>M>1$.*

(a) Given $\varepsilon>0$, there exist T_1 large enough and $\delta,\delta_1>0$ such that the following holds. If $u=u(t,x)$ is another solution such that

$$\int_{M't}^{-M't} \big|u(t,x) - \tilde u(t,x)\big|\,dx < \frac{\delta}{t} \qquad \text{for all}\ \ t\in\big[-T_1,\ -1/2\big], \tag{5.1}$$

$$\Upsilon^{M'T_1}\big(u(-T_1)\big) - \Upsilon^{M'/2}\big(u(-1/2)\big) < \delta_1, \tag{5.2}$$

then, calling $\mu_{-1}^{i\pm}$, $\tilde\mu_{-1}^{i\pm}$ the measures of positive and negative i-waves in $u(-1,\cdot)$, $\tilde u(-1,\cdot)$ respectively, one has

$$d_M\big(\mu_{-1}^{i\pm},\ \tilde\mu_{-1}^{i\pm}\big) < \varepsilon \qquad\qquad i=1,\dots,n. \tag{5.3}$$

(b) Given $\delta_1 > 0$, there exist $T_2,\ \delta_2, \delta > 0$ such that

$$\int_{M''t}^{-M''t} \big|u(t,x) - \tilde u(t,x)\big|\ dx < \frac{\delta}{t} \qquad \textit{for all}\ \ t \in \big[-T_2,\ -1/2\big], \tag{5.4}$$

$$\Upsilon^{M''T_2}\big(u(-T_2)\big) - \Upsilon^{M''/2}\big(u(-1/2)\big) < \delta_2, \tag{5.5}$$

together imply

$$\Upsilon^{M'}\big(u(-1)\big) - \Upsilon^{M'}\big(\tilde u(-1)\big) < \frac{\delta_1}{2}\,. \tag{5.6}$$

Indeed, consider part (a). If the conclusion fails, there would exist a sequence of functions u_ν and numbers $T_\nu \to \infty$, $\delta_\nu \to 0$ such that

$$\int_{-M'}^{M'} \big|u_\nu(-1,x) - \tilde u(-1,x)\big|\ dx \to 0, \tag{5.7}$$

$$\Upsilon^{M'T_\nu}\big(u_\nu(-T_\nu)\big) - \Upsilon^{M'/2}\big(u(-1/2)\big) < \delta_\nu, \tag{5.8}$$

but the corresponding wave measures satisfy

$$d_M\big(\mu^{i\pm}_{-1,\nu},\ \tilde\mu^{i\pm}_{-1}\big) \geq \varepsilon$$

for some index $i\pm$ and all ν. We shall obtain a contradiction using Lemma 15. Indeed, by the decay of positive waves (1.30), from (5.8) we deduce

$$\mu^{i+}_{-1,\nu} \rightharpoonup 0 = \tilde\mu^{i+}_{-1}.$$

Moreover, the assumption (2.2) also holds. To see this, let x be given. If (2.2) failed, we could construct a sequence of positive numbers ρ_ν, strictly decreasing to zero, such that

$$\limsup_{\nu\to\infty} Q\big(u_\nu;\ [x - \rho_n u,\ x + \rho_\nu]\big) > 0.$$

Therefore, each solution $u_n u$ would contain a uniformly positive amount of interaction within the time interval $[-1,\ -1/2]$, a contradiction with (5.8). By part **(1)** in Lemma 15, we conclude

$$\lim_{\nu\to\infty} d_M\big(\mu^{i-}_{-1,\nu},\ \tilde\mu^{i-}_{-1}\big) = 0,$$

proving (a).

Part (b) of the lemma is proved by the same arguments as (a), observing that all waves in $\tilde u$ are negative and using **(2)** in Lemma 15.

