Global BV solutions and relaxation limit for a system of conservation laws *

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Abstract

We consider the Cauchy problem for the (strictly hyperbolic, genuinely non-linear) system of conservation laws with relaxation

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \frac{1}{\varepsilon}\, r(u,v). \end{cases} \tag{0.1}$$

Assume there exists an equilibrium curve $A(u)$, such that $r(u, A(u)) = 0$. Under some assumptions on σ and r, we prove the existence of global (in time) solutions of bounded variation, u^ε, v^ε, for $\varepsilon > 0$ fixed.

As $\varepsilon \to 0$, we prove the convergence of a subsequence of u^ε, v^ε to some u, v that satisfy the equilibrium equations

$$u_t - A(u)_x = 0, \qquad v(t,\cdot) = A(u(t,\cdot)) \;\; \forall\, t \geq 0. \tag{0.2}$$

Keywords: p–system with relaxation, **BV** solutions, global in time solutions.

AMS Subject Classification: primary 35L65; secondary 35L45, 58G16, 58G18

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1 Introduction

We consider the Cauchy problem for the following system of conservation laws with relaxation term

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = \frac{1}{\varepsilon}\ r(u,v) \end{cases} \tag{1.1}$$

where $x \in \mathbf{R}$, $u(t,x), v(t,x) \in \mathbf{R}$, $\sigma : \mathbf{R} \to \mathbf{R}$ and $r : \mathbf{R}^2 \to \mathbf{R}$ are smooth functions, $\varepsilon > 0$ is a fixed parameter.

System (1.1) arises in one-dimensional elasticity. In the particular case of $r(u,v) = -v$, one recovers the damped model arising in nonlinear wave equations, studied in many papers (see for instance [7] and references therein)

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = -\alpha v \end{cases} \tag{1.2}$$

where $\alpha = \varepsilon^{-1}$. In [7], assuming that σ is a smooth increasing functions, the author proves a global existence result for weak solutions to (1.2) provided that the initial data

$$u(0,x) = u_o(x), \qquad v(0,x) = v_o(x) \tag{1.3}$$

have total variation and $\mathbf{L}^1$– norm sufficiently small.

Here we are concerned with the system (1.1) where the stress function σ satisfies the assumptions

$$\begin{array}{ll} \sigma'(u) > 0, & \text{(strict hyperbolicity)} \\ \sigma''(u) \neq 0. & \text{(genuine nonlinearity)} \end{array} \tag{1.4}$$

Moreover, we require that

• either σ satisfies the condition

$$2\sigma'(u)\sigma'''(u) - 3\sigma''(u)^2 > 0 \tag{1.5}$$

for u in some open interval $\mathcal{U}$ of $\mathbf{R}$,

• or σ has the specific form

$$\sigma(u) = -\frac{1}{u}\ . \tag{1.6}$$

Condition (1.5) is a technical assumption and is satisfied, for instance, in the nonlinear cases

$$\begin{array}{lll} \sigma(u) = u^\gamma, & u > 0, & 0 < \gamma < 1, \\ \sigma(u) = -u^{-\gamma}, & u > 0, & 0 < \gamma < 1. \end{array} \tag{1.7}$$

However, (1.5) is not satisfied in the main case of interest for isentropic gas dynamics, namely $\sigma(u) = -p(u) = -u^{-\gamma}$ with $1 < \gamma < 3$.

Concerning (1.6), it corresponds to the case of $\gamma = 1$ (isothermal flow). Observe that, for σ given by (1.6), conditions (1.4) are satisfied for all $u \neq 0$, while the quantity in (1.5) vanishes identically.

On the source term, we require that

$$r_v \leq 0, \qquad \left| r_u(u,v) \right| \leq \left| r_v(u,v) \right| \sqrt{\sigma'(u)} \qquad \forall\ (u,v) \in \mathcal{U} \times \mathbf{R} \tag{1.8}$$

and assume that there exists a $\mathbf{C}^1$ equilibrium curve $A(u)$ such that

$$r(u, A(u)) = 0 \qquad \forall\ u \in \mathcal{U}. \tag{1.9}$$

Condition (1.8) amounts to require that the source term in (1.1) is *weakly dissipative*, in the sense that satisfies the weak diagonally dominant condition (see [8]).
Indeed, denote by $G = G(u,v)$ the vector source term and by R the invertible matrix whose columns are the (normalized) right eigenvectors. Consider the matrix $B = R^{-1} \cdot DG \cdot R$, that in our case reads

$$R^{-1} \cdot DG \cdot R = (2\varepsilon \sqrt{\sigma'(u)})^{-1} \begin{pmatrix} r_u + r_v \sqrt{\sigma'(u)} & -r_u + r_v \sqrt{\sigma'(u)} \\ r_u + r_v \sqrt{\sigma'(u)} & -r_u + r_v \sqrt{\sigma'(u)} \end{pmatrix}. \tag{1.10}$$

In [8], the authors prove a global existence result assuming that $B = (B_{ij})$ is strictly diagonally dominant, in the sense that there exists a constant $\nu > 0$ such that

$$B_{ii} + \sum_{j \neq i} \left| B_{ji} \right| \leq -\nu < 0, \qquad i = 1,\ 2. \tag{1.11}$$

In our case, (1.11) is clearly not satisfied since, by (1.8), there holds $B_{ii} + \sum_{j \neq i} \left| B_{ji} \right| = 0$.
On the other hand, we can state the following existence theorem.

Theorem 1.1 *Let σ be a $\mathbf{C}^4$ map, satisfying (1.4)–(1.5) on some open interval $\mathcal{U} \subset \mathbf{R}$. Assume that r is $\mathbf{C}^1$ and satisfy (1.8), (1.9).*

Then for every compact $\mathcal{K} \subset \mathcal{U}$, there exist constants $C, \delta, L > 0$ (independent on ε) such that the following holds.

The Cauchy problem for (1.1) with initial data (u_o, v_o), such that

$$\lim_{x \to -\infty} u_o(x) \in \mathcal{K}, \qquad \text{Tot.Var. } (u_o, v_o) \leq \delta, \tag{1.12}$$

has a global weak (entropic) solution $(u,v)(t)$, that satisfies

$$\text{Tot.Var. } (u,v)(t) \leq C \cdot \text{Tot.Var. } (u,v)(0). \tag{1.13}$$

Moreover, for any $t \geq s \geq 0$ and for any $a < b$, there exists a constant L_ε (possibly depending on ε) such that

$$\int_a^b |u(t,x) - u(s,x)|\, dx \leq L|t-s|, \tag{1.14}$$

$$\int_a^b |v(t,x) - v(s,x)|\, dx \leq (L + L_\varepsilon)\, |t-s|. \tag{1.15}$$

We remark that δ, L and the Lipschitz estimate (1.14) do not depend on ε. Moreover, we do not require any assumptions on the $\mathbf{L}^1$– norm of the initial data.

Let us now turn to the case of (1.6), namely to the system

$$\begin{cases} u_t - v_x = 0 \\ v_t + (\frac{1}{u})_x = \frac{1}{\varepsilon}\, r(u,v). \end{cases} \tag{1.16}$$

For this system, we can drop the smallness assumptions on the data. Indeed we prove the existence (globally in time) of weak entropic solutions for data of arbitrary (but finite) total variation.

Theorem 1.2 *Let r be a $\mathbf{C}^1$-map satisfying (1.8), (1.9) with $\mathcal{U} = (0,\infty)$ and $\sigma = -u^{-1}$.*

Then for every compact $\mathcal{K} \subset (0,\infty)$ and constant $M > 0$ there exist constants $C, L > 0$ (independent on ε) such that the following holds. The Cauchy problem for (1.16) with initial data (u_o, v_o), such that

$$u_o(x) \in \mathcal{K} \quad \forall x \in \mathbf{R}, \qquad \text{Tot.Var. } (u_o, v_o) \leq M, \tag{1.17}$$

has a global weak (entropic) solution $(u,v)(t)$, satisfying (1.13), (1.14), (1.15).

The proofs of Theorem 1.1, 1.2 are based on a wave-front tracking algorithm (see [2, 3]) combined with a fractional step method (see [8, 6]). The key point is the special definition (2.5) (see also [11]) for the amplitude of waves, which fits well both with the geometry of the curves for the homogeneous system

$$\begin{cases} u_t - v_x = 0 \\ v_t - \sigma(u)_x = 0 \end{cases} \tag{1.18}$$

and with the presence of the source term in (1.1). This allows us to define global in time approximate solutions for the system (1.1). As remarked before, it is not possible to apply here the methods and results in [8].

More precisely, using this choice of the wave size, it is possible to control the growth of total variation due to the source term, across the time steps. This works for a general increasing and strictly convex (or concave) σ, provided that the dissipativity condition (1.8) is satisfied.

On the other hand, if σ satisfies either (1.4)–(1.5) or (1.6), only the linear part of the Glimm functional is enough to control the total variation for the approximate solutions of the homogeneous system. We remark that if either (1.4)–(1.5) or (1.6) are satisfied, then the homogeneous system (1.18) belongs to the class cited in Liu [11, Sect. 2] (previously defined in a paper in Russian by Bakhvarov).

In the case $\gamma = 1$, our definition (2.5) of the waves amplitude reduces to the one given in Luskin and Temple [14], where the authors studied a problem for a dissipative p–system in Eulerian coordinates. In their paper, the source term in Lagrangian form was given by $-u^2 K(\frac{v}{u})$, together with some assumptions on K; it was also assumed $|v| < 1$. We remark these hypotheses imply that condition (1.8) is satisfied.

