$\mathbf{L}^1$ Stability Estimates for n x n Conservation Laws

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Abstract. Let $u_t + f(u)_x = 0$ be a strictly hyperbolic $n \times n$ system of conservation laws, each characteristic field being linearly degenerate or genuinely nonlinear. In this paper we explicitly define a functional $\Phi = \Phi(u, v)$, equivalent to the $\mathbf{L}^1$ distance, which is "almost decreasing" i.e.

$$\Phi\big(u(t),\ v(t)\big) - \Phi\big(u(s),\ v(s)\big) \leq \mathcal{O}(\varepsilon)\cdot(t-s) \qquad \text{for all } \ t > s \geq 0,$$

for every couple of ε-approximate solutions u, v with small total variation, generated by a wave front tracking algorithm. The small parameter ε here controls the errors in the wave speeds, the maximum size of rarefaction fronts and the total strength of all non-physical waves in u and in v. From the above estimate, it follows that front-tracking approximations converge to a unique limit solution, depending Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ norm. This provides a new proof of the existence of the Standard Riemann Semigroup generated by a $n \times n$ system of conservation laws.

1 - Introduction.

Aim of this paper is to provide a new, concise proof of the $\mathbf{L}^1$ stability of solutions to the Cauchy problem

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x), \tag{1.2}$$

for a strictly hyperbolic $n\times n$ system of conservation laws. Within a domain of small BV functions, the existence of a globally Lipschitz flow, whose trajectories are entropy weak solutions to (1.1), was conjectured in [2] and first proved in [7] for systems of two equations and in [8] for general $n \times n$ systems. In all these works, the basic idea is to connect two solutions u, v of (1.1) by a one-parameter family of solutions u^θ, and study how the length of the path $\gamma_t : \theta \mapsto u^\theta(t)$ varies in time. As long as all solutions u^θ remain sufficiently regular, the length of γ_t can be computed by integrating the norm of a generalized tangent vector. By studying the linearized evolution equation for these tangent vectors [11], an a-priori estimate on their norm is derived [5]. In turn, this provides a bound on the length of γ_t and hence on the distance $\big\|u(t)-v(t)\big\|_{\mathbf{L}^1}$. Unfortunately, this approach is hampered by the possible loss of regularity of the solutions u^θ. In order to retain the minimal regularity (piecewise Lipschitz continuity) required for the existence of tangent vectors, in [4, 7, 8] various approximation and restarting procedures had to be devised. These eventually led to entirely rigorous proofs, but at the price of heavy technicalities.

A different approach has been proposed in [18, 20, 21, 22]. It relies on the explicit construction of a functional $\Phi = \Phi(u,v)$ which is equivalent to the $\mathbf{L}^1$ distance:

$$\frac{1}{C}\|u-v\|_{\mathbf{L}^1} \le \Phi(u,v) \le C\big\|u-v\|_{\mathbf{L}^1}\,, \tag{1.3}$$

and which is decreasing in time along any pair of solutions of (1.1), i.e.

$$\Phi\big(u(t),\ v(t)\big) \le \Phi\big(u(s),\ v(s)\big) \qquad \text{for every } t > s \ge 0. \tag{1.4}$$

For $n \times n$ systems with coinciding shock and rarefaction curves, a functional having these properties was introduced in [20]. In the case of 2×2 systems without this coincidence property, the construction of an appropriate functional was carried out in [21]. The method can also be applied to general $n \times n$ systems [22].

In the present paper we show that the functional constructed in [21] and [22] can be simplified considerably when the distance between the two solutions u and v are measured along shock, instead of rarefaction curves. The new functional is then a part of the original ones in [21] and [22]. Indeed, for piecewise constant u, v, the value of $\Phi(u,v)$ is defined as follows. For each $x \in \mathbb{R}$, connect $u(x)$ with $v(x)$ moving always along shock curves. Call $q_i(x)$ the size of the i-th shock in the jump thus determined by $u(x)$ and $v(x)$. We then define

$$\Phi(u,v) \doteq \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|q_i(x)\big| W_i(x)\ dx, \tag{1.5}$$

where the weights W_i have the following form:

$$\begin{aligned} W_i(x) \doteq 1 + \kappa_1 \cdot \big[&\text{total strength of waves in } u \text{ and in } v \text{ which approach the } i\text{-wave } q_i(x)\big] \\ &+ \kappa_2 \cdot \big[\text{wave interaction potentials of } u \text{ and of } v\big]. \end{aligned} \tag{1.6}$$

See (2.16) for a precise definition. This functional consists of the linear, quadratic and generalized entropy functionals of [21] and [22]. Here the functional is expressed in a form so that the weights W_i look very similar to the ones used in [2, 4, 5, 7, 8]. In the case of systems with coinciding shock and rarefaction curves, our functional coincides with the one in [20]. In the opposite case, for a given genuinely nonlinear family $i \in \{1, \ldots, n\}$, it is interesting to observe that the contribution to W_i given by the i-waves in u and v approaching $q_i(x)$ yields precisely the nonlinear entropy functional studied in [19].

Most of our analysis will be concerned with ε-approximate solutions constructed by a wave-front tracking algorithm [1, 3, 13, 23]. These are piecewise constant functions in the t-x plane with a finite number of wave fronts, classified as shocks, rarefactions and non-physical waves. The small parameter ε controls three types of errors:

- The errors in the speeds of shock and rarefaction fronts
- The maximum strength of rarefaction fronts
- The total strength of all non-physical waves

As $\varepsilon \to 0$, every strong limit of ε-approximate solutions provides an entropy weak solution to (1.1). Studying the behavior of the functional $\Phi(t) \doteq \Phi\big(u(t),\ v(t)\big)$ in connection with front tracking approximations offers considerable advantages. Indeed, the maps $t \mapsto u(t),\ \ t \mapsto v(t)$ are continuous with values in $\mathbf{L}^1$. Hence the same is true for the corresponding wave strengths q_i in (1.5). Moreover, at every time τ where two fronts in u or in v interact, by the Glimm interaction estimates [14, 24] all weights W_i will decrease. This trivially implies $\Phi(\tau+) < \Phi(\tau-)$. Therefore, to prove the basic inequality

$$\Phi(t) \le \Phi(s) + \mathcal{O}(1) \cdot \varepsilon(t-s) \qquad\qquad 0 \le s < t, \tag{1.7}$$

it suffices to show that $\dot{\Phi} \le \mathcal{O}(1) \cdot \varepsilon$ outside interaction times. This can be checked by a direct calculation, relying again on standard interaction estimates.

Using (1.7) in connection with sequences of approximate solutions u_ν, v_ν and letting $\varepsilon_\nu \to 0$, it now becomes clear that front tracking approximations converge to a unique limit, depending Lipschitz continuously on the initial data. This provides a new, much simpler proof of the existence of a Lipschitz semigroup generated by the $n \times n$ system of conservation laws (1.1). We recall that the existence of such a semigroup plays a key role in various uniqueness proofs [9, 10] for entropy weak solutions of the Cauchy problem. As a further consequence, thanks the front-tracing technique developed in [16], sharp error estimates on approximate solutions constructed by the Glimm scheme can also be derived [12].

The paper is organized as follows. After a review of basic material, in Section 2 we give the definition of the functional Φ and state the main results. Section 3 contains an outline of the proof. The calculations involved in the main estimate are then performed in Section 4 for the case with coinciding shock and rarefaction curves, and in Section 5 for the general genuinely nonlinear case.

2 - Statement of the main results.

Let (1.1) be a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension, and assume that each characteristic field is either linearly degenerate or genuinely nonlinear [15, 24]. In the following we denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $A(u) \doteq Df(u)$. Moreover,

$$\sigma \mapsto S_i(\sigma)(u_0), \qquad \sigma \mapsto R_i(\sigma)(u_0) \tag{2.1}$$

indicate respectively the i-shock and i-rarefaction curves through the point u_0. If the i-th field is linearly degenerate, or else if the i-rarefaction curves are straight lines, then the above shock and rarefaction curves coincide [25]. In this case, we parametrize them simply by arc-length. On the other hand, in the general genuinely nonlinear case without this coincidence property, we choose the parameter σ in (2.1) so that

$$\frac{d}{d\sigma}\lambda_i\big(S_i(\sigma)(u_0)\big) \equiv 1, \qquad \frac{d}{d\sigma}\lambda_i\big(R_i(\sigma)(u_0)\big) \equiv 1,$$

$$\lambda_i\big(S_i(\sigma)(u_0)\big) - \lambda_i(u_0) \;=\; \sigma \;=\; \lambda_i\big(R_i(\sigma)(u_0)\big) - \lambda_i(u_0).$$

In all cases, it is well known [24] that the two curves S_i, R_i have a second order tangency at u_0. By $\lambda_i(u^+, u^-)$ we shall denote the i-th eigenvalue of the averaged matrix

$$A(u^+, u^-) \doteq \int_0^1 A\big(\theta u^+ + (1-\theta)u^-\big)\, d\theta.$$

When $u^+ = S_i(\sigma)(u^-)$, this eigenvalue coincides with the Rankine-Hugoniot speed of the i-shock joining u^- with u^+. In the following we shall consider approximate solutions of (1.1) with small total variation, obtained by a wave-front tracking algorithm.