To prove the local forward structural stability, we shall also need:

Lemma 18. *Fix $M > 1, \varepsilon > 0$. For $t \geq 0$, let $\tilde{u}$ be the self-similar solution of the Riemann problem with data u^-, u^+. Then there exists $\delta > 0$ such that, for every solution u of (1.1) satisfying*

$$V\Big(u(0);\ \ [-M-2,\ -\delta] \cup [\delta,\ M+2]\Big) < \delta, \tag{5.9}$$

$$\big|u(0,-M) - u^-\big| < \delta, \qquad\qquad \big|u(0,M) - u^+\big| < \delta, \tag{5.10}$$

one has

$$d_M\big(\mu^{i\pm}(u(1)),\ \mu^{i\pm}\big(\tilde{u}(1)\big)\big) < \varepsilon. \tag{5.11}$$

Indeed, if the conclusion of the lemma fails, there would exist a sequence of solutions $u_\nu \to \tilde{u}$ and numbers $\delta_\nu \to 0+$ satisfying the corresponding relations (5.9)-(5.10), together with

$$d_M\Big(\mu^{i\pm}\big(u_\nu(1)\big),\ \mu^{i\pm}\tilde{\big(}\tilde{u}(1)\big)\Big) \geq \varepsilon \qquad \text{for all} \ \ \nu \geq 1. \tag{5.12}$$

By showing that

$$\lim_{\nu\to\infty} Q\Big(u_\nu(1);]-M-1, M+1[\Big) = 0, \tag{5.13}$$

we can then use part (3) of Lemma 15, and obtain the weak convergence

$$\mu^{i\pm}\big(u_\nu(1)\big) \rightharpoonup \mu^{i\pm}\tilde{\big(}\tilde{u}(1)\big)$$

on the interval $[-M, M]$, in contradiction with (5.12). To prove the lemma, it thus remains to establish (5.13).

For each fixed ν, construct a sequence of front-tracking approximations $u_{\nu,m} \to u_\nu$, with

$$\mu^{i\pm}\big(u_{\nu,m}(0)\big) \rightharpoonup \mu^{i\pm}\big(u_\nu(0)\big) \qquad\qquad \text{as} \ \ m \to \infty. \tag{5.14}$$

The lower semicontinuity of the functional Q, stated at (1.32), implies

$$\liminf_{m\to\infty} Q\Big(u_{\nu,m}(1);\ [-M-1,\ M+1]\Big) \geq Q\big(u_\nu(1);\ [-M,\ M]\big).$$

Hence, if (5.11) fails, for some $\varepsilon > 0$ we must have

$$Q\big(u_{\nu,m}(1);\ [-M-1,\ M+1]\big) > \varepsilon_0,$$

for all ν and all $m \geq m(\nu)$ suitably large. By diagonalization argument, we thus obtain a sequence of front-tracking approximations $u_{\nu,\ell(\nu)} \to \tilde{u}$ such that

$$V\Big(u_{\nu,\ell(\nu)}(0);\ \ [-M-2,\ -\delta_\nu] \cup [\delta_\nu,\ M+2]\Big) < 2\delta_\nu, \tag{5.15}$$

but

$$Q\Big(u_{\nu,\ell(\nu)}(1);\ [-M-1,\ M+1]\Big) > \varepsilon_0 > 0 \tag{5.16}$$

for all ν. We now apply Lemma 14 to the sequence $u_{\nu,\ell(\nu)}$, choosing $\varepsilon = \varepsilon_0/2$, $(\tau,\xi) = (0,0)$, $t = \rho/3$. By possibly taking a subsequence, this yields

$$Q\Big(u_{\nu,\ell(\nu)}(\rho/3);\ [-2\rho/3,\ 2\rho/3]\Big) < \varepsilon_0/2 \tag{5.17}$$

for every ν. In turn, (5.17) and (5.15) imply

$$\begin{aligned}\limsup_{\nu\to\infty} Q\Big(u_{\nu,\ell(\nu)}(1);\ [-M-1,\ M+1]\Big)&\\ \leq \limsup_{\nu\to\infty} Q\Big(u_{\nu,\ell(\nu)}(\rho/3);\ [-M-2+\rho/3,\ M+2-\rho/3]\Big)&\\ \leq \varepsilon_0/2.&\end{aligned} \tag{5.18}$$

Indeed, the total strength of waves in $u_{\nu,\ell(\nu)}(\rho/3)$ outside the interval $[-2\rho/3,\ 2\rho/3]$ approaches zero, and one can always assume that $\rho/3 < 1$. From (5.18) we obtain a contradiction with (5.16), proving (5.13) and hence the lemma.

We can now give a proof of Theorem 6, in several steps, starting with the case $t < 0$. To simplify the notation, we shall assume $\xi = 0$, which is not restrictive.