Let us now turn to the relaxation problem. Assume that, in addition to (1.8), (1.9), there holds

$$r_v(u, v) \leq -c < 0 \qquad \forall\ (u, v) \in \mathcal{U} \times \mathbf{R} \tag{1.19}$$

for some constant $c > 0$. A typical form for r is given by $r(u, v) = A(u) - v$. One can easily verify that (1.8), (1.9) and (1.19) together imply the weak sub-characteristic condition

$$-\sqrt{\sigma'(u)} \leq A'(u) \leq \sqrt{\sigma'(u)}. \tag{1.20}$$

As $\varepsilon \to 0$, we consider a sequence $(u^\varepsilon, v^\varepsilon)$ obtained in either Theorem 1.1 or Theorem 1.2, and study the convergence of a subsequence to the equilibrium equations

$$v = A(u), \qquad u_t - A(u)_x = 0. \tag{1.21}$$

The result is the following

Theorem 1.3 *In the assumptions of Theorem 1.1 (resp. of Theorem 1.2), for $\varepsilon > 0$ fixed, and assuming in addition (1.19), let $(u^\varepsilon, v^\varepsilon)$ be a solution of (1.1) (resp. (1.16)) with initial data $u_o^\varepsilon, v_o^\varepsilon$ satisfying (1.12) (resp. (1.17)) for the compact $\mathcal{K}$ (and the constant M) independent on ε.*
Then the constant L_ε at (1.15) takes the form

$$L_\varepsilon \doteq \frac{2c}{\varepsilon} \exp(-\frac{c \cdot s}{\varepsilon}) \cdot \int_a^b |r(u_o, v_o)(x)|\, dx. \tag{1.22}$$

Moreover, assume that as $\varepsilon \to 0$ there holds

$$u_o^\varepsilon \to u_o \qquad \text{in } \mathbf{L}^1{}_{loc} \tag{1.23}$$

and that the sequence $\{v_o^\varepsilon\}$ is uniformly bounded, in the sup–norm, as $\varepsilon \to 0$. Then there exists a subsequence $\varepsilon_k \to 0$ such that

$$(u^{\varepsilon_k}, v^{\varepsilon_k}) \to (\widetilde{u}, \widetilde{v}) \qquad \text{in } \mathbf{L}^1{}_{loc}\left([0,\infty) \times (-\infty,\infty)\right) \tag{1.24}$$

where $\widetilde{u}(t,\cdot), \widetilde{v}(t,\cdot) \in \mathbf{BV}(\mathbf{R})$, $\widetilde{u}$ is a weak solution of

$$u_t - A(u)_x = 0, \qquad u(0,\cdot) = u_o, \tag{1.25}$$

and, for all $t \geq 0$

$$\widetilde{v}(t,\cdot) = A(\widetilde{u}(t,\cdot)). \tag{1.26}$$

Moreover, for any $t \geq s \geq 0$ there holds

$$\int_a^b |\widetilde{u}(t,x) - \widetilde{u}(s,x)|\, dx \leq L|t-s|. \tag{1.27}$$

where L is the Lipschitz constant in (1.14).

It is still not clear, at the moment, if the limit $\widetilde{u}$ of Theorem 1.3 is the unique entropy solution of (1.25). A partial answer in this direction can be given following the lines of [5] and [10].

A previous results of relaxation in the BV framework was obtained in [12], where the authors considered a quasilinear system arising in viscoelasticity. We mention also [13] for a contemporary and independent approach to the case $\gamma = 1$. For a general review on relaxation problems, we refer to [15].

2 Proof of Theorem 1.1

By condition (1.4), if $u \in \mathcal{U}$, then $\sigma''(u)$ has constant sign and the system (1.1) becomes genuinely nonlinear. Now we assume $\sigma''(u) < 0$ (in the other case, a completely similar procedure can be followed). The rarefaction-shock curve of the first characteristic family, starting at the point (u_o, v_o), is given by the equations (see Figure 1)

$$\begin{array}{ll} u < u_o, & v = v_o + \Phi_1(u; u_o) = v_o - \sqrt{(u-u_o)\left(\sigma(u) - \sigma(u_o)\right)}, \\ u > u_o, & v = v_o + \Phi_1(u; u_o) = v_o + \int_{u_o}^u \sqrt{\sigma'(s)}\, ds. \end{array} \tag{2.1}$$

while the equations for the rarefaction-shock curves, of the second family, are the following

$$
\begin{array}{ll}
u < u_o, & v = v_o + \Phi_2(u; u_o) = v_o - \int_{u_o}^{u} \sqrt{\sigma'(s)}\, ds, \\
u > u_o, & v = v_o + \Phi_2(u; u_o) = v_o - \sqrt{(u - u_o)\left(\sigma(u) - \sigma(u_o)\right)}.
\end{array}
\tag{2.2}
$$

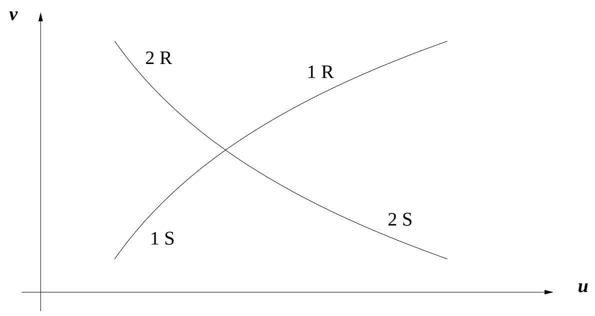

Figure 1: Shock-rarefaction curves

Observe that, for any $u > u_o$, there holds

$$
\int_{u_o}^{u} \sqrt{\sigma'(s)}\, ds \le \sqrt{(u - u_o)\left(\sigma(u) - \sigma(u_o)\right)}
\tag{2.3}
$$

and moreover, for any u, u_o, $i = 1, 2$, one has

$$
\sqrt{\sigma'(u)} \le \left| \frac{\partial \Phi_i(u; u_o)}{\partial u} \right|.
\tag{2.4}
$$

Given a wave-front with left and right state $U_\ell = (u_\ell, v_\ell)$, $U_r = (u_r, v_r)$ respectively, connected by a rarefaction-shock curve, we define the size of the wave as follows

$$
\varepsilon(U_\ell, U_r) = \left| \int_{u_\ell}^{u_r} \sqrt{\sigma'(s)}\, ds \right|
\tag{2.5}
$$

which, on compact sets, is clearly equivalent, along (2.1) and (2.2), to the distance in $\mathbf{R}^2$, $|U_\ell - U_r|$. Note that the size can be written in terms of the Riemann invariants

$$
z(u, v) = \int^{u} \sqrt{\sigma'(s)}\, ds + v, \qquad w(u, v) = \int^{u} \sqrt{\sigma'(s)}\, ds - v.
$$

Moreover, for a rarefaction wave, the size corresponds to the distance of the v coordinate, $|v_\ell - v_r|$.

Similarly to [1, 3], in order to have piecewise constant approximate solutions, we adopt a piecewise constant Riemann solver for the homogeneous system (1.18). Shocks are not modified and satisfy exactly Rankine-Hugoniot relations. For a fixed parameter $\eta > 0$, a rarefaction of size ε is approximated by a fan of N waves, $N = \left[\frac{\varepsilon}{\eta}\right] + 1$, of equal size $\frac{\varepsilon}{N}$ (which is smaller than η), and speed equal to the characteristic speed of the state at the right.

This approximation is applied for all newly generated rarefactions, while preexisting rarefactions can be simply prolonged by a single discontinuity with speed, again, equal to the characteristic speed of the state at the right.

The Rankine-Hugoniot relations are approximately satisfied by each rarefaction: denoting by $U_\ell = (u_\ell, v_\ell)$, $U_r = (u_r, v_r)$ the left and right state respectively, separated by a discontinuity $x(t)$, one has

$$\dot{x}(u_r - u_l) + (v_r - v_l) = \mathcal{O}(1)(u_r - u_l)^2, \tag{2.6}$$

$$\dot{x}(v_r - v_l) + (\sigma(u_r) - \sigma(u_l)) = \mathcal{O}(1)(u_r - u_l)^2. \tag{2.7}$$

Let us construct a sequence of approximate solutions $(u, v) = (u^\nu, v^\nu)(t, x)$ as follows. Fix a time step $\Delta t = \Delta t^\nu > 0$, a parameter $\eta = \eta_\nu$ for rarefactions, $(\Delta t^\nu,\ \eta_\nu \to 0$ as $\nu \to \infty)$ and take a sequence of piecewise constant functions (u_o^ν, v_o^ν), $\nu \in \mathbf{N}$, such that

$$\text{Tot.Var. } (u_o^\nu, v_o^\nu) \leq \text{Tot.Var. } (u_o, v_o), \quad \lim_{x\to-\infty} (u_o^\nu, v_o^\nu)(x) = \lim_{x\to-\infty} (u_o, v_o)(x), \tag{2.8}$$

$$\|(u_o^\nu, v_o^\nu) - (u_o, v_o)\|_\infty \leq \frac{1}{\nu}, \quad \int_{-\nu}^{\nu} |(u_o^\nu, v_o^\nu)(x) - (u_o, v_o)(x)|\, dx \leq \frac{1}{\nu}. \tag{2.9}$$

Solve each Riemann problem, arising at the points of jump, with the approximate Riemann solver for (1.18) introduced before. Then $(u, v)(t, \cdot)$ is defined until no interactions occur.

By slightly changing the speed of some waves, we can assume that only two wave-fronts interact at any single time. When this occurs, the solution is prolonged by solving the Riemann problem arising at the interaction point.

Assuming that the approximate solution is defined at some time $k\Delta t$, $k \geq 1$, the damping term is added, which affects only the v variable

$$\begin{aligned} u(k\Delta t+, x) &= u(k\Delta t-, x), \\ v(k\Delta t+, x) &= v(k\Delta t-, x) + \tfrac{\Delta t}{\varepsilon} \cdot r(u, v)(k\Delta t-, x). \end{aligned} \tag{2.10}$$

Observe that, according to (2.10), (1.8), (1.9), the functions $|r(u, v)|$, $|v - A(u)|$ decrease across time steps:

$$|r(u, v)(k\Delta t+, x)| \leq |r(u, v)(k\Delta t-, x)| \cdot \left(1 - \frac{\Delta t}{\varepsilon} \cdot \inf_w \left| r_v \left(u(k\Delta t-, x), w\right)\right|\right), \tag{2.11}$$

$$|[v-A(u)](k\Delta t+,x)| \leq |[v-A(u)](k\Delta t-,x)| \cdot \left(1-\frac{\Delta t}{\varepsilon}\cdot \inf_{w} \left| r_v\left(u(k\Delta t-,x),w\right)\right|\right). \quad (2.12)$$

The solution is then prolonged by solving the Riemann problems arising at the points of jump. Again we can assume that no interactions occur at a time $k\Delta t$.

