Uniformly Accurate Diffusive Relaxation Schemes for Multiscale Transport Equations

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Abstract

Many transport equations, such as the neutron transport, radiative transfer, and transport equations for waves in random media, have a diffusive scaling that leads to the diffusion equations. In many physical applications, the scaling parameter (mean free path) may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes within one problem, and it is desirable to develop a class of robust numerical schemes that can work uniformly with respect to this relaxation parameter. In an earlier work [JPT] we handled this numerical problem for discrete-velocity kinetic models by reformulating the system into a form commonly used for a relaxation scheme for conservation laws [JX]. Such a reformulation allows us to use the splitting technique for relaxation schemes to design a class of implicit, yet explicitly implementable, schemes that work with high resolution unformly with respect to the relaxation parameter. In this paper we show that such a numerical technique can be applied to a large class of transport equations with continuous velocities, when one uses the even and odd parities of the transport equation.

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1. Introduction

Many transport equations, such as the neutron transport [CZ], radiative transfer [Cha], and transport equations for waves in random media [RPK], have a diffusive scaling that leads to diffusion equations. In many physical applications, the scaling parameter (mean free path) may differ in several orders of magnitude from the rarefied regimes to the hydrodynamic (diffusive) regimes within one problem, and it is desirable to develop a class of robust numerical schemes that can work uniformly with respect to this parameter.

In an earlier work on discrete-velocity models [JPT] we handled this challenging numerical problem by reformulating the system into a form commonly used for a relaxation scheme for conservation laws [JX]. Such a reformulation allows us to use the splitting technique for relaxation schemes to design a class of implicit, yet *explicitly implementable*, schemes that work with high resolution unformly with respect to the scaling parameter. In this paper we show that such a numerical technique can be applied to a large class of transport equations with continuous velocities, when one uses the even and odd parities of the transport equation.

Although we are aiming at developing numerical schemes for transport equations, the study of the diffusive limit has a critical importance. In order for a scheme to be useful not only in the rarefied regime (where the mean free path ϵ is big) but also in the hydrodynamic regime (where ϵ is small) allowing the use of $\Delta t, \Delta x >> \epsilon^2$, where Δt and Δx are the time steps and grid size respectively, the scheme should be asymptotic preserving (AP), in the sense that the asymptotic limit that leads from the transport equations to the diffusion equations should be preserved at the discrete level. Past progress on developing robust numerical schemes for transport equations that also work in the diffusive regimes has been strongly guided by the study of the diffusion limit [Ada, Jin, JL1, JL2, JPT, Kl1, Kl2, LMM, LM, Mil, NP].

To motivate the idea we first illustrate the major ingredient that leads to the development of the diffusive relaxation schemes in [JPT] for discrete-velocity kinetic models. Consider the one dimensional Goldstein-Taylor model [Go, Ta] under the diffusive scaling

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{1}{\epsilon}\frac{\partial u}{\partial x} &= \frac{1}{\epsilon^2}(v-u)\,, \\ \frac{\partial v}{\partial t} - \frac{1}{\epsilon}\frac{\partial v}{\partial x} &= \frac{1}{\epsilon^2}(u-v)\,, \end{aligned} \tag{1.1}$$

where $(x,t) \in \mathbf{R} \times \mathbf{R}^+$. In (1.1) ϵ is the positive relaxation parameter. The macroscopic (fluid) variables for this model are the mass density ρ and the flux j, defined by

$$\rho = u + v\,, \qquad j = \epsilon^{-1}(u-v)\,. \tag{1.2}$$

Since u and v can be expressed in terms of ρ and j, system (1.1) is equivalent to the following system for the mass density and the flux

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{1}{\epsilon^2}\frac{\partial \rho}{\partial x} &= -\frac{2}{\epsilon^2} j\,. \end{aligned} \tag{1.3}$$

In the zero relaxation (or diffusion) limit ($\epsilon \to 0^+$), system (1.3) can be approximated to leading order by

$$j = -\frac{1}{2}\frac{\partial \rho}{\partial x}\,, \tag{1.4a}$$

$$\frac{\partial \rho}{\partial t} - \frac{\partial}{\partial x}\frac{1}{2}\frac{\partial \rho}{\partial x} = 0\,. \tag{1.4b}$$

The state satisfying (1.4a) will be called the *local equilibrium,* while (1.4b) is the equation of continuum mechanics generated by the kinetic model, that in this case is the heat equation.

