Construction of BGK models with a family of kinetic entropies for a given system of conservation laws

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Abstract

We introduce a general framework for kinetic BGK models. We assume to be given a system of hyperbolic conservation laws with a family of Lax entropies, and we characterize the BGK models that lead to this system in the hydrodynamic limit, and that are compatible with the whole family of entropies. This is obtained by a new characterization of maxwellians as entropy minimizers, that can take into account the simultaneous minimization problems corresponding to the family of entropies. We deduce a general procedure to construct such BGK models, and we give examples and applications in gas dynamics.

Key-words: conservation laws – BGK models – hydrodynamic limit – kinetic entropy – space of maxwellians

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1 Introduction

The rarefied gas dynamics is described by the kinetic Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \frac{Q(f)}{\varepsilon}, \tag{1.1}$$

where $f(t,x,v) \geq 0$ is the particle density in the phase space $(x,v) \in \mathbb{R}^N \times \mathbb{R}^N$, ε is the mean free path and Q is the Boltzmann collision operator. This integral operator acts in the velocity variable v only, satisfies the moment relations

$$\int \phi(v) Q(f)(v)\, dv = 0, \tag{1.2}$$

$\phi \in Span(1, v_j, |v|^2)$, and the entropy inequality

$$\int \ln f(v) Q(f)(v)\, dv \leq 0. \tag{1.3}$$

These properties ensure the local conservation of mass, momentum and energy by integrating (1.1) with respect to v,

$$\partial_t \int \phi(v) f\, dv + \operatorname{div}_x \int v\phi(v) f\, dv = 0, \tag{1.4}$$

$\phi \in Span(1, v_j, |v|^2)$, and the decrease of entropy

$$\partial_t \int f \ln f\, dv + \operatorname{div}_x \int v f \ln f\, dv \leq 0. \tag{1.5}$$

Another striking property of Boltzmann's equation is that $Q(f) = 0$ if and only if f is a maxwellian, that is

$$f(v) = M(v) \equiv \frac{\rho}{(2\pi T)^{N/2}} e^{-|v-u|^2/2T}, \tag{1.6}$$

for some $\rho \geq 0$, $T > 0$, $u \in \mathbb{R}^N$. When time and space dependence are allowed as in (1.1), ρ, T, u can depend on t, x also. When $\varepsilon \to 0$ in (1.1), f therefore goes formally to a maxwellian of parameters $\rho(t,x)$, $T(t,x)$ and $u(t,x)$, which satisfies the conservation laws (1.4), and entropy inequality (1.5), with f given by (1.6). This system is the Euler system of monatomic perfect gas dynamics.

In their paper [4], Bhatnagar, Gross, and Krook introduced a simplified Boltzmann-like model (called the BGK model) which satisfies all the above cited properties. It is written under the form (1.1) with

$$Q(f) = M_f - f, \tag{1.7}$$

and

$$M_f(v) = \frac{\rho_f}{(2\pi T_f)^{N/2}} e^{-|v-u_f|^2/2T_f}, \tag{1.8}$$

$$\rho_f \left(1, u_f, \frac{1}{2}|u_f|^2 + \frac{N}{2} T_f\right) = \int (1, v, |v|^2/2) f(v)\, dv. \tag{1.9}$$

The existence of a global solution to the BGK model has been proved in [36], and regularity properties are given in [39]. It is important to notice that the entropy inequality (1.5) is closely related to a minimization principle that asserts that for given (ρ, u, T), the maxwellian M in (1.6) realizes

$$\min_{\substack{f(v) \geq 0 \\ (\rho_f, u_f, T_f) = (\rho, u, T)}} \int f(v) \ln f(v)\, dv, \tag{1.10}$$

and that the corresponding value $\eta(\rho, u, T)$ is a convex entropy for the system of gas dynamics.

We have to mention that the kinetic BGK equation can also appear under a time discrete form, that is described in Section 2.3. With a space discretization overall, it leads to the so called kinetic or Boltzmann schemes. Many works have been devoted to the use of BGK models in numerical methods for gas dynamics, and to the possible generalizations, in order to provide a natural kinetic description of other hyperbolic systems of conservation laws, see [41], [42], [31], [30] [6], [5], [37], [38], [16], [24], [46], [17], [25], [23], [32], [3].

In the previous papers, only a single entropy is taken into account, as in the above described model, and the main tools used in order to obtain a minimization property such as (1.10) are Lagrange multipliers and symmetrization of hyperbolic systems.

On the other hand, for scalar conservation laws, it is known that it is possible to have a minimization principle for all entropies at the same time. This has been proved in [7], [8], [18] and [40]. It is also the case for the systems considered in [9], [10]. These two families of systems of conservation laws admit a so called kinetic formulation, that is an equation like (1.1), but with $\varepsilon = 0$, in the sense that f is a maxwellian and the right-hand side is replaced by a suitable term, see [26], [9]. However, as we shall see, the property of having a BGK model with a large family of entropy inequalities is quite independent of the property of having a kinetic formulation. Other systems have a kinetic formulation, but which is not "purely kinetic", in the sense that the transport coefficient is not a function of v only, see [27], [21].

BGK models can be seen as a subclass of the general class of relaxation models, described in [28], [14], [13], [35]. These have been intensively studied in recent years, for example in [20], [1], [33], [44], [22], [19], [2], [34], [45], [43].

The aim of this paper is to provide a general framework for BGK models, that describes nearly all the known ones, and also some relaxation models that can be interpreted as BGK models with finitely many velocities, as described in [2]. We assume to be given a system of hyperbolic conservation laws and a family of Lax entropies, and we describe BGK models compatible with this whole family of entropies, in the sense that there exists a corresponding family of kinetic entropies. An important point is to take vector valued kinetic functions f. Then, known scalar models enter the framework by the property that f takes its values in a one-dimensional submanifold (at fixed ξ). We call this a rank-one model. Similarly, models with two functions such as those of [38] correspond here to rank-two models, and the discrete-velocities relaxation models are merely full-rank models. General full-rank models have been introduced by Serre in [43]. The general framework is presented in Section 2.1.

The corner stone of this paper is the reformulation in Section 2.2 of the simultaneous minimization problems corresponding to each entropy of the family by first order linear partial differential equations and inequations. These were already written down in [11] and [43], and the inequalities give natural stability conditions generalizing the so called subcharacteristic conditions in the case of finitely many velocities. This reformulation is based on elementary properties in optimization theory, and avoids symmetrizing the system, which is undesirable here because the new variable would depend on the entropy. The characterization of the space of maxwellians allows to give a practical method to build models with many kinetic entropies.

In Section 2.3 we briefly explain how to write time discrete models, while in Section 2.4 we perform the so called Chapman-Enskog expansion. Then in Section 3 we show how classical examples enter the framework, in Section 4 we consider isentropic gas dynamics and obtain new BGK models satisfying all entropy inequalities, and in Section 5 we apply our theory to the full system of gas dynamics. The appendix is devoted to some basic properties on bilinear forms and duality that are used in the paper.

Since the present paper is mainly intended to study very general models, the results provided are not completely rigorous. Technical arguments should be examined somewhere else, depending on the model considered. In particular, we never discuss measurability or integrability of functions.

2 The BGK framework

We consider a system of conservation laws

$$\partial_t u + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} F_j(u) = 0, \tag{2.1}$$

where $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $u = u(t,x) \in \mathcal{U}$ a convex subset of $\mathbb{R}^p$, and $F_j : \mathcal{U} \to \mathbb{R}^p$ are given (smooth) functions. We recall that an entropy for (2.1) is a function $\eta : \mathcal{U} \to \mathbb{R}$ such that there exist functions $G_j : \mathcal{U} \to \mathbb{R}$ satisfying

$$G_j' = \eta' F_j', \qquad j = 1, \ldots, N. \tag{2.2}$$

Here and all throughout the paper, prime denotes differentiation with respect to the conservative variable (u here). With this definition, η is not necessarily convex. We recall that (2.2) ensures that any smooth solution u to (2.1) satisfies

$$\partial_t [\eta(u)] + \sum_{j=1}^{N} \frac{\partial}{\partial x_j} G_j(u) = 0, \tag{2.3}$$

which is obtained by left-multiplying (2.1) by $\eta'(u)$. The existence of functions G_j satisfying (2.2) means that the differential forms $\eta' F_j'$ are exact, and this can be characterized by the property that the $(\eta' F_j')'$ are symmetric (see Appendix). Since $(\eta' F_j')' = (F_j')^t \eta'' + \eta' F_j''$, this can be written as

$$(F_j')^t \eta'' \quad \text{is symmetric}, \qquad j = 1, \ldots, N, \tag{2.4}$$

or equivalently as the well-known condition that $\eta'' F_j'$ is symmetric. If η is strictly convex, an interpretation of (2.4) is to say that F_j' is self-adjoint for the scalar product defined by η''. Then, for any direction $\omega \in \mathbb{R}^N$, $\sum_j \omega_j F_j'$ is also self-adjoint, and is therefore diagonalizable. Thus, we recover the well-known result that says that if (2.1) admits a strictly convex entropy, then it is hyperbolic.

Another result that we shall use in applications is how (2.4) is transformed by a change of variable $v = \varphi(u)$. Equation (2.1) becomes (for smooth solutions)

$$\partial_t v + \sum_{j=1}^N A_j \frac{\partial v}{\partial x_j} = 0, \qquad A_j(u) = \varphi'(u) F_j'(u) \varphi'(u)^{-1}, \tag{2.5}$$

while (2.2) becomes by denoting $\eta = \eta(\varphi^{-1}(v))$, $G_j = G_j(\varphi^{-1}(v))$,

$$\partial_v G_j = (\partial_v \eta) A_j, \tag{2.6}$$

and again this can be characterized by asking that $\partial_v((\partial_v\eta)A_j) = A_j^t \partial_{vv}^2 \eta + (\partial_v \eta)(\partial_v A_j)$ is symmetric.

In the following section, $\mathcal{E}$ denotes a non-empty family $(\eta)_{\eta\in\mathcal{E}}$ of convex (smooth) entropies for (2.1). For technical reasons we assume that $\mathcal{E}$ is separable, that is it contains a countable set which is dense in a suitable topology, that we are not going to precise. In particular, $\mathcal{E}$ can be just a single convex entropy.

2.1 Axioms for vector models

This section is devoted to the general setting of our BGK framework. It generalizes those proposed in [34], [2] and [43]. We assume to be given a system of conservation laws and a non-empty family $\mathcal{E}$ of convex entropies, as described in the previous paragraph. A BGK model will consist in an equation

$$\partial_t f + a(\xi) \cdot \nabla_x f = \frac{M_f - f}{\varepsilon}, \tag{2.7}$$

where $t > 0$, $x \in \mathbb{R}^N$,

$$\xi \in \Xi, \quad \text{a measure space with measure } d\xi, \tag{$H1$}$$

$f(t,x,\xi) \in \mathbb{R}^p$ is the unknown,

$$a : \Xi \to \mathbb{R}^N \tag{$H2$}$$

is the velocity, $a(\xi) \cdot \nabla_x = \sum_j a_j(\xi) \partial/\partial x_j$, and

$$M_f(t,x,\xi) = M(u(t,x),\xi), \qquad u(t,x) = \int f(t,x,\xi)\, d\xi - k, \tag{2.8}$$

where the equilibrium state $M : \mathcal{U} \times \Xi \to \mathbb{R}^p$ is assumed to satisfy the moment equations

$$\int M(u,\xi)\, d\xi = u + k, \qquad u \in \mathcal{U}, \tag{$M0$}$$

$$\int a_j(\xi) M(u,\xi)\, d\xi = F_j(u) + k_j', \qquad u \in \mathcal{U},\ j = 1, \ldots, N. \tag{$M1$}$$

Here $k, k'_1, \dots, k'_N \in \mathbb{R}^p$ are arbitrary constants. In (2.8), we have to assume that

$$u(t,x) \in \mathcal{U}, \tag{2.9}$$

and this could be obtained by an a priori estimate, but in general this will be a difficult question to say whether or not (2.9) holds.

It is quite obvious that (M0), (M1) are the necessary consistency relations for the hydrodynamic limit $\varepsilon \to 0$ to coincide with (2.1). Indeed, if a solution f_ε with average u_ε is bounded independently of ε, and if $u_\varepsilon \to u$, then we get formally from (2.7) and (2.8) that $M_{f_\varepsilon} \to M(u,\xi)$ and $f_\varepsilon \to M(u,\xi)$. Then, since by integrating (2.7) with respect to ξ

$$\partial_t \int f_\varepsilon(t,x,\xi)\,d\xi + \sum_{j=1}^N \frac{\partial}{\partial x_j} \int a_j(\xi) f_\varepsilon(t,x,\xi)\,d\xi = 0, \tag{2.10}$$

we get (2.1) at the limit by using (M0) and (M1).

Now, before introducing axioms for kinetic entropies, we have to remark that the kinetic equation (2.7) leaves invariant any family of convex sets indexed by ξ. Actually, if we choose for each ξ a convex set $D_\xi \subset \mathbb{R}^p$ such that

$$a.e.\ \xi \quad \forall u \in \mathcal{U} \quad M(u,\xi) \in D_\xi \text{ convex}, \tag{$H3$}$$

then we have that

$$\forall t \geq 0 \quad a.e.\ x,\xi \quad f(t,x,\xi) \in D_\xi, \tag{2.11}$$

as soon as it is true at $t=0$ (which is the case for example if we take $f^0(x,\xi) = M(u^0(x),\xi)$ for some initial state u^0). We now assume that such sets D_ξ are chosen, and it is always possible, for example take $D_\xi = \underset{u\in\mathcal{U}}{\text{conv}}\, M(u,\xi)$.

Let us now postulate that for any $\eta \in \mathcal{E}$ there exists a kinetic entropy $H_\eta(f,\xi)$ satisfying

$$a.e.\ \xi \qquad H_\eta(.,\xi) : D_\xi \to \mathbb{R} \text{ is convex}, \tag{$E0$}$$

$$\int H_\eta(M(u,\xi),\xi)\,d\xi = \eta(u) + c_\eta, \qquad u \in \mathcal{U}, \tag{$E1$}$$

for any $f : \Xi \to \mathbb{R}^p$ such that $a.e.\ \xi\ f(\xi) \in D_\xi$ and $u_f \equiv \int f(\xi) d\xi - k \in \mathcal{U}$,

$$\int H_\eta(M(u_f,\xi),\xi)\,d\xi \leq \int H_\eta(f(\xi),\xi)\,d\xi. \tag{$E2$}$$

These properties are sufficient to obtain the Lax entropy inequalities in the hydrodynamic limit. Indeed, multiplying (2.7) on the left by $\partial_f H_\eta(f(t,x,\xi),\xi)$ and using (E0) we get

$$\partial_t[H_\eta(f,\xi)] + a(\xi)\cdot\nabla_x[H_\eta(f,\xi)] = \partial_f H_\eta(f,\xi)\frac{M_f - f}{\varepsilon} \leq \frac{H_\eta(M_f,\xi) - H_\eta(f,\xi)}{\varepsilon}. \tag{2.12}$$

Here we need H_η to be at least C^1, but this condition should be relaxed if we just retain the last inequality. Then, integrating (2.12) with respect to ξ and using (E2), we obtain

$$\partial_t \int H_\eta(f(t,x,\xi),\xi)\, d\xi + \mathrm{div}_x \int a(\xi) H_\eta(f(t,x,\xi),\xi)\, d\xi \le 0. \tag{2.13}$$

Finally, if as above we put f_ε and let $\varepsilon \to 0$, by (E1) we end up with

$$\partial_t[\eta(u)] + \mathrm{div}_x\, G(u) \le 0, \tag{2.14}$$

with

$$G(u) \equiv \int a(\xi) H_\eta(M(u,\xi),\xi)\, d\xi. \tag{2.15}$$

By the way we get that if there exists H_η satisfying (E0), (E1), (E2), then η is necessarily an entropy for (2.1), with flux given by (2.15) (and the relations (2.2) will follow from the analysis of Section 2.2). We have to notice also that η is necessarily convex, because by (E1) and (E2),

$$\eta(u) + c_\eta = \min_{\substack{f(\xi)\in D_\xi \\ u_f = u}} \int H_\eta(f(\xi),\xi)\, d\xi, \tag{2.16}$$

and thanks to (E0) this is a convex function of u (see [29], §8.3).

Remark 2.1 One can always add to H_η an arbitrary function of ξ only, (E0), (E1), (E2) will remain valid. Also, we can transform a BGK model into another by defining $\widetilde{M} = M + \varphi(\xi)$ with $\varphi(\xi)$ an arbitrary function. Then $\widetilde{f} = f + \varphi(\xi)$, $\widetilde{D}_\xi = D_\xi + \varphi(\xi)$, $\widetilde{k} = k + \int \varphi(\xi) d\xi$, $\widetilde{k}'_j = k'_j + \int a_j(\xi)\varphi(\xi) d\xi$. Therefore, it is always possible to reduce to the case $k = 0$.

Example 2.1 1. If $\eta(u) = cst$, we can take $H_\eta \equiv 0$.
2. If $\eta(u) = \ell u$, with $\ell \in (\mathbb{R}^p)'$, we can take $H_\eta(f,\xi) = \ell f$.
3. In the case of the scalar model presented in the introduction, we have $\xi \in \Xi = \mathbb{R}^N$ with the Lebesgue measure, $a(\xi) = \xi$, $M(u,\xi) = (1,\xi,|\xi|^2/2) G(u,\xi)$, with G the gaussian defined by (1.6), and $f(t,x,\xi) = (1,\xi,|\xi|^2/2) g(t,x,\xi)$ where g solves (1.1). Here $D_\xi = (1,\xi,|\xi|^2/2)\mathbb{R}_+$ is a half-line. We call this a rank-one model.