Let us denote by $x_i < x_{i+1}$, $i = 1, \ldots, N(t)$, the points at which $(u,v)(t,\cdot)$ is discontinuous, for some integer $N(t)$. Using the definition (2.5) for the strength of the waves, we introduce the functional

$$V(t) = \sum_{i=1}^{N(t)} \varepsilon_i \quad (2.13)$$

for any time $t > 0$ at which no interactions occur. We set $V(0) \doteq \lim_{t\to 0+} V(t)$.

Remark 2.1 Since the strength of the waves solving a Riemann problem are $\mathbf{C}^2$ functions of the right and the left state (u^-,v^-), (u^+,v^+), one has that

$$V(0) \leq C^* \,\text{Tot.Var.}\ (u,v)(0,\cdot) \quad (2.14)$$

provided that $(u,v)(0,\cdot)$ is contained in a compact set $\mathcal{Q} \subset \mathcal{U} \times \mathbf{R}$ and has sufficiently small total variation. The constant C^* depends only on the compact set $\mathcal{Q}$. Moreover, for $t > 0$, $V(t)$ is equivalent to the total variation of $(u,v)(t,\cdot)$, on every set in which σ' remains bounded and away from 0. In other words, for any compact set $\mathcal{K}' \subset \mathcal{U}$ there exists a constant $C(\mathcal{K}') > 1$ such that if $u(t,x) \in \mathcal{K}'$ $\forall x \in \mathbf{R}$, there holds

$$\frac{1}{C(\mathcal{K}')}\,\text{Tot.Var.}\ (u,v)(t,\cdot) \leq V(t) \leq C(\mathcal{K}')\,\text{Tot.Var.}\ (u,v)(t,\cdot), \qquad t > 0. \quad (2.15)$$

We claim that, until $(u,v)(t,\cdot)$ is defined, the functional V is non-increasing. Indeed, let us consider the different cases (for a complete description of waves interactions, see [4]).

Assume that, at the time t, $(k-1)\Delta t < t < k\Delta t$, two wave-fronts interact. Denote by (u_ℓ, v_ℓ), (u_m, v_m), (u_r, v_r) the left, middle and right states respectively (see Figure 2), and by (u^*, v^*) the middle state after the interaction time. Then one has

$$\Delta V = \varepsilon(U_\ell, U^*) + \varepsilon(U^*, U_r) - \varepsilon(U_\ell, U_m) - \varepsilon(U_m, U_r). \quad (2.16)$$

In the case of a 2-shock and a 1-shock interacting (see Figure 3), we have the following

Lemma 2.1 *For any compact $\mathcal{K}' \subset \mathcal{U}$, there exists a $\delta_1 > 0$ such that, if $u_\ell \in \mathcal{K}'$, $|u_\ell - u_m| \leq \delta_1$, $|u_m - u_r| \leq \delta_1$, and if the two wave-fronts are shocks of different family, then $\Delta V \leq 0$.*

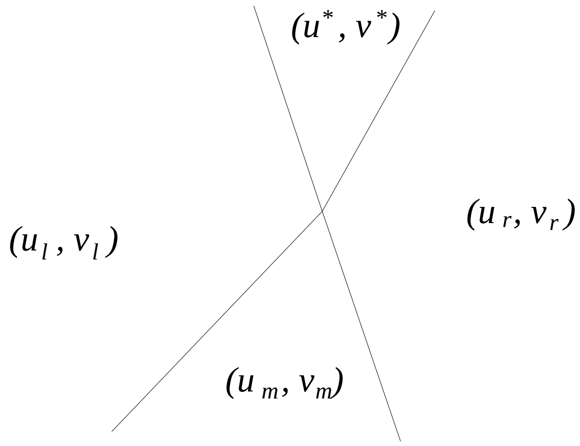

Figure 2: Interaction of wave-fronts

Proof. Set $\Psi(u_\ell, a) \doteq \int_{u_\ell}^{u_\ell+a} \sqrt{\sigma'(s)}\, ds$, and denote by x, y, z the positive quantities $u_m - u_\ell$, $u_m - u_r$ and $u_\ell - u^*$, respectively. Consider the application

$$\begin{aligned} f(u_\ell, x, y, z) &= \int_{u_\ell - z}^{u_\ell} \sqrt{\sigma'(s)}\, ds + \int_{u_\ell - z}^{u_\ell + x - y} \sqrt{\sigma'(s)}\, ds \\ &\quad - \int_{u_\ell}^{u_\ell + x} \sqrt{\sigma'(s)}\, ds - \int_{u_\ell + x - y}^{u_\ell + x} \sqrt{\sigma'(s)}\, ds \\ &= 2\left[\Psi(u_\ell, x - y) - \Psi(u_\ell, -z) - \Psi(u_\ell, x)\right] \end{aligned} \tag{2.17}$$

where $z = z(u_\ell, x, y)$ is implicitly defined by

$$\begin{aligned} &\sqrt{x\left(\sigma(u_\ell + x) - \sigma(u_\ell)\right)} + \sqrt{y\left(\sigma(u_\ell + x) - \sigma(u_\ell + x - y)\right)} = \\ &\quad = \sqrt{z\left(\sigma(u_\ell) - \sigma(u_\ell - z)\right)} + \sqrt{(x - y + z)\left(\sigma(u_\ell + x - y) - \sigma(u_\ell - z)\right)}. \end{aligned} \tag{2.18}$$

The function $g(u_\ell, x, y) \doteq f(u_\ell, x, y, z(u_\ell, x, y))$ is continuous up to the fourth derivatives. Applying the implicit function theorem, after lengthy but straightforward calculations, one obtains: $g(u_\ell, x, 0) = 0$, $g(u_\ell, 0, y) = 0$,

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} g(u_\ell, 0, 0) = 0 \tag{2.19}$$

for every α, β satisfying $\alpha + \beta \leq 4$, $(\alpha, \beta) \neq (1, 3), (3, 1)$, and

$$\frac{\partial^4}{\partial x^3 \partial y} g(u_\ell, 0, 0) = \frac{\partial^4}{\partial x \partial y^3} g(u_\ell, 0, 0) = \frac{\sigma''(u_\ell)\left(2\sigma'(u_\ell)\sigma'''(u_\ell) - 3\sigma''(u_\ell)^2\right)}{16\ \sigma'(u_\ell)^{5/2}}. \tag{2.20}$$

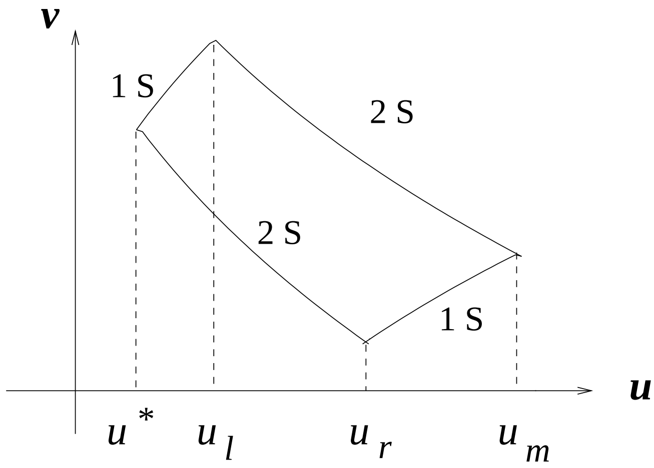

Figure 3: Interaction of two shocks

Recalling that $\sigma''(u_\ell) < 0$, by assumption (1.5) and Proposition A.1, there exists $\delta_1 > 0$ such that g is non positive in $\mathcal{K}' \times [0, \delta_1] \times [0, \delta_1]$. □

In the case of two rarefactions of different family interacting, by (2.1),(2.2) one has

$$\begin{aligned} v_r &= v_\ell - \int_{u_\ell}^{u_m} \sqrt{\sigma'(s)}\, ds + \int_{u_m}^{u_r} \sqrt{\sigma'(s)}\, ds \\ &= v_\ell + \int_{u_\ell}^{u^*} \sqrt{\sigma'(s)}\, ds - \int_{u^*}^{u_r} \sqrt{\sigma'(s)}\, ds. \end{aligned} \tag{2.21}$$

Since $u_\ell > u_m$, $u^* > u_r$, one simply gets $\Delta V = 0$. Note that, in this case, the two wave-fronts have the same size before and after the interaction. Indeed, by (2.21), one has

$$2\int_{u_m}^{u_\ell} \sqrt{\sigma'(s)}\, ds + \int_{u_\ell}^{u_r} \sqrt{\sigma'(s)}\, ds = \int_{u_\ell}^{u_r} \sqrt{\sigma'(s)}\, ds + 2\int_{u_r}^{u^*} \sqrt{\sigma'(s)}\, ds \tag{2.22}$$

that gives $\varepsilon(U_\ell, U_m) = \varepsilon(U^*, U_r)$; the other equality follows by $\Delta V = 0$. This is not surprising since the size defined in (2.5) is strictly related to Riemann invariants.

In the case of a rarefaction and a shock of the same i-family ($i = 1, 2$) interacting, one can use Glimm estimates (see [9]). For any compact set $\mathcal{K}' \subset \mathcal{U}$ there exist $C, \delta_2' > 0$ such that the following hold: If $u_\ell \in \mathcal{K}'$ and $\varepsilon(U_\ell, U_m) \le \delta_2'$, $\varepsilon(U_m, U_r) \le \delta_2'$, then there holds:

$$|\varsigma\varepsilon_i' - \varepsilon(U_\ell, U_m) + \varepsilon(U_m, U_r)| + \varepsilon_j' \le C\varepsilon(U_m, U_r)\varepsilon(U_\ell, U_m) \tag{2.23}$$

where ε_i' is the strength of the outgoing wave of the i family, ε_j' is the strength of the outgoing wave of the other family and ς is $+1$ or -1 depending on the interaction. Suppose $\varsigma = +1$

(the other case is similar), from (2.23) one gets

$$\begin{aligned}\Delta V &= \varepsilon_i' + \varepsilon_j' - \varepsilon(U_m, U_r) - \varepsilon(U_\ell, U_m) \\ &\le \varepsilon_j' + |\varepsilon_i' - \varepsilon(U_\ell, U_m) + \varepsilon(U_m, U_r)| - 2\varepsilon(U_m, U_r) \\ &\le \varepsilon(U_m, U_r)[C\varepsilon(U_\ell, U_m) - 2] \le 0\end{aligned} \tag{2.24}$$

for $\varepsilon(U_\ell, U_m) \le \delta_2 = \min\{\delta_2', \frac{1}{C}\}$.