1. Let $\varepsilon > 0$ and $M > 1$ be given. Choose any $M'' > M' > M$. We remark that the conclusions of Lemma 17 remain valid under rescalings. More precisely, for any $\theta > 0$ the assumptions

$$\int_{\theta M' t}^{-\theta M' t} \big|u(t,x) - \tilde u(t,x)\big|\, dx < \frac{\delta}{t} \qquad \text{for all } t \in \big[-\theta T_1,\ -\theta/2\big], \tag{5.19}$$

$$\Upsilon^{\theta M' T_1}\big(u(-\theta T_1)\big) - \Upsilon^{\theta M'/2}\big(u(-\theta/2)\big) < \delta_1, \tag{5.20}$$

imply

$$d_M\big(\widehat\mu^{i\pm}_{-\theta},\ \tilde\mu^{i\pm}_{-1}\big) < \varepsilon \qquad\qquad i = 1,\ldots,n. \tag{5.21}$$

Here $\widehat\mu^{i\pm}_{-\theta}$ denotes the normalized measure of positive or negative i-waves in $u(-\theta)$, i.e.

$$\int \phi(x)\, d\widehat\mu^{i\pm}_{-\theta} = \int \phi(x/\theta)\, d\mu^{i\pm}_{-\theta}.$$

The same remark applies to part (b) of the lemma.

2. Choose T_1, δ_1, δ according to part (a) of Lemma 17, then choose T_2, δ_2 and a possibly smaller $\delta > 0$ so that part (b) of the lemma also holds. By Theorem 2, there exists $\tau_0 < \tau$ such that

$$\int_{-M''}^{M''} \Big|u\big(t,\ \xi + (\tau - t)x\big) - \tilde u(-1,x)\Big|\, dx < \frac{\delta}{2} \qquad \text{for all } t \in [\tau_0,\ \tau[\,. \tag{5.22}$$

3. By (1.14)-(1.15), all functions $t \mapsto \Upsilon\big(u_\nu(t)\big)$ are non-increasing and uniformly bounded by a constant δ_0. We now choose an integer N so large that

$$\delta_0/N < \delta_2. \tag{5.23}$$

Next, with τ_0 as in step 2, we define the finite sequence of times τ_ℓ inductively by

$$\tau - \tau_\ell = \frac{\tau - \tau_{\ell-1}}{2 + 2T_2} \qquad \ell = 1, \ldots, N. \tag{5.24}$$

Moreover, we consider the intermediate times $\tau'_\ell \in [\tau_{\ell-1},\ \tau_\ell]$ such that

$$\tau - \tau'_\ell = 2(\tau - \tau_\ell). \tag{5.25}$$

Finally, we define the times $\tau^*, \tau^{**} \in [\tau_N,\ \tau]$ as

$$(\tau - \tau^{**}) = \frac{\tau - \tau'_N}{2 + 2T_1}, \qquad \tau - \tau^* = 2(\tau - \tau^{**}). \tag{5.26}$$

4. We now show that (1.23) holds in the case $t = \tau^*$. Indeed, for all ν large enough there holds

$$\int_{-M''}^{M''} \big|u_\nu\big(t,\ (\tau - t)x\big) - u\big(t,\ (\tau - t)x\big)\big|\, dx < \frac{\delta}{2} \qquad \text{for all } t \in \big[\tau_0,\ \tau^{**}\big]. \tag{5.27}$$

This, together with (5.22), implies

$$\int_{M''s}^{-M''s} \big|u_\nu(\tau + s,\ x) - \tilde{u}(s, x)\big|\ dx < \frac{\delta}{s} \qquad \text{for all}\ \ s \in [\tau_0 - \tau,\ \tau^{**} - \tau].$$

By (5.23) there exists at least one time interval, say $[\tau_{j-1}, \tau_j]$, where the drop in the functional $\Upsilon(u_\nu)$ is $< \delta_2$. By part (b) in Lemma 17, this implies

$$\Upsilon^{M'(\tau - \tau'_N)}\big(u_\nu(\tau'_N)\big) \le \Upsilon^{M'(\tau - \tau'_j)}\big(u_\nu(\tau'_j)\big) < \Upsilon\big(\tilde{u}(-1)\big) + \frac{\delta_1}{2}\,. \tag{5.28}$$

Since the functional Υ is lower semicontinuous and $t \mapsto \Upsilon\big(\tilde{u}(t)\big)$ is obviously constant for $t < 0$, for all ν large enough we have

$$\Upsilon^{M'(\tau - \tau^{**})}\big(u_\nu(\tau^{**})\big) > \Upsilon(\tilde{u}(-1)) - \frac{\delta_1}{2}\,. \tag{5.29}$$