From now on, by BGK model we will mean a model as above. This can be summarized by equations (H1), (H2), (H3), (M0), (M1), (E0), (E1), (E2).

2.2 Characterization of the space of maxwellians

This section is devoted to the analysis of the entropy axioms (E0), (E1), (E2). Let us assume that $(\Xi, d\xi)$, a, M and D_ξ are given satisfying (H1), (H2), (H3), (M0), (M1). The problem is to find kinetic entropies H_η satisfying (E0), (E1), (E2).

If we want just a single entropy, a possible strategy is to find a non-trivial function $H(f,\xi)$ satisfying (E0) and (E2), and then to define η by (E1). As noticed in the previous paragraph, η is then a convex entropy for (2.1). This was done in most of the previous works (see the introduction) by reformulating (E2) by saying that for any $u \in \mathcal{U}$,

$$\inf_{\substack{f(\xi)\in D_\xi \\ u_f = u}} \int H_\eta(f(\xi),\xi)\, d\xi \tag{2.17}$$

must be attained for $f(\xi) = M(u,\xi)$. This can be characterized by writing Lagrange multipliers (see [29], §8.3-4), and together with (M0), (M1), it is possible to determine $M(u,\xi)$ if H is given. However, since here we want to have this property for each H_η, $\eta \in \mathcal{E}$, this formulation is bad because the minimizer will depend on H_η, which is not acceptable. Therefore, we rather consider M to be given, and we are going to characterize the existence of functions H_η satisfying (E0), (E1), (E2).

Let us first introduce the "microscopic entropy"

$$G_\eta(u,\xi) \equiv H_\eta(M(u,\xi),\xi). \tag{2.18}$$

With this notation, (E1) can be written

$$\int G_\eta(u,\xi)\, d\xi = \eta(u) + c_\eta, \qquad u \in \mathcal{U}. \tag{2.19}$$

We define the vector space of maxwellians

$$\mathcal{M}^{\mathcal{E}} = \left\{ M : \mathcal{U} \to \mathbb{R}^p \ ; \ \forall \eta \in \mathcal{E} \quad (M')^t \eta'' \text{ is symmetric everywhere in } \mathcal{U} \right\}, \tag{2.20}$$

and the convex cone of nondecreasing maxwellians

$$\mathcal{M}^{\mathcal{E}}_+ = \left\{ M \in \mathcal{M}^{\mathcal{E}} \ ; \ \forall \eta \in \mathcal{E} \quad (M')^t \eta'' \geq 0 \text{ everywhere in } \mathcal{U} \right\}. \tag{2.21}$$

Here, we assume implicitly some regularity for M, and "symmetric" has to be understood in the sense of the appendix. The main result of this section is the following.

Theorem 2.1 *Let us assume that $\mathcal{U}$ is open, and that for a.e. $\xi \in \Xi$, $M(.,\xi) \in C^1(\mathcal{U})$ and $D_\xi = \operatorname{conv}_{u\in\mathcal{U}} M(u,\xi)$. We also assume that either condition (CH1) or (CH2) below is satisfied. Then, the existence of functions $(H_\eta)_{\eta\in\mathcal{E}}$ satisfying (E0), (E1), (E2) and such that the G_η defined in (2.18) are C^1 in $\mathcal{U}$, is equivalent to*

$$a.e.\ \xi \in \Xi \quad M(.,\xi) \in \mathcal{M}^{\mathcal{E}}_+. \tag{2.22}$$

Moreover, if this is true, we have

$$a.e.\ \xi \quad \forall u \in \mathcal{U} \quad G'_\eta(u,\xi) = \eta'(u) M'(u,\xi). \tag{2.23}$$

The convexity assumptions (CH1) and (CH2) are defined as follows.

(CH1) For η in a dense subset of $\mathcal{E}$, $\eta'' > 0$ and $\eta'(\mathcal{U})$ is convex.

(CH2) For *a.e.* ξ, $M(.,\xi)$ is a C^1 diffeomorphism from $\mathcal{U}$ onto a *convex* open set (that is necessarily D_ξ).

We have to remark that these assumptions are rather technical, even if (CH2) is natural for full-rank models (i.e. $\dim D_\xi = p$); this assumption is made in [43]. Actually, if they are not satisfied, Theorem 2.1 could fail only with severe geometric pathology. Indeed what could only occur is that some function has nonnegative hessian, but is not convex, this is possible if its domain is not convex. Thus we could say that nevertheless, the theorem is true locally. It is also possible to replace (CH1) or (CH2) by other conditions, see Remark 2.7.

Let us postpone the proof of the theorem, and make some comments. It is noticeable that the minimization problems (E0), (E1), (E2) can be written as (2.22), a condition that is written at fixed ξ. Only the dependence in u is involved, while it is the ξ dependence that is found by the Lagrange multipliers method. Thus we only need to know what are $\mathcal{M}^{\mathcal{E}}$ and $\mathcal{M}^{\mathcal{E}}_+$. The space $\mathcal{M}^{\mathcal{E}}$ is defined by linear first-order partial differential equations with variable coefficients, and there are as many as ηs in $\mathcal{E}$. Here we see that if $\mathcal{E}$ is very large, there will be many conditions, and we can hope to describe $\mathcal{M}^{\mathcal{E}}$ in a simple way. On the contrary, if $\mathcal{E}$ is just a single entropy, then $\mathcal{M}^{\mathcal{E}}$ is very large and condition (2.22) will be useless. In this case the classical approach should be preferred.

Now, since $\mathcal{M}^{\mathcal{E}}$ is a vector space and $\mathcal{M}^{\mathcal{E}}_+$ is a convex cone, we see by (2.22) and (M0), (M1) that we must have $F_j \in \mathcal{M}^{\mathcal{E}}$ and $\mathrm{Id} \in \mathcal{M}^{\mathcal{E}}_+$. By definition of $\mathcal{M}^{\mathcal{E}}$ and $\mathcal{M}^{\mathcal{E}}_+$, this means exactly that η is an entropy of (2.1) and is convex.

We have to notice that (2.20) can be interpreted by asking that any $\eta \in \mathcal{E}$ needs to be an entropy for the one-dimensional system

$$\partial_t u + \partial_x[M(u)] = 0. \tag{2.24}$$

In (2.22), $M(.,\xi)$ has to satisfy this property for a.e. ξ, and (2.23) exactly says that $G_\eta(.,\xi)$ is the associated entropy-flux. This property can be used to rewrite (2.22) in new coordinates (see (2.5) and (2.6)).

Concerning $\mathcal{M}^{\mathcal{E}}_+$, we have an important characterization.

Proposition 2.2 *Assume that $\mathcal{E}$ contains at least a strictly convex entropy η_0, and let $M \in \mathcal{M}^{\mathcal{E}}$ be of class C^1 on $\mathcal{U}$, assumed to be open. Then, for all $u \in \mathcal{U}$, $M'(u)$ is diagonalizable (and thus has only real eigenvalues). Moreover, $M \in \mathcal{M}^{\mathcal{E}}_+$ if and only if*

$$\forall u \in \mathcal{U} \quad \sigma(M'(u)) \subset [0,\infty[, \tag{2.25}$$

where σ denotes the spectrum.

Remark 2.2 Condition (2.25) is independent of $\mathcal{E}$, and justifies the terminology "nondecreasing maxwellian". It generalizes the conditions introduced by Natalini [34] in the scalar case.

Remark 2.3 In (2.25) it is not obvious that $\mathcal{M}_+^{\mathcal{E}}$ is a convex cone.

Proof of Proposition 2.2 For any $u \in \mathcal{U}$, $M'(u)^t\eta_0''(u)$ is symmetric, which means that $M'(u)$ is self-adjoint for the scalar product $\eta_0''(u)$. Thus $M'(u)$ is diagonalizable. Moreover, $M'(u)^t\eta_0''(u) \geq 0$ if and only if $M'(u)$ is self-adjoint nonnegative for this scalar product, i.e. $\sigma(M'(u)) \subset \mathbb{R}_+$. Thus the condition is necessary. Conversely, if $\sigma(M'(u)) \subset \mathbb{R}_+$, then for any $\eta \in \mathcal{E}$ and $\varepsilon > 0$, $M'(u)^t(\eta''(u)+\varepsilon\eta_0''(u))$ is symmetric, $M'(u)$ is self-adjoint for the scalar product $\eta''(u) + \varepsilon\eta_0''(u)$. Since by assumption $\sigma(M'(u)) \subset \mathbb{R}_+$, we conclude that $M'(u)$ is self-adjoint nonnegative for this scalar product, $M'(u)^t(\eta''(u) + \varepsilon\eta_0''(u)) \geq 0$, and we get that $M'(u)^t\eta''(u) \geq 0$ by letting $\varepsilon \to 0$. $\square$

From Theorem 2.1 and Proposition 2.2, we deduce the following practical procedure in four steps to build BGK models for given $\mathcal{E}$.

1. Find the general solution of the (possibly infinite) system of first-order linear partial differential equations which define $\mathcal{M}^{\mathcal{E}}$ in (2.20), or at least a subspace of it (as large as possible), containing Id and the functions F_j. On the contrary, it is not necessary that this subspace contains the constants (see Remark 2.1). This space will be described by arbitrary parameters and functions.

2. Introduce in these arbitrary parameters and functions a dependence in ξ, and look for a measure space Ξ and a function $a : \Xi \to \mathbb{R}^N$ so that the moment equations (M0) and (M1) are satisfied. This should be possible since Id, $F_j \in \mathcal{M}^{\mathcal{E}}$.

3. Write the stability condition (2.22), or equivalently (2.25). It determines the validity domain of the model. Then, after having eventually reduced $\mathcal{U}$, try to check the a priori estimate (2.9). This is the most difficult step.

4. Eventually, compute G_η by (2.23), find D_ξ satisfying (H3) and try to explicit H_η from (2.18).

This procedure is used in Sections 4 and 5. We have to notice that in practice, the regularity assumptions of Theorem 2.1 will often not be satisfied. Then, one should directly check (E0), (E1), (E2).

The remainder of this section is devoted to the proof of Theorem 2.1. It is obtained in several steps. Let us first recall the definition of the subdifferential of a convex function. If H_η satisfies (E0), $\xi \in \Xi$ and $f_0 \in D_\xi$,

$$\partial_f H_\eta(f_0,\xi) = \{\ell \in (\mathbb{R}^p)' \; ; \; \forall f \in D_\xi \;\; H_\eta(f,\xi) \geq H_\eta(f_0,\xi) + \ell(f-f_0)\} \,. \tag{2.26}$$

Of course, if H_η is strongly differentiable at f_0, it reduces to a single point.

Proposition 2.3 *If H_η satisfies (E0) and if*

$$\forall u \in \mathcal{U} \quad a.e.\ \xi \quad \partial_f H_\eta(M(u,\xi),\xi) \ni \eta'(u) \tag{2.27}$$

then (E2) is satisfied.

Proof. By the definition (2.26), (2.27) writes

$$\forall u \in \mathcal{U} \quad a.e.\ \xi \quad \forall f \in D_\xi \quad H_\eta(f,\xi) \geq H_\eta(M(u,\xi),\xi) + \eta'(u)(f - M(u,\xi)). \tag{2.28}$$

Now, taking $u = u_f$ and $f = f(\xi)$, we integrate this inequality with respect to ξ and obtain (E2). □

We have also a converse result.

Proposition 2.4 *If (E0), (E1), (E2) are satisfied, and if either*

$$\text{any } f : \Xi \to \mathbb{R}^p \text{ such that a.e. } \xi\ f(\xi) \in D_\xi \text{ verifies } u_f \in \mathcal{U}, \tag{2.29}$$

or

$$\mathcal{U} \text{ is open,} \tag{2.30}$$

then (2.27) holds.

Proof. We have for any function f as in (E2)

$$\int H_\eta(f(\xi),\xi)\, d\xi - \eta(u_f) - c_\eta \geq 0, \tag{2.31}$$

and it is an equality if $f(\xi) = M(u,\xi)$ for some $u \in \mathcal{U}$. Therefore,

$$\int H_\eta(f(\xi),\xi)\, d\xi - \int H_\eta(M(u,\xi),\xi)\, d\xi \geq \eta(u_f) - \eta(u). \tag{2.32}$$

For any $0 < t < 1$, let us apply this inequality to $(1-t)M(u,\xi) + tf(\xi)$. We obtain thanks to the convexity of H_η

$$t\left(\int H_\eta(f(\xi),\xi)\, d\xi - \int H_\eta(M(u,\xi),\xi)\, d\xi\right) \geq \eta\left((1-t)u + tu_f\right) - \eta(u). \tag{2.33}$$

Then, dividing by t and letting $t \to 0$, this yields

$$\int H_\eta(f(\xi),\xi)\, d\xi - \int H_\eta(M(u,\xi),\xi)\, d\xi \geq \eta'(u)(u_f - u), \tag{2.34}$$

or equivalently

$$\int \Big[H_\eta(f(\xi),\xi) - H_\eta(M(u,\xi),\xi) - \eta'(u)(f(\xi) - M(u,\xi))\Big] d\xi \geq 0. \tag{2.35}$$

Now, if (2.28) is not verified, there exists a set A with $\text{meas}(A) > 0$, and for any $\xi \in A$ a vector $f_\xi \in D_\xi$ such that

$$H_\eta(f_\xi,\xi) < H_\eta(M(u,\xi),\xi) + \eta'(u)(f_\xi - M(u,\xi)). \tag{2.36}$$

By setting

$$f(\xi) = \begin{cases} f_\xi & \text{if } \xi \in A, \\ M(u,\xi) & \text{if } \xi \notin A, \end{cases} \tag{2.37}$$

we obtain a contradiction with (2.35) in the case where (2.29) holds. In the case of (2.30), we take

$$f(\xi)=\begin{cases} M(u,\xi)+t(f_\xi-M(u,\xi)) & \text{if } \xi\in A,\\ M(u,\xi) & \text{if } \xi\notin A,\end{cases} \tag{2.38}$$

so that

$$u_f=u+t\int_A (f_\xi-M(u,\xi))\,d\xi, \tag{2.39}$$

and we get that $u_f\in\mathcal{U}$ if t is small enough. But for $\xi\in A$

$$\begin{aligned} &H_\eta(f(\xi),\xi)-H_\eta(M(u,\xi),\xi)-\eta'(u)(f(\xi)-M(u,\xi))\\ &\le t\,[H_\eta(f_\xi,\xi)-H_\eta(M(u,\xi),\xi)-\eta'(u)(f_\xi-M(u,\xi))]<0,\end{aligned} \tag{2.40}$$

and this contradicts (2.35). □

Corollary 2.5 *Under either condition (2.29) or (2.30), we can replace (E0), (E1), (E2) by (E0), (E1), (2.27).*

This result indeed is a particular case of the following one. Consider the problem

$$\inf_{Lf=u} J(f), \tag{2.41}$$

with J convex and L linear. Then a family $(M(u))_u$ satisfying the constraint $LM(u)=u$, and such that

$$\eta(u)\equiv J(M(u)) \quad \text{is differentiable}, \tag{2.42}$$

is a family of minimizers for (2.41) if and only if

$$\forall u \qquad J'(M(u))\ni \eta'(u)L. \tag{2.43}$$

The proof can be obtained by following the one above, and is indeed a variant of the sensitivity property ([29] §8.5). Let us give also a more simple proof of the necessary condition if J is differentiable. Since $M(u)$ is a minimizer, the value of (2.41) is $\eta(u)$ and we have

$$\forall f \qquad J(f)-\eta(Lf)\ge 0, \tag{2.44}$$

and it is an equality for $f=M(u)$. By writing that the derivative of (2.44) with respect to f at $M(u)$ vanishes, we get $J'(M(u))=\eta'(u)L$, which is (2.43).

Remark 2.4 It is important in (2.27) not to assume H_η differentiable, because it is only defined in D_ξ, which has empty interior in many applications. If D_ξ is a submanifold of $\mathbb{R}^p$ of dimension r (the rank of the model), and if H_η is C^1 on it, then (2.27) is equivalent to

$$\forall u\in\mathcal{U}\quad a.e.\ \xi\quad \forall\delta f\in TD_\xi\quad H_\eta'(M(u,\xi),\xi)\delta f=\eta'(u)\delta f, \tag{2.45}$$

where TD_ξ is the tangent space of D_ξ, which does not depend on the point considered since D_ξ is convex.

Remark 2.5 The classical approach of (2.17) by Lagrange multipliers is to assert that there exists $\lambda(u) \in (\mathbb{R}^p)'$ such that

$$\forall u \in \mathcal{U} \quad a.e.\ \xi \quad \partial_f H_\eta(M(u,\xi),\xi) \ni \lambda(u). \tag{2.46}$$

The proof of the sufficiency of this condition is the same as that of Proposition 2.3. From Proposition (2.4) it is obviously also necessary, indeed the sensitivity property exactly says that we must have $\lambda(u) = \eta'(u)$.