Definition 1. *Given $\varepsilon > 0$, we say that $u : [0,\infty[\,\mapsto \mathbf{L}^1(I\!R;\ I\!R^n)$ is an ε-approximate front tracking solution of (1.1) if the following holds:*

1. *As a function of two variables, $u = u(t,x)$ is piecewise constant, with discontinuities occurring along finitely many lines in the t-x plane. Only finitely many wave-front interactions occur, each involving exactly two incoming fronts. Jumps can be of three types: shocks (or contact discontinuities), rarefactions and non-physical waves, denoted as $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP}$.*

2. *Along each shock (or contact discontinuity) $x = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the values $u^- \doteq u(t, x_\alpha-)$ and $u^+ \doteq u(t, x_\alpha+)$ are related by*

$$u^+ = S_{k_\alpha}(\sigma_\alpha)(u^-), \tag{2.2}$$

for some $k_\alpha \in \{1,\ldots,n\}$ and some wave size σ_α. If the k_α-th family is genuinely nonlinear, then the entropy admissibility condition $\sigma_\alpha < 0$ also holds. Moreover, the speed of the shock front satisfies

$$\big|\dot{x}_\alpha - \lambda_{k_\alpha}(u^+, u^-)\big| \leq \varepsilon. \tag{2.3}$$

3. *Along each rarefaction front $x = x_\alpha(t)$, $\alpha \in \mathcal{R}$, one has*

$$u^+ = R_{k_\alpha}(\sigma_\alpha)(u^-), \qquad\qquad \sigma_\alpha \in \,]0, \varepsilon] \tag{2.4}$$

for some genuinely nonlinear family k_α. Moreover,

$$\big|\dot{x}_\alpha(t) - \lambda_{k_\alpha}(u^+)\big| \leq \varepsilon. \tag{2.5}$$

4. *All non-physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ have the same speed:*

$$\dot{x}_\alpha(t) \equiv \hat{\lambda}, \tag{2.6}$$

where $\hat{\lambda}$ is a fixed constant strictly greater than all characteristic speeds. The total strength of all non-physical fronts in $u(t,\cdot)$ remains uniformly small, namely

$$\sum_{\alpha \in \mathcal{NP}} \big|u(t, x_\alpha+) - u(t, x_\alpha-)\big| \leq \varepsilon \qquad\qquad \textit{for all } t \geq 0. \tag{2.7}$$

If, in addition, the initial value of u satisfies

$$\big\|u(0,\cdot) - \bar{u}\big\|_{\mathbf{L}^1} < \varepsilon, \tag{2.8}$$

we say that u is an ε-approximate solution to the Cauchy problem (1.1)-(1.2).

In the following, we simply call

$$\sigma_\alpha \doteq \big|u(t, x_\alpha+) - u(t, x_\alpha-)\big| \qquad\qquad \alpha \in \mathcal{NP} \tag{2.9}$$

the strength of the non-physical front at $x_\alpha(t)$. By convention, we regard non-physical fronts as belonging to a fictitious linearly degenerate $(n+1)$-th characteristic family, so that $k_\alpha \doteq n+1$ for every $\alpha \in \mathcal{NP}$. As customary, the *total strength of waves* in u is measured by

$$V(u) \doteq \sum_\alpha |\sigma_\alpha|, \tag{2.10}$$

where the summation runs over all wave fronts of u. The *wave interaction potential* is

$$Q(u) = \sum_{(\alpha,\beta)\in\mathcal{A}} |\sigma_\alpha \cdot \sigma_\beta|, \tag{2.11}$$

where the summation runs over all couples of approaching waves. With the above convention on non-physical fronts, we recall that two fronts, of the families $k_\alpha, k_\beta \in \{1, \ldots, n+1\}$ and located at $x_\alpha < x_\beta$ respectively, are *approaching* if either $k_\alpha > k_\beta$ or else if $k_\alpha = k_\beta$ and at least one of them is a genuinely nonlinear shock.

The existence of front tracking approximate solutions was proved in [13] for systems of two equations and in [1, 3, 23] for general $n \times n$ systems. More precisely, for each $\varepsilon > 0$ and every initial data $\bar u \in \mathbf{L}^1$ with sufficiently small total variation, there exists an ε-approximate solution $u = u(t,x)$ to the Cauchy problem of (1.1)-(1.2), defined for all $t \geq 0$. For a suitable constant C_0, the function

$$t \mapsto \Upsilon\big(u(t)\big) \doteq V\big(u(t)\big) + C_0 Q\big(u(t)\big), \tag{2.12}$$

bounding the total variation of u, is non-increasing.

Now let v be another ε-approximate solution of (1.1) with small total variation. We wish to estimate how the distance $\big\|v(t) - u(t)\big\|_{\mathbf{L}^1}$ changes in time. For this purpose, we define the scalar functions q_i implicitly by

$$v(x) = S_n\big(q_n(x)\big) \circ \ \cdots \ \circ S_1\big(q_1(x)\big)\big(u(x)\big). \tag{2.13}$$

Intuitively, $q_i(x)$ can be regarded as the strength of the i-shock wave in the jump $\big(u(x),\ v(x)\big)$. On a compact neighborhood of the origin, we clearly have

$$\frac{1}{C_1} \cdot \big|v(x) - u(x)\big| \leq \sum_{i=1}^n \big|q_i(x)\big| \leq C_1 \cdot \big|v(x) - u(x)\big| \tag{2.14}$$

for some constant C_1. We now consider the functional

$$\Phi(u,v) \doteq \sum_{i=1}^{n} \int_{-\infty}^{\infty} \big|q_i(x)\big| W_i(x)\ dx, \tag{2.15}$$

where the weights W_i are defined by setting:

$$\begin{aligned} W_i(x) &\doteq 1 + \kappa_1 \cdot \big[\text{total strength of waves in } u \text{ and in } v \text{ which approach the } i\text{-wave } q_i(x)\big] \\ &\qquad + \kappa_2 \cdot \big[\text{wave interaction potentials of } u \text{ and of } v\big] \\ &\doteq 1 + \kappa_1 A_i(x) + \kappa_2 \big[Q(u) + Q(v)\big]. \end{aligned} \tag{2.16}$$

The amount of waves approaching $q_i(x)$ is defined as follows. If the i-shock and i-rarefaction curves coincide (Temple class), we simply take

$$A_i(x) \doteq \left[\sum_{x_\alpha < x,\ i < k_\alpha \le n} + \sum_{x_\alpha > x,\ 1 \le k_\alpha < i} \right] |\sigma_\alpha|. \tag{2.17}$$

The summations here extend to waves both of u and of v. By [25], the definition (2.17) applies if the i-th field is linearly degenerate, or else if all i-rarefaction curves are straight lines. On the other hand, if the i-th field is genuinely nonlinear with shock and rarefactions curves not always coinciding, our definition of A_i will contain an additional term, accounting for waves in u and in v of the same i-th family:

$$\begin{aligned} A_i(x) \doteq & \left[\sum_{\substack{\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v) \\ x_\alpha < x,\ i < k_\alpha \le n}} + \sum_{\substack{\alpha \in \mathcal{J}(u) \cup \mathcal{J}(v) \\ x_\alpha > x,\ 1 \le k_\alpha < i}} \right] |\sigma_\alpha| \\ & + \begin{cases} \left[\displaystyle\sum_{\substack{k_\alpha = i \\ \alpha \in \mathcal{J}(u),\ x_\alpha < x}} + \sum_{\substack{k_\alpha = i \\ \alpha \in \mathcal{J}(v),\ x_\alpha > x}} \right] |\sigma_\alpha| & \text{if } \ q_i(x) < 0, \\ \left[\displaystyle\sum_{\substack{k_\alpha = i \\ \alpha \in \mathcal{J}(v),\ x_\alpha < x}} + \sum_{\substack{k_\alpha = i \\ \alpha \in \mathcal{J}(u),\ x_\alpha > x}} \right] |\sigma_\alpha| & \text{if } \ q_i(x) > 0. \end{cases} \end{aligned} \tag{2.18}$$

Here and in the sequel, $\mathcal{J}(u)$ and $\mathcal{J}(v)$ denote the sets of all jumps in u and in v, while $\mathcal{J} \doteq \mathcal{J}(u) \cup \mathcal{J}(v)$. We recall that $k_\alpha \in \{1, \ldots, n+1\}$ is the family of the jump located at x_α with size σ_α. Notice that the strengths of non-physical waves do enter in the definition of Q. Indeed, a non-physical front located at x_α approaches all shock and rarefaction fronts located at points $x_\beta > x_\alpha$. On the other hand, non-physical fronts play no role in the definition of A_i.

The values of the large constants κ_1, κ_2 in (2.16) will be specified later. Observe that, as soon as these constants have been assigned, we can then impose a suitably small bound on the total variation of u, v so that

$$1 \le W_i(x) \le 2 \qquad \text{for all } \ i, x. \tag{2.19}$$

From (2.14), (2.15) and (2.19) it thus follows

$$\frac{1}{C_1}\cdot\big\|v-u\big\|_{\mathbf{L}^1}\leq\Phi(u,v)\leq 2C_1\cdot\big\|v-u\big\|_{\mathbf{L}^1}. \tag{2.20}$$

Recalling (2.12), our $\mathbf{L}^1$ stability estimate for front tracking approximations can now be stated as follows.