By (5.28)-(5.29), the drop in $\Upsilon(u_\nu)$ on the interval $[\tau'_N,\ \tau^{**}]$ is $< \delta_1$. We can thus apply part (a) of Lemma 17, in its rescaled version, and conclude that, at time $t = \tau^*$,

$$d_M(\widehat{\mu}^{i\pm}_{\nu,t},\ \tilde{\mu}^{i\pm}_{-}) < \varepsilon.$$

This proves the local backward stability of wave-fronts in u, stated in (1.23), at the time $t = \tau^*$.

5. The general case of (1.23), where $t = \theta\tau^* + (1 - \theta)\tau$ for some $\theta \in]0, 1]$ can be treated in the same way, simply replacing the points τ_ℓ, τ^* with

$$\tau_{\ell,\theta} \doteq \theta\tau_\ell + (1 - \theta)\tau, \qquad \tau^*_\theta \doteq \theta\tau^* + (1 - \theta)\tau,$$

and defining $\tau'_{\ell,\theta}$, τ^{**}_θ similarly.

6. The remaining two steps will provide a proof of (1.24), always assuming $\xi = 0$. Let $\varepsilon > 0$ be given. For a fixed $t > 0$, by Lemma 18 the relation

$$d_M(\widehat{\mu}^{i\pm}_{\nu,t},\ \tilde{\mu}^{i\pm}_{+}) < \varepsilon \tag{5.30}$$

holds provided that the rescaled solution

$$u^*_\nu(t^*, x^*) \doteq u_\nu(\tau + t^*/t,\ x^*/t)$$

satisfies the corresponding estimates (5.7)-(5.8). In terms of the original function u_ν, these estimates hold if

$$V\big(u_\nu(\tau);\ \ [(-M-2)t,\ -\delta t] \cup [\delta t,\ (M+2)t]\big) < \delta, \tag{5.31}$$

$$\big|u_\nu(\tau, -Mt) - u^-\big| < \delta, \qquad\qquad \big|u_\nu(\tau, Mt) - u^+\big| < \delta, \tag{5.32}$$

7. By the backward stability property proved in steps 1–5, for any given $M' > 0$, $\varepsilon' > 0$ there exists $\rho' > 0$ such that

$$\limsup_{\nu\to\infty} d_{M'}(\widehat{\mu}^{i\pm}_{\nu,t'},\ \tilde{\mu}^{i\pm}_{-}) < \varepsilon' \qquad\qquad t' \in [\tau - \rho',\ \tau[\tag{5.33}$$

We claim that, for a suitable choice of M', ε', t', (5.33) implies (5.31). Indeed, since all waves in $\tilde{u}(t')$ lie inside the interval $\big[(t'-\tau),\ (\tau - t')\big]$, from (5.33) we deduce

$$V\Big(u_\nu(t');\ \ \big[(1-M')(\tau-t'),\ (M'-1)(\tau-t')\big] \setminus \big[2(t'-\tau),\ 2(\tau-t')\big]\Big) < C\varepsilon'. \tag{5.34}$$

In turn, since all wave speeds are < 1 in absolute value, from (5.34) it follows

$$V\Big(u_\nu(t');\ \ \big[(2-M')(\tau-t'),\ (M'-2)(\tau-t')\big] \setminus \big[3(t'-\tau),\ 3(\tau-t')\big]\Big) < C\varepsilon', \tag{5.35}$$

with a possibly different constant C.

Given $\varepsilon > 0$, $M > 1$, let δ be the constant in (5.31)-(5.32), determined in Lemma 18. We now choose M', ε' such that

$$C\varepsilon' = \delta, \qquad M' - 2 = 3(M+2)/\delta. \tag{5.36}$$

Then (5.33) holds for a suitable $\rho' > 0$. Set $\rho \doteq \delta\rho'/3$. For $t \in]\tau,\ \tau+\rho]$ we then have

$$t'(t) \doteq \tau - 3(t-\tau)/\delta \in [\tau - \rho',\ \tau[\,.$$

By the choices in (5.36), from (5.35) it thus follows (5.31). For ν large, the validity of (5.32) is clear. This completes the proof of Theorem 6.