Proof of the necessary condition in Theorem 2.1. Let us assume that $(H_\eta)_\eta$ satisfies (E0), (E1), (E2) and that G_η in (2.18) is C^1. Thanks to Proposition 2.4, (2.27), i.e. (2.28), holds, and let us take in it $f = M(v,\xi)$, $v \in \mathcal{U}$. We get $\forall u \in \mathcal{U}$, $a.e.\ \xi$, $\forall v \in \mathcal{U}$,

$$\varphi(v) \equiv G_\eta(v,\xi) - G_\eta(u,\xi) - \eta'(u)(M(v,\xi) - M(u,\xi)) \geq 0, \tag{2.47}$$

and by continuity in u, we can replace $\forall u \in \mathcal{U}$ $a.e.\ \xi$ by $a.e.\ \xi$ $\forall u \in \mathcal{U}$. The function φ is C^1, nonnegative, and cancels at u. Writing that its derivative must vanish at u, we get

$$G'_\eta(u,\xi) = \eta'(u)M'(u,\xi), \tag{2.48}$$

that is (2.23). Then, we have

$$\begin{aligned} \varphi'(v) &= G'_\eta(v,\xi) - \eta'(u)M'(v,\xi) \\ &= (\eta'(v) - \eta'(u))M'(v,\xi) \\ &= [\eta''(u)(v-u)]M'(u,\xi) + \underset{v\to u}{o}(v-u). \end{aligned} \tag{2.49}$$

This proves that φ' is differentiable at u, with

$$\varphi''(u) = (M'(u,\xi))^t \eta''(u). \tag{2.50}$$

Now Schwarz' theorem says that $\varphi''(u)$ is symmetric, and since φ has a minimum at u, we have $\varphi''(u) \geq 0$, thus $(M'(u,\xi))^t\eta''(u)$ is symmetric nonnegative. Finally, we obtain (2.22) by replacing $\forall \eta \in \mathcal{E}$ $a.e.\ \xi$ by $a.e.\ \xi$ $\forall \eta \in \mathcal{E}$ (this is possible because $\mathcal{E}$ is separable). $\square$

Remark 2.6 The previous proof also gives that

$$a.e.\ \xi \quad \forall u,v \in \mathcal{U} \quad G_\eta(v,\xi) \geq G_\eta(u,\xi) + \eta'(u)(M(v,\xi) - M(u,\xi)). \tag{2.51}$$

By exchanging u and v, it yields the necessary condition

$$a.e.\ \xi \quad \forall u,v \in \mathcal{U} \quad (\eta'(v) - \eta'(u))(M(v,\xi) - M(u,\xi)) \geq 0. \tag{2.52}$$

Proof of the sufficient condition in Theorem 2.1. Let us now assume that (2.22) holds. Thus we can define for any $\eta \in \mathcal{E}$ a function $G_\eta(u,\xi)$, which is C^1 in u, such that

$$a.e.\ \xi \quad \forall u \in \mathcal{U} \quad G'_\eta(u,\xi) = \eta'(u)M'(u,\xi) \tag{2.53}$$

(see the remarks on the system (2.24)). Let us assume for a moment that (2.51) holds true with this G_η. Then we define for any $f \in \mathbb{R}^p$

$$H_\eta(f,\xi) = \sup_{u\in\mathcal{U}}\Big(G_\eta(u,\xi) + \eta'(u)(f - M(u,\xi))\Big) \in]-\infty,\infty]. \tag{2.54}$$

It is easy to see that $H_\eta(.,\xi)$ is convex, and thanks to (2.51),

$$\forall v \in \mathcal{U} \quad H_\eta(M(v,\xi),\xi) = G_\eta(v,\xi) < \infty. \tag{2.55}$$

By convexity, $H_\eta(.,\xi)$ is therefore finite on conv $M(.,\xi) = D_\xi$ by assumption, thus (E0) is true. Then, by definition of H_η and by (2.55), (2.27), i.e. (2.28) is obviously satisfied, and Proposition 2.3 ensures that (E2) is true. Finally, thanks to (M0),

$$\begin{aligned}
\partial_u \int G_\eta(u,\xi)\,d\xi &= \int G'_\eta(u,\xi)\,d\xi \\
&= \int \eta'(u) M'(u,\xi)\,d\xi \\
&= \eta'(u)\int M'(u,\xi)\,d\xi \\
&= \eta'(u),
\end{aligned} \tag{2.56}$$

thus (E1) holds for some constant c_η. Notice also that a computation similar to (2.56) but using (M1) gives

$$\partial_u \int a_j(\xi) G_\eta(u,\xi)\,d\xi = \eta'(u)F'_j(u), \tag{2.57}$$

which means that G in (2.15) is an entropy-flux associated to η.

Now, in order to complete the proof, it only remains to prove that each condition (CH1) or (CH2) imply that G_η defined through (2.53) verifies (2.51).

Let us first assume (CH1). It is enough to obtain (2.51) for η in a dense subset of $\mathcal{E}$, and thus let us assume that $\eta'' > 0$ and $\eta'(\mathcal{U})$ is convex. Since $\eta'' > 0$, η' is a C^1 diffeomorphism from $\mathcal{U}$ onto an open set $\mathcal{V}$ of $\mathbb{R}^p$, which is convex by assumption. Now take $u, v \in \mathcal{U}$, and define $v(t)$ and $\varphi(t)$, $0 \le t \le 1$, by

$$\eta'(v(t)) = (1-t)\eta'(u) + t\eta'(v), \tag{2.58}$$

$$\varphi(t) = G_\eta(v(t),\xi) - G_\eta(u,\xi) - \eta'(u)(M(v(t),\xi) - M(u,\xi)). \tag{2.59}$$

Then

$$\begin{aligned}
\varphi'(t) &= \Big(\eta'(v(t)) - \eta'(u)\Big)M'(v(t),\xi)v'(t) \\
&= t\Big(\eta''(v(t))v'(t)\Big)M'(v(t),\xi)v'(t) \\
&\ge 0
\end{aligned} \tag{2.60}$$

because $M(.,\xi) \in \mathcal{M}_+^{\mathcal{E}}$. Therefore, $\varphi(1) \ge \varphi(0)$, which is (2.51).

Assume now that (CH2) holds, and let $u, v \in \mathcal{U}$. We define $\varphi(t)$ as in (2.59), but with $v(t)$ defined by

$$M(v(t),\xi) = (1-t)M(u,\xi) + tM(v,\xi). \tag{2.61}$$

Then

$$\begin{aligned}\varphi'(t) &= \Big(\eta'(v(t)) - \eta'(u)\Big) M'(v(t),\xi) v'(t) \\ &= \Big(\eta'(v(t)) - \eta'(u)\Big)\Big(M(v,\xi) - M(u,\xi)\Big),\end{aligned} \tag{2.62}$$

$$\begin{aligned}\varphi''(t) &= \Big(\eta''(v(t))v'(t)\Big)\Big(M(v,\xi) - M(u,\xi)\Big) \\ &= \Big(\eta''(v(t))v'(t)\Big) M'(v(t),\xi) v'(t) \\ &\geq 0\end{aligned} \tag{2.63}$$

because $M(.,\xi) \in \mathcal{M}_+^{\mathcal{E}}$. Thus $\varphi'(t) \geq \varphi'(0) = 0$, and $\varphi(t) \geq \varphi(0)$, which is (2.51). □

Remark 2.7 As the above proof shows, the optimal sufficient condition which should be checked instead of (CH1) or (CH2) is that (2.51) holds, with G_η defined by (2.23). Another possibility is also to find directly a function H_η satisfying (2.18), with G_η defined by (2.23), and to check (E0) and (2.28) (here (E1) is automatically satisfied, see (2.56)).

Remark 2.8 In the case of assumption (CH2), H_η is given by

$$H_\eta(f,\xi) = G_\eta(U(f,\xi),\xi), \tag{2.64}$$

where G_η is defined by (2.23) and $U(.,\xi)$ is the inverse of $M(.,\xi)$, and it is easy to check that $H_\eta(.,\xi) \in C^2$. Indeed, the definition of H_η by (2.64) provides a shorter proof, that is given in [43], of the sufficient part in Theorem 2.1 under (CH2). Just differentiate (2.64) to obtain (2.27) and (E0), while (2.56) gives (E1).

2.3 Discrete times

There exists an alternate time discrete form of the BGK models, which is used in numerical methods for solving systems of conservation laws. It appears in the literature as the transport-collapse method [8], and leads to kinetic or Boltzmann schemes (see the introduction). Let us write it down in the context of the assumptions of Section 2.1, i.e. (H1), (H2), (H3), (M0), (M1), (E0), (E1), (E2).

The algorithm gives $u^{n+1}(x)$ in terms of $u^n(x)$, providing an approximation of the solution to (2.1) $u(t_n,x) \simeq u^n(x)$, where (t_n) is an increasing sequence of times.

In the transport step, we solve

$$\partial_t f + a(\xi)\cdot\nabla_x f = 0 \qquad \text{in} \quad]t_n, t_{n+1}[\times\mathbb{R}^N\times\Xi, \tag{2.65}$$

with initial data

$$f(t_n,x,\xi) = M(u^n(x),\xi). \tag{2.66}$$

Indeed the solution is given by

$$f(t,x,\xi) = f(t_n, x-(t-t_n)a(\xi),\xi), \qquad t_n < t < t_{n+1}. \tag{2.67}$$

Then, u^{n+1} is obtained by

$$u^{n+1}(x) = \int f(t_{n+1}-, x, \xi)\, d\xi - k, \tag{2.68}$$

where $f(t_{n+1}-, .) = \lim f(t, .)$ when $t \to t_{n+1}$, $t < t_{n+1}$. The collapse step, or projection step, consists in the averaging procedure (2.68) followed by the reconstruction (2.66).

We see easily that with these definitions and $f^n = f(t_n, .)$, $f^{n-} = f(t_n-, .)$, we have

$$\begin{aligned}\partial_t f + a(\xi)\cdot\nabla_x f &= \sum_{n=1}^{\infty} \delta(t-t_n)(f^n - f^{n-})\\ &= \sum_{n=1}^{\infty} \delta(t-t_n)(M_{f^{n-}} - f^{n-}) \qquad \text{in} \quad]0,\infty[\times\mathbb{R}^N\times\Xi.\end{aligned} \tag{2.69}$$

Thus if $t_{n+1} - t_n = \Delta t$, since $\Delta t \sum_n \delta(t-t_n) \rightharpoonup 1$ when $\Delta t \to 0$, (2.69) is similar to (2.7) with $\varepsilon = \Delta t$.

In order to obtain discrete entropy inequalities, we write

$$\partial_t[H_\eta(f,\xi)] + a(\xi)\cdot\nabla_x[H_\eta(f,\xi)] = \sum_{n=1}^{\infty} \delta(t-t_n)\Big(H_\eta(f^n,\xi) - H_\eta(f^{n-},\xi)\Big), \tag{2.70}$$

and after integration with respect to ξ, thanks to (E2),

$$\begin{aligned}&\partial_t \int H_\eta(f,\xi)\, d\xi + \mathrm{div}_x \int a(\xi) H_\eta(f,\xi)\, d\xi\\ &= \sum_{n=1}^{\infty} \delta(t-t_n)\left[\int H_\eta(M_{f^{n-}},\xi)d\xi - \int H_\eta(f^{n-},\xi)d\xi\right] \le 0.\end{aligned} \tag{2.71}$$

We have to notice that here the entropy production takes the form

$$\begin{aligned}&\int H_\eta(M_{f^{n-}},\xi)d\xi - \int H_\eta(f^{n-},\xi)d\xi\\ &= \int \Big[H_\eta(M(u^n,\xi),\xi) - H_\eta(f^{n-},\xi) + \eta'(u^n)(f^{n-} - M(u^n,\xi))\Big]\, d\xi,\end{aligned} \tag{2.72}$$

and the integrand is nonpositive thanks to (2.27), or more explicitly (2.28). The integration of (2.71) on a time interval leads finally to time discrete entropy inequalities.

For space discretization by finite volume methods, we also integrate in x over a control domain, and we obtain at lowest order the following flux decomposition. For any $\omega \in \mathbb{R}^N$, if

$$F^\omega(u) \equiv \sum_{j=1}^{N} \omega_j F_j(u), \tag{2.73}$$

then from (M1)

$$F^\omega(u) + \sum_j \omega_j k_j' = F^{\omega+}(u) + F^{\omega-}(u), \tag{2.74}$$

where

$$F^{\omega\pm}(u) = \int_{\pm\omega\cdot a(\xi)>0} \omega\cdot a(\xi)\, M(u,\xi)\, d\xi, \tag{2.75}$$

and we observe that since $M(.,\xi)$ belongs to the convex cone $\mathcal{M}_+^{\mathcal{E}}$ by Theorem 2.1, we have also

$$\pm F^{\omega\pm} \in \mathcal{M}_+^{\mathcal{E}}, \tag{2.76}$$

which is a required property for stability. Decompositions of the type (2.74), (2.76) were studied in [11].

2.4 Chapman-Enskog expansion

Let us consider a solution f to (2.7). We have seen in Section 2.1 that when $\varepsilon\to 0$, the moment u of f tends (at least formally) to a solution of (2.1). The so called Chapman-Enskog expansion consists in writing the higher-level terms in ε. Writing f and u instead of f_ε and u_ε in order to simplify notations, we have the following result.

Proposition 2.6 *Up to terms in ε^2, we have*

$$\partial_t u + \sum_{j=1}^N \frac{\partial}{\partial x_j} F_j(u) = \varepsilon \sum_{1\le i,j\le N} \frac{\partial}{\partial x_j}\Big[D_{ji}(u)\frac{\partial u}{\partial x_i}\Big], \tag{2.77}$$

with

$$D_{ji}(u) = Q'_{ji}(u) - F'_j(u)F'_i(u), \qquad Q_{ij}(u) = \int a_i(\xi)a_j(\xi)M(u,\xi)\, d\xi. \tag{2.78}$$

Proof. Since $M_f - f = O(\varepsilon)$, we have from (2.7)

$$M_f - f = \varepsilon\Big(\partial_t M_f + a(\xi)\cdot\nabla_x M_f\Big) + O(\varepsilon^2), \tag{2.79}$$

and thus

$$\begin{aligned}\int a_j(\xi) f d\xi &= \int a_j(\xi) M_f\, d\xi - \varepsilon \int a_j(\xi)\Big(\partial_t M_f + a(\xi)\cdot\nabla_x M_f\Big) d\xi + O(\varepsilon^2)\\ &= F_j(u) + k'_j - \varepsilon\partial_t(F_j(u)+k'_j) - \varepsilon\sum_i \frac{\partial}{\partial x_i} Q_{ji}(u) + O(\varepsilon^2).\end{aligned} \tag{2.80}$$

Putting this in (2.10) this yields

$$\partial_t u + \sum_j \frac{\partial}{\partial x_j} F_j(u) = \varepsilon \sum_j \frac{\partial}{\partial x_j}\Big[\partial_t(F_j(u)) + \sum_i \frac{\partial}{\partial x_i} Q_{ji}(u)\Big] + O(\varepsilon^2), \tag{2.81}$$

and since

$$\partial_t[F_j(u)] = F'_j(u)\partial_t u = -\sum_i F'_j(u)F'_i(u)\frac{\partial u}{\partial x_i} + O(\varepsilon), \tag{2.82}$$

we get

$$\partial_t u + \sum_j \frac{\partial}{\partial x_j} F_j(u) = \varepsilon \sum_{ij} \frac{\partial}{\partial x_j}\Big[\Big(Q'_{ji}(u) - F'_j(u)F'_i(u)\Big)\frac{\partial u}{\partial x_i}\Big] + O(\varepsilon^2), \quad (2.83)$$

and the proof is complete. □

Remark 2.9 The approximation is second-order in ε if and only if for any smooth solution to (2.1) we have

$$\sum_j \frac{\partial}{\partial x_j}\Big[\partial_t(F_j(u)) + \sum_i \frac{\partial}{\partial x_i} Q_{ji}(u)\Big] = 0. \quad (2.84)$$

We see that we need all the components of the F_j to be entropies of (2.1). It is the case for scalar multidimensional equations, or for the models of [9].

An important property is that the reduced system (2.77) obtained by dropping terms in ε^2 is still compatible with the family $\mathcal{E}$.