Theorem 1. *For suitable constants $C_2,\kappa_1,\kappa_2,\delta_0>0$ the following holds. Let u,v be ε-approximate front tracking solutions of (1.1), with*

$$\Upsilon\big(u(t)\big)<\delta_0,\qquad \Upsilon\big(v(t)\big)<\delta_0\qquad \textit{for all}\ \ t\geq 0. \tag{2.21}$$

Then the functional Φ in (2.15)-(2.18) satisfies

$$\Phi\big(u(t),\ v(t)\big)-\Phi\big(u(s),\ v(s)\big)\leq C_2\varepsilon(t-s)\qquad \textit{for all}\ \ 0\leq s<t. \tag{2.22}$$

From the above result, the existence of a Lipschitz semigroup generated by (1.1) can be easily proved. Indeed, recalling (2.12), consider the domain

$$\mathcal{D}\doteq cl\Big\{u\in\mathbf{L}^1(I\!R;\,I\!R^n);\ \ u\ \text{is piecewise constant},\ \ V(u)+C_0\cdot Q(u)<\delta_0\Big\}, \tag{2.23}$$

where cl denotes $\mathbf{L}^1$-closure. We then have

Theorem 2. *For every initial data $\bar u\in\mathcal{D}$, as $\varepsilon\to 0$ any sequence of ε-approximate front tracking solutions of the Cauchy problem (1.1)-(1.2) converges to a unique limit $u=u(t,x)$. The map $(\bar u,t)\mapsto u(t,\cdot)\doteq S_t\bar u$ defines a uniformly Lipschitz continuous semigroup, whose trajectories are entropy weak solutions of (1.1).*

Proof of Theorem 2. Let $\bar u\in\mathcal{D}$ be given. Consider any sequence $\{u_\nu\}_{\nu\geq 1}$, such that each u_ν is a front tracking ε_ν-approximate solution of (1.1) with

$$\big\|u_\nu(0)-\bar u\big\|_{\mathbf{L}^1}<\varepsilon_\nu,\qquad \lim_{\nu\to\infty}\varepsilon_\nu=0, \tag{2.24}$$

$$\Upsilon\big(u_\nu(t)\big)<\delta_0\qquad \text{for all}\ \ t\geq 0. \tag{2.25}$$

For every $\mu,\nu\geq 1$ and $t\geq 0$, by (2.20) and (2.22) it now follows

$$\begin{aligned}\big\|u_\mu(t)-u_\nu(t)\big\|_{\mathbf{L}^1}&\leq C_1\cdot\Phi\big(u_\mu(t),\ u_\nu(t)\big)\\&\leq C_1\cdot\Phi\big(u_\mu(0),\ u_\nu(0)\big)+C_1C_2t\cdot\max\{\varepsilon_\mu,\ \varepsilon_\nu\}\\&\leq 2C_1^2\big\|u_\mu(0)-u_\nu(0)\big\|_{\mathbf{L}^1}+C_1C_2t\cdot\max\{\varepsilon_\mu,\ \varepsilon_\nu\}.\end{aligned} \tag{2.26}$$

Since the right hand side of (2.26) approaches zero as $\mu, \nu \to \infty$, the sequence is Cauchy and converges to a unique limit. The semigroup property

$$S_s\big(S_t\bar u\big) = S_{s+t}\bar u$$

is an immediate consequence of uniqueness. Finally, let $\bar u, \bar v \in \mathcal{D}$ be given. For each $\nu \geq 1$, let u_ν, v_ν be front tracking ε_ν-approximate solutions of (1.1) with

$$\big\|u_\nu(0) - \bar u\big\|_{\mathbf{L}^1} < \varepsilon_\nu, \qquad \big\|v_\nu(0) - \bar v\big\|_{\mathbf{L}^1} < \varepsilon_\nu, \qquad \lim_{\nu\to\infty} \varepsilon_\nu = 0. \tag{2.27}$$

Using again (2.20) and (2.22) we deduce

$$\begin{aligned}
\big\|u_\nu(t) - v_\nu(t)\big\|_{\mathbf{L}^1} &\leq C_1 \cdot \Phi\big(u_\nu(t),\ v_\nu(t)\big) \\
&\leq C_1 \cdot \Big[\Phi\big(u_\nu(0),\ v_\nu(0)\big) + C_2 t\varepsilon_\nu\Big] \\
&\leq 2C_1^2\big\|u_\nu(0) - v_\nu(0)\big\|_{\mathbf{L}^1} + C_1C_2t\varepsilon_\nu.
\end{aligned} \tag{2.28}$$

Letting $\nu \to \infty$, by (2.27) it follows

$$\big\|u(t) - v(t)\big\|_{\mathbf{L}^1} \leq 2C_1^2 \cdot \|\bar u - \bar v\|_{\mathbf{L}^1}\,. \tag{2.29}$$

This establishes the Lipschitz continuity of the semigroup, completing the proof.

For the uniqueness of the semigroup S and a characterization of its trajectories we refer to [6].

3 - The basic estimate.

To prove Theorem 1, we need to examine how the functional Φ evolves in time. In connection with (2.13), at each x define the intermediate states $\omega_0(x) = u(x),\ \omega_1(x),\ \ldots,\ \omega_n(x) = v(x)$ by setting

$$\omega_i(x) \doteq S_i\big(q_i(x)\big) \circ\ S_{i-1}\big(q_{i-1}(x)\big) \circ\ \cdots\ \circ S_1\big(q_1(x)\big)\big(u(x)\big). \tag{3.1}$$

Moreover, call

$$\lambda_i(x) \doteq \lambda_i\big(\omega_{i-1}(x),\ \omega_i(x)\big) \tag{3.2}$$

the speed of the i-shock connecting $\omega_{i-1}(x)$ with $\omega_i(x)$. A direct computation now yields

$$\begin{aligned}
\frac{d}{dt}\Phi\big(u(t),\ v(t)\big) &= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^n \Big\{\big|q_i(x_\alpha-)\big|W_i(x_\alpha-) - \big|q_i(x_\alpha+)\big|W_i(x_\alpha+)\Big\}\cdot \dot x_\alpha \\
&= \sum_{\alpha\in\mathcal{J}}\sum_{i=1}^n \Big\{\big|q_i^{\alpha+}\big|W_i^{\alpha+}\big(\lambda_i^{\alpha+} - \dot x_\alpha\big) - \big|q_i^{\alpha-}\big|W_i^{\alpha-}\big(\lambda_i^{\alpha-} - \dot x_\alpha\big)\Big\},
\end{aligned} \tag{3.3}$$

with obvious meaning of notations. We regard the quantity $\big|q_i(x)\big|\lambda_i(x)$ as the flux of the i-th component of $|v-u|$ at x. For $x_{\alpha-1}<x<x_\alpha$, one clearly has

$$\big|q_i^{(\alpha-1)+}\big|\lambda_i^{(\alpha-1)+}W_i^{(\alpha-1)+}=\big|q_i(x)\big|\lambda_i(x)W_i(x)=\big|q_i^{\alpha-}\big|\lambda_i^{\alpha-}W_i^{\alpha-}.$$

Moreover, the assumption $u(t),v(t)\in\mathbf{L}^1$ and piecewise constant implies $q_i(t,x)\equiv 0$ for x outside a bounded interval. This allowed us to add and subtract the above terms in (3.3), without changing the overall sum.

In connection with (3.3), for each jump point $\alpha\in\mathcal{J}$ and every $i=1,\ldots,n$, define

$$E_{\alpha,i}\doteq\big|q_i^{\alpha+}\big|W_i^{\alpha+}\big(\lambda_i^{\alpha+}-\dot x_\alpha\big)-\big|q_i^{\alpha-}\big|W_i^{\alpha-}\big(\lambda_i^{\alpha-}-\dot x_\alpha\big). \tag{3.4}$$

Our main goal will be to establish the bounds

$$\sum_{i=1}^n E_{\alpha,i}\leq\mathcal{O}(1)\cdot|\sigma_\alpha|\qquad\qquad\alpha\in\mathcal{NP}, \tag{3.5}$$

$$\sum_{i=1}^n E_{\alpha,i}\leq\mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha|\qquad\qquad\alpha\in\mathcal{R}\cup\mathcal{S}. \tag{3.6}$$

Here and throughout the following, by the Landau symbol $\mathcal{O}(1)$ we denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (1.1). In particular, this bound does not depend on ε or on the functions u,v. It is also independent of the choice of the constants κ_1,κ_2 in (2.16).