By the same steps 1–5 in the proof of Theorem 6, using part (2) of Lemma 15 one obtains Theorem 7.

The results stated in Theorem 8 are a straightforward consequence of backward structural stability. Indeed, in cases (1)–(3), our assumptions imply the existence of $\rho, \delta > 0$ with the following property. If w is any other solution of (1.1) satisfying the integral estimates (1.26), then the restriction of $w(\tau - \rho, \cdot)$ to the interval $[\xi - 2\rho,\ \xi + 2\rho]$ contains respectively 0, 1 or 2 large incoming shocks, plus possibly other waves of total strength $< C\varepsilon$. Therefore, the structure of the solution w is determined in the domain of dependency:

$$\big\{(t,x);\quad t \in [\tau-\rho,\ \tau+\rho/3],\quad |x-\xi| \le \rho - (t-\tau)\big\}$$

as described in Theorem 8.

6 - Properties of wave measures.

This section contains a proof of Propositions 11–13 concerning wave measures.

To establish Proposition 11, we fix $0 \le \tau < \tau'$ and prove (1.35)-(1.36) with $(t,t') = (\tau,\tau')$. By a standard procedure we can construct a sequence of piecewise constant functions $x \mapsto u_\nu(\tau,x)$ satisfying

$$u_\nu(\tau) \to u(\tau), \qquad\qquad \mu^{i\pm}(u_\nu,\tau) \rightharpoonup \mu^{i\pm}(u,\tau), \quad i = 1,\ldots,n. \tag{6.1}$$

$$Q\big(u_\nu(\tau)\big) \to Q\big(u(\tau)\big), \qquad \Upsilon\big(u_\nu(\tau)\big) \to \Upsilon\big(u(\tau)\big). \tag{6.2}$$

Then, from this initial data $u_\nu(\tau)$, we construct a sequence of front tracking approximations u_ν, defined for $t \ge \tau$, such that $u_\nu \to u$. For a given ν, call $\tau = t_0 < t_1 < \cdots < t_N$ the interaction times of wave fronts in u_ν. On each interval $[t_{j-1},\ t_j[$ we have

$$d\big(\mu^{i\pm}(u_\nu,t_j-),\ \mu^{i\pm}(u_\nu,t_{j-1})\big) \le \delta_0(t_j - t_{j-1}) \tag{6.3}$$

because all wave speeds are < 1 in absolute value and the total strength of waves is $< \delta_0$. Moreover, at interaction times there holds (Cf. Proposition 1)

$$d\big(\mu^{i}(u_\nu,t_j),\ \mu^{i}(u_\nu,t_j-)\big) \le C_3\Big(Q\big(u_\nu(t_j-)\big) - Q\big(u_\nu(t_j)\big)\Big), \tag{6.4}$$

$$d\big(\mu^{i\pm}(u_\nu,t_j),\ \mu^{i\pm}(u_\nu,t_j-)\big) \le C_3\Big(\Upsilon\big(u_\nu(t_j-)\big) - \Upsilon\big(u_\nu(t_j)\big)\Big), \tag{6.5}$$

for some constant $C_3 > 0$. From (6.3)–(6.5) it follows

$$d\big(\mu^{i}(u_\nu,t),\ \mu^{i}(u_\nu,\tau)\big) \le C_3\Big(Q\big(u_\nu(\tau)\big) - Q\big(u_\nu(t)\big)\Big) + (t-\tau)\delta_0, \tag{6.6}$$

$$d\big(\mu^{i\pm}(u_\nu,t),\ \mu^{i\pm}(u_\nu,\tau)\big)\leq C_3\Big(\Upsilon\big(u_\nu(\tau)\big)-\Upsilon\big(u_\nu(t)\big)\Big)+(t-\tau)\delta_0, \tag{6.7}$$

for all $\nu\geq 1$, $t\geq\tau$.