Theorem 2.7 *The reduced system (2.77) is dissipating all entropies $\eta \in \mathcal{E}$, and the tensor of operators of $\mathcal{L}(\mathbb{R}^p, (\mathbb{R}^p)')$ defined by*

$$\sigma_{ij} = D_{ji}(u)^t \eta''(u), \qquad 1 \le i, j \le N \quad (2.85)$$

is symmetric nonnegative, in the sense that

$$\sigma^t_{ij} = \sigma_{ji}, \quad (2.86)$$

$$\forall v_1, \ldots, v_N \in \mathbb{R}^p \qquad \sum_{ij} \sigma_{ij} \cdot v_j \cdot v_i \ge 0. \quad (2.87)$$

Proof. Here we are not going to use (E0), (E1), (E2), but rather the characterization (2.22). If u is a smooth solution to (2.77) and $\eta \in \mathcal{E}$, we have by left-multiplying by $\eta'(u)$

$$\partial_t[\eta(u)] + \operatorname{div}_x G(u) = \varepsilon \sum_{ij} \frac{\partial}{\partial x_j}\Big[\eta'(u) D_{ji}(u)\frac{\partial u}{\partial x_i}\Big] - \varepsilon \sum_{ij} \eta''(u) \cdot \frac{\partial u}{\partial x_j} \cdot D_{ji}(u)\frac{\partial u}{\partial x_i}, \quad (2.88)$$

where we have used the notation (A.7). Thus we only need to prove (2.86), (2.87), which is equivalent to the fact that we have a bilinear form which is symmetric nonnegative on $(\mathbb{R}^p)^N$. This bilinear form can be written for $v = (v_1, \ldots, v_N) \in (\mathbb{R}^p)^N$, $w = (w_1, \ldots, w_N) \in (\mathbb{R}^p)^N$,

$$\begin{aligned}
&\sigma \cdot v \cdot w \\
&= \sum_{ij} \sigma_{ij} \cdot v_j \cdot w_i \\
&= \sum_{ij} \Big[Q'_{ji}(u)^t - F'_i(u)^t F'_j(u)^t\Big]\eta''(u) \cdot v_j \cdot w_i \\
&= \sum_{ij} \left[\int a_i(\xi)a_j(\xi)M'(u,\xi)^t \eta''(u) d\xi - F'_i(u)^t F'_j(u)^t \eta''(u)\right] \cdot v_j \cdot w_i \\
&= \int M'(u,\xi)^t \eta''(u) \cdot a(\xi)v \cdot a(\xi)w \; d\xi - \sum_j F'_j(u)^t \eta''(u) \cdot v_j \cdot I(w),
\end{aligned} \quad (2.89)$$

with
$$a(\xi)w = \sum_i a_i(\xi)w_i, \qquad I(w) = \sum_i F_i'(u)w_i. \tag{2.90}$$

Now since $F_j \in \mathcal{M}^{\mathcal{E}}$, $F_j'(u)^t\eta''(u)$ is symmetric and

$$\begin{aligned}&\sum_j F_j'(u)^t\eta''(u)\cdot v_j\cdot I(w)\\ &= \sum_j F_j'(u)^t\eta''(u)\cdot I(w)\cdot v_j\\ &= \eta''(u)\cdot I(w)\cdot I(v),\end{aligned} \tag{2.91}$$

so that

$$\sigma\cdot v\cdot w = \int M'(u,\xi)^t\eta''(u)\cdot a(\xi)v\cdot a(\xi)w\; d\xi - \eta''(u)\cdot I(v)\cdot I(w). \tag{2.92}$$

Now since $M(.,\xi)\in\mathcal{M}^{\mathcal{E}}$, this expression is symmetric in v and w. Thus σ is symmetric. Let us now take $w = v$. Since $M(.,\xi)\in\mathcal{M}_+^{\mathcal{E}}$, we have in (2.92) the difference of two nonnegative terms. But by applying the Cauchy-Schwarz inequality twice

$$\begin{aligned}&\eta''(u)\cdot I(v)\cdot I(v)\\ &= \int M'(u,\xi)^t\eta''(u)\cdot I(v)\cdot a(\xi)v\; d\xi\\ &\le \int \Big(M'(u,\xi)^t\eta''(u)\cdot I(v)\cdot I(v)\Big)^{1/2}\Big(M'(u,\xi)^t\eta''(u)\cdot a(\xi)v\cdot a(\xi)v\Big)^{1/2}d\xi\\ &\le \left[\int M'(u,\xi)^t\eta''(u)\cdot I(v)\cdot I(v)\, d\xi\right]^{1/2}\left[\int M'(u,\xi)^t\eta''(u)\cdot a(\xi)v\cdot a(\xi)v\, d\xi\right]^{1/2},\end{aligned} \tag{2.93}$$

and the first term between brackets is $\eta''(u)\cdot I(v)\cdot I(v)$, which proves the needed inequality. □

Remark 2.10 Let us define the linear operator $D(u) : (\mathbb{R}^p)^N \to (\mathbb{R}^p)^N$ by

$$\big(D(u)v\big)_j = \sum_i D_{ji}(u)v_i, \tag{2.94}$$

where $v = (v_1,\ldots,v_N)\in(\mathbb{R}^p)^N$. If $\mathcal{E}$ contains a strictly convex entropy η_0, then (2.86), (2.87) mean that D is self-adjoint nonnegative for the scalar product on $(\mathbb{R}^p)^N$ defined by

$$(w/v) = \sum_{j=1}^N \eta_0''(u)\cdot v_j\cdot w_j. \tag{2.95}$$

Therefore, D has nonnegative real eigenvalues, and (2.77) is parabolic (possibly degenerate).

3 Classical examples

In this section, we show how classical examples enter the model presented in Section 2.1.

3.1 Gas dynamics with single entropy

Here we write down models with a single entropy, as they were settled down in [5] and [37], [38] (see also [16], [46], [3]), by using the method of Lagrange multipliers: we find H satisfying (E0), then we write (2.46) to have (E2), and together with (M0), (M1) this determines M. Finally, (E1) is read as a definition of η.

3.1.1 Full gas dynamics

The first BGK models were those of the Euler system of perfect polytropic gas dynamics

$$\begin{gathered}\partial_t \rho + \mathrm{div}_x(\rho u) = 0,\\ \partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u + p\,\mathrm{Id}) = 0,\\ \partial_t E + \mathrm{div}_x\Big((E+p)u\Big) = 0,\end{gathered} \tag{3.1}$$

where $x \in \mathbb{R}^N$, $\rho \geq 0$, $u \in \mathbb{R}^N$, $p \geq 0$, $E = \rho|u|^2/2 + p/(\gamma-1)$, $\gamma > 1$. Here the state is $\underline{u} = (\rho, \rho u_1, \dots, \rho u_N, E)$. Let us define the number d of degrees of freedom by

$$N + d = \frac{2}{\gamma - 1}. \tag{3.2}$$

We assume to be in the physically relevant case $d \geq 0$. We take

$$\xi = (v, I) \in \mathbb{R}^N \times [0, \infty[, \qquad d\xi = dv\, c_0 I^{d-1} dI, \tag{3.3}$$

with $c_0 = 2\pi^{d/2}/\Gamma(d/2)$. If d is an integer, I can be interpreted as the modulus of a variable in $\mathbb{R}^d$, then $c_0 I^{d-1} dI$ is the Lebesgue measure in $\mathbb{R}^d$ for radial functions. When $d = 0$, the variable I is not necessary, one should replace formally $c_0 I^{d-1} dI$ by $\delta(I)$. We define

$$a(\xi) = v, \tag{3.4}$$

and set

$$M(\underline{u}, \xi) = M_0(\underline{u}, \xi) K(\xi), \qquad K(\xi) = (1, v_1, \dots, v_N, \frac{|v|^2 + I^2}{2}), \tag{3.5}$$

$$D_\xi = K(\xi)\mathbb{R}_+. \tag{3.6}$$

Thus (H3) holds as soon as $M_0 \geq 0$. Next, we choose

$$H(f, \xi) = S(f_0), \qquad f = K(\xi) f_0 \in D_\xi, \tag{3.7}$$

where S is a given strictly convex function on $[0, \infty[$. Then, H is differentiable on $K(\xi)\mathbb{R}_+^*$,

$$\partial_f H(f, \xi)\delta f = S'(f_0)\delta f_0 \qquad \text{if } f_0 > 0, \qquad \delta f = K(\xi)\delta f_0 \in TD_\xi, \tag{3.8}$$

while on the boundary the subdifferential (2.26) is given by

$$\partial_f H(0, \xi) = \{\lambda \in (\mathbb{R}^p)' \ ;\ \lambda K(\xi) \leq S'(0)\}. \tag{3.9}$$

Writing down (2.46) (under the form (2.45) in $K(\xi)\mathbb{R}_+^*$), we obtain the conditions

$$\begin{aligned} S'(M_0(\underline{u},\xi)) = \lambda(\underline{u})K(\xi) \qquad & \text{if } M_0(\underline{u},\xi) > 0, \\ S'(0) \geq \lambda(\underline{u})K(\xi) \qquad & \text{if } M_0(\underline{u},\xi) = 0. \end{aligned} \tag{3.10}$$

Thus we can take

$$M_0(\underline{u},\xi) = R(\lambda(\underline{u})K(\xi)), \tag{3.11}$$

$$R(y) = \begin{cases} (S')^{-1}(y) & \text{if } y > S'(0), \\ 0 & \text{if } y \leq S'(0). \end{cases} \tag{3.12}$$

Finally, $\lambda(\underline{u})$ is chosen so that (M0), (M1) are satisfied, and this is possible under technical assumptions [37]. Now all assumptions are satisfied, we only need to define η by (E1) (it remains to check (2.9), this is done in Section 5). By the way we can check that Theorem 2.1 is verified. Here $M(\underline{u},\xi) = R(\eta'(\underline{u})K(\xi))K(\xi)$ since $\lambda(\underline{u}) = \eta'(\underline{u})$, and $M'(\underline{u},\xi)d\underline{u} = R'(\eta'(\underline{u})K(\xi))(\eta''(\underline{u})\cdot d\underline{u}\cdot K(\xi))K(\xi)$,

$$M'(\underline{u},\xi)^t\eta''(\underline{u})\cdot d\underline{u}\cdot d\underline{v} = R'(\eta'(\underline{u})K(\xi))\Big(\eta''(\underline{u})\cdot d\underline{u}\cdot K(\xi)\Big)\Big(\eta''(\underline{u})\cdot d\underline{v}\cdot K(\xi)\Big) \tag{3.13}$$

which is symmetric nonnegative. Also the eigenvalues of $M'(\underline{u},\xi)$ are 0 with multiplicity $N+1$ and $\operatorname{tr} M'(\underline{u},\xi) = R'(\eta'(\underline{u})K(\xi))(\eta''(\underline{u})\cdot K(\xi)\cdot K(\xi)) \geq 0$.

Example 3.1 Take $S(f_0) = f_0 \ln f_0$. Then $S'(f_0) = \ln f_0 + 1$,

$$M_0(\underline{u},\xi) = \exp\Big(\lambda(\underline{u})K(\xi) - 1\Big) = \frac{\rho}{(2\pi T)^{1/(\gamma-1)}} e^{-\frac{|v-u|^2+I^2}{2T}} \tag{3.14}$$

with $p = \rho T$, $\eta(\underline{u}) = \rho\ln(\rho/T^{1/(\gamma-1)}) - \rho(1+\ln 2\pi)/(\gamma-1)$,

$$\lambda(\underline{u}) = \eta'(\underline{u}) = \left(\ln\frac{\rho}{T^{1/(\gamma-1)}} - |u|^2/2T + 1 - \frac{\ln 2\pi}{\gamma-1}, \frac{u}{T}, -\frac{1}{T}\right). \tag{3.15}$$

Example 3.2 Take $S(f_0) = f_0^{m+1}/(m+1)$, $m > 0$. Then $S'(f_0) = f_0^m$,

$$M_0(\underline{u},\xi) = \Big(\lambda(\underline{u})K(\xi)\Big)_+^{1/m} = c_1\frac{\rho}{T^{1/(\gamma-1)}}\left(1 - \frac{|v-u|^2+I^2}{2T(\frac{\gamma}{\gamma-1}+\frac{1}{m})}\right)_+^{1/m}, \tag{3.16}$$

with

$$c_1 = \left[2\pi\left(\frac{\gamma}{\gamma-1}+\frac{1}{m}\right)\right]^{-1/(\gamma-1)} \frac{\Gamma\left(\frac{\gamma}{\gamma-1}+\frac{1}{m}\right)}{\Gamma\left(\frac{1}{m}+1\right)}, \tag{3.17}$$

and

$$\eta(\underline{u}) = \frac{c_1^m}{1+m\gamma/(\gamma-1)}\ \rho\left(\frac{\rho}{T^{1/(\gamma-1)}}\right)^m. \tag{3.18}$$

An interesting limit exists when $m \to \infty$,

$$M_0(\underline{u},\xi) \underset{m\to\infty}{\longrightarrow} \left(2\pi\frac{\gamma}{\gamma-1}\right)^{-1/(\gamma-1)} \Gamma\left(\frac{\gamma}{\gamma-1}\right)\ \frac{\rho}{T^{1/(\gamma-1)}} \mathbb{1}_{|v-u|^2+I^2<\frac{\gamma}{\gamma-1}2T}, \tag{3.19}$$

which gives a model that does not enter our framework, one should replace the minimization of $\int H(f,\xi)d\xi$ by that of $\operatorname{ess\,sup}_\xi f_0(\xi)$. This model is studied in [24], it has the interesting property to give the maximum principle on the specific entropy.

3.1.2 First model for isentropic gas dynamics

A similar model can also be built for the isentropic case, which is

$$\begin{aligned} \partial_t \rho + \operatorname{div}_x(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u + \kappa\rho^\gamma \,\mathrm{Id}) = 0, \end{aligned} \tag{3.20}$$

where $x \in \mathbb{R}^N$, $\rho \geq 0$, $u \in \mathbb{R}^N$, and $\kappa > 0$, $\gamma > 1$. The state is $\underline{u} = (\rho, \rho u_1, \dots, \rho u_N)$. Let us first assume that d defined in (3.2) satisfies $d > 0$. We take $\Xi = \mathbb{R}^N$ with Lebesgue measure, $a(\xi) = \xi$,

$$M(\underline{u},\xi) = M_0(\underline{u},\xi)K(\xi), \quad K(\xi) = (1, \xi_1, \dots, \xi_N), \quad D_\xi = K(\xi)\mathbb{R}_+, \tag{3.21}$$

thus we need that $M_0 \geq 0$. We choose

$$H(f,\xi) = f_0|\xi|^2/2 + \frac{1}{2c_2^{2/d}} \frac{f_0^{1+2/d}}{1+2/d}, \qquad f = K(\xi)f_0 \in D_\xi \tag{3.22}$$

for some constant $c_2 > 0$. A computation similar to the one above gives that

$$M_0(\underline{u},\xi) = c_2 \left(2\lambda(\underline{u})K(\xi) - |\xi|^2\right)_+^{d/2}. \tag{3.23}$$

Next, we want to obtain $\eta = \rho|u|^2/2 + \kappa\rho^\gamma/(\gamma-1)$, thus we put $\lambda(\underline{u}) = \eta'(\underline{u}) = \left(\frac{\gamma}{\gamma-1}\kappa\rho^{\gamma-1} - |u|^2/2, u\right)$, we can check that the moment equations hold if

$$c_2 = \left(\frac{2\gamma\kappa}{\gamma-1}\right)^{-1/(\gamma-1)} \frac{\Gamma\left(\frac{\gamma}{\gamma-1}\right)}{\pi^{N/2}\Gamma(d/2+1)}. \tag{3.24}$$

Thus we obtain

$$M_0(\underline{u},\xi) = c_2 \left(\frac{2\gamma}{\gamma-1}\kappa\rho^{\gamma-1} - |\xi-u|^2\right)_+^{d/2}. \tag{3.25}$$

We notice that this function is not that of the kinetic formulation (in one dimension), the normalization is different (see Section 4.1).

It is also possible to treat the isothermal case, by taking

$$H(f,\xi) = f_0|\xi|^2/2 + \kappa f_0 \ln f_0, \qquad M_0(\underline{u},\xi) = \frac{\rho}{(2\pi\kappa)^{N/2}} e^{-\frac{|\xi-u|^2}{2\kappa}}. \tag{3.26}$$

Then $\eta = \rho|u|^2/2 + \kappa\left(\rho\ln\rho - \rho\ln(2\pi\kappa)^{N/2}\right)$.

3.1.3 Second model for isentropic gas dynamics

Let us finally give another model for the same system (3.20) of isentropic gas dynamics, which works for $d \geq 0$. We take Ξ and a as in (3.3), (3.4), and

$$M(\underline{u}, \xi) = M_0(\underline{u}, \xi) K(\xi), \qquad K(\xi) = (1, v_1, \ldots, v_N), \tag{3.27}$$

but now

$$D_\xi = K(\xi)[0, c_3] \tag{3.28}$$

for some $c_3 > 0$. Thus we need $0 \leq M_0 \leq c_3$. Then we choose

$$H(f, \xi) = f_0 \frac{|v|^2 + I^2}{2}, \qquad f = K(\xi) f_0 \in D_\xi, \tag{3.29}$$

and we can check that the subdifferential at endpoints is given by

$$\begin{aligned} \partial_f H(f, \xi) &= \left\{ \lambda \in (\mathbb{R}^p)' \; ; \; \lambda K(\xi) \leq \frac{|v|^2 + I^2}{2} \right\} \quad \text{if } f = 0, \\ \partial_f H(f, \xi) &= \left\{ \lambda \in (\mathbb{R}^p)' \; ; \; \lambda K(\xi) \geq \frac{|v|^2 + I^2}{2} \right\} \quad \text{if } f = c_3 K(\xi). \end{aligned} \tag{3.30}$$

Therefore, in order to satisfy (2.46), it is enough to take

$$M_0(\underline{u}, \xi) = c_3 \mathbb{1}_{|v|^2 + I^2 - 2\lambda(\underline{u}) K(\xi) < 0}. \tag{3.31}$$

As above we want to get $\eta = \rho |u|^2/2 + \kappa \rho^\gamma/(\gamma - 1)$, thus we take $\lambda(\underline{u}) = \eta'(\underline{u})$, and the moments are satisfied if

$$c_3 = \left(\frac{2\pi\gamma\kappa}{\gamma - 1} \right)^{-1/(\gamma-1)} \Gamma \left(\frac{\gamma}{\gamma - 1} \right). \tag{3.32}$$

Therefore, we obtain

$$M_0(\underline{u}, \xi) = c_3 \mathbb{1}_{|v-u|^2 + I^2 < \frac{2\gamma}{\gamma-1} \kappa \rho^{\gamma-1}}. \tag{3.33}$$

The previous model in Section 3.1.2 is actually related to this one, it can be obtained by integration in I with respect to the measure $c_0 I^{d-1} dI$.