In the remaining cases, one has

$$u_m,\ u^* \in [\min\{u_\ell, u_r\}, \max\{u_\ell, u_r\}], \tag{2.25}$$

that implies $\Delta V = 0$.

Assume now that $t = k\Delta t$. Denote by (u_l, v_l) and (u_r, v_r) the states at the left and at the right of a wave approaching the time step, and by (u_l, v_l^+), (u^*, v^*), (u_r, v_r^+) respectively the left, middle and right states after the time step (see Figure 4).

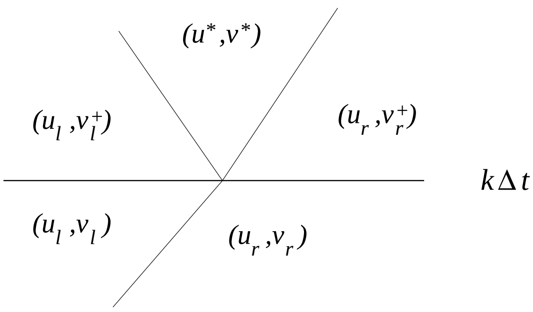

Figure 4: Wave–fronts through the step

We claim that

$$|v_l^+ - v_r^+| \le |v_l - v_r|. \tag{2.26}$$

Indeed, recalling (2.10) and (2.1),(2.2), we can write:

$$\begin{aligned}|v_r^+ - v_l^+| &= |v_r - v_l| \cdot \left| 1 + \frac{\Delta t}{\varepsilon} \cdot \frac{[r(u_r, v_r) - r(u_l, v_l)]}{v_r - v_l} \right| \\ &= |v_r - v_l| \cdot \left| 1 + \frac{\Delta t}{\varepsilon} \Phi_i(u_r; u_l)^{-1} \int_{u_l}^{u_r} [r_u + r_v \Phi_{i,s}(s; u_l)]\, ds \right|.\end{aligned} \tag{2.27}$$

Let us check the sign of the quantity, in (2.27), that multiplies $\frac{\Delta t}{\varepsilon}$. Observe that, using (1.8) and (2.4), either it vanishes or it has the same sign of $-\Phi_{i,s} \cdot \Phi_i(u_r; u_l) \cdot (u_r - u_l)$. From

(2.1),(2.2), one can check that this last quantity is always negative. Hence the smallness of Δt implies (2.26).

Inequality (2.26) and the invariance of shock–rarefaction curves with respect to translations along the v direction ensure that $u^* \in [\min\{u_l, u_r\}, \max\{u_l, u_r\}]$ (see Figure 5) which implies $\Delta V = 0$. As a consequence there hold:

i. a rarefaction of the 1^{th} family becomes a smaller 1-rarefaction followed by a small 2-shock;

ii. a shock wave of the 1^{th} family becomes a smaller 1-shock followed by a 2-rarefaction;

iii. a rarefaction of the 2^{th} family produces a small 1-shock followed by a 2-rarefaction;

iv. a shock of the 2^{th} family produces a small 1-rarefaction followed by a 2-shock.

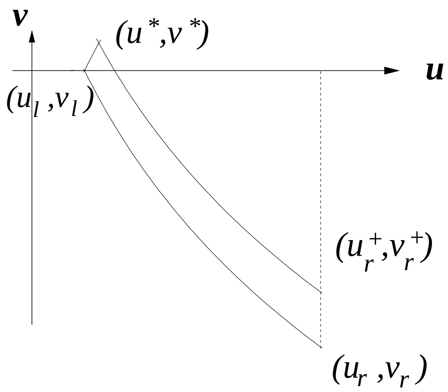

Figure 5: A 2–shock (with $v_l = 0$) through the step

Hence the following lemma is proved

Lemma 2.2 *For any closed interval $\mathcal{K}' \subset \mathcal{U}$ containing the compact set $\mathcal{K}$ in its interior there exists a $\delta_3 > 0$, independent on ν, such that the following holds. If $V(0) \le \delta_3$ and until the approximate solution $(u,v)(t,\cdot)$ is defined, there holds $\Delta V(t) \le 0$, and consequently $V(t) \le \delta_3$. Moreover, $u(t,x) \in \mathcal{K}'$ $\forall x \in \mathbf{R}$.*

Indeed, consider a closed interval $\mathcal{K}' \subset \mathcal{U}$, such that the compact set $\mathcal{K}$ is contained in its interior. Define

$$\delta_4 = \inf_{\substack{u_1 \in \mathcal{K} \\ u_2 \in \mathcal{U}\setminus\mathcal{K}'}} \left| \int_{u_1}^{u_2} \sqrt{\sigma'(s)}\, ds \right| .$$

Let $C(\mathcal{K}')$ be the constant in Remark 2.1. Then if

$$\delta_3 \le \min\{\delta_1/C(\mathcal{K}'), \delta_2, \delta_4\},$$

the Lemma 2.1, (2.24) and the subsequent consideration hold. Moreover, since $\delta_3 \le \delta_4$, there holds $u(t,x) \in \mathcal{K}'\ \forall x \in \mathbf{R}$.

Remark 2.2 The size of a rarefaction wave does not increase in time. Indeed, this is not the case for $(2r : 1r \to 1r : 2r)$, due to (2.22).
The other cases to be considered are $(2s : 1r \to 1r : 2s)$ and $(2r : 1s \to 1s : 2r)$ (when a shock and a rarefaction of the same family interact, the rarefaction cannot increase because there is a compensation that can be seen with a computation similar to (2.24)). In the first one, for instance, proceed as in the proof of Lemma 2.1. Consider the application

$$f(u_\ell, x, y, z) = \Psi(u_\ell, 0, z) - \Psi(u_\ell, x, x+y) \tag{2.28}$$

where $\Psi(u_\ell, a, b) \doteq \int_{u_\ell+a}^{u_\ell+b} \sqrt{\sigma'(s)}\, ds$, $x = u_m - u_l$, $y = u_r - u_m$ and $z = z(u_\ell, x, y) = u^* - u_l$ is implicitly defined by

$$\begin{aligned} h(u_\ell, x, y, z) = \Psi(u_\ell, 0, z) - \sqrt{(x+y-z)\left(\sigma(u+x+y) - \sigma(u+z)\right)} - \\ - \Psi(u_\ell, x, x+y) + \sqrt{x\left(\sigma(u+x) - \sigma(u)\right)} \equiv 0, \end{aligned} \tag{2.29}$$

hence $z(u_\ell, x, 0) = 0$, $z(u_\ell, 0, y) = y$. Clearly f gives the difference of the amplitude of the rarefaction, after and before the interaction. We prove that $g(u_\ell, x, y) \doteq f(u_\ell, x, y, z(u_\ell, x, y))$ is negative in a sufficiently small neighborhood of zero.

Indeed, g is continuous up to the fourth derivatives in the variables x, y. It is easy to check that

$$g(u_\ell, x, 0) \equiv 0, \quad g(u_\ell, 0, y) \equiv 0, \tag{2.30}$$

Moreover,

$$h_x(u_\ell, x, y, z(u_\ell, x, y))|_{x=0,y=0} = 2g_x(u_\ell, 0, y) \equiv 0, \tag{2.31}$$

$$h_{xx}(u_\ell, x, y, z(u_\ell, x, y))|_{x=0,y=0} = 2g_{xx}(u_\ell, 0, y) \equiv 0 \tag{2.32}$$

and, applying the implicit function theorem,

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} g(u_\ell, 0, 0) = 0 \tag{2.33}$$

for every α, β, with $\alpha + \beta \leq 4$, $(\alpha, \beta) \neq (3,1)$; moreover

$$\frac{\partial^4}{\partial x^3 \partial y} g(u_\ell, 0, 0) = \frac{\sigma''(u_\ell)}{32\, \sigma'(u_\ell)^{5/2}} \left(2\sigma'(u_\ell)\sigma'''(u_\ell) - 3\sigma''(u_\ell)^2 \right). \tag{2.34}$$

By assumption (1.5), by (2.30)–(2.32) and Proposition A.2, g is negative $\forall (u_\ell, x, y) \in \mathcal{K}' \times [0, \delta'] \times [0, \delta']$ for a suitable $\delta' > 0$. By eventually taking a smaller δ_3 in Lemma 2.2, the claim is proved.

Lemma 2.2 provides a uniform a-priori bound on the total variation of the approximate solutions. Now we prove that the total number of interactions remains finite in finite time, hence every approximate solution can be defined for all times $t > 0$. We need the following lemma.

Lemma 2.3 *Let a wave front tracking pattern be given in the strip $[0, T) \times \mathbf{R}$, made of segments (waves) of two families. Assume that the velocities of the segments of the first family lay between two constants $a_1 < a_2$. Analogously the velocities of the segments of the second family lay between two constants $b_1 < b_2$ with $a_2 < b_1$. Assume that the wave front tracking pattern has also the following properties:*

- *At $t = 0$ there is a finite number N_0 of waves.*
- *The interactions occur only between two wave fronts at any single time.*
- *Except a finite number of interactions, there is at most one outgoing wave of each family for each interaction.*

Then the number of interactions, in the region $[0, T) \times \mathbf{R}$, is finite.

Proof. It is not restrictive to assume $a_2 < 0 < b_1$. Suppose by contradiction that there is an infinite number of interactions. We can assume that in $[0, t]$, $t < T$ there is a finite number of them and that T is an accumulation point. Therefore there is a sequence $\mathcal{I}$ of interactions which occur at the points (t_i, x_i), $i = 1, 2, \ldots$. Without loss of generality we can assume that $t_1 < t_2 < \ldots$ and that (t_i, x_i) tends to a point $(T, \overline{x})$ as i tends to infinity.