From (3.5)-(3.6), recalling (2.7), (2.9) and the uniform bounds at (2.12) on the total strength of waves, one obtains the key estimate

$$\frac{d}{dt}\Phi\big(u(t),\ v(t)\big)\leq\mathcal{O}(1)\cdot\varepsilon. \tag{3.7}$$

If the constant κ_2 in (2.16) is chosen large enough, by the Glimm interaction estimates [15, 23] all weight functions $W_i(x)$ decrease at each time τ where two fronts of u or two fronts of v interact. Integrating (3.7) over any interval $[0,t]$ we therefore obtain

$$\Phi\big(u(t),\ v(t)\big)\leq\Phi\big(u(0),\ v(0)\big)+\mathcal{O}(1)\cdot\kappa_1\varepsilon t, \tag{3.8}$$

proving the theorem. All the remaining work is thus aimed at establishing (3.5)-(3.6).

If $\alpha\in\mathcal{NP}$, calling σ_α the strength of this jump as in (2.9), for $i=1,\ldots,n$ one has the easy estimates

$$\begin{aligned} q_i^{\alpha+}-q_i^{\alpha-}&=\mathcal{O}(1)\cdot\sigma_\alpha,\\ W_i^{\alpha+}-W_i^{\alpha-}&=0,\\ \lambda_i^{\alpha+}-\lambda_i^{\alpha-}&=\mathcal{O}(1)\cdot\sigma_\alpha. \end{aligned} \tag{3.9}$$

Therefore, writing

$$E_{\alpha,i} = \big(|q_i^{\alpha+}| - |q_i^{\alpha-}|\big) W_i^{\alpha+}(\lambda_i^{\alpha+} - \dot x_\alpha) \ + \ |q_i^{\alpha-}|(W_i^{\alpha+} - W_i^{\alpha-})(\lambda_i^{\alpha+} - \dot x_\alpha) \\ + \ |q_i^{\alpha-}| W_i^{\alpha-}(\lambda_i^{\alpha+} - \lambda_i^{\alpha-}), \tag{3.10}$$

the estimate (3.5) is clear.

Proving (3.6) will require more work. The case where the jump at x_α occurs in a family with coinciding shock and rarefaction curves is somewhat easier, and will be covered in Section 4. The general case will then be studied in Section 5.

4 - The case of coinciding shock and rarefaction curves.

The goal of this section is to establish (3.6) in the case where the jump at x_α occurs in a family with coinciding shock and rarefaction curves. We recall that σ_α denotes the size of this jump, occurring in the k_α-th characteristic family. In the following, since all computations refer to a fixed jump $\alpha \in \mathcal{R} \cup \mathcal{S}$, we shall drop the superscript $^\alpha$ and simply write $W_i^+ \doteq W_i^{\alpha+}$, $q_{k_\alpha}^- \doteq q_{k_\alpha}^{\alpha-}$, etc... The definition of the weights at (2.16)-(2.17) implies

$$W_{k_\alpha}^+ = W_{k_\alpha}^-, \qquad W_i^+ - W_i^- = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if} \quad i < k_\alpha, \\ -\kappa_1|\sigma_\alpha| & \text{if} \quad i > k_\alpha. \end{cases} \tag{4.1}$$

By strict hyperbolicity, we can assume that in the (suitably small) neighborhood Ω of the origin where u and v take values, one has

$$\lambda_j(\omega) - \lambda_i(\omega') \ge c > 0 \qquad \text{for all} \ \ i < j, \quad \omega, \omega' \in \Omega. \tag{4.2}$$

For $\alpha \in \mathcal{R} \cup \mathcal{S}$, thanks to the assumption of coinciding shock and rarefaction curves, we have the estimates

$$\big|q_{k_\alpha}^+ - q_{k_\alpha}^- \pm \sigma_\alpha\big| + \sum_{i \neq k_\alpha} |q_i^+ - q_i^-| = \mathcal{O}(1) \cdot \sum_{j \neq k_\alpha} \big|q_j^-\big||\sigma_\alpha| \\ = \mathcal{O}(1) \cdot \sum_{j \neq k_\alpha} \big|q_j^+\big||\sigma_\alpha| \,. \tag{4.3}$$

In the first term, the plus or minus sign is taken in the case where the jump occurs in u or in v, respectively. Indeed, to derive (4.3), we can regard the waves $q_i^\pm$ as generated by the interaction of the waves $q_i^\mp$ with a single k_α-wave of strength $|\sigma_\alpha|$. More precisely, if the jump occurs in v, we consider the interactions:

$$(q_1^-, \dots, q_n^-) \ \diamond \ (0, \dots, 0, \sigma_\alpha, 0, \dots, 0) \quad \mapsto \quad (q_1^+, \dots, q_n^+),$$
$$(q_1^+, \dots, q_n^+) \ \diamond \ (0, \dots, 0, -\sigma_\alpha, 0, \dots, 0) \quad \mapsto \quad (q_1^-, \dots, q_n^-).$$

If the jump occurs in u, we consider the interactions:

$$
\begin{aligned}
(0,\dots,0,\,-\sigma_\alpha\,,0,\dots,0) \;\diamond\; (q_1^-,\dots,q_n^-) \quad &\mapsto \quad (q_1^+,\dots,q_n^+),\\
(0,\dots,0,\,\sigma_\alpha\,,0,\dots,0) \;\diamond\; (q_1^+,\dots,q_n^+) \quad &\mapsto \quad (q_1^-,\dots,q_n^-).
\end{aligned}
$$

In all cases, the difference between the total amount of i-waves before and after interaction is bounded by an interaction potential, which contains only products of waves of distinct families. We now estimate the left hand side of (3.6), separating the terms related to the k_α-th family from all the others. For $i \neq k_\alpha$, writing $E_{\alpha,i}$ in the form (3.10) and using (4.1)-(4.2), we deduce

$$
E_{\alpha,i} \leq \mathcal{O}(1)\cdot W^{\max}\sum_{j\neq k_\alpha}\big|q_j^-\big||\sigma_\alpha| \;-\; c\kappa_1\big|q_i^-\big||\sigma_\alpha| + \mathcal{O}(1)\cdot\big|q_i^-\big||\sigma_\alpha|W_i^-. \tag{4.4}
$$

Here $W^{\max}$ denotes an upper bound for all weight functions W_i. As remarked at (2.19), one can assume $W^{\max} \leq 2$. Concerning the k_α-th component, we claim

$$
\begin{aligned}
E_{\alpha,k_\alpha} &= W_{k_\alpha}^-\Big\{|q_{k_\alpha}^+|(\lambda_{k_\alpha}^+ - \dot{x}_\alpha) \;-\; |q_{k_\alpha}^-|(\lambda_{k_\alpha}^- - \dot{x}_\alpha)\Big\}\\
&\leq \mathcal{O}(1)\cdot\Big\{\varepsilon|\sigma_\alpha| + \sum_{j\neq k_\alpha}\big|q_j^-\big||\sigma_\alpha|\Big\}.
\end{aligned} \tag{4.5}
$$

To prove (4.5), assume that the jump occurs in v, the other case being entirely similar. As in (3.1), consider the intermediate states

$$
\begin{aligned}
&\omega_0^- = u(x_\alpha-),\ \ \dots\ \ ,\ \omega_i^- = S_i(q_i^-)(\omega_{i-1}^-),\ \ \dots,\ \omega_n^- = v(x_\alpha-),\\
&\omega_0^+ = u(x_\alpha+),\ \ \dots\ \ ,\ \omega_i^+ = S_i(q_i^+)(\omega_{i-1}^+),\ \ \dots\ \ ,\ \omega_n^+ = v(x_\alpha+),
\end{aligned} \tag{4.6}
$$

Moreover, define the quantities

$$
\widetilde{q}_{k_\alpha} \doteq q_{k_\alpha}^- + \sigma_\alpha, \tag{4.7}
$$

$$
\lambda_{k_\alpha}^* \doteq \int_0^1 \lambda_{k_\alpha}\big(S_{k_\alpha}(\theta\sigma_\alpha)(\omega_{k_\alpha}^-)\big)\; d\theta. \tag{4.8}
$$

$$
\widetilde{\lambda}_{k_\alpha} \doteq \int_0^1 \lambda_{k_\alpha}\big(S_{k_\alpha}(\theta\widetilde{q}_{k_\alpha})(\omega_{k_\alpha-1}^-)\big)\; d\theta. \tag{4.9}
$$

By assumption, either the k_α-th family is linearly degenerate, or else its rarefaction curves are straight lines. In both cases, the shock speed defined according to (3.2) satisfies

$$
\lambda_{k_\alpha}^- \doteq \int_0^1 \lambda_{k_\alpha}\Big(S_{k_\alpha}(\theta q_{k_\alpha}^-)(\omega_{k_\alpha-1}^-)\big)\Big)\; d\theta. \tag{4.10}
$$

From (4.8)–(4.10) it follows the identity

$$
q_{k_\alpha}^-(\lambda_{k_\alpha}^- - \lambda_{k_\alpha}^*) = \widetilde{q}_{k_\alpha}(\widetilde{\lambda}_{k_\alpha} - \lambda_{k_\alpha}^*). \tag{4.11}
$$