Note that solving (1.3) numerically in challenging due to the stiffness of the problem for both the convection and collision terms. If an explicit scheme is used one needs a CFL condition $\Delta t < \epsilon \Delta x$, which is very expensive when ϵ is small. If an implicit scheme is used, due to the nonlinear nature of the numerical flux for the convection term, a major ingredient for all modern high order shock capturing schemes for hyperbolic conservation laws, one needs an iterative method, such as the Newton's method, to solve the resulting nonlinear systems, which significantly complicates the algorithm.

Our key idea in [JPT] to attack this problem is to reformulate (1.3) as a (nonstiff) linear hyperbolic system with stiff relaxation term (hereafter called *the diffusive relaxation system*)

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0 \\ &\frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} = -\frac{2}{\epsilon^2}\left[j + \frac{1-\epsilon^2}{2}\frac{\partial \rho}{\partial x}\right]\,. \end{aligned} \tag{1.5}$$

It is immediately recognizable that system (1.5) has the form used to construct relaxation schemes for conservation laws by Jin and Xin [JX]. There a given one dimensional hyperbolic scalar conservation law of the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}F(\rho) = 0\,, \tag{1.6}$$

with initial data

$$\rho(x,0) = \rho_0(x) \geq 0\,,$$

is approximated by the following linear hyperbolic system with a stiff relaxation term

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} &= 0\,, \\ \frac{\partial j}{\partial t} + \frac{\partial \rho}{\partial x} &= -\frac{1}{\epsilon}\left[j - F(\rho)\right]\,, \end{aligned} \tag{1.7}$$

with initial data

$$\rho^\epsilon(x,0) = \rho_0(x)\,, \quad j^\epsilon(x,0) = j_0(x) = F(\rho_0(x))\,.$$

In such a problem only the source term is stiff, thus a proper splitting of an explicit convection and an implicit source term suffices to give a scheme with a formal uniform accuracy in ϵ, yet such a scheme can be implemented *explicitly.*

In [JPT] we developed a class of numerical schemes, called *diffusive relaxation schemes*, for (1.1) and other discrete-velocity models in any space dimensions, that have the following features. First, they have an ϵ-independent, parabolic CFL condition $\Delta t \approx (\Delta x)^2$, which is the only natural condition in the diffusive regime. The schemes also capture shocks and other types of discontinuities with high resolutions, and can be implemented explicitly. Moreover, they capture the correct diffusive behavior even when $\Delta t, \Delta x >> \epsilon$, thus are AP. Our schemes utilized the second order nonlinear upwind discretization on the convection term, an implicit center difference for the source term near the parabolic regime, and a second order Runge-Kutta splitting method developed in [CJR].

In this paper we show how to develop schemes that have similar properties for time-dependent, continuous-velocity transport equations that work uniformly with respect to the relaxation parameter. Of course, if we discretize the velocity, all transport equations become the discrete-velocity equations, and the technique used in [JPT] could apply. However, we want to emphasize that the formulation presented in this paper, based on the even and odd parities of the equation, allows one to write the transport equation in the form such that the diffusion equation arises at the leading order expansion, while the Hilbert expansion directly performed on the original transport equation does not lead to the diffusion equation until $O(\epsilon^2)$. A numerical advantage is that with the new form, the AP property can be easily obtained. After this formulation, any way of discretizing the velocity applies.