3.2 Models with finitely many velocities

Some relaxation models have been written in [2] and [43] under the form of BGK equations with finitely many velocities. We briefly explain here how it enters the framework.

Following the general method introduced in Section 2.2, in order to find BGK models with a large family of kinetic entropies, we need to find a subspace of $\mathcal{M}^{\mathcal{E}}$ containing Id and the F_j. The most simple way to achieve that is to take linear combinations of these functions. Thus we look for M under the form

$$M(u, \xi) = \alpha_0(\xi) u + \sum_{j=1}^{N} \alpha_j(\xi) F_j(u), \tag{3.34}$$

with $\alpha_0,\dots,\alpha_N$ real. Then the moment equations (M0), (M1) are realized as soon as

$$\begin{aligned} \int \alpha_0(\xi)d\xi = 1, \qquad & \int \alpha_j(\xi)d\xi = 0, \\ \int \alpha_0(\xi)a_j(\xi)d\xi = 0, \qquad & \int \alpha_j(\xi)a_i(\xi)d\xi = \delta_{ij}, \end{aligned} \tag{3.35}$$

for $1 \le i,j \le N$. The stability condition (2.25) writes

$$a.e.\ \xi \qquad \forall u \qquad \sigma\left(\alpha_0(\xi)\,\mathrm{Id} + \sum_{j=1}^N \alpha_j(\xi)F_j'(u)\right) \subset [0,\infty[, \tag{3.36}$$

and the microscopic entropies are given by

$$G_\eta(u,\xi) = \alpha_0(\xi)\eta(u) + \sum_{j=1}^N \alpha_j(\xi)G_j(u), \tag{3.37}$$

where η is any convex entropy of (2.1), and $(G_1,\dots,G_N)$ is its associated entropy-flux. Here we can take for $\mathcal{E}$ the set of all convex entropies of (2.1). If we exclude the value 0 in (3.36), the kinetic entropies H_η are then given by (2.64). We have to precise that it is not possible to check condition (2.9) in general. A possible issue is to study invariant domains, as is done in [43].

It remains to find Ξ and a so as to verify (3.35), and a possible choice is as follows. We take Ξ to be a set of $N+1$ elements with the uniform probability, and $(a(\xi))_{\xi\in\Xi}$ are $N+1$ independent vectors of $\mathbb{R}^N$ in the affine sense. Then (3.35) determines $\alpha_0,\dots\alpha_N$ in a unique way.

Example 3.3 Take $N=1$, $\Xi=\{-1,1\}$, $a(\xi)=\xi c$, $c>0$. This yields $\alpha_0(\xi)=1$, $\alpha_1(\xi)=\xi/c$,

$$M(u,\xi) = u + \frac{\xi}{c}F(u). \tag{3.38}$$

The BGK equation is

$$\partial_t f(t,x,\xi) + \xi c\partial_x f(t,x,\xi) = \frac{1}{\varepsilon}\left(u(t,x) + \frac{\xi}{c}F\big(u(t,x)\big) - f(t,x,\xi)\right), \tag{3.39}$$

$$u(t,x) = \int f(t,x,\xi)\,d\xi = \frac{1}{2}\Big[f(t,x,1)+f(t,x,-1)\Big]. \tag{3.40}$$

By defining $v(t,x) = \frac{c}{2}[f(t,x,1)-f(t,x,-1)]$ we get the equivalent formulation

$$\left\{\begin{array}{l} \partial_t u + \partial_x v = 0, \\ \partial_t v + c^2\partial_x u = \dfrac{1}{\varepsilon}\Big(F(u)-v\Big), \end{array}\right. \tag{3.41}$$

while the stability condition (3.36) becomes

$$\sigma\Big(F'(u)\Big) \subset [-c,c]. \tag{3.42}$$

The system (3.41) was studied in [20], and (3.42) is known as the subcharacteristic condition. However, as noticed in [2], in higher dimension N the model is different from that of [20].

3.3 Scalar equations

The case $p = 1$ of a single equation in (2.1) has been the object of a lot of studies, especially in [34] where discrete BGK models are written down. We here give more general models which enter the framework of Section 2.

We assume $\mathcal{U}$ to be an interval, it can be the whole real line for example. Since all functions are entropies, we take for $\mathcal{E}$ the set of all globally Lipschitz convex functions. Then $\mathcal{M}^{\mathcal{E}}$ contains all functions $\mathcal{U} \to \mathbb{R}$, and $\mathcal{M}_+^{\mathcal{E}}$ contains all the nondecreasing ones. Therefore, a BGK model is provided by the knowledge of Ξ, $a(\xi) \in \mathbb{R}^N$ and $M(u,\xi)$ nondecreasing in u, and satisfying the moment equations

$$\int M(u,\xi)\,d\xi = u + k, \qquad \int a(\xi) M(u,\xi)\,d\xi = F(u) + k'. \tag{3.43}$$

By Theorem 2.1 we know that if we have enough regularity, the monotonicity of M in u ensures the existence of kinetic entropies H_η for any η, with $D_\xi = \operatorname*{conv}_{u\in\mathcal{U}} M(u,\xi)$. In the case when $M(.,\xi) \in C^1$ and $M'(u,\xi) > 0$, we have $D_\xi = M(\mathcal{U},\xi)$ and H_η is given by $H_\eta(f,\xi) = G_\eta(U(f,\xi),\xi)$, $f \in D_\xi$, with $U(.,\xi)$ the inverse of $M(.,\xi)$ (see Remark 2.8), and

$$G_\eta'(u,\xi) = \eta'(u) M'(u,\xi). \tag{3.44}$$

In general, the microscopic entropy G_η is still defined by (3.44), and H_η can be obtained by an integral formula. Let us first consider a Kruzkov entropy

$$\eta_v(u) = |u - v|. \tag{3.45}$$

We have

$$G_v(u,\xi) = |M(u,\xi) - M(v,\xi)| - |M(v,\xi)| \tag{3.46}$$

(the constant $|M(v,\xi)|$ is not really necessary, see Remark 2.1), and we can take

$$H_v(f,\xi) = |f - M(v,\xi)| - |M(v,\xi)|. \tag{3.47}$$

This yields for any η

$$H_\eta(f,\xi) = \frac{1}{2}\int \Big(|f - M(v,\xi)| - |M(v,\xi)|\Big)\eta''(v)\,dv + \frac{1}{2}\Big(\eta'(-\infty) + \eta'(+\infty)\Big) f. \tag{3.48}$$

One can check directly (E0), (E1), (E2) or (2.27) with this formula. For example, we can prove (E2) for a Kruzkov entropy as in [22],

$$\begin{aligned}
&\int H_v(M(u_f,\xi),\xi)\,d\xi \\
&= \int G_v(u_f,\xi)\,d\xi \\
&= \int \Big\{ \mathrm{sgn}(u_f - v)\,[M(u_f,\xi) - M(v,\xi)] - |M(v,\xi)| \Big\} d\xi \\
&= \mathrm{sgn}(u_f - v) \int [M(u_f,\xi) - M(v,\xi)]\,d\xi - \int |M(v,\xi)|\,d\xi \\
&= |u_f - v| - \int |M(v,\xi)|\,d\xi \\
&= \left| \int [f(\xi) - M(v,\xi)]\ d\xi \right| - \int |M(v,\xi)|\,d\xi \\
&\le \int \Big[|f(\xi) - M(v,\xi)| - |M(v,\xi)| \Big]\,d\xi \\
&= \int H_v(f(\xi),\xi)\,d\xi.
\end{aligned} \tag{3.49}$$

Actually from the monotonicity of M, $\int |M(u,\xi) - M(v,\xi)|\,d\xi = |u - v|$.

In order to have (2.9), it is enough to have the maximum principle. Let us assume for example that initially $u_m \le u^0(x) \le u_M$, where u_m and u_M are constants. Then from the monotonicity $M(u_m,\xi) \le M(u^0(x),\xi) \le M(u_M,\xi)$, and therefore by convexity (we can take $D_\xi = [M(u_m,\xi), M(u_M,\xi)]$), we have for any time

$$M(u_m,\xi) \le f(t,x,\xi) \le M(u_M,\xi). \tag{3.50}$$

By integration in ξ we get $u_m + k \le u(t,x) + k \le u_M + k$, which is the needed a priori estimate.

The model also satisfies the good properties of scalar models. For example, the contraction property can be obtained, as in [22], by taking two solutions $f(t,x,\xi)$ and $g(t,x,\xi)$, and by writing down the equation satisfied by $|f-g|$,

$$\begin{aligned}
\partial_t |f-g| + a(\xi)\cdot\nabla_x |f-g| &= \frac{1}{\varepsilon}\,\mathrm{sgn}(f-g)(M_f - f - M_g + g) \\
&\le \frac{|M_f - M_g| - |f-g|}{\varepsilon}.
\end{aligned} \tag{3.51}$$

Integrating with respect to ξ this gives

$$\partial_t \int |f-g|\,d\xi + \mathrm{div}_x \int a(\xi)|f-g|\,d\xi \le 0, \tag{3.52}$$

and thus

$$\frac{d}{dt} \iint |f-g|\,d\xi dx \le 0. \tag{3.53}$$

In particular, if we take $g(t,x,\xi) = f(t,x+h,\xi)$, it yields

$$\frac{d}{dt} \iint |f(t,x+h,\xi) - f(t,x,\xi)|\,d\xi dx \le 0, \tag{3.54}$$

and by choosing $h = \lambda e_i$ with e_i the i^{th} basis vector and $\lambda \to 0_+$, we obtain the TVD property

$$\frac{d}{dt} \iint \left| \frac{\partial f}{\partial x_i}(t,x,\xi) \right| d\xi dx \le 0. \tag{3.55}$$

Example 3.4 The model of [40] is obtained by taking $\Xi = \mathbb{R}$ with the Lebesgue measure, $a(\xi) = F'(\xi)$, and $M(u, \xi) = \chi(u, \xi)$,

$$\begin{aligned}\chi(u,\xi) &\equiv \frac{1}{2}\Big[\mathrm{sgn}(u-\xi) + \mathrm{sgn}(\xi)\Big] \\ &= \begin{cases} 1 & \text{if } \; 0 < \xi < u, \\ -1 & \text{if } \; u < \xi < 0, \\ 0 & \text{otherwise.} \end{cases}\end{aligned} \tag{3.56}$$

We have then $M'(u, \xi) = \delta(\xi - u) \geq 0$. Here M is not C^1, but it is possible to check directly (E0), (E1), (E2) and (2.27), with $D_\xi = [0, \mathrm{sgn}\,\xi]$ and H_η given by (3.48), which gives

$$H_\eta(f, \xi) = \eta'(\xi) f. \tag{3.57}$$

Indeed, our approach provides with Proposition 2.3 a new proof of Brenier's lemma [7], which is exactly (E2). We only need to prove (2.28), which writes $(\eta'(u) - \eta'(\xi))(\chi(u, \xi) - f) \geq 0$, and since it is linear in f, it is enough to consider the endpoints $f = 0$ or $f = \mathrm{sgn}\,\xi$, and both are obviously satisfied since η' is nondecreasing.

The previous model naturally converges to the kinetic formulation of [26] when $\varepsilon \to 0$. However, in the case of a general maxwellian, we have all entropy inequalities, and thus we have an approximate kinetic formulation, that can be obtained as follows. Let us define $m(t, x, v)$ by

$$-2m(t,x,v) = \int \Big\{ \partial_t \Big[H_v(f(t,x,\xi),\xi) \Big] + a(\xi) \cdot \nabla_x \Big[H_v(f(t,x,\xi),\xi) \Big] \Big\} d\xi. \tag{3.58}$$

Then $m \geq 0$, and

$$\partial_t \left[-\partial_v \int H_v(f(t,x,\xi),\xi)\, d\xi \right] + \mathrm{div}_x \left[-\partial_v \int a(\xi) H_v(f(t,x,\xi),\xi)\, d\xi \right] = 2\partial_v m. \tag{3.59}$$

Since

$$\partial_v [H_v(f,\xi)] = -\Big[\mathrm{sgn}(f - M(v,\xi)) + \mathrm{sgn}(M(v,\xi))\Big] M'(v,\xi), \tag{3.60}$$

we obtain

$$\begin{aligned} &\partial_t \int \frac{1}{2}\Big[\mathrm{sgn}(f(t,x,\xi) - M(v,\xi)) + \mathrm{sgn}\, v\Big] M'(v,\xi)\, d\xi \\ &\quad + \mathrm{div}_x \int a(\xi) \frac{1}{2}\Big[\mathrm{sgn}(f(t,x,\xi) - M(v,\xi)) + \mathrm{sgn}\, v\Big] M'(v,\xi)\, d\xi = \partial_v m. \end{aligned} \tag{3.61}$$

When f is a maxwellian $f(t, x, \xi) = M(u(t, x), \xi)$ in (2.61), we recover the kinetic formulation

$$\partial_t \Big[\chi(u(t,x),v)\Big] + \mathrm{div}_x \Big[F'(v)\chi(u(t,x),v)\Big] = \partial_v m(t,x,v). \tag{3.62}$$

4 Isentropic gas dynamics

In this section we apply our general method to find BGK models having many kinetic entropies to the one-dimensional system of isentropic gas dynamics, in either Euler or Lagrange coordinates. In each case we obtain a model satisfying all entropy inequalities. The model is global for the Euler coordinates, while in the Lagrange case it is only defined on a bounded positively invariant region. Both systems can be handled by the following result which is valid for square systems that are "symmetric" in the sense defined below.

Theorem 4.1 *Let us consider a square system (2.1) with $p = N+1$, and let us denote $u = (u_0, \dots, u_N)$ and $F_0(u) = u$. If the system if symmetric in the sense that*

$$F_j^{(i)} = F_i^{(j)}, \qquad 0 \le i, j \le N, \tag{4.1}$$

where $F_j^{(i)}$ denotes the i^{th} component of F_j, then whatever is the family $\mathcal{E}$, the space $\mathcal{M}^{\mathcal{E}}$ contains all functions

$$M = (M_0, \dots, M_N) : \mathcal{U} \longrightarrow \mathbb{R}^{N+1} \tag{4.2}$$

such that M_0 is an entropy of (2.1) and $(M_1, \dots, M_N)$ is its associated entropy flux.

Proof. Let $\eta_0, \eta_1, \dots, \eta_N$ a system entropy, entropy-flux. This means that we have the relations

$$\eta_j' = \eta_0' F_j', \qquad 0 \le j \le N, \tag{4.3}$$

and the property for η_0 to be an entropy writes

$$\frac{\partial}{\partial u_l}\Big(\eta_0' F_j'\Big)^{(m)} = \frac{\partial}{\partial u_m}\Big(\eta_0' F_j'\Big)^{(l)}, \qquad 0 \le l, m \le N, \tag{4.4}$$

and this must hold for $0 \le j \le N$. More explicitly, it is equivalent to the property that

$$\sum_{n=0}^{N} \frac{\partial^2 \eta_0}{\partial u_l \partial u_n} \frac{\partial F_j^{(n)}}{\partial u_m} \qquad \text{is symmetric in } l, m. \tag{4.5}$$

Let now $M_0, M_1, \dots, M_N$ be another system entropy entropy-flux. We have

$$\begin{aligned} & \sum_{n=0}^{N} \frac{\partial^2 \eta_0}{\partial u_l \partial u_n} \frac{\partial M_n}{\partial u_m} \\ &= \sum_{n=0}^{N} \frac{\partial^2 \eta_0}{\partial u_l \partial u_n} \Big(M_0' F_n'\Big)^{(m)} \\ &= \sum_{n=0}^{N} \frac{\partial^2 \eta_0}{\partial u_l \partial u_n} \left(\sum_{j=0}^{N} \frac{\partial M_0}{\partial u_j} \frac{\partial F_n^{(j)}}{\partial u_m} \right) \\ &= \sum_{j=0}^{N} \frac{\partial M_0}{\partial u_j} \left(\sum_{n=0}^{N} \frac{\partial^2 \eta_0}{\partial u_l \partial u_n} \frac{\partial F_n^{(j)}}{\partial u_m} \right), \end{aligned} \tag{4.6}$$

and since by (4.1) and (4.5) the last expression is symmetric in l, m, we get that $(M')^t \eta''$ is symmetric. $\square$

Under the assumption of symmetry, the first step of our method described in Section 2.2 is now achieved, it is enough to take for $\mathcal{E}$ the set of all convex entropies, and for the subspace of $\mathcal{M}^{\mathcal{E}}$ all systems entropy, entropy-flux (as far as they are known explicitly). Notice that by symmetry, this space contains Id, $F_1, \ldots, F_N$.