Indeed, introducing the scalar flux function

$$g(\sigma) \doteq \int_0^\sigma \lambda_{k_\alpha}\big(S_{k_\alpha}(s)(\omega^-_{k_\alpha-1})\big)\ ds$$

and calling $q^-_{k_\alpha} \doteq a$, $\quad \widetilde{q}_{k_\alpha} \doteq b$, one has

$$\lambda^-_{k_\alpha} = \frac{g(a)}{a}, \qquad \widetilde{\lambda}_{k_\alpha} = \frac{g(b)}{b}, \qquad \lambda^*_{k_\alpha} = \frac{g(b)-g(a)}{b-a},$$

and (4.11) follows easily. In addition, we have the bounds

$$|\dot{x}_\alpha - \lambda^*_{k_\alpha}| = \mathcal{O}(1)\cdot\varepsilon + \mathcal{O}(1)\cdot|\omega^-_{k_\alpha} - \omega^-_n| = \mathcal{O}(1)\cdot\Big(\varepsilon + \sum_{j\neq k_\alpha}|q^-_j|\Big), \tag{4.12}$$

$$|q^+_{k_\alpha} - \widetilde{q}_{k_\alpha}| = \mathcal{O}(1)\cdot\sum_{j\neq k_\alpha}|q^-_j||\sigma_\alpha|, \tag{4.13}$$

$$|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| = \mathcal{O}(1)\cdot\big(|\omega^+_{k_\alpha-1} - \omega^-_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\big) = \mathcal{O}(1)\cdot\sum_{j\neq k_\alpha}|q^-_j||\sigma_\alpha|. \tag{4.14}$$

We now write

$$\begin{aligned} E_{\alpha,k_\alpha} \leq W^-_{k_\alpha}\cdot\Big\{&|\widetilde{q}_{k_\alpha}|(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) - |q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}) \ + \ |q^+_{k_\alpha} - \widetilde{q}_{k_\alpha}||\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}| \\ &+ |q^+_{k_\alpha} - q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^*_{k_\alpha}| \ + \ |q^+_{k_\alpha}||\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}|\Big\} \end{aligned} \tag{4.15}$$

and observe that, by (4.12),

$$|q^+_{k_\alpha} - q^-_{k_\alpha}||\dot{x}_\alpha - \lambda^*_{k_\alpha}| = \mathcal{O}(1)\cdot\Big(\varepsilon + \sum_{j\neq k_\alpha}|q^-_j|\Big)|\sigma_\alpha|. \tag{4.16}$$

Moreover, (4.13) and (4.14) imply

$$|q^+_{k_\alpha} - \widetilde{q}_{k_\alpha}||\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}| + |q^+_{k_\alpha}||\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| = \mathcal{O}(1)\cdot\sum_{j\neq k_\alpha}|q^-_j||\sigma_\alpha|. \tag{4.17}$$

To estimate the term

$$E^*_\alpha \doteq |\widetilde{q}_{k_\alpha}|(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) - |q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}) \tag{4.18}$$

we shall distinguish four cases.

CASE 1: $q^-_{k_\alpha}$, $\widetilde{q}_{k_\alpha}$ have the same sign. In this case (4.11) implies $E^*_\alpha = 0$.

CASE 2: The k_α-field is linearly degenerate. In this case $\lambda^-_{k_\alpha} = \lambda^*_{k_\alpha} = \widetilde{\lambda}_{k_\alpha}$, hence $E^*_\alpha = 0$.

CASE 3: The k_α-field is genuinely nonlinear and $q^-_{k_\alpha} < 0 < \widetilde{q}_{k_\alpha}$, hence $\sigma_\alpha > 0$. In this case, a basic property of front tracking approximations requires $\sigma_\alpha \leq \varepsilon$. Therefore, from the estimates

$$|\widetilde{q}_{k_\alpha}| + |q^-_{k_\alpha}| = \sigma_\alpha, \qquad |\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}| = \mathcal{O}(1)\cdot\varepsilon, \qquad |\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}| = \mathcal{O}(1)\cdot\varepsilon,$$

we deduce $E^*_\alpha = \mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha|$.

CASE 4: The k_α-field is genuinely nonlinear and $\widetilde{q}_{k_\alpha} < 0 < q^-_{k_\alpha}$, hence $\sigma_\alpha < 0$. In this case, we have $\widetilde{\lambda}_{k_\alpha} < \lambda^*_{k_\alpha} < \lambda^-_{k_\alpha}$, therefore $E^*_\alpha < 0$.

The discussion of these four cases completes the proof of (4.5).

By (4.4)-(4.5), we can now choose the constant κ_1 in (2.16) large enough and the total variation of u, v sufficiently small so that

$$\begin{aligned} E_{\alpha,k_\alpha} + \sum_{i\neq k_\alpha} E_{i,\alpha} &\leq \mathcal{O}(1)\cdot\sum_{j\neq k_\alpha}\left|q^{\alpha-}_j\right||\sigma_\alpha| - c\kappa_1\cdot\sum_{i\neq k_\alpha}\left|q^{\alpha-}_i\right||\sigma_\alpha| + \mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha| \\ &\leq \mathcal{O}(1)\cdot\varepsilon|\sigma_\alpha|. \end{aligned} \tag{4.19}$$

This establishes (3.6).

5 - The genuinely nonlinear case.

We shall now prove (3.6) in the genuinely nonlinear case, dropping the assumption that shock and rarefaction curves coincide. To fix the ideas, let $\alpha \in \mathcal{J}(v)$, the other case being similar. As usual, let σ_α be the size of the jump at x_α, occurring in the k_α-th characteristic family. According to (2.16), (2.18), the weights $W^{\alpha\pm}_i$ satisfy

$$W^+_i - W^-_i = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if} \quad i < k_\alpha, \\ -\kappa_1|\sigma_\alpha| & \text{if} \quad i > k_\alpha, \end{cases} \tag{5.1}$$

$$W^+_{k_\alpha} - W^-_{k_\alpha} = \begin{cases} \kappa_1|\sigma_\alpha| & \text{if} \quad \min\{q^+_{k_\alpha},\ q^-_{k_\alpha}\} > 0, \\ -\kappa_1|\sigma_\alpha| & \text{if} \quad \max\{q^+_{k_\alpha},\ q^-_{k_\alpha}\} < 0. \end{cases} \tag{5.2}$$

We first seek an estimate relating the quantities $q^+_i, q^-_i, \sigma_\alpha$, which will replace (4.3). Toward this goal, by an easy modification of Theorem 3.3 in [17] one obtains

Lemma 1. *Given any state $u^* \in \Omega$, if the values $\sigma_i, \sigma'_i, \sigma''_i$ satisfy*

$$S_n(\sigma_n)\circ\ \cdots\ \circ S_1(\sigma_1)(u^*) = S_n(\sigma'_n)\circ\ \cdots\ \circ S_1(\sigma'_1)\circ S_n(\sigma''_n)\circ\ \cdots\ \circ S_1(\sigma''_1)(u^*),$$

then

$$\sum_{i=1}^n |\sigma_i - \sigma'_i - \sigma''_i| = \mathcal{O}(1)\cdot\left(\sum_j |\sigma'_j\sigma''_j|\big(|\sigma'_j| + |\sigma''_j|\big) + \sum_{j\neq k}|\sigma'_j\sigma''_k|\right).$$

If the values σ'_i, σ are related by

$$R_{k_\alpha}(\sigma)(u^*) = S_n(\sigma'_n)\circ\ \cdots\ \circ S_1(\sigma'_1)(u^*),$$

then

$$|\sigma - \sigma'_{k_\alpha}| + \sum_{i\neq k_\alpha} |\sigma'_i| = \mathcal{O}(1)\cdot \left(|\sigma'_{k_\alpha}\sigma| \big(|\sigma'_{k_\alpha}| + |\sigma|\big) + \sum_{j\neq k_\alpha} |\sigma'_j \sigma| \right).$$

In the case where the jump σ_α in v is a shock, using the first part of Lemma 1 we obtain

$$\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| = \mathcal{O}(1)\cdot \left(|q^-_{k_\alpha}|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{j\neq k_\alpha} \big|q^-_j\big| \right) |\sigma_\alpha|, \tag{5.3}$$

$$= \mathcal{O}(1)\cdot \left(|q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{j\neq k_\alpha} \big|q^+_j\big| \right) |\sigma_\alpha|. \tag{5.4}$$

On the other hand, in case of a rarefaction, recalling that $\sigma_\alpha \in]0,\varepsilon]$ and using both parts of Lemma 1 we recover the estimates

$$\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big| + \sum_{i\neq k_\alpha} |q^+_i - q^-_i| = \mathcal{O}(1)\cdot \left(\varepsilon + |q^-_{k_\alpha}|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{j\neq k_\alpha} \big|q^-_j\big| \right) |\sigma_\alpha|, \tag{5.5}$$

$$= \mathcal{O}(1)\cdot \left(\varepsilon + |q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{j\neq k_\alpha} \big|q^+_j\big| \right) |\sigma_\alpha|. \tag{5.6}$$

The following simple lemma will be repeatedly used in the sequel.