To simplify the remaining arguments, we assume that all functions u, u_ν have uniformly bounded support, which is not restrictive. More precisely, let

$$u(t,x)=u_\nu(t,x)=0 \qquad \text{for all } \nu\geq 1,\quad |x|>R,\quad t\in[\tau,\ \tau'+1]$$

Consider a sequence $\varepsilon_k\to 0+$. Given $\varepsilon_1>0$, for each $\xi\in[-R,R]$ by Lemma 14 there exists $\rho>0$ such that

$$\limsup_{\nu\to\infty} Q\big(u_\nu(\tau'),\ [\xi-\rho,\ \xi+\rho]\big)<\varepsilon_1 \qquad \text{for each } t\in\,]\tau',\ \tau'+\rho].$$

Covering the compact set $[-R,R]$ with finitely many such intervals, we can find a constant $r_1>0$ and a subsequence $u_\nu^{(1)}$ such that

$$\limsup_{\nu\to\infty} Q\big(u_\nu^{(1)}(t),\ J\big)<\varepsilon_1$$

for each $t\in\,]\tau',\ \tau'+r_1]$ and every interval J of length $<r_1$. Repeating the previous argument, by induction for each ε_k we obtain a constant $r_k>0$ and a further subsequence $(u_\nu^{(k)})_{\nu\geq 1}$ such that

$$\limsup_{\nu\to\infty} Q\big(u_\nu^{(k)}(t),\ J\big)<\varepsilon_k$$

for each $t\in\,]\tau',\ \tau'+r_k]$ and every interval J of length $<r_k$. By a diagonalization procedure, we can now construct a subsequence u_ν^* and a sequence of times $t_\nu\to\tau'+$ such that

$$Q\big(u_\nu^*(t_\nu),\ J\big)<\varepsilon_k$$

whenever $\text{meas}(J)<r_k$ and $\nu\geq k$. By Lemma 15, this implies

$$\lim_{\nu\to\infty} d\Big(\mu^{i\pm}(u_\nu^*,t_\nu),\ \mu^{i\pm}(u,\tau')\Big)=0. \tag{6.8}$$

Since $u_\nu^*(t_\nu)\to u(\tau')$, the estimates (1.35)-(1.36) are now obtained from (6.6)-(6.7) (with $t=\tau'$ and u_ν replaced with u_ν^*), using (6.8), (6.2) and the lower semicontinuity of the functionals Q,Υ (Proposition 10).

To prove Proposition 12, consider a sequence of solutions u_ν to (1.1). Observe that the functions $t\mapsto Q\big(u_\nu(t)\big)$, $t\mapsto\Upsilon\big(u_\nu(t)\big)$ are uniformly bounded and non-increasing. Hence, by Helly's theorem, there exists a subsequence for which the limits (1.37) hold pointwise everywhere.

Call J the countable set of times t where Υ_∞ has a jump. By weak compactness, we can extract a further subsequence such that

$$\mu^{i\pm}(u_\nu, t) \rightharpoonup \mu^{i\pm}_\infty(t) \qquad \text{for all} \;\; t \in \mathbb{Q} \cup J.$$

Since the map $t \mapsto \mu^{i\pm}(u_\nu, t)$ has been shown to be continuous for $t \notin J$, then by (1.36) (with again u replaced with u_ν) the convergence (1.38) and the estimates (1.39) hold for all $t' > t \geq 0$.

Proposition 13 is again a consequence of local structural stability. Let a point (τ, ξ) be given. To shorten the notations, we write

$$\mu^{i\pm}_t \doteq \mu^{i\pm}(u,t), \qquad \mu^{i\pm}_{\nu,t} \doteq \mu^{i\pm}(u_\nu, t), \qquad \mu^{i\pm}_{\infty,t} \doteq \mu^{i\pm}_\infty(t).$$

Assume

$$\mu^{i\pm}_\tau\big(\{\xi\}\big) = b_{i\pm}.$$

By the convergence $\mu^{i\pm}_{\nu,\tau} \rightharpoonup \mu^{i\pm}_{\infty,\tau}$, for any given $\varepsilon > 0$ there exists $\rho_\varepsilon > 0$ and a sequence $r_\nu \to 0+$ such that for all $i = 1, \ldots, n$,

$$\mu^{i\pm}_{\nu,\tau}\big([\xi - \rho_\varepsilon,\ \xi + \rho_\varepsilon] \setminus [\xi - r_\nu,\ \xi + r_\nu]\big) < \varepsilon \qquad\qquad \nu \geq 1,$$

and hence

$$\mu^{i\pm}_{\nu,\tau+h}\big([\xi - \rho_\varepsilon/2,\ \xi + \rho_\varepsilon/2] \setminus [\xi - r_\nu - h,\ \xi + r_\nu + h]\big) < C\varepsilon \qquad\qquad \nu \geq 1. \tag{6.9}$$