4.1 Euler coordinates

The one-dimensional system of isentropic gas dynamics in Euler coordinates can be written

$$\begin{aligned} &\partial_t \rho + \partial_x(\rho u) = 0, \\ &\partial_t(\rho u) + \partial_x(\rho u^2 + \kappa \rho^\gamma) = 0, \end{aligned} \tag{4.7}$$

with $\rho \geq 0$, $u \in \mathbb{R}$, and $\kappa > 0$, $\gamma > 1$ are constants. The eigenvalues of the system are

$$\lambda_1 = u - \sqrt{\gamma \kappa \rho^{\gamma-1}}, \qquad \lambda_2 = u + \sqrt{\gamma \kappa \rho^{\gamma-1}}, \tag{4.8}$$

and the (strong) Riemann invariants $w_1 < w_2$ are given by

$$w_1 = u - \frac{2}{\gamma - 1} \sqrt{\gamma \kappa} \rho^{(\gamma-1)/2}, \qquad w_2 = u + \frac{2}{\gamma - 1} \sqrt{\gamma \kappa} \rho^{(\gamma-1)/2}. \tag{4.9}$$

Thus the system can be written under nonconservative form

$$\begin{aligned} &\partial_t w_1 + \lambda_1 \partial_x w_1 = 0, \\ &\partial_t w_2 + \lambda_2 \partial_x w_2 = 0. \end{aligned} \tag{4.10}$$

Since the system is symmetric in the sense of Theorem 4.1, let us take for $\mathcal{E}$ the family of all convex entropies. Then it contains at least one that is strictly convex, the physical energy

$$\eta = \rho u^2/2 + \frac{\kappa}{\gamma - 1} \rho^\gamma, \tag{4.11}$$

which entropy-flux is given by $G = (\eta + \kappa \rho^\gamma) u$. As suggested by Theorem 4.1, we take $M = (M_0, M_1)$ with M_0 an entropy and M_1 the associated entropy-flux. Actually, it is possible here by using a general approach which is detailed in Section 5 to prove that $\mathcal{M}^{\mathcal{E}}$ exactly coincides with the space of such couples. Moreover, it is still true if we take for $\mathcal{E}$ the set of only the weak entropies that are considered in [27].

For the characterization of $\mathcal{M}^{\mathcal{E}}_+$, we have the following result.

Proposition 4.2 *The couple entropy, entropy-flux (M_0, M_1) belongs to $\mathcal{M}^{\mathcal{E}}_+$ if and only if*

$$\frac{\partial M_0}{\partial w_2} \geq 0 \qquad \textit{and} \qquad \frac{\partial M_0}{\partial w_1} \leq 0. \tag{4.12}$$

Proof. We take w_1, w_2 as new variables. By (2.6), the property for (M_0, M_1) to be a couple entropy, entropy-flux writes

$$\frac{\partial M_1}{\partial w_2} = \lambda_2 \frac{\partial M_0}{\partial w_2}, \qquad \frac{\partial M_1}{\partial w_1} = \lambda_1 \frac{\partial M_0}{\partial w_1}. \tag{4.13}$$

Thanks to Proposition 2.2, we only have to write that $\sigma(M') \subset [0, \infty[$. Let us denote

$$B \equiv \partial_{\rho,\rho u}(w_1, w_2) = \frac{1}{\rho} \begin{pmatrix} -\lambda_2 & 1 \\ -\lambda_1 & 1 \end{pmatrix}. \tag{4.14}$$

Then

$$\begin{aligned} BM'B^{-1} &= B \begin{pmatrix} \partial M_0/\partial w_1 & \partial M_0/\partial w_2 \\ \partial M_1/\partial w_1 & \partial M_1/\partial w_2 \end{pmatrix} \\ &= \frac{1}{\rho} \begin{pmatrix} \partial M_1/\partial w_1 - \lambda_2 \partial M_0/\partial w_1 & \partial M_1/\partial w_2 - \lambda_2 \partial M_0/\partial w_2 \\ \partial M_1/\partial w_1 - \lambda_1 \partial M_0/\partial w_1 & \partial M_1/\partial w_2 - \lambda_1 \partial M_0/\partial w_2 \end{pmatrix} \\ &= \frac{\lambda_2 - \lambda_1}{\rho} \begin{pmatrix} -\partial M_0/\partial w_1 & 0 \\ 0 & \partial M_0/\partial w_2 \end{pmatrix}, \end{aligned} \tag{4.15}$$

and the result follows. $\square$

Remark 4.1 We have diagonalized M' in a basis which is independent of the choice of (M_0, M_1). In particular, it diagonalizes F'. Moreover, w_1, w_2 are common Riemann invariants for all systems $\partial_t u + \partial_x[M(u)] = 0$.

Now, in order to build a BGK model which is compatible with all convex entropies, it is enough to find $(M_0(\rho, u, \xi), M_1(\rho, u, \xi))$ couple entropy, entropy-flux for each ξ, satisfying (4.12) and such that the moment equations (M0), (M1) are verified. Moreover, a sufficient condition to preserve the positiveness of ρ is that the first component k_0 of k in (M0) vanishes and

$$M_0(\rho, u, \xi) \geq 0, \tag{4.16}$$

because then we can choose $D_\xi \subset \{(f_0, f_1); f_0 \geq 0\}$, and thus (2.9) is satisfied. Moreover, the commutation of M' and F' (see Remark 4.1) ensures that $M(\mathcal{U}, \xi)$ is convex (see [43]).

Let us now assume that $\gamma \leq 3$. Then we have a solution to all these constraints, which is given by the fundamental solution of the weak entropies used in [27] (see also [12]). We take $\Xi = \mathbb{R}$ with Lebesgue measure, $a(\xi) = \xi$ and

$$M_0(\rho, u, \xi) = \chi(\rho, \xi - u), \qquad M_1(\rho, u, \xi) = \Big[(1-\theta)u + \theta\xi\Big]\chi(\rho, \xi - u), \tag{4.17}$$

with

$$\chi(\rho, \xi) = c_{\gamma,\kappa} \left(\frac{4\gamma\kappa}{(\gamma-1)^2} \rho^{\gamma-1} - \xi^2 \right)_+^\lambda, \tag{4.18}$$

and

$$\theta = \frac{\gamma-1}{2}, \qquad \lambda = \frac{1}{\gamma-1} - \frac{1}{2}, \qquad c_{\gamma,\kappa} = \left[\frac{4\gamma\kappa}{(\gamma-1)^2}\right]^{-1/(\gamma-1)} / J_\lambda,$$
$$J_\lambda = \int_{-1}^{1} (1-z^2)^\lambda dz = \pi^{1/2}\Gamma(\lambda+1)/\Gamma(\lambda+3/2). \tag{4.19}$$

We have the moments

$$\int M_0\, d\xi = \rho, \qquad \int M_1\, d\xi = \rho u,$$
$$\int \xi M_0\, d\xi = \rho u, \qquad \int \xi M_1\, d\xi = \rho u^2 + \kappa\rho^\gamma, \tag{4.20}$$

so that (M0), (M1) are satisfied, and since

$$M_0(\rho, u, \xi) = c_{\gamma,\kappa}\left[(w_2-\xi)(\xi-w_1)\right]_+^\lambda, \tag{4.21}$$

the monotonicity conditions (4.12) are fulfilled provided that $\gamma \leq 3$. On the contrary, if $\gamma > 3$, M_0 is not monotone with respect to w_1 or w_2, and this construction does not work. One can also obtain the same conclusion by computing the diffusion matrix D of the Chapman-Enskog expansion defined in (2.94), which is

$$D(\rho, u) = \frac{3-\gamma}{\gamma-1}\gamma\kappa\rho^{\gamma-1}\,\mathrm{Id}\,. \tag{4.22}$$

Notice also that if $\gamma = 3$, we have $M_1 = \xi M_0$, and the model is the one of [10], which has already be written down in Section 3.1.3, with the domain $D_\xi = \left\{(f_0, \xi f_0); 0 \leq f_0 \leq (2\sqrt{3\kappa})^{-1}\right\}$.

Let us now express the kinetic entropies for this model if $\gamma < 3$. We are in the full-rank case of Remark 2.8. Actually

$$D_\xi = \left\{(f_0, f_1) \in \mathbb{R}^2\ ;\ f_0 > 0 \text{ or } f_0 = f_1 = 0\right\}, \tag{4.23}$$

and the relation $(f_0, f_1) = (M_0(\rho, u, \xi), M_1(\rho, u, \xi))$ can be inverted at fixed ξ, the solution is (if $f_0 > 0$)

$$u(f,\xi) = \frac{f_1/f_0 - \theta\xi}{1-\theta},$$
$$\rho(f,\xi) = \left[\frac{4\gamma\kappa}{(\gamma-1)^2}\right]^{-1/(\gamma-1)} \left(\left(\frac{f_1/f_0-\xi}{1-\theta}\right)^2 + \left(f_0/c_{\gamma,\kappa}\right)^{1/\lambda}\right)^{1/(\gamma-1)}. \tag{4.24}$$

The kinetic entropy is given by

$$H_\eta(f,\xi) = G_\eta(\rho(f,\xi), u(f,\xi), \xi), \tag{4.25}$$

and it only remains to compute the microscopic entropy $G_\eta(\rho, u, \xi)$, by $G_\eta' = \eta' M'$. Let us give the result for the weak entropies η_S defined by

$$\eta_S(\rho, u) = \int \chi(\rho, v-u) S(v)\, dv, \tag{4.26}$$

where $S:\mathbb{R}\to\mathbb{R}$ is an arbitrary convex function. We write $H_S \equiv H_{\eta_S}$, $G_S \equiv G_{\eta_S}$ to simplify. Since η_S is defined through a kernel, it is also true for G_S, and we have

$$G_S(\rho,u,\xi) = \int \Phi(\rho,u,\xi,v)S(v)\,dv, \tag{4.27}$$

with Φ defined via the Riemann invariants (4.9) by

$$\begin{aligned}\Phi(\rho,u,\xi,v) &= \frac{(1-\theta)^2}{\theta}\frac{c_{\gamma,\kappa}}{J_\lambda}\mathbb{I}_{w_1<\xi<w_2}\mathbb{I}_{w_1<v<w_2}|\xi-v|^{2\lambda-1}\Upsilon_{\lambda-1}\Big(z(w_1,w_2,\xi,v)\Big),\\ z(w_1,w_2,\xi,v) &= \frac{(\xi+v)(w_1+w_2)-2(w_1w_2+\xi v)}{(w_2-w_1)|\xi-v|},\\ \Upsilon_{\lambda-1}(z) &= \int_1^z (y^2-1)^{\lambda-1}dy, \qquad z\geq 1.\end{aligned} \tag{4.28}$$

The relation $G_S' = \eta_S' M'$ can be obtained from (4.27), (4.28) by using the identity

$$\begin{aligned}4(w_2-\xi)(\xi-w_1)(w_2-v)(v-w_1)+(w_2-w_1)^2(\xi-v)^2\\ =\left[(\xi+v)(w_1+w_2)-2(w_1w_2+\xi v)\right]^2.\end{aligned} \tag{4.29}$$

Finally, H_S is given by

$$H_S(f,\xi) = \int \Psi(f,\xi,v)S(v)\,dv, \tag{4.30}$$

where $\Psi(f,\xi,v)=\Phi(\rho(f,\xi),u(f,\xi),\xi,v)$ and $\rho(f,\xi)$ and $u(f,\xi)$ are given by (4.24), or equivalently $\Phi(\rho,u,\xi,v)=\Psi(M(\rho,u,\xi),\xi,v)$.

We notice that Φ is nonnegative, symmetric in ξ,v

$$\Phi(\rho,u,\xi,v)=\Phi(\rho,u,v,\xi), \tag{4.31}$$

and that

$$\int \Phi(\rho,u,\xi,v)\,dv = M_0(\rho,u,\xi), \qquad \int v\Phi(\rho,u,\xi,v)\,dv = M_1(\rho,u,\xi). \tag{4.32}$$

For example, for the physical energy (4.11), which is obtained for $S(v)=v^2/2$, the associated kinetic entropy is

$$H_{\mathrm{Id}^2/2}(f,\xi) = \frac{\theta}{1-\theta}\frac{\xi^2}{2}f_0 + \frac{\theta}{2c_{\gamma,\kappa}^{1/\lambda}}\frac{f_0^{1/\lambda+1}}{1/\lambda+1} + \frac{1}{1-\theta}\frac{1}{2}f_1^2/f_0 - \frac{\theta}{1-\theta}\xi f_1. \tag{4.33}$$

On the contrary, in the case $\gamma=3$, we have a one-rank model, $M(.,\xi)$ is not invertible, but we have directly

$$H_S(f,\xi)=S(\xi)f_0, \tag{4.34}$$

with $f=(f_0,\xi f_0)\in D_\xi=(1,\xi)[0,(2\sqrt{3}\kappa)^{-1}]$, and η_S is still given by (4.26) for any convex S. Formula (4.30) is still valid with $\Psi(f,\xi,v)=\delta(\xi-v)f_0$, and

(4.27) is true with $\Phi(\rho, u, \xi, v) = \Psi(M(\rho, u, \xi), \xi, v) = \delta(\xi - v)\chi(\rho, u, \xi)$.

Approximate kinetic formulation

Let us again assume that $\gamma < 3$. When $\varepsilon \to 0$, the above BGK model does not tend to the kinetic formulation of [27] because there the kinetic velocity is not purely kinetic. However, the property of having a complete set of kinetic entropies enables, as in the scalar case, to have an approximate equation which tends to the kinetic formulation. In order to obtain it, let us write down the equation satisfied by $\Psi(f(t,x,\xi), \xi, v)$ at fixed v,

$$\partial_t[\Psi(f, \xi, v)] + \xi\partial_x[\Psi(f, \xi, v)] = \Psi'(f, \xi, v)\frac{M_f - f}{\varepsilon}. \tag{4.35}$$

Integrating with respect to ξ we get

$$\partial_t \int \Psi(f, \xi, v)\, d\xi + \partial_x \int \xi\Psi(f, \xi, v)\, d\xi = \int \Psi'(f, \xi, v)\frac{M_f - f}{\varepsilon} d\xi \equiv R_\varepsilon(t, x, v). \tag{4.36}$$

We have for any convex S

$$\begin{aligned}\int R_\varepsilon(t, x, v)S(v)\, dv &= \iint \Psi'(f, \xi, v)\frac{M_f - f}{\varepsilon} S(v)\, dv d\xi \\ &= \int H_S'(f, \xi)\frac{M_f - f}{\varepsilon} d\xi \\ &\le \int \frac{H_S(M_f, \xi) - H_S(f, \xi)}{\varepsilon} d\xi \\ &\le 0,\end{aligned} \tag{4.37}$$

thus

$$R_\varepsilon(t, x, v) = -\partial^2_{vv} m_\varepsilon(t, x, v), \qquad m_\varepsilon \ge 0, \tag{4.38}$$

and (4.36), (4.38) is an approximate kinetic formulation. When $\varepsilon \to 0$, f becomes a maxwellian in (4.36), and we obtain with $f = M(\rho, u, \xi)$

$$\begin{aligned}\Psi(f, \xi, v) = \Psi(M(\rho, u, \xi), \xi, v) &= \Phi(\rho, u, \xi, v), \\ \int \Psi(f, \xi, v)\, d\xi = \int \Phi(\rho, u, \xi, v)\, d\xi &= \chi(\rho, u, v), \\ \int \xi\Psi(f, \xi, v)\, d\xi = \int \xi\Phi(\rho, u, \xi, v)\, d\xi &= [(1-\theta)u + \theta v]\chi(\rho, u, v),\end{aligned} \tag{4.39}$$

and (4.36) becomes

$$\partial_t[\chi(\rho, u, v)] + \partial_x\Big\{[(1-\theta)u + \theta v]\chi(\rho, u, v)\Big\} = -\partial^2_{vv} m, \qquad m \ge 0. \tag{4.40}$$

Remark 4.2 The kinetic formulation only takes into account the weak entropies, while the BGK model is compatible with all entropies.

Maximum principle

The system (4.7) has the property to verify the maximum principle on the Riemann invariants, which is

$$w_{min} \le w_1 \le w_2 \le w_{max} \tag{4.41}$$

with w_1, w_2 defined by (4.9) and $w_{min} < w_{max}$ given such that (4.41) holds true initially.

The above BGK model (with $\gamma \le 3$), has a kinetic counterpart of it, which can be obtained as follows. Let us first define the domain

$$\widetilde{D}_\xi = \Big\{ f \in D_\xi \ ; \ w_{min} \le w_1(f,\xi) \le w_2(f,\xi) \le w_{max} \Big\}, \tag{4.42}$$

where $w_1(f,\xi)$ and $w_2(f,\xi)$ are defined by (4.9) and by replacing ρ, u by $\rho(f,\xi)$ and $u(f,\xi)$ defined in (4.24). A different formula has to be taken for $\gamma = 3$, $\widetilde{D}_\xi = \{ f \in D_\xi \ ; \ 0 \le f_0(\xi) \le \chi(w_{min}, w_{max}, \xi) \}$. It is possible to check that $\widetilde{D}_\xi$ is convex (this can also be seen by Remark 4.1 and by the results of [43]), and thus the property $f(t,x,\xi) \in \widetilde{D}_\xi$ is preserved if it is satisfied initially. Thus we can conclude and obtain the bounds on w_1 and w_2 with the following lemma.

Lemma 4.3 *Let $f(\xi) = (f_0(\xi), f_1(\xi))$, $\xi \in \mathbb{R}$ be a function satisfying*

$$a.e.\ \xi \qquad f(\xi) \in \widetilde{D}_\xi. \tag{4.43}$$

Then w_1, w_2 defined by (4.9) with

$$\rho = \int f_0(\xi)\, d\xi, \qquad \rho u = \int f_1(\xi)\, d\xi \tag{4.44}$$

satisfy (4.41).