Lemma 2. *Let $\bar\omega \in \Omega$, $\sigma, \sigma' \in I\!R$, $k \in \{1\ldots,n\}$. Define the states and wave speeds*

$$\begin{array}{ll} \omega \doteq S_k(\sigma)(\bar\omega) & \lambda \doteq \lambda_k(\bar\omega,\ \omega), \\ \omega' \doteq S_k(\sigma')(\omega) & \lambda' \doteq \lambda_k(\omega,\ \omega'), \\ \omega'' \doteq S_k(\sigma+\sigma')(\bar\omega) & \lambda'' \doteq \lambda_k(\bar\omega,\ \omega''). \end{array}$$

One then has the estimate

$$\big|(\sigma+\sigma')(\lambda''-\lambda') - \sigma(\lambda-\lambda')\big| = \mathcal{O}(1)\cdot|\sigma\sigma'|\big(|\sigma|+|\sigma'|\big). \tag{5.7}$$

Indeed, the function

$$\Psi(\sigma,\sigma') \ \doteq\ (\sigma+\sigma')\lambda'' - \sigma\lambda - \sigma'\lambda' = (\sigma+\sigma')(\lambda''-\lambda') - \sigma(\lambda-\lambda')$$

is smooth and satisfies

$$\Psi(\sigma,0) \equiv \Psi(0,\sigma') \equiv 0, \qquad \frac{\partial^2\Psi}{\partial\sigma\,\partial\sigma'}(0,0) = 0. \tag{5.8}$$

Therefore,

$$\Psi(\sigma,\sigma') = \int_0^\sigma \int_0^{\sigma'} \frac{\partial^2\Psi}{\partial\sigma\,\partial\sigma'}(r,s)\ drds = \mathcal{O}(1)\cdot \int_0^{|\sigma|}\int_0^{|\sigma'|} \big(|r|+|s|\big)\ drds,$$

proving the lemma.

We can now begin the proof of the basic estimate (3.6), considering first the case of a rarefaction front: $\alpha \in \mathcal{R}$. Since the total variation is small, by (5.5)-(5.6) we can assume

$$\sigma_\alpha \in]0, \varepsilon], \qquad 0 < q^+_{k_\alpha} - q^-_{k_\alpha} < 2\sigma_\alpha \leq 2\varepsilon. \tag{5.9}$$

Three cases must be studied, depending on the signs of $q^-_{k_\alpha}, q^+_{k_\alpha}$.

CASE 1: $0 < q^-_{k_\alpha} < q^+_{k_\alpha}$.

We first seek an estimate similar to (4.4), valid for $i \neq k_\alpha$ Writing again $E_{\alpha,i}$ in the form (3.10) but using (5.5) in place of (4.3), we obtain

$$E_{\alpha,i} \leq \mathcal{O}(1) \cdot \left(\varepsilon + |q^-_{k_\alpha}| \big(|q^-_{k_\alpha}| + |\sigma_\alpha| \big) + \sum_{j \neq k_\alpha} |q^-_j| \right) |\sigma_\alpha| - c\kappa_1 |q^-_i| |\sigma_\alpha| \qquad i \neq k_\alpha. \tag{5.10}$$

Next, for $i = k_\alpha$, we seek an estimate replacing (4.5). Consider the intermediate states $\omega^\pm_i$ as in (4.6). We then define the states

$$\widetilde{\omega}_{k_\alpha} \doteq S_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha)(\omega^-_{k_\alpha - 1}), \qquad \omega^*_{k_\alpha} \doteq S_{k_\alpha}(\sigma_\alpha)(\omega^-_{k_\alpha}), \tag{5.11}$$

and the shock speeds

$$\lambda^*_{k_\alpha} \doteq \lambda_{k_\alpha}(\omega^-_{k_\alpha},\ \omega^*_{k_\alpha}), \qquad \widetilde{\lambda}_{k_\alpha} \doteq \lambda_{k_\alpha}(\omega^-_{k_\alpha - 1},\ \widetilde{\omega}_{k_\alpha}). \tag{5.12}$$

We recall that, according to (3.2),

$$\lambda^-_{k_\alpha} \doteq \lambda^-_{k_\alpha}(\omega^-_{k_\alpha - 1},\ \omega^-_{k_\alpha}). \tag{5.13}$$

One can easily check that, if shock and rarefaction curves coincided, then $\widetilde{\omega}_{k_\alpha} = \omega^*_{k_\alpha}$ and the above definitions would reduce to (4.8)–(4.10). An application of Lemma 2 with $\bar{\omega} = \omega^-_{k_\alpha - 1}$, $\sigma = q^-_{k_\alpha}$, $\sigma' = \sigma_\alpha$ yields

$$\left| (q^-_{k_\alpha} + \sigma_\alpha)(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) - q^-_{k_\alpha}(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}) \right| = \mathcal{O}(1) \cdot |q^-_{k_\alpha}| |\sigma_\alpha| \big(|q^-_{k_\alpha}| + |\sigma_\alpha| \big). \tag{5.14}$$

Moreover, one has

$$\begin{aligned} |\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| &= \mathcal{O}(1) \cdot \big(|\omega^-_{k_\alpha - 1} - \omega^+_{k_\alpha - 1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha| \big) \\ &= \mathcal{O}(1) \cdot \left(\varepsilon + |q^-_{k_\alpha}| \big(|q^-_{k_\alpha}| + |\sigma_\alpha| \big) + \sum_{i \neq k_\alpha} |q^-_i| \right) |\sigma_\alpha|, \end{aligned} \tag{5.15}$$

$$\begin{aligned} |\lambda^*_{k_\alpha} - \dot{x}_\alpha| &\leq \varepsilon + \mathcal{O}(1) \cdot |\omega^-_{k_\alpha} - \omega^-_n| \\ &\leq \varepsilon + \mathcal{O}(1) \cdot \sum_{i \neq k_\alpha} |q^-_i|. \end{aligned} \tag{5.16}$$

The assumption of genuine nonlinearity implies

$$\lambda^*_{k_\alpha} - \widetilde{\lambda}_{k_\alpha} > c'|q^-_{k_\alpha}| \tag{5.17}$$

for some constant $c' > 0$. In this case $W^+_{k_\alpha} = W^-_{k_\alpha} + \kappa_1|\sigma_\alpha|$. Using (5.14)–(5.17) and recalling that the quantities $\sigma_\alpha, q^+_{k_\alpha}, q^-_{k_\alpha}$ are all positive, we now compute

$$\begin{aligned}
E_{\alpha,k_\alpha} &= \big(W^-_{k_\alpha} + \kappa_1\sigma_\alpha\big) q^+_{k_\alpha}(\lambda^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha} q^-_{k_\alpha}(\lambda^-_{k_\alpha} - \dot{x}_\alpha) \\
&\leq \kappa_1\sigma_\alpha(q^-_{k_\alpha} + \sigma_\alpha)(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) + \kappa_1\sigma_\alpha(q^-_{k_\alpha} + \sigma_\alpha)\big(|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| + |\dot{x}_\alpha - \lambda^*_{k_\alpha}|\big) \\
&\qquad + \kappa_1\sigma_\alpha\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^+_{k_\alpha} - \dot{x}_\alpha| + W^-_{k_\alpha}\big\{q^+_{k_\alpha}(\lambda^+_{k_\alpha} - \dot{x}_\alpha) - q^-_{k_\alpha}(\lambda^-_{k_\alpha} - \dot{x}_\alpha)\big\} \\
&\leq -c'\kappa_1 q^-_{k_\alpha}\sigma_\alpha\big(q^-_{k_\alpha} + \sigma_\alpha\big) + \mathcal{O}(1)\cdot\kappa_1\sigma_\alpha(q^-_{k_\alpha} + \sigma_\alpha)\left(\varepsilon + q^-_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha) + \sum_{i\neq k_\alpha}|q^-_i|\right) \\
&\qquad + W^-_{k_\alpha}\bigg\{\Big|(q^-_{k_\alpha} + \sigma_\alpha)(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) - q^-_{k_\alpha}(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha})\Big| + \big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^+_{k_\alpha} - \dot{x}_\alpha| \\
&\qquad\qquad + \sigma_\alpha|\lambda^*_{k_\alpha} - \dot{x}_\alpha| + (q^-_{k_\alpha} + \sigma_\alpha)|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}|\bigg\} \\
&\leq -c'\kappa_1 q^-_{k_\alpha}\sigma_\alpha(q^-_{k_\alpha} + \sigma_\alpha) + \mathcal{O}(1)\cdot\left(\varepsilon + q^-_{k_\alpha}(q^-_{k_\alpha} + \sigma_\alpha) + \sum_{i\neq k_\alpha}|q^-_i|\right)\sigma_\alpha.
\end{aligned} \tag{5.18}$$

Choosing the constant κ_1 large enough, (5.10) and (5.18) together imply (3.6).

CASE 2: $q^-_{k_\alpha} < q^+_{k_\alpha} < 0$.