Denote by $\mathcal{F}$ the set of all the segments that can be joined to some point of $\mathcal{I}$, forward in time, by a continuous path along the wave-fronts. For instance, all the segments interacting at the points (t_i, x_i), $i \in \mathbf{N}$, belong to $\mathcal{F}$. Call $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) the set of all elements of $\mathcal{F}$ which belong to the first (resp. second) family.
Then we partition all the interaction points in the following sets.

- $\mathcal{I}_1$: the set of all interaction points in which there are exactly two outgoing segment belonging to $\mathcal{F}$, one for each family.

- $\mathcal{I}_2$: the set of all interaction points in which the two in-going waves belong to $\mathcal{F}$ and there is at most one outgoing wave belonging to $\mathcal{F}$.
- $\mathcal{I}_3$: the set of all interaction points in which no in-going wave belongs to $\mathcal{F}$.
- $\mathcal{I}_4$: the set of all interaction points in which the two in-going waves both belong to $\mathcal{F}$ and there are at least two outgoing waves of the same family belonging to $\mathcal{F}$.

We remark that, by assumption, $\mathcal{I}_4$ is finite, that all the in-going waves of the interactions of $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{I}_4$ belong to $\mathcal{F}$ and that the out-going waves of the interactions of $\mathcal{I}_3$ do not belong to $\mathcal{F}$. Note also that the points (t_i, x_i), $i \in \mathbf{N}$, do not belong to $\mathcal{I}_3$.
We define the potential $\mathcal{V}(t) \geq 0$ as the number of all the segments belonging to $\mathcal{F}$ present at the time t. Observe that, except from the points of the finite set $\mathcal{I}_4$, $\mathcal{V}(t)$ is non increasing across interaction points, moreover it decreases of one or two across the points of $\mathcal{I}_2$. Since $\mathcal{V}(0)$ is bounded, then $\mathcal{I}_2$ is finite.

As a consequence, all the points (t_i, x_i), $i \in \mathbf{N}$, except at most a finite number, belong to $\mathcal{I}_1$.

We start now from (t_1, x_1), and try to go forward in time with two continuous lines: the first one made of segments of $\mathcal{F}_1$ (first family) and the second one using segments of $\mathcal{F}_2$ (second family). In doing this possibly we have to stop when we reach an interaction point $(\widetilde{t}, \widetilde{x})$, with $\widetilde{t} < T$, belonging to $\mathcal{I}_2$ or $\mathcal{I}_4$ (obviously we cannot reach points of $\mathcal{I}_3$). If this happens we take a point (t_j, x_j) with $t_j > \widetilde{t}$, and start again.

Since $\mathcal{I}_2$ and $\mathcal{I}_4$ are finite sets we have to find a point $(t^*, x^*) \in \mathcal{I}$ from which we can draw two lines $\gamma_1(t)$, $\gamma_2(t)$ until the time T: the first one made of segments of $\mathcal{F}_1$ and the second using segments of $\mathcal{F}_2$. The bounds on the velocities imply

$$\gamma_1(t) \leq x^* + a_2(t - t^*), \qquad \gamma_2(t) \geq x^* + b_1(t - t^*). \tag{2.35}$$

Define $\rho = (T - t^*)(b_1 - a_2)/5 > 0$ and fix t_n with $(t_n, x_n) \in \mathcal{I}$ satisfying

$$|a_1|(T - t_n) \leq \rho, \qquad |b_2|(T - t_n) \leq \rho. \tag{2.36}$$

Since γ_1 is composed by segments of $\mathcal{F}$, the point $(t_n, \gamma_1(t_n))$ can be joined to some point of $\mathcal{I}$. Therefore the bounds on the velocities imply that there must be a point $(t_h, x_h) \in \mathcal{I}$ satisfying

$$x_h \leq \gamma_1(t_n) + b_2(t_h - t_n), \qquad h \geq n. \tag{2.37}$$

Analogously there must be a point $(t_k, x_k) \in \mathcal{I}$ satisfying

$$x_k \geq \gamma_2(t_n) + a_1(t_k - t_n), \qquad k \geq n. \tag{2.38}$$

From (2.37), (2.38), (2.36) and (2.35) we get

$$\begin{aligned} x_k - x_h &\geq \gamma_2(t_n) - |a_1|(T - t_n) - \gamma_1(t_n) - |b_2|(T - t_n) \\ &\geq (b_1 - a_2)(t_n - t^*) - 2\rho \\ &= (b_1 - a_2)(T - t^*) + (b_1 - a_2)(t_n - T) - 2\rho \geq \rho. \end{aligned} \tag{2.39}$$

Since n can be chosen arbitrarily large, (2.39) contradicts the fact that x_i tends to $\overline{x}$. □

To apply the lemma we need to prove that, between two time steps, the number of interactions in which there are more than one outgoing wave of the same family is finite. In each strip, the interactions in which there are more than one outgoing wave of the same family can occur only in the case of two shocks of the same family interacting (of size ε_1 and ε_2), because a rarefaction (of size ε) of the other family appears and we have possibly to split it. If this happens, there holds

$$\eta \leq \varepsilon \leq C\varepsilon_1\varepsilon_2. \tag{2.40}$$

for a suitable $C > 0$. Recalling the usual definition of the interaction potential Q, we use the fact that, eventually taking δ_3 smaller in Lemma 2.2, Q decreases at each interaction of waves between times $k\Delta t$ and $(k+1)\Delta t$. Due to (2.40), the previous situation can occur only a finite number of times. Therefore, by Lemma 2.2, if $V(0) \leq \delta_3$, we can construct the approximate solution $(u, v)(t, \cdot)$ for all $t \geq 0$.

Recalling Remark 2.1, we define $\delta \doteq \delta_3/C^*$. Hence if Tot.Var. $(u_o, v_o) \leq \delta$, then the approximate solutions have equibounded total variation and there holds

$$\text{Tot.Var. } (u^\nu, v^\nu)(t, \cdot) \leq C^* C(K')\, \text{Tot.Var. } (u_o, v_o). \tag{2.41}$$

Observe that last inequality, together with (2.12), ensures that the approximate solutions remain equibounded in $\mathbf{L}^\infty$–norm.

To apply Helly's theorem we need estimates on the dependence on t of the approximating functions. These estimates are given by the following lemma.

Lemma 2.4 *Let $(u, v)(t, x)$ be an approximate solution defined by the previous algorithm, such that* Tot.Var. $(u, v)(t, \cdot) \leq K$, $\|u\|_\infty$, $\|v\|_\infty \leq K^\dagger$, $\forall t \geq 0$ *and for some positive K, $K^\dagger$. Then there exists a constant L^*, independent on ε, such that $\forall a < b$, $0 \leq s < t$ there holds:*

$$\int_a^b |u(t, x) - u(s, x)|\, dx \leq L^* K(t - s) \tag{2.42}$$

$$\int_a^b |v(t, x) - v(s, x)|\, dx \leq (t - s + \Delta t)\,(L^* K + L_\varepsilon) \tag{2.43}$$

where L_ε depends on ε and on the interval $[a, b]$.

Proof. Fix a constant $L^* > 1$ satisfying $\sqrt{\sigma'(u)} \leq L^* \ \forall u \in \mathcal{K}'$, and take two numbers $0 \leq s < t$. If there are no time steps between s and t, (2.42) and (2.43) obviously hold with any $L_\varepsilon \geq 0$. Suppose now that there are steps between s and t:

$$s \leq k_o\Delta t < (k_o+1)\Delta t < \ldots < \widetilde{k}\Delta t \leq t, \qquad 1 \leq k_o \leq \widetilde{k}, \tag{2.44}$$

so that there holds $(\widetilde{k} - k_o)\Delta t \leq t - s$. Since u does not change through the steps we have

$$\begin{aligned} \int_a^b |u(t,x) - u(s,x)|\, dx &\leq L^* K\left[t - \widetilde{k}\Delta t + \sum_{i=k_o+1}^{\widetilde{k}} \Delta t + k_o\Delta t - s\right] \\ &\leq L^* K(t-s), \end{aligned} \tag{2.45}$$

and (2.42) is proved. On the other hand, for the estimate (2.43) we have to consider a supplementary term due to the fact that v is discontinuous through the steps:

$$\begin{aligned} \sum_{i=k_0}^{\widetilde{k}} \int_a^b |v(i\Delta t+, x) - v(i\Delta t-, x)|\, dx &= \sum_{i=k_0}^{\widetilde{k}} \frac{\Delta t}{\varepsilon} \int_a^b |r(u,v)(i\Delta t-, x)|\, dx \\ &\leq \Delta t(\widetilde{k} - k_o + 1)L_\varepsilon \leq L_\varepsilon(t - s + \Delta t) \end{aligned} \tag{2.46}$$

where $L_\varepsilon = (b-a)\varepsilon^{-1}\sup_{|u|,|v|\leq K^\dagger} \left|r(u,v)\right|$. This proves the lemma. □

By Helly's theorem, there exists a subsequence that converges in $\mathbf{L}^1{}_{loc}$ to a limit (u,v). By Remark 2.1, the maximum size of the rarefactions tends to zero as the order of approximation increases; moreover the approximated solutions do not have non-physical waves. Then, by standard procedure [3, 6], one can show that (u,v) is a weak solution to (1.1)-(1.3). Moreover lemma 2.4 ensures that the solution (u,v) satisfies (1.14) and (1.15) with $L = L^*K$.

Finally, let η be a convex entropy, with flux q. Following the same arguments as in [6], one can prove that the entropy inequality

$$\begin{aligned} \int_0^\infty \int_{-\infty}^\infty \eta(u,v)\phi_t + q(u,v)\phi_x + \varepsilon^{-1}\eta_v(u,v)r(u,v)\phi\, dxdt \ + \\ + \int_{-\infty}^\infty \eta(u_o,v_o)(x)\phi(0,x)\, dx \geq 0 \end{aligned} \tag{2.47}$$

holds for any test function $\phi \geq 0$.