The even parity equation is a classical method for steady state calculation of the neutron transport, since in the steady state problem one can eliminate the odd-parity and derives a second order equation for the even-parity [LMi]. In the time-independent case, it possesses the diffusion limit, even in the discrete case [Ada, JL2, Mil]. However, to solve the time-independent transport equation, in particularly in the diffusive regimes, require delicated iteration techniques, such as the Diffusion Synthetic Acceleration [Alc], where the diffusion equations are used as preconditioners. Although such acceleration techniques are available for many existing methods, they are problem dependent, and lack a generality for nonlinear problems and transport problems where the diffusion approximations are not available. Our method, besides it suitability for time-dependent problems, can also be used to compute steady state problems with ease due to its explicit nature. The main idea here is that with the equations of even- and odd-parities, we put the transport equation in a form similar to the Goldstein-Taylor equations (1.3), thus one can easily apply the numerical methods developed in [JPT] to solve these transport equations.

The rest of the paper is organized as follows. In section 2 we review the diffusion limit for the general transport equation in $\mathbb{R}^d$ using the Hilbert expansion. The reformulation to a diffusive relaxation system, and the construction of diffusive relaxation schemes for the one-dimensional equation is described in detail in section 3. In section 4 we derive two dimensional diffusive relaxation problems. A uniform consistency error of such a scheme for smooth solutions is proved in section 5. Numerical results are given in section 6. We conclude the paper with some remarks in section 7, followed by a separate Appendix in which we give the diffusive relaxation scheme for two-dimensional transport equation in polar coordinates.

2. The Transport Equation and its Diffusion Limit

We consider the multidimensional transport equation under the diffusive scaling. Let $f(t, \mathbf{x}, \mathbf{v})$ be the probability density distribution for particles at space point $\mathbf{x} \in \mathbb{R}^d$, time t, and travel in direction $\mathbf{v} \in \Omega \subset \mathbb{R}^d$, with $\int_\Omega d\mathbf{v} = S$. Here Ω is symmetric in $\mathbf{v}$, meaning that $\int_\Omega g(\mathbf{v}) d\mathbf{v} = 0$ for any function g odd in $\mathbf{v}$. Then f solves the linear transport equation

$$\epsilon\, \partial_t f + \mathbf{v} \cdot \nabla_x f = \frac{1}{\epsilon} \left[\frac{\sigma_s}{S} \int_\Omega f\, d\mathbf{v}' - \sigma f \right] + \epsilon Q, \tag{2.1}$$

where $\sigma = \sigma(\mathbf{x})$ is the total transport coefficient, $\sigma_S = \sigma_S(\mathbf{x})$ is the scattering coefficient, $Q = Q(\mathbf{x})$ is the source term, ϵ is the mean free path. Notice that in (2.1), the time variable has been rescaled to the long-time scale $O(1/\epsilon)$. Typically,

$$\sigma_S = \sigma - \epsilon^2 \sigma_A\,, \tag{2.2}$$

where $\sigma_A = \sigma_A(\mathbf{x})$ is the absorption coefficient. Such equation arises in neutron transport [CZ], radiative transfer [Cha], and wave propagation in random media [RPK], etc. In all these applications, the scaling appeared in (2.1) is typical, and give rise to a diffusion equation as $\epsilon \to 0$.

To study the diffusion approximation, one may use the Hilbert expansion [LK]. Introduce the ansatz

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots \tag{2.3}$$

in (2.1) and balance the terms by the order in ϵ.

To the leading order $O(1/\epsilon)$

$$f^{(0)}(t, \mathbf{x}, \mathbf{v}) = \frac{1}{S} \int_\Omega f^{(0)} d\mathbf{v}' \equiv \rho^{(0)}(t, \mathbf{x}). \tag{2.4}$$

Thus the leading term is isotropic (independent of $\mathbf{v}$). To $O(1)$,

$$\mathbf{v} \cdot \nabla_x f^{(0)} = \frac{\sigma}{S} \int_\Omega f^{(1)} d\mathbf{v}' - \sigma f^{(1)} . \tag{2.5}$$

Let

$$\rho^{(1)}(t, \mathbf{x}) = \frac{1}{S} \int_\Omega f^{(1)} d\mathbf{v}',$$

then

$$f^{(1)} = \rho^{(1)} - \frac{\mathbf{v}}{\sigma} \cdot \nabla_x \rho^{(0)}. \tag{2.6}$$

Clearly, the solvability condition is satisfied by (2.6) since $(\mathbf{v}/\sigma) \cdot \nabla_x \rho^{(0)}$ is odd in $\mathbf{v}$.