Proof. We have by (E2) for any convex S

$$\eta_S(\rho, u) \le \int H_S(f(\xi), \xi)\, d\xi. \tag{4.45}$$

But thanks to (4.43), $\mathrm{supp}_{\xi,v} \Psi(f(\xi), \xi, v) \subset [w_{min}, w_{max}]^2$, and from (4.30), $H_S(f(\xi), \xi)$ vanishes if $\xi \notin [w_{min}, w_{max}]$ or if S is identically 0 in $[w_{min}, w_{max}]$. From (4.45) and since $\eta_S \ge 0$, $\eta_S(\rho, u)$ vanishes as soon as S vanishes in $[w_{min}, w_{max}]$. But thanks to (4.26), this implies that $\mathrm{supp}_v \chi(\rho, v-u) \subset [w_{min}, w_{max}]$, and thus (4.41) holds. □

Remark 4.3 The flux splitting (2.74), (2.75) associated to the model presented here has already be obtained in [11].

4.2 Lagrange coordinates

Let us now consider the one-dimensional system of isentropic gas dynamics in Lagrange coordinates, or p-system,

$$\begin{aligned} &\partial_\tau \frac{1}{\rho} - \partial_y u = 0, \\ &\partial_\tau u + \partial_y (\kappa \rho^\gamma) = 0, \end{aligned} \tag{4.46}$$

with $\rho > 0$, $u \in \mathbb{R}$, and $\kappa > 0$, $\gamma > 1$ are constants. It can be obtained from (4.7) by a change of variables. Indeed the first equation in (4.7) ensures that there exists a function $m(t,x)$ satisfying

$$\partial_x m = \rho, \qquad \partial_t m = -\rho u, \tag{4.47}$$

and by setting

$$\tau = t, \qquad y = m(t,x), \tag{4.48}$$

we have the relations

$$\partial_\tau = \partial_t + u\partial_x, \qquad \partial_y = \frac{1}{\rho}\partial_x, \tag{4.49}$$

and we obtain (4.46). This change of variables also enables to obtain the entropies of (4.46), which are given by

$$E = \frac{\eta}{\rho}, \tag{4.50}$$

with η an entropy of (4.7), and its entropy flux is $G - \eta u$, with G the entropy flux of η. Moreover, the convexity of E in the variables $(1/\rho, u)$ is equivalent to the convexity of η in $(\rho, \rho u)$. The (strong) Riemann invariants w_1, w_2 for (4.46) are still given by (4.9), and the eigenvalues are now $\mp c(\rho)$, with

$$c(\rho) = \sqrt{\gamma\kappa}\rho^{(\gamma+1)/2}. \tag{4.51}$$

We again take for $\mathcal{E}$ the family of all convex entropies, and since the system is "symmetric" in the sense of Theorem 4.1 (replace the second equation by its opposite), we take $M = (M_0, M_1)$ with M_0 an entropy for (4.46) and M_1 the opposite of its entropy-flux.

Proposition 4.4 *The couples entropy, opposite of entropy-flux (M_0, M_1) belongs to $\mathcal{M}_+^{\mathcal{E}}$ if and only if*

$$\frac{\partial M_0}{\partial w_2} \leq 0 \qquad \textit{and} \qquad \frac{\partial M_0}{\partial w_1} \geq 0. \tag{4.52}$$

Proof. Since M_1 is the opposite of the entropy-flux of M_0, we have

$$\frac{\partial M_1}{\partial w_2} = -c(\rho)\frac{\partial M_0}{\partial w_2}, \qquad \frac{\partial M_1}{\partial w_1} = c(\rho)\frac{\partial M_0}{\partial w_1}. \tag{4.53}$$

By defining

$$B \equiv \partial_{1/\rho, u}(w_1, w_2) = \begin{pmatrix} c(\rho) & 1 \\ -c(\rho) & 1 \end{pmatrix}, \tag{4.54}$$

we get

$$\begin{aligned} BM'B^{-1} &= B\begin{pmatrix} \partial M_0/\partial w_1 & \partial M_0/\partial w_2 \\ \partial M_1/\partial w_1 & \partial M_1/\partial w_2 \end{pmatrix} \\ &= \begin{pmatrix} \partial M_1/\partial w_1 + c(\rho)\partial M_0/\partial w_1 & \partial M_1/\partial w_2 + c(\rho)\partial M_0/\partial w_2 \\ \partial M_1/\partial w_1 - c(\rho)\partial M_0/\partial w_1 & \partial M_1/\partial w_2 - c(\rho)\partial M_0/\partial w_2 \end{pmatrix} \\ &= 2c(\rho)\begin{pmatrix} \partial M_0/\partial w_1 & 0 \\ 0 & -\partial M_0/\partial w_2 \end{pmatrix}, \end{aligned} \tag{4.55}$$

and we conclude by Proposition 2.2. $\square$

Now, if $\gamma \leq 3$, we can build a BGK model by taking $\Xi = \mathbb{R}$ with the Lebesgue measure, $a(\xi) = \rho_m \xi$, for some constant $\rho_m > 0$, and

$$\begin{aligned} M_0(\rho, u, \xi) &= \frac{1}{\rho_m \rho}\Big[\chi(\rho_m, 0, \xi) - \chi(\rho, u, \xi)\Big], \\ M_1(\rho, u, \xi) &= \chi(\rho_m, 0, \xi)\frac{u}{\rho_m} + \frac{\theta}{\rho_m}(\xi - u)\chi(\rho, u, \xi), \end{aligned} \tag{4.56}$$

and χ is still defined by (4.18), (4.19). Then, from the characterization (4.50), M_1 is the opposite of the entropy-flux of M_0, and (M0), (M1) are satisfied since

$$\begin{aligned} &\int M_0 \, d\xi = \frac{1}{\rho} - \frac{1}{\rho_m}, \qquad &&\int M_1 \, d\xi = u, \\ &\int \rho_m \xi M_0 \, d\xi = -u, \qquad &&\int \rho_m \xi M_1 \, d\xi = \kappa \rho^\gamma. \end{aligned} \tag{4.57}$$

It is easy to check that the stability conditions (4.52) are satisfied provided that

$$-\frac{2}{\gamma - 1}\sqrt{\gamma\kappa}\rho_m^{(\gamma-1)/2} \leq w_1 \leq w_2 \leq \frac{2}{\gamma - 1}\sqrt{\gamma\kappa}\rho_m^{(\gamma-1)/2}. \tag{4.58}$$

The big difference between this model and that provided in Section 4.1 is that we have now a parameter $\rho_m > 0$, and that the model is only valid in the domain defined by (4.58). Therefore, D_ξ is a non-trivial convex set; one of its extremal points is $(0, \theta(\rho_m)^{-1}\xi\chi(\rho_m, 0, \xi))$. However, this model satisfies all the good properties of the preceding one. In particular, the maximum principle on Riemann invariants ensures that (4.58) remains true, and the a priori estimate (2.9) is satisfied (here $\mathcal{U}$ is defined by $\rho > 0$ and (4.58)). For the weak entropies $E_S = \eta_S/\rho$, with η_S defined in (4.26) and S any convex function, the microscopic entropy $G_S(\rho, u, \xi)$ is now given by

$$\begin{aligned} &G_S(\rho, u, \xi) \\ &= \theta\frac{c_{\gamma,\kappa}}{J_\lambda}\frac{1}{\rho_m}\int S(v)\frac{\partial^2}{\partial\xi\partial v}\Big\{\mathbb{I}_{w_1<\xi<w_2}\mathbb{I}_{w_1<v<w_2}|\xi - v|^{2\lambda+1}\Upsilon_\lambda(z(w_1, w_2, \xi, v))\Big\}dv \\ &\qquad\qquad\qquad\qquad + E_S(\rho, u)\frac{\chi(\rho_m, 0, \xi)}{\rho_m}. \end{aligned} \tag{4.59}$$

Example 4.1 If $\gamma = 3$, (4.59) gives

$$\begin{aligned} G_S(\rho, u, \xi) = \frac{1}{2\sqrt{3\kappa}\rho_m}\Bigg[&\frac{1}{w_2 - w_1}\left(\int_{w_1}^{w_2} S\right)\left(\mathbb{I}_{|\xi|<\sqrt{3\kappa}\rho_m} - \mathbb{I}_{w_1<\xi<w_2}\right) \\ &+ S(\xi)\mathbb{I}_{w_1<\xi<w_2}\Bigg]. \end{aligned} \tag{4.60}$$

In this case, the model is full-rank, but condition (CH2) in Theorem 2.1 is not satisfied.

Remark 4.4 There is no change of variables that transforms the BGK model written above into the one of Section 4.1.

5 Full system of gas dynamics

In this section we apply our method to the full Euler system of perfect polytropic gas dynamics, in one dimension to simplify, but this could be done also in multidimension. We obtain a large class of models having a complete set of kinetic entropies. Unfortunately, these models are defined on domains that are not positively invariant. Thus they are inappropriate for theoretical studies, because depending on initial data, the solution could leave the domain in finite time. However, these models can be used in numerical methods, because in this situation it is possible to adapt the parameters of the model to the computed solution. Indeed in this context it is possible to choose the parameters of the model at each cell interface where a numerical flux has to be computed, as is done in [15], and moreover, this technique has the advantage of being able to minimize the numerical diffusion.

The system can be written

$$\begin{aligned}
&\partial_t\rho + \partial_x(\rho u) = 0,\\
&\partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0,\\
&\partial_t E + \partial_x(Eu + pu) = 0,
\end{aligned} \tag{5.1}$$

with $\rho \geq 0$, $u \in \mathbb{R}$, $p \geq 0$ and $E = \rho u^2/2 + p/(\gamma - 1)$, where $\gamma > 1$. Let us first give a criterion which ensures that ρ and p remain nonnegative. It generalizes the argument used in the classical models of Section 3.1.

Proposition 5.1 *Assume to be given a BGK model for (5.1), in the sense of Section 2.1, such that the constant k of (M0) vanishes and such that the maxwellian $M(\rho, \rho u, E, \xi) = (M_0, M_1, M_2)$ satisfies*

$$\forall\rho \geq 0 \quad \forall u \in \mathbb{R} \quad \forall p \geq 0 \quad a.e.\ \xi \qquad M_0 \geq 0,\ M_2 \geq 0,\ |M_1|^2 \leq 2M_0M_2. \tag{5.2}$$

Then this model preserves the positiveness of ρ and p.

Proof. The set D of all $(f_0, f_1, f_2) \in \mathbb{R}^3$ such that $f_0 \geq 0$, $f_2 \geq 0$ and $|f_1|^2 \leq 2f_0f_2$ is convex (because $|f_1|^2/f_0$ is a convex function of (f_0, f_1)). Thanks to (5.2), we can thus take $D_\xi \subset D$, and (H3) will be satisfied. Therefore, the solution f of the BGK equation will satisfy $f(t, x, \xi) \in D$ at any time, and we conclude with the following lemma.

Lemma 5.2 *Let $f = (f_0, f_1, f_2) : \Xi \to \mathbb{R}^3$ satisfy*

$$a.e.\ \xi \qquad f_0(\xi) \geq 0,\ f_2(\xi) \geq 0,\ |f_1(\xi)|^2 \leq 2f_0(\xi)f_2(\xi), \tag{5.3}$$

and let us define ρ, u, p by

$$\rho = \int f_0 d\xi, \quad \rho u = \int f_1 d\xi, \quad \frac{1}{2}\rho|u|^2 + \frac{p}{\gamma - 1} = \int f_2 d\xi. \tag{5.4}$$

Then $\rho \geq 0$ and $p \geq 0$.

Proof. Of course we have $\rho \geq 0$. If $\rho = 0$, then $f_0 = 0$ a.e., and $f_1 = 0$ a.e. thanks to (5.3). Thus (5.4) gives that $p \geq 0$. Let us now assume that $\rho > 0$. Then by the Cauchy-Schwarz inequality

$$\begin{aligned}
\rho^2|u|^2 &= \left| \int f_1 d\xi \right|^2 \\
&= \left| \int_{f_0>0} \frac{f_1}{\sqrt{f_0}} \sqrt{f_0} d\xi \right|^2 \\
&\leq \int_{f_0>0} \frac{|f_1|^2}{f_0} d\xi \int_{f_0>0} f_0 d\xi \\
&\leq 2 \int f_2 d\xi \int f_0 d\xi \\
&= \rho^2|u|^2 + \frac{2}{\gamma-1}\rho p,
\end{aligned} \tag{5.5}$$

and this yields $p \geq 0$. □

We now introduce the variable

$$w = \frac{p^{1/\gamma}}{\rho}, \tag{5.6}$$

which is a function of the specific entropy. It is well-known that the functions

$$\eta_\phi = \rho\phi(w) \tag{5.7}$$

are entropies of (5.1) for any function ϕ.

Lemma 5.3 *The function η_ϕ is convex with respect to the conservative variables if and only if*

$$\phi' \leq 0 \quad \textit{and} \quad \phi'' \geq 0. \tag{5.8}$$

The proof of this lemma will be provided below, together with the proof of Theorem 5.4. We choose now the family $\mathcal{E}$ of all entropies η_ϕ of the form (5.7) with ϕ satisfying (5.8), we introduce another variable

$$\sigma = p^{1-1/\gamma}, \tag{5.9}$$

and we take σ, u, w as new variables.

Theorem 5.4 *The general solution $M = (M_0, M_1, M_2)$ of the equations (2.20) defining $\mathcal{M}^{\mathcal{E}}$ is given by*

$$\begin{aligned}
M_0(\sigma, u, w) &= \frac{1}{w}\Big[(1-1/\gamma)\partial_\sigma\psi(\sigma, u) + \iota(w)\Big], \\
M_1(\sigma, u, w) &= uM_0(\sigma, u, w) + \partial_u\psi(\sigma, u), \\
M_2(\sigma, u, w) &= \frac{u^2}{2}M_0(\sigma, u, w) + \sigma\partial_\sigma\psi(\sigma, u) + u\partial_u\psi(\sigma, u) - \psi(\sigma, u),
\end{aligned} \tag{5.10}$$

where $\psi(\sigma,u)$ and $\iota(w)$ are arbitrary functions, and the couple (ψ,ι) is defined up to a constant times $(\sigma,-(1-1/\gamma))$. The associated flux G_ϕ defined by $G'_\phi=\eta'_\phi M'$ is given by

$$G_\phi(\sigma,u,w)=M_0(\sigma,u,w)\phi(w)+\Upsilon(w),\qquad \Upsilon'(w)=-\phi'(w)\iota'(w). \tag{5.11}$$

The identity Id *is obtained for $\psi(\sigma,u)=\sigma^{\gamma/(\gamma-1)}$, $\iota(w)=0$, and F is obtained for $\psi(\sigma,u)=\sigma^{\gamma/(\gamma-1)}u$, $\iota(w)=0$. Moreover, $M\in\mathcal{M}^{\mathcal{E}}_+$ if and only if*

$$\partial_w M_0\le 0,\quad \partial^2_{\sigma\sigma}\psi\ge 0,\quad \partial^2_{uu}\psi+M_0\ge 0,\quad \Big(\partial^2_{\sigma u}\psi\Big)^2\le\partial^2_{\sigma\sigma}\psi\left(\partial^2_{uu}\psi+M_0\right). \tag{5.12}$$

The conditions (5.2) are satisfied if and only if

$$M_0\ge 0,\quad \sigma\partial_\sigma\psi-\psi\ge 0,\quad (\partial_u\psi)^2\le 2M_0(\sigma\partial_\sigma\psi-\psi). \tag{5.13}$$

Proof. Since here we have no symmetry such as in Section 4, we are going to use a very general method to obtain $\mathcal{M}^{\mathcal{E}}$. We have to write that the bilinear forms $(M')^t\eta''_\phi$ are symmetric (see Appendix) for all $\phi\in\mathcal{E}$ (respectively symmetric nonnegative for $\mathcal{M}^{\mathcal{E}}_+$). We write the matrix of this bilinear form in the basis corresponding to the variable $v=(\sigma,u,w)$,

$$\begin{aligned}\mathrm{matrix}\Big((M')^t\eta''_\phi\Big)&=\Big(\partial_v M\Big)^t\Big(\partial_v\eta'_\phi\Big)\\&=\partial_v\Big(\eta'_\phi\partial_v M\Big)-\eta'_\phi\partial^2_{vv}M.\end{aligned}\tag{5.14}$$

This formula is actually true for any change of variables, and in order to compute the last expression, we just need to differentiate with respect to v the product $\eta'_\phi\partial_v M$, and to drop the second order derivatives of M. We have to recall that prime denotes differentiation with respect to the conservative variables. Here one can check that

$$\eta'_\phi=\phi(w)(1,0,0)+\phi'(w)\Big\{-w(1,0,0)+(1-\frac{1}{\gamma})\frac{1}{\sigma}(u^2/2,-u,1)\Big\}. \tag{5.15}$$

Therefore,

$$\begin{aligned}\eta'_\phi\partial_v M=\phi(w)\partial_v M_0+\phi'(w)\bigg\{&-w\partial_v M_0\\&+(1-\frac{1}{\gamma})\frac{1}{\sigma}\Big(\frac{u^2}{2}\partial_v M_0-u\partial_v M_1+\partial_v M_2\Big)\bigg\},\end{aligned}\tag{5.16}$$

and by (5.14) we obtain

$$\begin{aligned}(M')^t\eta''_\phi=\phi'(w)(1-\frac{1}{\gamma})\bigg\{&-\Big(\frac{u^2}{2}dM_0-udM_1+dM_2\Big)\otimes\frac{d\sigma}{\sigma^2}\\&+\frac{1}{\sigma}\Big(dM_0\otimes(udu)-dM_1\otimes du\Big)\bigg\}\\+\phi''(w)\bigg\{&-wdM_0+(1-\frac{1}{\gamma})\frac{1}{\sigma}\Big(\frac{u^2}{2}dM_0-udM_1+dM_2\Big)\bigg\}\otimes dw.\end{aligned}\tag{5.17}$$