3 Proof of Theorem 1.2

This proof is similar to the previous one. We have only to change the points in which we used the hypothesis (1.5) or the smallness of the total variation. We remark that with $\sigma(u) = -\frac{1}{u}$, one has $2\sigma'(u)\sigma'''(u) - 3\sigma''(u) \equiv 0$; moreover the waves measure, defined at (2.5), here reads as follows

$$\varepsilon(U_\ell, U_m) = \left|\log \frac{u_\ell}{u_r}\right|$$

(see also [13] for an independent approach, using the Glimm scheme, to the case $\gamma = 1$).

First of all we observe that we can always solve the Riemann problem, even if the data are arbitrarily large [4] and that Remark 2.1 holds without requiring the smallness of the total variation of $(u, v)(0, \cdot)$. The first thing that does not work here is Lemma 2.1. We substitute it with Lemma 3.1. In the following, as before, we will denote by $U_l = (u_l, v_l)$, $U_m = (u_m, v_m)$, $U_r = (u_r, v_r)$ respectively the left, middle and right state before an interaction of two waves, and by $U^* = (u^*, v^*)$, the middle state after the interaction (see Figure 2).

Lemma 3.1 *Waves of different families cross each other without changing their strength.*

Proof. It is easy to see that in the interactions of waves of different families one has

$$u^* = \frac{u_l u_r}{u_m}. \tag{3.1}$$

For example let us check the case of a 2–shock and a 1–rarefaction interacting (2s–1r). The equation defining u^* is given by

$$\begin{aligned} -\sqrt{(u_m - u_l)[\sigma(u_m) - \sigma(u_l)]} + \int_{u_m}^{u_r} \sqrt{\sigma'(s)}\, ds \quad &= \\ = -\sqrt{(u^* - u_r)[\sigma(u^*) - \sigma(u_r)]} + \int_{u_l}^{u^*} \sqrt{\sigma'(s)}. & \end{aligned} \tag{3.2}$$

With $\sigma(u) = -\frac{1}{u}$ we get

$$\sqrt{(u^* - u_r)\left(\frac{1}{u_r} - \frac{1}{u^*}\right)} - \sqrt{(u_m - u_l)\left(\frac{1}{u_l} - \frac{1}{u_m}\right)} = \ln \frac{u^* u_m}{u_l u_r}. \tag{3.3}$$

It is easy to verify that (3.1) represents the unique solution of (3.3).
Using (3.1) it is easy to check that

$$\epsilon(U_l, U_m) = \epsilon(U^*, U_r), \qquad \epsilon(U_m, U_r) = \epsilon(U_l, U^*). \tag{3.4}$$

This completes the proof. □

Next, we need to prove that V does not increase when a shock and a rarefaction of the same family interact, since in the previous proof we used the smallness of the interacting waves. Therefore we state the following more general Lemma that will be useful later and whose proof is given in the appendix. We denote by V_s and V_r the splitting of V into shocks and rarefactions respectively:

$$V_s(t) = \sum_{i:\ \epsilon_i \text{ is a shock}} \epsilon_i\,, \tag{3.5}$$

$$V_r(t) = \sum_{i:\ \epsilon_i \text{ is a rarefaction}} \epsilon_i\,. \tag{3.6}$$

Lemma 3.2 *Let $\mathcal{K}' \subset (0, +\infty)$ be a compact set. If, in a interaction between a shock and a rarefaction of the same family, there holds $u_l, u_m, u_r, u^* \in \mathcal{K}'$, then there exists $\xi_0 > 1$, depending only on the compact set $\mathcal{K}'$, such that the potentials*

$$V_\xi(t) = \xi V_s(t) + V_r(t) \qquad \xi \in [1, \xi_0], \tag{3.7}$$

are not increasing.

We remark that in Lemma 3.2 taking $\xi = 1$ one has that independently of u_l, u_m, u_r, u^* the potential $V(t)$ is not increasing across the interaction of a shock and a rarefaction of the same family.

Hence we can state the corresponding of Lemma 2.2.

Lemma 3.3 *Until the approximate solution $(u, v)(t, \cdot)$ is defined, there holds $\Delta V(t) \leq 0$, and consequently $V(t) \leq V(0)$.*

Suppose now to have an initial data (u_o, v_o) satisfying (1.17). Take then an approximating sequence (u^ν, v^ν) satisfying (2.8), (2.9). For ν sufficiently large (u^ν, v^ν) is contained in a compact set $\mathcal{Q} = \mathcal{K}'' \times [-N, N]$. Therefore Remark 2.1 implies

$$V(0) \leq C^* \text{ Tot.Var. } (u_o, v_o) \tag{3.8}$$

where C^* depends only on $\mathcal{Q}$.

Lemma 3.3 ensures that until $(u^\nu(t, \cdot), v^\nu(t, \cdot))$ is defined, then $V^\nu(t) \leq C^* M$. But if $u_\infty = \lim_{x \to +\infty} u_o(x) = \lim_{x \to +\infty} u_o^\nu(x)$ then it must hold

$$u^\nu(t, x) \in [u_\infty \cdot \exp(-C^* M), u_\infty \cdot \exp(C^* M)] = \mathcal{K}'. \tag{3.9}$$

In other words, until the approximate solution exists, $u^\nu(t, x)$ belongs to a compact set $\mathcal{K}' \subset (0, +\infty)$ that depends only on $\mathcal{K}$ and M. Therefore, since $V^\nu(t)$ is equibounded, also the total variation is equibounded.

Finally to prove that the approximate solutions can be defined for all times $t > 0$ we apply Lemma 2.3. To apply this Lemma we have to show that except a finite number of interactions, there is at most one outgoing wave of each family for each interaction. We consider the potential defined in Lemma 3.2: $V_{\xi_0}(t)$, where ξ_0 depends on the compact set $\mathcal{K}'$. We know that this potential is not increasing across interactions of shocks and rarefactions of the same family. Moreover due to Lemma 3.1 it does not change across interactions of waves of different families. The only case in which we have more then one outgoing wave for each family is the interaction of two shocks of the same family. In this case we have to split the rarefaction if its strength is greater than $\eta > 0$. Since in these interactions there holds $\Delta V = \Delta V_s + \Delta V_r = 0$, we have $\Delta V_s \leq -\eta$ and therefore $\Delta V_{\xi_0} = (\xi_0 - 1)\Delta V_s + \Delta V \leq -(\xi_0 - 1)\eta$. But this can happen only a finite number of times because V_{ξ_0} is non increasing and it is finite after any time steps.

Lemma 2.4 and the subsequent considerations hold for the present case without any change.

4 Zero relaxation limit

In this section we study the convergence to zero of the relaxation parameter ε, proving Theorem 1.3. Let $\mathcal{K}$ be a compact subset of $\mathcal{U}$ and denote with L, δ, C the constants in Theorem 1.1 (resp. L, M, C in Theorem 1.2).

Consider a family of initial data, $(u_o^\varepsilon, v_o^\varepsilon)$, either satisfying $\lim_{x\to-\infty} u_o^\varepsilon(x) \in \mathcal{K}$ and

$$\text{Tot.Var. } (u_o^\varepsilon, v_o^\varepsilon) \leq \delta, \tag{4.1}$$

or, respectively, $u_o^\varepsilon(x) \in \mathcal{K}$ for all x and

$$\text{Tot.Var. } (u_o^\varepsilon, v_o^\varepsilon) \leq M, \tag{4.2}$$

for any $\varepsilon > 0$. Moreover, assume that $u_o^\varepsilon \to u_o$ in $\mathbf{L}^1{}_{loc}$ and that $\{v_o^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in the $\mathbf{L}^\infty$–norm. We remark that it is not required to be $A(u_o^\varepsilon) = v_o^\varepsilon$, and that v_o^ε does not need to have limit as $\varepsilon \to 0$.

By Theorem 1.1 (resp. Theorem 1.2) there exists a family of corresponding solutions $(u^\varepsilon, v^\varepsilon)$ to (1.1)–(1.3) (corresp. to (1.16)–(1.3)) in a weak sense. The following equalities are satisfied

$$\int_{-\infty}^{\infty}\int_0^{\infty} u^\varepsilon \phi_t - v^\varepsilon \phi_x \, dxdt = 0, \tag{4.3}$$

$$\int_{-\infty}^{\infty}\int_0^{\infty} \varepsilon v^\varepsilon \phi_t - \varepsilon\sigma(u^\varepsilon)\phi_x + \phi \; r(u^\varepsilon, v^\varepsilon) \, dxdt = 0 \tag{4.4}$$

for all $\phi \in \mathbf{C}_o^\infty((0,\infty)\times\mathbf{R})$. Moreover, for any $a<b$, $t\geq s\geq 0$, there holds

$$\int_a^b |u^\varepsilon(t,x)-u^\varepsilon(s,x)|\,dx \leq L|t-s|, \qquad \int_a^b |v^\varepsilon(t,x)-v^\varepsilon(s,x)|\,dx \leq (L+L_\varepsilon)|t-s| \tag{4.5}$$

To pass to the limit as $\varepsilon\to 0$, we need a better estimate on the Lipschitz constant L_ε at (1.15), hence the proof of Lemma 2.4 must be refined.