By the property (1.24) of local forward stability, we also have

$$\limsup_{\nu\to\infty} \left| \mu^{i\pm}_{\nu,t}\Big(\big[\xi - 2(t-\tau),\ \xi + 2(t-\tau)\big]\Big) - b_{i\pm} \right| < C\varepsilon \qquad\qquad t \in]\tau,\ \tau + \rho\varepsilon]. \tag{6.10}$$

Concerning the measures $\mu^{i\pm}_{\infty,t}$, from (6.9)-(6.10) we deduce

$$\mu^{i\pm}_{\infty,\tau+}\big([\xi - \rho_\varepsilon/2,\ \xi + \rho_\varepsilon/2] \setminus \{\xi\}\big) < C\varepsilon \qquad\qquad \nu \geq 1. \tag{6.11}$$

$$\left| \mu^{i\pm}_{\infty,\tau+}\Big(\big[\xi - 2(t-\tau),\ \xi + 2(t-\tau)\big]\Big) - b_{i\pm} \right| < C\varepsilon. \tag{6.12}$$

Since $\varepsilon > 0$ is arbitrary, from (6.11)-(6.12) it follows

$$\mu^{i\pm}_{\infty,\tau+}\big(\{\xi\}\big) = b_{i\pm},$$

proving the first statement in Proposition 13.

Finally, observe that the function $t \mapsto Q\big(u(t)\big)$ is continuous at time τ if and only if at every point (τ, ξ), $\xi \in I\!R$, the corresponding self-similar solution

$$\tilde{u}(t,x) \doteq \lim_{\eta \to 0+} u(\tau + \eta t,\ \xi + \eta x)$$

constructed in Theorem 2 either is a constant or consists of a single shock. We thus distinguish two cases.

Case 1: At some point ξ, the corresponding function $\tilde{u}$ is not constant and does not consist of a single shock.

In this case, define $m_0 \doteq Q\big(\tilde{u}(-1)\big) > 0$. By the local stability properties (1-23)-(1.24), for every $\varepsilon > 0$ there exists $\rho > 0$ such that

$$\limsup_{\nu \to \infty} Q\big(u_\nu(\tau + h),\ [\xi - 2h,\ \xi + 2h]\big) \leq \varepsilon, \tag{6.13}$$

$$\liminf_{\nu \to \infty} Q\big(u_\nu(\tau - h),\ [\xi - 4h,\ \xi + 4h]\big) \geq m_0 - \varepsilon, \tag{6.14}$$

for all $h \in]0, \rho]$. Since $\varepsilon > 0$ can be taken arbitrarily small, from (6.13)-(6.14) we deduce

$$Q_\infty(\tau-) - Q_\infty(\tau+) \geq c\, m_0$$

for some positive constant c. Hence Q_∞ also has a jump at $t = \tau$.

Case 2: At every point ξ, the corresponding function $\tilde{u}$ either is constant or consists of a single shock.

It is then clear that the atomic parts of the measures $\mu^i(u, \tau)$ are purely negative. To show that the map $t \mapsto Q_\infty(t)$ is also continuous at time τ, fix $\varepsilon > 0$ and let the supports of all functions $u(\tau, \cdot)$, $u_\nu(\tau, \cdot)$ be contained in the interval $[-R, R]$.

By the local structural stability property (1.23), for each ξ there exists $\rho_\xi > 0$ such that

$$\limsup_{\nu \to \infty} Q\big(u_\nu(\tau - h);\ [\xi - 4h,\ \xi + 4h]\big) < \varepsilon \quad \text{for all } h \in]0, \rho_\xi].$$

Taking a finite covering of the interval $[-R, R]$, we deduce the existence of some $\rho^* > 0$ such that

$$\limsup_{\nu \to \infty} Q\big(u_\nu(\tau - \rho^*);\ [x - 4\rho^*,\ \xi + 4\rho^*]\big) < \varepsilon \qquad \text{for all } x \in [-R, R]. \tag{6.15}$$

From (6.15) it follows that the amount of interaction in u_ν near the time τ must be small. More precisely

$$\limsup_{\nu \to \infty} \Big(Q\big(u_\nu(\tau - \rho^*)\big) - Q\big(u_\nu(\tau + \rho^*)\big) \Big) < C\varepsilon. \tag{6.16}$$

Since ε is arbitrary, (6.16) implies the continuity of the function Q_∞ at time τ, completing the proof of Proposition 13.

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