Now in order to prove Lemma 5.3, we just take $M = \mathrm{Id}$, and we get

$$\begin{aligned}\eta''_\phi = \phi'(w)(1-\frac{1}{\gamma})\Big\{-\frac{\gamma}{(\gamma-1)^2}\sigma^{1/(\gamma-1)}d\sigma\otimes\frac{d\sigma}{\sigma^2}-\frac{1}{\sigma}\frac{\sigma^{1/(\gamma-1)}}{w}du\otimes du\Big\}\\ +\phi''(w)\sigma^{1/(\gamma-1)}\frac{dw}{w}\otimes dw.\end{aligned}\tag{5.18}$$

This gives obviously the result. Then, for Theorem 5.4, we have to write that $(M')^t\eta''_\phi$ in (5.17) is symmetric (respectively symmetric nonnegative) for any ϕ satisfying (5.8). It means that both bilinear forms

$$\begin{aligned}&\Big\{-wdM_0+(1-\frac{1}{\gamma})\frac{1}{\sigma}\left(\frac{u^2}{2}dM_0-udM_1+dM_2\right)\Big\}\otimes dw,\\ &\left(\frac{u^2}{2}dM_0-udM_1+dM_2\right)\otimes\frac{d\sigma}{\sigma}+(dM_1-udM_0)\otimes du,\end{aligned}\tag{5.19}$$

must be symmetric (respectively nonnegative). Let us denote

$$M_3 = M_1 - uM_0, \qquad M_4 = M_2 - uM_1 + \frac{u^2}{2}M_0.\tag{5.20}$$

Then

$$dM_3 = dM_1 - udM_0 - M_0du, \qquad dM_4 = dM_2 - udM_1 + \frac{u^2}{2}dM_0 - M_3du,\tag{5.21}$$

and the bilinear forms of (5.19) can be written

$$\begin{aligned}&\Big\{-wdM_0+(1-\frac{1}{\gamma})\frac{1}{\sigma}(dM_4+M_3du)\Big\}\otimes dw,\\ &(dM_4+M_3du)\otimes\frac{d\sigma}{\sigma}+(dM_3+M_0du)\otimes du.\end{aligned}\tag{5.22}$$

Thus we obtain the symmetry conditions

$$\begin{gathered}-w\partial_\sigma M_0+(1-\frac{1}{\gamma})\frac{1}{\sigma}\partial_\sigma M_4=0,\\ -w\partial_u M_0+(1-\frac{1}{\gamma})\frac{1}{\sigma}(\partial_u M_4+M_3)=0,\\ \partial_w M_4=0, \qquad \partial_w M_3=0,\\ \partial_u M_4+M_3=\sigma\partial_\sigma M_3.\end{gathered}\tag{5.23}$$

This gives that $M_3 = M_3(\sigma, u)$, $M_4 = M_4(\sigma, u)$. Then the last condition gives

$$\partial_\sigma(\partial_\sigma M_3) = \partial_u\left(\frac{1}{\sigma}\partial_\sigma M_4\right),\tag{5.24}$$

thus there exists a function $\varphi(\sigma, u)$ such that

$$\partial_\sigma M_3 = \partial_u\varphi, \qquad \frac{1}{\sigma}\partial_\sigma M_4 = \partial_\sigma\varphi.\tag{5.25}$$

Therefore, there exists $\psi(\sigma, u)$ such that

$$\partial_\sigma \psi = \varphi, \qquad \partial_u \psi = M_3, \tag{5.26}$$

and we obtain

$$\begin{aligned} \partial_\sigma M_4 = \sigma \partial^2_{\sigma\sigma} \psi = \partial_\sigma \left(\sigma \partial_\sigma \psi - \psi \right), \\ \partial_u M_4 = \sigma \partial_\sigma M_3 - M_3 = \sigma \partial^2_{\sigma u} \psi - \partial_u \psi = \partial_u \left(\sigma \partial_\sigma \psi - \psi \right). \end{aligned} \tag{5.27}$$

By eventually adding a constant to ψ, we can assume that $M_4 = \sigma \partial_\sigma \psi - \psi$. Together with (5.26), the relations

$$M_3 = \partial_u \psi, \qquad M_4 = \sigma \partial_\sigma \psi - \psi, \tag{5.28}$$

with $\psi(\sigma, u)$ arbitrary, give the general solution to the two last lines of (5.23). Finally, the two first equations of (5.23) give

$$\partial_\sigma (w M_0) = (1 - 1/\gamma) \partial^2_{\sigma\sigma} \psi, \qquad \partial_u (w M_0) = (1 - 1/\gamma) \partial^2_{\sigma u} \psi, \tag{5.29}$$

which general solution is

$$w M_0 = (1 - 1/\gamma) \partial_\sigma \psi + \iota(w), \tag{5.30}$$

with $\iota(w)$ arbitrary. By using (5.20), we finally obtain (5.10). Concerning positiveness, we have to write that the bilinear forms of (5.22) are nonnegative. The first gives $\partial_w M_0 \leq 0$, while the second gives the matrix

$$\begin{pmatrix} \partial^2_{\sigma\sigma} \psi & \partial^2_{\sigma u} \psi \\ \partial^2_{\sigma u} \psi & \partial^2_{uu} \psi + M_0 \end{pmatrix} \tag{5.31}$$

and the conditions (5.12) follow. The remaining assertions are left to the reader. □

Proposition 5.5 *Let $M(\sigma, u, w, \xi)$ be such that $M(., \xi) \in \mathcal{M}^{\mathcal{E}}$ for a.e. $\xi \in \Xi$, a measure space, with corresponding functions $\psi(\sigma, u, \xi)$ and $\iota(w, \xi)$ obtained by Theorem 5.4. Then the moment equations (M0), (M1) are satisfied with constants $k = (k_0, k_1, k_2)$, $k' = (k'_0, k'_1, k'_2)$ if and only if*

$$\begin{aligned} \int \psi(\sigma, u, \xi) d\xi = \sigma^{\gamma/(\gamma-1)} - k_0 |u|^2/2 + k_1 u - k_2 + \ell \sigma, \\ \int \iota(w, \xi) d\xi = k_0 w - \ell (1 - 1/\gamma), \\ \int a(\xi) \psi(\sigma, u, \xi) d\xi = \sigma^{\gamma/(\gamma-1)} u - k'_0 |u|^2/2 + k'_1 u - k'_2 + \ell' \sigma, \\ \int a(\xi) \iota(w, \xi) d\xi = k'_0 w - \ell' (1 - 1/\gamma), \end{aligned} \tag{5.32}$$

for some constants ℓ and ℓ'.

Proof. Let us denote

$$\Psi(\sigma, u) = \int \psi(\sigma, u, \xi) d\xi, \qquad I(w) = \int \iota(w, \xi) d\xi. \tag{5.33}$$

Then (5.10) gives

$$\begin{aligned}
&\int M_0\,d\xi = \frac{1}{w}\Big[(1-1/\gamma)\partial_\sigma\Psi + I\Big],\\
&\int M_1\,d\xi = u\int M_0\,d\xi + \partial_u\Psi,\\
&\int M_2\,d\xi = \frac{|u|^2}{2}\int M_0\,d\xi + \sigma\partial_\sigma\Psi + u\partial_u\Psi - \Psi,
\end{aligned}\tag{5.34}$$

and (M0) is equivalent to

$$\begin{aligned}
\frac{1}{w}\Big[(1-1/\gamma)\partial_\sigma(\Psi - \sigma^{\gamma/(\gamma-1)}) + I\Big] &= k_0,\\
\partial_u\Psi &= k_1 - k_0 u,\\
\sigma\partial_\sigma(\Psi - \sigma^{\gamma/(\gamma-1)}) + u\partial_u\Psi - (\Psi - \sigma^{\gamma/(\gamma-1)}) &= k_2 - k_0\frac{|u|^2}{2}.
\end{aligned}\tag{5.35}$$

The two last equations give

$$\begin{aligned}
\partial_u(\Psi - \sigma^{\gamma/(\gamma-1)}) &= k_1 - k_0 u,\\
\sigma\partial_\sigma(\Psi - \sigma^{\gamma/(\gamma-1)}) - (\Psi - \sigma^{\gamma/(\gamma-1)}) &= k_2 - k_1 u + k_0\frac{|u|^2}{2},
\end{aligned}\tag{5.36}$$

and the general solution is given by

$$\Psi(\sigma,u) - \sigma^{\gamma/(\gamma-1)} = -k_2 + k_1 u - k_0|u|^2/2 + \ell\sigma,\tag{5.37}$$

with ℓ arbitrary. This yields $I(w) = k_0 w - \ell(1-1/\gamma)$ by the first equation of (5.35). The two last equations in (5.32) are obtained in a similar way. □

The results of Theorem 5.4 and Proposition 5.5 describe all the BGK models having a whole set of kinetic entropies. The only difficult task is then to find one such that the stability conditions (5.12) (and also (5.13), but we have not proved that these conditions were really necessary) are satisfied in a reasonable domain. As announced in the beginning of this section, we have a negative result.

Proposition 5.6 *There is no BGK model compatible with the family $\mathcal{E}$ which is defined on a domain $\mathcal{U}$ of the form $(\sigma,u)\in\Omega$, $w\in I$ with I an interval which is not bounded on the right.*

Proof. Let us assume that such a model exists. The condition $\partial_w M_0 \le 0$ in (5.12) ensures that $M_0(\sigma,u,w,\xi)$ tends to a limit $L(\sigma,u,\xi)$ when $w\to\infty$. Then the constraint $\partial^2_{uu}\psi(\sigma,u,\xi) + M_0(\sigma,u,w,\xi) \ge 0$ in (5.12) implies that $L(\sigma,u,\xi)$ is finite, and that

$$\partial^2_{uu}\psi(\sigma,u,\xi) + L(\sigma,u,\xi) \ge 0.\tag{5.38}$$

But from the first equation in (5.10), we get that

$$\frac{\iota(w,\xi)}{w} \underset{w\to\infty}{\longrightarrow} L(\sigma,u,\xi),\tag{5.39}$$

and thus $L(\sigma, u, \xi) = L(\xi)$. Thanks to the monotonicity of M_0 in w, this yields

$$L(\xi) \le M_0(\sigma, u, w, \xi), \qquad \int L(\xi) d\xi \le \int M_0 d\xi = \frac{p^{1/\gamma}}{w} + k_0, \tag{5.40}$$

and by letting $w \to \infty$,

$$\int L(\xi) d\xi \le k_0. \tag{5.41}$$

Now thanks to the first equation in (5.32), we get $\int \partial^2_{uu}\psi(\sigma, u, \xi) d\xi = -k_0$, thus

$$\int \left(\partial^2_{uu}\psi(\sigma, u, \xi) + L(\xi)\right) d\xi \le 0. \tag{5.42}$$

Together with (5.38) we obtain $\partial^2_{uu}\psi(\sigma, u, \xi) + L(\xi) = 0$, and by letting $w \to \infty$ in the last inequality of (5.12) this gives $\partial^2_{\sigma u}\psi(\sigma, u, \xi) = 0$, which contradicts the third equation of (5.32). □

Remark 5.1 The above proof indicates more precisely that we must have a bound on w at fixed (σ, u) in the domain where a BGK model exists. Unfortunately, in (5.1), we do not have any such a priori bound; the maximum principle only gives the converse inequality $w \ge w_{min}$ if it is true initially. Thus (2.9) will not be satisfied.

In order to illustrate our purpose, let us finally give a concrete example of such a BGK model satisfying all entropy inequalities (other than those with finitely many velocities that are described in Section 3.2). We take $\Xi = \mathbb{R}$ with Lebesgue measure, $a(\xi) = \xi$, $\iota(w, \xi) = 0$, and

$$\psi(\sigma, u, \xi) = c_{\gamma,\kappa} \frac{\theta}{2} \frac{1}{\lambda + 1} \left[\frac{4\gamma\kappa}{(\gamma - 1)^2} \frac{\sigma}{\kappa^{1-1/\gamma}} - (\xi - u)^2 \right]_+^{\lambda+1}, \tag{5.43}$$

where $\kappa > 0$ is a constant, and θ, λ, $c_{\gamma,\kappa}$ are defined in (4.19). One can check that (5.12) holds provided that $\gamma \le 3$ and

$$w \le \frac{\kappa^{1/\gamma}}{\theta}. \tag{5.44}$$

The other variables only have their natural limitations $\sigma \ge 0$, $u \in \mathbb{R}$ (and $w \ge 0$). One can also verify that $k = 0$ and that the inequalities (5.13) also hold under the same limitation (5.44), ensuring the positiveness of ρ and p.

In the case $\gamma = 3$, this model takes a more simple form. The stability condition becomes

$$w \le w_m, \tag{5.45}$$

with $w_m = \kappa^{1/3} > 0$, and we have $\sigma = p^{2/3}$, $w = p^{1/3}/\rho$,

$$\begin{aligned}
M_0(\sigma, u, w, \xi) &= \frac{1}{2\sqrt{3w_m}} \frac{1}{w} \mathbb{1}_{(\xi-u)^2 < 3w_m\sigma}, \\
M_1(\sigma, u, w, \xi) &= \left[\left(1 - \frac{w}{w_m}\right) u + \frac{w}{w_m}\xi\right] M_0(\sigma, u, w, \xi), \\
M_2(\sigma, u, w, \xi) &= \left[\left(1 - \frac{w}{w_m}\right) \frac{u^2}{2} + \frac{w}{w_m}\frac{\xi^2}{2}\right] M_0(\sigma, u, w, \xi).
\end{aligned} \tag{5.46}$$

We can take

$$D_\xi = \Big\{(f_0, f_1, f_2)\ ;\ f_0 > 0,\ 2f_0 f_2 - f_1^2 \geq 0, f_2 - \xi f_1 + \frac{\xi^2}{2} f_0 \geq 0\Big\} \cup \{(0,0,0)\}\,, \tag{5.47}$$

and for any function ϕ such that $\phi' \leq 0$ and $\phi'' \geq 0$

$$H_\phi(f_0, f_1, f_2, \xi) = f_0\ \phi\left(\min\left\{\frac{1}{2\sqrt{3w_m}f_0}, w_m\left[1 - \frac{(f_1 - \xi f_0)^2}{2f_0(f_2 - \xi f_1 + \frac{\xi^2}{2} f_0)}\right]\right\}\right). \tag{5.48}$$

Appendix: bilinear forms and duality

In this appendix we recall some basic facts in linear algebra, and we introduce notations that are used in all the paper.

Let E be a finite-dimensional real vector space, and E' its dual space. We denote by $\mathcal{B}(E)$ the space of bilinear forms on E, and by $\mathcal{L}(E,E')$ the space of linear operators $E \to E'$. Then $\mathcal{L}(E,E')$ and $\mathcal{B}(E)$ are isomorphic, via the relation between $u \in \mathcal{L}(E,E')$ and $b \in \mathcal{B}(E)$ given by

$$u(x)\cdot y = b(y,x), \qquad x,y \in E. \tag{A.1}$$

With this correspondence, in any base e of E, u and b have the same matrix,

$$\underset{e\to e^*}{\mathrm{matrix}}(u) = \underset{e,e}{\mathrm{matrix}}(b), \tag{A.2}$$

where e^* denotes the dual base of e.

For any $u \in \mathcal{L}(E,E')$, we can define its transpose $u^t \in \mathcal{L}(E'',E') = \mathcal{L}(E,E')$, by identifying E'' and E. Thus it is defined by

$$u^t(y)\cdot x = u(x)\cdot y, \qquad x,y \in E. \tag{A.3}$$

By (A.1), the corresponding notion of transposition for bilinear forms is therefore

$$b^t(x,y) = b(y,x), \qquad x,y \in E, \tag{A.4}$$

and this corresponds to the usual transposition for matrices. Then, we say that u (respectively b) is symmetric if $u^t = u$ (respectively $b^t = b$). We also say that u is symmetric nonnegative if the corresponding b is, that is

$$u^t = u \qquad \text{and} \qquad u(x)\cdot x \geq 0, \quad x \in E. \tag{A.5}$$

For example, if $u(x) = \ell_1\, \ell_2(x)$, for some ℓ_1, $\ell_2 \in E'$, then $b(y,x) = \ell_1(y)\ell_2(x)$, which is denoted by $b = \ell_1 \otimes \ell_2$, and $b^t = \ell_2 \otimes \ell_1$. Thus u is symmetric nonnegative if and only if

$$\ell_1 = 0 \qquad \text{or} \qquad \ell_2 = \lambda \ell_1, \quad \lambda \geq 0. \tag{A.6}$$

We also use the notation

$$u(x)\cdot y = u\cdot x\cdot y, \qquad x,y \in E, \tag{A.7}$$

where $u \in \mathcal{L}(E,E')$, and this yields for any $A \in \mathcal{L}(E,E)$

$$u\cdot x\cdot Ay = A^t u\cdot x\cdot y, \tag{A.8}$$

where $A^t \in \mathcal{L}(E',E')$.

Finally, let us recall that a differential form $\omega \in C^1(\Omega,E')$, with Ω an open subset of E, is (locally) an exact form (that is $\omega = dV$ for some $V \in C^2(\Omega,\mathbb{R})$) if and only if $d\omega \in C(\Omega,\mathcal{L}(E,E'))$ is symmetric everywhere in Ω, in the sense defined above. Moreover, V is (locally) convex if and only if $d\omega$ is symmetric nonnegative.

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