Lemma 4.1 *In the same assumptions of Lemma 2.4 and assuming moreover (1.19), the constant L_ε takes the form*

$$L_\varepsilon \doteq \frac{2c}{\varepsilon}\exp(-\frac{sc}{\varepsilon})\cdot\int_a^b |r(u_o^\varepsilon,v_o^\varepsilon)(x)|\,dx. \tag{4.6}$$

Proof. For any $k\geq 1$ define

$$g_k^\pm = \int_a^b |r(u,v)(k\Delta t\pm,x)|\,dx. \tag{4.7}$$

From (2.11) and (1.19), it follows that

$$g_k^+ \leq \left(1-\frac{c\Delta t}{\varepsilon}\right) g_k^-, \qquad \forall k\geq 1. \tag{4.8}$$

where c is the constant at (1.19); moreover,

$$\begin{aligned} g_k^- - g_{k-1}^+ &\leq \int_a^b \left|r(u,v)(k\Delta t-,x) - r(u,v)((k-1)\Delta t+,x)]\right|\,dx \\ &\leq \sup_{|u|,|v|\leq K^\dagger} |r_v| \left(L^*\int_a^b \left|u(k\Delta t-,x)-u((k-1)\Delta t+,x)\right|\,dx\right. \\ &\qquad \left. + \int_a^b \left|v(k\Delta t-,x)-v((k-1)\Delta t+,x)\right|\,dx\right) \\ &\leq K_1\Delta t \end{aligned} \tag{4.9}$$

for a suitable constant $K_1 = 2(L^*)^2 K\cdot\sup|r_v|$, with L^*, $K^\dagger$ as in the proof of Lemma 2.4. Together, (4.8) and (4.9) give:

$$\begin{aligned} g_k^- &\leq K_1\Delta t\cdot\sum_{i=0}^{k-1}\left(1-\frac{c\Delta t}{\varepsilon}\right)^i + \left(1-\frac{c\Delta t}{\varepsilon}\right)^{k-1} g_o^+ \\ &\leq K_1\frac{\varepsilon}{c} + \left(1-\frac{c\Delta t}{\varepsilon}\right)^{k-1} g_o^+. \end{aligned} \tag{4.10}$$

As in (2.46), using(4.10) we get

$$
\begin{aligned}
\sum_{i=k_0}^{\tilde{k}} \int_a^b |v(i\Delta t+,x) - v(i\Delta t-,x)|\,dx &= \frac{\Delta t}{\varepsilon} \sum_{i=k_0}^{\tilde{k}} g_i^- \\
&\leq \Delta t \sum_{i=k_0}^{\tilde{k}} \frac{K_1}{c} + \frac{\Delta t}{\varepsilon} \sum_{i=k_0}^{\tilde{k}} \left(1 - \frac{c\Delta t}{\varepsilon}\right)^{i-1} g_0^+ \\
&\leq \frac{K_1}{c}(t - s + \Delta t) + g_o^+ \left(1 - \frac{c\Delta t}{\varepsilon}\right)^{k_o - 1} \left[1 - \left(1 - \frac{c\Delta t}{\varepsilon}\right)^{\tilde{k} - k_o + 1}\right] \\
&\leq \frac{K_1}{c}(t - s + \Delta t) + 2g_o^+ \left(1 - \frac{c\Delta t}{\varepsilon}\right)^{\frac{k_o \Delta t}{\Delta t}} (\tilde{k} - k_o + 1)\frac{c\Delta t}{\varepsilon} \\
&\leq (t - s + \Delta t) \left[\frac{K_1}{c} + 2\frac{c}{\varepsilon} g_o^+ \exp(-sc/\varepsilon)\right].
\end{aligned}
\tag{4.11}
$$

Hence the lemma is proved and (2.42), (2.43) hold with a larger constant L^*. □

We recall that the constant L at (4.5) does not depend on ε. By the assumptions on the initial data, the constant L_ε, at (4.5), can be bounded uniformly, as $\varepsilon \to 0$, on any set of the type $[\frac{1}{n}, \infty) \times [-n, n]$, for any fixed $n \in \mathbf{N}$: if $t, s \geq \frac{1}{n}$, one has

$$
L_\varepsilon \leq 2C_1 \cdot (b - a)\varepsilon^{-1} \exp\left(-\frac{1}{n\varepsilon}\right) \tag{4.12}
$$

where $C_1 > 0$ depends only on the (uniform) bounds on the initial data. For n fixed, the second term of the right hand side in (4.12) goes to zero with ε.

By Helly's Theorem, there exists a subsequence $\varepsilon_k \to 0$, as $k \to \infty$, such that u^{ε_k} converges to some $\tilde{u}$ in $\mathbf{L}^1{}_{loc}\left([0, \infty) \times (-\infty, \infty)\right)$. Moreover, $\tilde{u}(0, \cdot) = u_o$ and the Lipschitz inequality for u^ε in (4.5) holds also for the limit.

By eventually extracting a subsequence ε_k^1 from ε_k, the sequence $v^{\varepsilon_k^1}$ converges to a limit $\tilde{v}$ in $\mathbf{L}^1{}_{loc}\left((1, \infty) \times (-1, 1)\right)$. Passing to the limit in (4.3)–(4.4) with ε_k^1, one has

$$
\tilde{v}(t, \cdot) = A(\tilde{u}(t, \cdot)), \qquad \tilde{u}_t - A(\tilde{u})_x = 0 \tag{4.13}
$$

on the set $(1, \infty) \times (-1, 1)$.

For any $n \in \mathbf{N}$, one can extract a subsequence of ε_k^{n-1}, ε_k^n, such that $v^{\varepsilon_k^n}$ converges to $\tilde{v}$ in $\mathbf{L}^1{}_{loc}\left((\frac{1}{n}, \infty) \times (-n, n)\right)$. Due to (4.12), (4.5), for any $t, s > \frac{1}{n}$ there holds

$$
\int_{-n}^{n} |\tilde{v}(t, x) - \tilde{v}(s, x)|\,dx \leq L|t - s|. \tag{4.14}
$$

With a diagonalization argument, the sequences $u^{\epsilon_n^n}$, $v^{\epsilon_n^n}$ converge, as $n \to \infty$, to $\widetilde{u}$, $\widetilde{v}$ respectively, on $(0,\infty) \times \mathbf{R}$, then (4.13) holds on the region $(0,\infty) \times \mathbf{R}$. Since the sequence $v^{\epsilon_n^n}$ is equibounded, the convergence takes place in $[0,\infty) \times \mathbf{R}$. Inequality (4.14) is satisfied for any $t, s > 0$ and for all $n \in \mathbf{N}$. Hence there exists, in $\mathbf{L^1}_{loc}$, the limit

$$\lim_{t\to 0} \widetilde{v}(t,\cdot) = \lim_{t\to 0} A(\widetilde{u}(t,\cdot)) = A(u_o).$$

We can therefore define $\widetilde{v}(0,\cdot) \doteq \lim_{t\to 0} \widetilde{v}(t,\cdot)$. The function $\widetilde{u}$ is a weak solution of the scalar equation

$$u_t - A(u)_x = 0, \qquad u(0,\cdot) = u_o, \tag{4.15}$$

with Lipschitz dependence on time.

Concerning entropy conditions, following [5], let $\eta = \eta(u,v)$ be a $\mathbf{C^2}$ convex entropy, with flux q, for the relaxing system (1.1). We require the entropy to be *dissipative* in the sense that

$$\eta_v(u,v) \cdot r(u,v) \le 0, \tag{4.16}$$

for all (u,v) in an open neighborhood of the equilibrium curve $(u, A(u))$. This clearly implies $\eta_v = 0$ along the equilibrium curve. Moreover, let us set

$$\eta^o(u) \doteq \eta(u, A(u)), \qquad q^o(u) \doteq q(u, A(u)). \tag{4.17}$$

In these assumptions, if $A \in \mathbf{C^2}$, the function $\eta^o(u)$ is found to be a convex entropy for the limit equation (4.15), with corresponding flux $q^o(u)$. Using (2.47) and (4.16), one obtains that the limit function $\widetilde{u}$ satisfies the entropy inequality for the scalar equation

$$\int_0^\infty \int_{-\infty}^\infty \eta^o(\widetilde{u})\phi_t + q^o(\widetilde{u})\phi_x \, dxdt \; + \; \int_{-\infty}^\infty \eta^o(u_o)(x)\phi(0,x)\, dx \ge 0. \tag{4.18}$$

However, one would like to find out if $\widetilde{u}$ is the unique entropy solution of (4.15) or not; this is still not clear at the moment and requires further investigation. A partial answer can be given following the lines of [5] and [10].

A Appendix

Proposition A.1 *Let $\mathcal{K} \subset \mathcal{R}^n$ be a compact set and $g : \mathcal{K} \times [0,\delta_1] \times [0,\delta_2] \to \mathbf{R}$ be a $\mathbf{C^4}$ function satisfying for any $u \in \mathcal{K}$: $g(u,x,0) = g(u,0,y) = 0$,*

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} g(u,0,0) = 0 \tag{A.1}$$

for every α,β satisfying $\alpha+\beta\leq 4$, $(\alpha,\beta)\neq(1,3),(3,1)$, and

$$\frac{\partial^4}{\partial x^3\partial y}g(u,0,0)=\frac{\partial^4}{\partial x\partial y^3}g(u,0,0)=a(u)>0. \tag{A.2}$$

Then there exists $\delta>0$, such that $g(u,x,y)\geq 0$ for any $(u,x,y)\in\mathcal{K}\times[0,\delta)\times[0,\delta)$.

Proof. We can write

$$\begin{aligned} g(u,x,y) &= g(u,x,y)-g(u,x,0)=\int_0^y \frac{\partial}{\partial y'}g(u,x,y')\,dy' \\ &= \int_0^y \left[\frac{\partial}{\partial y'}g(u,x,y')-\frac{\partial}{\partial y'}g(u,0,y')\right]\,dy' \\ &= \int_0^y\int_0^x \frac{\partial^2}{\partial x'\partial y'}g(u,x',y')\,dx'\,dy'. \end{aligned} \tag{A.3}$$

Since $\frac{\partial^2}{\partial x\partial y}g(u,x,y)$ is $\mathbf{C}^2$, from the Taylor expansion in Lagrange form one has:

$$\begin{aligned} \frac{\partial^2}{\partial x\partial y}g(u,x,y) &= \frac{1}{2}\left[x^2\frac{\partial^4}{\partial x^3\partial y}g(u,\theta x,\theta y)\right. \\ &\left. +2xy\frac{\partial^4}{\partial x^2\partial y^2}g(u,\theta x,\theta y)+y^2\frac{\partial^4}{\partial x\partial y^3}g(u,\theta x,\theta y)\right] \end{aligned} \tag{A.4}$$

where $\theta\in(0,1)$. a is a continuous function on the compact set $\mathcal{K}$, hence it has a minimum a_0. Since the derivatives in the right hand side of (A.4) are uniformly continuous on the compact set $\mathcal{K}\times[0,\delta_1]\times[0,\delta_2]$, there exists $\delta>0$ such that $\forall(u,x,y)\in\mathcal{K}\times[0,\delta)\times[0,\delta)$ one has

$$\begin{aligned} &\frac{\partial^4}{\partial x^3\partial y}g(u,\theta x,\theta y)>\frac{a_0}{2} \quad , \quad \frac{\partial^4}{\partial x^2\partial y^2}g(u,\theta x,\theta y)>-\frac{a_0}{2}, \\ &\frac{\partial^4}{\partial x\partial y^3}g(u,\theta x,\theta y)>\frac{a_0}{2}. \end{aligned} \tag{A.5}$$