The estimates are almost the same as in Case 1, using (5.6) instead of (5.5). For $i \neq k_\alpha$, writing $E_{\alpha,i}$ in the form (3.10) we obtain

$$E_{\alpha,i} \leq \mathcal{O}(1)\cdot\left(\varepsilon + |q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{j\neq k_\alpha}\big|q^+_j\big|\right)|\sigma_\alpha| - c\kappa_1\big|q^+_i\big||\sigma_\alpha| \qquad i \neq k_\alpha. \tag{5.19}$$

To estimate E_{α,k_α}, consider the intermediate states $\omega^\pm_i$ as in (4.6) and define

$$\widetilde{\omega}_{k_\alpha} \doteq S_{k_\alpha}(q^+_{k_\alpha} - \sigma_\alpha)(\omega^+_{k_\alpha-1}), \qquad \omega^*_{k_\alpha} \doteq S_{k_\alpha}(-\sigma_\alpha)(\omega^+_{k_\alpha}), \tag{5.20}$$

and the shock speeds

$$\lambda^*_{k_\alpha} \doteq \lambda_{k_\alpha}(\omega^+_{k_\alpha},\ \omega^*_{k_\alpha}), \qquad \widetilde{\lambda}_{k_\alpha} \doteq \lambda_{k_\alpha}(\omega^+_{k_\alpha-1},\ \widetilde{\omega}_{k_\alpha}). \tag{5.21}$$

We recall that, according to (3.2),

$$\lambda^-_{k_\alpha} \doteq \lambda^-_{k_\alpha}(\omega^-_{k_\alpha-1},\ \omega^-_{k_\alpha}). \tag{5.22}$$

An application of Lemma 2 with $\bar\omega = \omega^+_{k_\alpha-1}$, $\sigma = q^+_{k_\alpha}$, $\sigma' = -\sigma_\alpha$ yields

$$\big|(q^+_{k_\alpha} - \sigma_\alpha)(\widetilde\lambda_{k_\alpha} - \lambda^*_{k_\alpha}) - q^+_{k_\alpha}(\lambda^+_{k_\alpha} - \lambda^*_{k_\alpha})\big| = \mathcal{O}(1)\cdot |q^+_{k_\alpha}||\sigma_\alpha|\big(|q^+_{k_\alpha}| + |\sigma_\alpha|\big). \tag{5.23}$$

Moreover, by (5.4) one has

$$\begin{aligned} |\lambda^-_{k_\alpha} - \widetilde\lambda_{k_\alpha}| &= \mathcal{O}(1)\cdot\big(|\omega^-_{k_\alpha-1} - \omega^+_{k_\alpha-1}| + |q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha|\big) \\ &= \mathcal{O}(1)\cdot\left(\varepsilon + |q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{i\neq k_\alpha}|q^+_i|\right)|\sigma_\alpha|, \end{aligned} \tag{5.24}$$

$$\begin{aligned} |\lambda^*_{k_\alpha} - \dot x_\alpha| &\le \varepsilon + \mathcal{O}(1)\cdot|\omega^+_{k_\alpha} - \omega^+_n| \\ &\le \varepsilon + \mathcal{O}(1)\cdot\sum_{i\neq k_\alpha}|q^+_i|. \end{aligned} \tag{5.25}$$

The assumption of genuine nonlinearity implies

$$\widetilde\lambda_{k_\alpha} - \lambda^*_{k_\alpha} > c'|q^+_{k_\alpha}|, \tag{5.26}$$

for some constant $c' > 0$. In this case $W^-_{k_\alpha} = W^+_{k_\alpha} + \kappa_1|\sigma_\alpha|$. Using (5.23)–(5.26) and recalling that $q^-_{k_\alpha} < q^+_{k_\alpha} < 0 < \sigma_\alpha$, we now compute

$$\begin{aligned} E_{\alpha,k_\alpha} &= W^+_{k_\alpha}|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot x_\alpha) - \big(W^+_{k_\alpha} + \kappa_1\sigma_\alpha\big)|q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot x_\alpha) \\ &\le -\kappa_1\sigma_\alpha\big(|q^+_{k_\alpha}| + \sigma_\alpha\big)(\widetilde\lambda_{k_\alpha} - \lambda^*_{k_\alpha}) + \kappa_1\sigma_\alpha\big(|q^+_{k_\alpha}| + \sigma_\alpha\big)\big(|\lambda^+_{k_\alpha} - \widetilde\lambda_{k_\alpha}| + |\dot x_\alpha - \lambda^*_{k_\alpha}|\big) \\ &\quad + \kappa_1\sigma_\alpha\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^-_{k_\alpha} - \dot x_\alpha| + W^+_{k_\alpha}\big\{|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot x_\alpha) - |q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot x_\alpha)\big\} \\ &\le -c'\kappa_1|q^+_{k_\alpha}|\sigma_\alpha\big(|q^+_{k_\alpha}| + \sigma_\alpha\big) + \mathcal{O}(1)\cdot\kappa_1\sigma_\alpha\big(|q^+_{k_\alpha}| + \sigma_\alpha\big)\left(\varepsilon + |q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + \sigma_\alpha\big) + \sum_{i\neq k_\alpha}|q^+_i|\right) \\ &\quad + W^+_{k_\alpha}\Big\{\Big||q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \lambda^*_{k_\alpha}) - \big(|q^+_{k_\alpha}| + \sigma_\alpha\big)(\widetilde\lambda_{k_\alpha} - \lambda^*_{k_\alpha})\Big| + \big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^-_{k_\alpha} - \dot x_\alpha| \\ &\qquad + \sigma_\alpha|\lambda^*_{k_\alpha} - \dot x_\alpha| + \big(|q^+_{k_\alpha}| + \sigma_\alpha\big)|\lambda^-_{k_\alpha} - \widetilde\lambda_{k_\alpha}|\Big\} \\ &\le -c'\kappa_1|q^+_{k_\alpha}|\sigma_\alpha\big(|q^+_{k_\alpha}| + \sigma_\alpha\big) + \mathcal{O}(1)\cdot\left(\varepsilon + |q^+_{k_\alpha}|\big(|q^+_{k_\alpha}| + \sigma_\alpha\big) + \sum_{i\neq k_\alpha}|q^+_i|\right)\sigma_\alpha. \end{aligned} \tag{5.27}$$

Choosing the constant κ_1 large enough, (5.19) and (5.27) together imply (3.6).

CASE 3: $q^-_{k_\alpha} < 0 < q^+_{k_\alpha}$.

For the terms $E_{\alpha,i}$ with $i \neq k_\alpha$, the estimates (5.10) remain valid. Using (2.19) and (5.9), for the term E_{α,k_α} one obtains the easy estimate

$$\begin{aligned} E_{\alpha,k_\alpha} &= W^+_{k_\alpha}|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot x_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot x_\alpha) \\ &\le 2\big(|q^+_{k_\alpha}| + |q^-_{k_\alpha}|\big)\big(|\lambda^+_{k_\alpha} - \dot x_\alpha| + |\lambda^-_{k_\alpha} - \dot x_\alpha|\big) \\ &= \mathcal{O}(1)\cdot\sigma_\alpha\left(\varepsilon + \sum_{i\neq k_\alpha}|q^-_i|\right). \end{aligned} \tag{5.28}$$

Choosing the constant κ_1 large enough, (5.10) and (5.28) together imply (3.6).

The final part of our analysis will be concerned with a shock front: $\alpha \in \mathcal{S}$, hence $\sigma_\alpha < 0$, $q^+_{k_\alpha} < q^-_{k_\alpha}$. As before, three cases will be considered, depending on the signs of $q^-_{k_\alpha}$ and $q^+_{k_\alpha}$.

CASE 1: $0 < q^+_{k_\alpha} < q^-_{k_\alpha}$.

For $i \neq k_\alpha$, using (5.4) we now obtain

$$E_{\alpha,i} \leq \mathcal{O}(1) \cdot \left(|q^+_{k_\alpha}| \big(|q^+_{k_\alpha}| + |\sigma_\alpha| \big) + \sum_{j \neq k_\alpha} |q^+_j| \right) |\sigma_\alpha| - c\kappa_1 |q^+_i| |\sigma_\alpha| \qquad i \neq k_\alpha. \tag{5.29}$$

Next, we seek an estimate on E_{α,k_α}. For this purpose, consider again the intermediate states $\omega^\pm_i$ as in (4.6). Moreover, define the states $\widetilde{\omega}_{k_\alpha}$, $\omega^*_{k_\alpha}$ and the shock speeds $\lambda^*_{k_\alpha}$, $\widetilde{\lambda}_{k_\alpha}$ as in (5.20)-(5.21). By (5.4) and Lemma 2, the estimates (5.23)–(5.25) still hold. The assumption of genuine nonlinearity now implies

$$\lambda^*_{k_\alpha} - \widetilde{\lambda}_{k_\alpha} > c' q^+_{k_\alpha}, \tag{5.30}$$

for some constant $c' > 0$. In this case, $W^-_{k_\alpha} = W^+_{k_\alpha} - \kappa_1 |\sigma_\alpha|$. Using (5.30) together with (5.23)–(5.25) and recalling that $\sigma_\alpha < 0 < q^+_{k_\alpha} < q^-_{k_\alpha}$, we compute