Consider the $O(\epsilon)$ balance

$$\partial_t \rho^{(0)} + \mathbf{v} \cdot \nabla_x f^{(1)} = \frac{\sigma}{S} \int_\Omega f^{(2)} d\mathbf{v}' - \sigma f^{(2)} - \sigma_A \rho^{(0)} + Q . \tag{2.7}$$

Now integrating (2.7) over Ω, and using (2.4) and (2.6), one gets

$$\partial_t \rho^{(0)} + \int_\Omega \mathbf{v} \cdot \nabla_x \left(\rho^{(1)} - \frac{\mathbf{v}}{\sigma} \cdot \nabla_x \rho^{(0)} \right) d\mathbf{v} = -\sigma_A \rho^{(0)} + Q . \tag{2.8}$$

This implies

$$\partial_t \rho^{(0)} - \frac{1}{S} \int_\Omega \mathbf{v} \cdot \nabla \left(\frac{\mathbf{v}}{\sigma} \cdot \nabla_x \rho^{(0)} \right) d\mathbf{v} = -\sigma_A \rho^{(0)} + Q . \tag{2.9}$$

For most transport problems, one can integrate (2.9) over Ω to get

$$\partial_t \rho^{(0)} = D \nabla_x \cdot \left(\frac{1}{\sigma} \nabla_x \rho^{(0)} \right) - \sigma_A \rho^{(0)} + Q , \tag{2.10}$$

which is the diffusion approximation of (2.1).

Remark 1 : For different collision operators, the diffusion coefficient D in (2.10) may be different. For examples, $D = 1/3$ in one-dimensional slab geometry as shown in section 3, and $D = 1/2$ when Ω is a unit sphere in two-dimensions as will be discussed in section 4.

Remark 2 : This diffusion approximation can also be derived using the so-called even- and odd-parities, as will be made specific in sections 3 and 4. One can write an equivalent system of equations for the two parities, which gives the diffusion equation in the leading order. This is instrumental in the derivation of the diffusive relaxation schemes.

In the next sections, for the sake of clarity we will derive the schemes for the one-dimensional and the two-dimensional transport equations in the simplified situation where $Q = 0$, $\sigma_A = 0$ and $\sigma = \sigma_S = 1$. The extension of the method to the general case does not present additional difficulties and will be considered at the end of each section.

3. 1-D Method

3.1. Reformulation to diffusive relaxation system

Consider the one-dimensional transport equation

$$\epsilon \partial_t f + v \partial_x f = \frac{1}{\epsilon} \left(\frac{1}{2} \int_{-1}^{1} f dv' - f \right), \qquad x_L < x < x_R, \quad -1 \leq v \leq 1, \tag{3.1}$$

with boundary conditions

$$\begin{aligned} f(t, x_L, v) &= F_L(v), \qquad \text{for} \quad v > 0, \\ f(t, x_R, -v) &= F_R(v), \qquad \text{for} \quad v > 0. \end{aligned}$$

Split this equation as two equations, each for $v > 0$:

$$\begin{aligned} \epsilon \partial_t f(v) + v \partial_x f(v) &= \frac{1}{\epsilon} \left(\frac{1}{2} \int_{-1}^{1} f dv - f(v) \right), \\ \epsilon \partial_t f(-v) - v \partial_x f(-v) &= \frac{1}{\epsilon} \left(\frac{1}{2} \int_{-1}^{1} f dv - f(-v) \right). \end{aligned} \tag{3.2}$$

In this case the even- and odd-parities are

$$\begin{aligned} r(t, x, v) &= \frac{1}{2} [f(t, x, v) + f(t, x, -v)], \\ j(t, x, v) &= \frac{1}{2\epsilon} [f(t, x, v) - f(t, x, -v)]. \end{aligned}$$