Therefore (A.4) becomes

$$\begin{aligned} \frac{\partial^2}{\partial x\partial y}g(u,x,y) &\geq \frac{1}{2}\left[\frac{a_0}{2}x^2-\frac{a_0}{2}xy+\frac{a_0}{2}y^2\right] \\ &= \frac{1}{4}a(u)(x-y)^2\geq 0 \end{aligned} \tag{A.6}$$

for any $(u,x,y)\in\mathcal{K}\times[0,\delta)\times[0,\delta)$. (A.6) and (A.3) imply the proposition. □

Proposition A.2 *Let $\mathcal{K}\subset\mathcal{R}^n$ be a compact set and $g:\mathcal{K}\times[0,\delta_1]\times[0,\delta_2]\to\mathbf{R}$ be a $\mathbf{C}^4$ function satisfying for any $u\in\mathcal{K}$: $g(u,x,0)=g(u,0,y)=g_x(u,0,y)=g_{xx}(u,0,y)=0$,*

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\partial y^\beta}g(u,0,0)=0 \tag{A.7}$$

for every α, β satisfying $\alpha + \beta \leq 4$ and $(\alpha, \beta) \neq (3,1)$. Moreover assume that for any $u \in \mathcal{K}$ there holds

$$\frac{\partial^4}{\partial x^3 \partial y} g(u,0,0) = a(u) > 0. \tag{A.8}$$

Then there exists $\delta > 0$, such that $g(u,x,y) \geq 0$ for any $(u,x,y) \in \mathcal{K} \times [0,\delta) \times [0,\delta)$.

Proof. The proposition is proved if one shows that $\frac{\partial^2}{\partial x \partial y} g(u,x,y)$ is nonnegative in $\mathcal{K} \times [0,\delta) \times [0,\delta)$ for some $\delta > 0$ (see the proof of Proposition A.1). Therefore we write

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} g(u,x,y) &= \frac{\partial^2}{\partial x \partial y} g(u,x,y) - \frac{\partial^2}{\partial x \partial y} g(u,0,y) \\ &= \int_0^x \frac{\partial^3}{\partial x'^2 \partial y} g(u,x',y)\, dx' \\ &= \int_0^x \left[\frac{\partial^3}{\partial x'^2 \partial y} g(u,x',y) - \frac{\partial^3}{\partial x'^2 \partial y} g(u,0,y) \right] dx' \\ &= \int_0^x \int_0^{x'} \frac{\partial^4}{\partial x'^3 \partial y} g(u,x'',y)\, dx'' dx'. \end{aligned} \tag{A.9}$$

Observing that the function $\frac{\partial^4}{\partial x^3 \partial y} g(u,x,y)$ is uniformly continuous in $\mathcal{K} \times [0,\delta_1] \times [0,\delta_2]$ and positive at $(u,0,0)$, for any $u \in \mathcal{K}$ the theorem is proved. □

Proof of Lemma 3.2. To simplify the notations we set

$$\Psi(u_1,u_2) = \sqrt{(u_1 - u_2)\left(\frac{1}{u_2} - \frac{1}{u_1}\right)}, \tag{A.10}$$

$$\Phi(u_1,u_2) = \ln \frac{u_2}{u_1}. \tag{A.11}$$

Consider a wave interaction pattern as in Figure 2. If the outgoing waves are both shocks, one for each family, the equation defining u^* are:

$$W(u_l,u_r,u_m,u^*) = 0 \qquad \text{for 1s–1r, 2s–2r}, \tag{A.12}$$

$$\widetilde{W}(u_l,u_r,u_m,u^*) = 0 \qquad \text{for 1r–1s, 2r–2s}, \tag{A.13}$$

where

$$W(u_l,u_r,u_m,u^*) = \Psi(u_l,u^*) + \Psi(u_r,u^*) - \Psi(u_l,u_m) + \left|\Phi(u_m,u_r)\right|, \tag{A.14}$$

$$\widetilde{W}(u_l,u_r,u_m,u^*) = W(u_r,u_l,u_m,u^*). \tag{A.15}$$

We claim that there exists $\kappa_o < 1$ depending only on the compact set $\mathcal{K}'$ such that for the first case (A.14) there holds:

$$u^* \geq u_l \left(\frac{u_r}{u_m}\right)^{\kappa}, \qquad \forall \kappa \in [\kappa_o, 1] \tag{A.16}$$

and for the second case (A.15)

$$u^* \geq u_r \left(\frac{u_l}{u_m}\right)^{\kappa}, \qquad \forall \kappa \in [\kappa_o, 1]. \tag{A.17}$$

We will prove only (A.16) (note that (A.17) is symmetric with respect to u_l and u_r). From Lagrange formula we get

$$\begin{aligned} W\left(u_l, u_r, u_m, u_l \left(\frac{u_r}{u_m}\right)^{\kappa}\right) &= W\left(u_l, u_r, u_m, u_l \frac{u_r}{u_m}\right) + \\ +u_l(\kappa-1)\left(\frac{u_r}{u_m}\right)^{\tilde{\kappa}} \ln \frac{u_r}{u_m} \cdot \frac{\partial}{\partial u^*} & W\left(u_l, u_r, u_m, u_l \left(\frac{u_r}{u_m}\right)^{\tilde{\kappa}}\right) \\ &\geq \left|\ln \frac{u_m}{u_r}\right| \left[1 + (1-\kappa) h(u_l, u_r, u_m, \tilde{\kappa})\right] \end{aligned} \tag{A.18}$$

where $\kappa \leq \tilde{\kappa} \leq 1$ and h is a suitable function bounded by a constant C^*, that can be chosen greater than 2, depending only on the compact set $\mathcal{K}'$. Hence if we take $\kappa_o = 1 - \frac{1}{C^*}$ we obtain that

$$W\left(u_l, u_r, u_m, u_l \left(\frac{u_r}{u_m}\right)^{\kappa}\right) \geq 0 \qquad \forall \kappa \in [\kappa_o, 1]. \tag{A.19}$$

The claim is proved by observing that the function $u^* \mapsto W(u_l, u_r, u_m, u_l, u^*)$ is decreasing. Now we fix $\xi_0 = \frac{1}{2\kappa_o - 1}$; using (A.16), consider the two cases (with $\xi \in [1, \xi_0]$):

- 2s–2r:

$$\begin{aligned} \Delta[\xi V_s + V_r] &= \xi \left[\ln \frac{u_l}{u^*} + \ln \frac{u_r}{u^*} - \ln \frac{u_m}{u_l}\right] - \xi \ln \left(\frac{u_m}{u_r}\right)^{\frac{1}{\xi}} \\ &= 2\xi \ln \left[\frac{u_l}{u^*} \left(\frac{u_r}{u_m}\right)^{(1+\xi)/2\xi}\right] \leq 2\xi \ln 1 = 0; \end{aligned} \tag{A.20}$$

- 1s–1r:

$$\begin{aligned} \Delta[\xi V_s + V_r] &= \xi \left[\ln \frac{u_l}{u^*} + \ln \frac{u_r}{u^*} - \ln \frac{u_l}{u_m}\right] - \xi \ln \left(\frac{u_r}{u_m}\right)^{\frac{1}{\xi}} \\ &= \xi \ln \left[\frac{u_m u_r}{u^{*2}} \left(\frac{u_m}{u_r}\right)^{1/\xi}\right] \leq \xi \ln \left[\frac{u_m^2}{u_l^2} \left(\frac{u_m}{u_r}\right)^{2/\xi}\right] \leq 0; \end{aligned} \tag{A.21}$$

where the last inequality in (A.21) is obtained observing that here there holds $u_l, u_r > u_m$.

The proof for the 1r–1s and 2r–2s is completely similar.

On the other hand, if the outgoing waves are a shock and a rarefaction then u^* is no longer defined by (A.12) and (A.13). We consider only the example 2r–2s with a 1s and a 2r outgoing from the interaction, the other cases being similar. The equation defining u^* is given by:

$$\Phi(u_m, u_l) - \Psi(u_m, u_r) = -\Psi(u^*, u_l) + \Phi(u_r, u^*), \tag{A.22}$$

that, recalling (A.11), can be written as

$$\Phi(u^*, u_l) + \Psi(u^*, u_l) = \Psi(u_m, u_r) - \Phi(u_m, u_r). \tag{A.23}$$

From definitions (A.10) and (A.11) one can see that there are constants $\overline{c}$ and $\overline{C}$ depending only on the compact set $\mathcal{K}'$ satisfying

$$\Phi(u_m, u_r) \geq \overline{c}(u_r - u_m), \qquad \Psi(u_m, u_r) \leq \overline{C}(u_r - u_m). \tag{A.24}$$

Hence if we choose $\xi_0 = 1 + \overline{c}/\overline{C}$, recalling (A.23) and the inequality (2.3), for any $\xi \in [1, \xi_0]$ we can write

$$\begin{aligned} \Delta\left[\xi V_s + V_r\right] &= \xi\Phi(u^*, u_l) + \Phi(u_r, u^*) - \Phi(u_m, u_l) - \xi\Phi(u_m, u_r) \\ &\leq (\xi - 1)\Phi(u^*, u_l) - \Phi(u_m, u_r) \\ &\leq \frac{(\xi - 1)}{2}\left[\Phi(u^*, u_l) + \Psi(u^*, u_l)\right] - \Phi(u_m, u_r) \\ &= \frac{(\xi - 1)}{2}\left[\Psi(u_m, u_r) - \Phi(u_m, u_r)\right] - \Phi(u_m, u_r) \\ &\leq \frac{u_r - u_m}{2}\left[(\xi - 1)\overline{C} - \overline{c}\right] \leq 0. \end{aligned} \tag{A.25}$$

The proof is complete. □

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