$$\begin{aligned}
E_{\alpha,k_\alpha} &= W^+_{k_\alpha} q^+_{k_\alpha} (\lambda^+_{k_\alpha} - \dot{x}_\alpha) - \big(W^+_{k_\alpha} - \kappa_1 |\sigma_\alpha| \big) q^-_{k_\alpha} (\lambda^-_{k_\alpha} - \dot{x}_\alpha) \\
&\leq \kappa_1 \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) |\sigma_\alpha| (\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) + \kappa_1 |\sigma_\alpha| \big| q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha \big| |\lambda^-_{k_\alpha} - \dot{x}_\alpha| \\
&\qquad + \kappa_1 q^+_{k_\alpha} |\sigma_\alpha| \big(|\lambda^-_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| + |\dot{x}_\alpha - \lambda^*_{k_\alpha}| \big) + W^+_{k_\alpha} \big\{ q^+_{k_\alpha} (\lambda^+_{k_\alpha} - \dot{x}_\alpha) - q^-_{k_\alpha} (\lambda^-_{k_\alpha} - \dot{x}_\alpha) \big\} \\
&\leq -c' \kappa_1 \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) |\sigma_\alpha| q^+_{k_\alpha} + \mathcal{O}(1) \cdot \kappa_1 |\sigma_\alpha| \left(q^+_{k_\alpha} \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) + \sum_{i \neq k_\alpha} |q^+_i| \right) |\sigma_\alpha| \\
&\qquad + \mathcal{O}(1) \cdot \kappa_1 |\sigma_\alpha| q^+_{k_\alpha} \left(\varepsilon + q^+_{k_\alpha} |\sigma_\alpha| \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) + \sum_{i \neq k_\alpha} |q^+_i| \right) \\
&\leq -c' \kappa_1 \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) |\sigma_\alpha| q^+_{k_\alpha} + \mathcal{O}(1) \cdot \left(\varepsilon + \big(q^+_{k_\alpha} + |\sigma_\alpha| \big) q^+_{k_\alpha} + \sum_{i \neq k_\alpha} |q^+_i| \right) |\sigma_\alpha|.
\end{aligned} \tag{5.31}$$

Choosing the constant κ_1 large enough, (5.29) and (5.31) together imply (3.6).

CASE 2: $q^+_{k_\alpha} < q^-_{k_\alpha} < 0$.

The estimates are almost the same as in Case 1, using (5.3) instead of (5.4). For $i \neq k_\alpha$, we obtain

$$E_{\alpha,i} \leq \mathcal{O}(1) \cdot \left(|q^-_{k_\alpha}| \big(|q^-_{k_\alpha}| + |\sigma_\alpha| \big) + \sum_{j \neq k_\alpha} |q^-_j| \right) |\sigma_\alpha| - c\kappa_1 |q^-_i| |\sigma_\alpha| \qquad i \neq k_\alpha. \tag{5.32}$$

To estimate E_{α,k_α}, consider the intermediate states $\omega_i^\pm$ as in (4.6) and define $\widetilde{\omega}_{k_\alpha}$, $\omega^*_{k_\alpha}$, $\lambda^*_{k_\alpha}$, $\widetilde{\lambda}_{k_\alpha}$ as in (5.11)-(5.12). By (5.3) and Lemma 2, the estimates (5.14)–(5.16) again hold. The assumption of genuine nonlinearity now implies

$$\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha} > c'|q^-_{k_\alpha}|, \tag{5.33}$$

for some constant $c' > 0$. In this case $W^+_{k_\alpha} = W^-_{k_\alpha} - \kappa_1|\sigma_\alpha|$. Using (5.33) together with (5.14)–(5.16) and recalling that $q^-_{k_\alpha} < q^+_{k_\alpha} < 0$ and $\sigma_\alpha < 0$, we now compute

$$\begin{aligned}
E_{\alpha,k_\alpha} &= \big(W^-_{k_\alpha} - \kappa_1|\sigma_\alpha|\big)|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot{x}_\alpha) \\
&\le -\kappa_1|\sigma_\alpha|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big)\big(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}\big) + \kappa_1|\sigma_\alpha|(|q^-_{k_\alpha}| + |\sigma_\alpha|)\big(|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| + |\dot{x}_\alpha - \lambda^*_{k_\alpha}|\big) \\
&\qquad + \kappa_1|\sigma_\alpha|\big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^+_{k_\alpha} - \dot{x}_\alpha| + W^-_{k_\alpha}\big\{|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot{x}_\alpha) - |q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot{x}_\alpha)\big\} \\
&\le -c'\kappa_1|q^-_{k_\alpha}||\sigma_\alpha|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \mathcal{O}(1)\cdot\kappa_1|\sigma_\alpha|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big)\left(\varepsilon + |q^-_{k_\alpha}|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{i\neq k_\alpha}|q^-_i|\right) \\
&\qquad + W^-_{k_\alpha}\bigg\{\Big|q^-_{k_\alpha}(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}) - \big(q^-_{k_\alpha} + \sigma_\alpha\big)\big(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}\big)\Big| + \big|q^+_{k_\alpha} - q^-_{k_\alpha} - \sigma_\alpha\big||\lambda^+_{k_\alpha} - \dot{x}_\alpha| \\
&\qquad\qquad + |\sigma_\alpha||\lambda^*_{k_\alpha} - \dot{x}_\alpha| + \big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big)|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}|\bigg\} \\
&\le -c'\kappa_1|q^-_{k_\alpha}||\sigma_\alpha|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \mathcal{O}(1)\cdot\left(\varepsilon + |q^-_{k_\alpha}|\big(|q^-_{k_\alpha}| + |\sigma_\alpha|\big) + \sum_{i\neq k_\alpha}|q^-_i|\right)|\sigma_\alpha|.
\end{aligned} \tag{5.34}$$

Choosing the constant κ_1 large enough, (5.32) and (5.34) together imply (3.6).

CASE 3: $q^+_{k_\alpha} < 0 < q^-_{k_\alpha}$.

To fix the ideas, assume $|q^-_{k_\alpha}| \le |q^+_{k_\alpha}|$, the other case being entirely similar. Since the total variation is small, the above assumption implies

$$|\sigma_\alpha| > \frac{3}{2}q^-_{k_\alpha}, \qquad\qquad q^-_{k_\alpha} + \sigma_\alpha < \frac{1}{3}\sigma_\alpha < 0. \tag{5.35}$$

For $i \neq k_\alpha$, by (5.3) the estimate (5.32) still holds. Defining $\omega_i^\pm$ as in (4.6) and $\widetilde{\omega}_{k_\alpha}$, $\omega^*_{k_\alpha}$, $\lambda^*_{k_\alpha}$, $\widetilde{\lambda}_{k_\alpha}$ as in (5.11)-(5.12), the estimates (5.14)–(5.16) also remain valid. In the present case, the assumption of genuine nonlinearity implies

$$\lambda^*_{k_\alpha} \ge \widetilde{\lambda}_{k_\alpha}, \qquad\qquad \lambda^-_{k_\alpha} - \lambda^*_{k_\alpha} \ge c'|q^-_{k_\alpha} + \sigma_\alpha| \tag{5.36}$$

for some constant $c' > 0$. Recalling that $q^+_{k_\alpha} < 0 < q^-_{k_\alpha} < |\sigma_\alpha|$ and using (2.19), (5.15)-(5.16) together with (5.35)-(5.36), we now compute

$$\begin{aligned} E_{\alpha,k_\alpha} &= W^+_{k_\alpha}|q^+_{k_\alpha}|(\lambda^+_{k_\alpha} - \dot{x}_\alpha) - W^-_{k_\alpha}|q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \dot{x}_\alpha) \\ &\leq W^+_{k_\alpha}|q^+_{k_\alpha}|(\widetilde{\lambda}_{k_\alpha} - \lambda^*_{k_\alpha}) - W^-_{k_\alpha}|q^-_{k_\alpha}|(\lambda^-_{k_\alpha} - \lambda^*_{k_\alpha}) \\ &\qquad + W^+_{k_\alpha}|q^+_{k_\alpha}|\big(|\lambda^+_{k_\alpha} - \widetilde{\lambda}_{k_\alpha}| + |\lambda^*_{k_\alpha} - \dot{x}_\alpha|\big) + W^-_{k_\alpha}|q^-_{k_\alpha}|\big|\lambda^*_{k_\alpha} - \dot{x}_\alpha\big| \\ &\leq -\frac{c'}{3}\cdot\big|q^-_{k_\alpha}\big||\sigma_\alpha| + \mathcal{O}(1)\cdot\left(\varepsilon + q^-_{k_\alpha}\big(q^-_{k_\alpha} + |\sigma_\alpha|\big) + \sum_{i\neq k_\alpha}|q^-_i|\right)|\sigma_\alpha|. \end{aligned} \tag{5.37}$$

Since the amplitude of the solution is small, (5.32) and (5.37) together imply (3.6). This completes the proof of the estimate (3.6) in all cases where $q^+_{k_\alpha}, q^-_{k_\alpha} \neq 0$. The remaining cases where $q^+_{k_\alpha} = 0$ or $q^-_{k_\alpha} = 0$ can be easily recovered as limits of the previous ones.

Acknowledgments. The research of the first author was partially supported by the European TMR Network ERBFMRXCT960033 on Hyperbolic Conservation Laws. The research of the second author was supported in part by NSF Grant DMS-9623025. The third author was supported in part by the RGC Competitive Earmarked Research Grant 9040290.

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