Adding and subtracting the two equations in (3.2) lead to

$$\begin{aligned} \partial_t r + v\partial_x j &= \frac{1}{\epsilon^2}(\rho - r), \\ \partial_t j + \frac{v}{\epsilon^2}\partial_x r &= -\frac{1}{\epsilon^2}\, j, \end{aligned} \tag{3.3}$$

where

$$\rho(t,x) = \int_0^1 r\,dv.$$

This system looks similar to the Goldstein-Taylor model, except the right hand side of the first equation which is not critical in the numerical implementation. Now, as was done in [JPT], we rewrite (3.3) into the following diffusive relaxation system

$$\begin{aligned} \partial_t r + v\partial_x j &= -\frac{1}{\epsilon^2}\,(r - \rho), \\ \partial_t j + \phi v\partial_x r &= -\frac{1}{\epsilon^2}\,(j + (1 - \epsilon^2\phi)v\partial_x r). \end{aligned} \tag{3.4}$$

where $\phi = \phi(\epsilon)$ is such that $0 \le \phi \le 1/\epsilon^2$. The simplest choiche of ϕ is

$$\phi(\epsilon) = \min\left\{ 1, \frac{1}{\epsilon} \right\}. \tag{3.5}$$

As $\epsilon \to 0$, system (3.4) gives

$$r = \rho, \qquad j = -v\partial_x r.$$

Applying this in the first equation of (3.4) and integrating over v we get the diffusion equation

$$\partial_t \rho = \frac{1}{3}\partial_{xx}\rho. \tag{3.6}$$

Thus (3.4) guarantees that we will get the correct diffusion limit numerically if we discretize it in an appropriate way.

3.2. The diffusive relaxation scheme

A natural splitting on (3.4) is consist of combining the relaxation step

$$\begin{aligned} \partial_t r &= -\frac{1}{\epsilon^2}\,(r - \rho), \\ \partial_t j &= -\frac{1}{\epsilon^2}\,(j + (1 - \epsilon^2\phi)v\partial_x r), \end{aligned} \tag{3.7}$$

with the transport step

$$\begin{aligned} \partial_t r + v\partial_x j &= 0, \\ \partial_t j + \phi v \partial_x r &= 0. \end{aligned} \tag{3.8}$$

Step (3.8) can be solved using an explicit scheme, whereas for step (3.7) we have to use an implicit scheme. For example, the backward Euler method

$$\begin{aligned} \frac{r^* - r^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left(r^* - \rho^*\right), \\ \frac{j^* - j^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left(j^* + (1-\epsilon^2\phi)v\partial_x r^*\right). \end{aligned} \tag{3.9}$$

Although we have an implicit relaxation term, (3.9)can be implemented explicitly, due to the observation that ρ is preserved in (3.9), i.e., $\rho^* = \rho^n$ if one integrates the first equation of (3.9) over v. Thus it can be replaced by

$$\frac{r^* - r^n}{\Delta t} = -\frac{1}{\epsilon^2}(r^* - \rho^n). \tag{3.10}$$

Now one can solve (3.10) for r^* explicitly, and then get j^* from (3.9) also explicitly provided the spatial derivative on the right hand side is discretized using linear schemes. Thus our implicit scheme, yielding an ϵ-independent numerical stability, can be implemented explicitly. The spatial derivative that appear in (3.7) is approximated using central difference as in [JPT].

To evaluate ρ, we use the Gaussian quadrature rule

$$\rho = \int_0^1 r\,dv \approx \sum_{i=1}^{N} r_i z_i,$$

where $r_i = r(\tau_i)$ and (τ_i, z_i) are the Gaussian quadrature points and weights on $(0,1)$.

During the convection step (3.8), we use the second order upwind scheme with slope limiters on its diagonal form

$$\begin{aligned} \partial_t U + \phi^{1/2} v \partial_x U &= 0, \\ \partial_t V - \phi^{1/2} v \partial_x V &= 0, \end{aligned} \tag{3.11}$$

where $U = r + \phi^{-1/2} j, V = r - \phi^{-1/2} j$ are the Riemann invariants.

To get a boundary condition for r and j, we use relations

$$r + \epsilon j\,|_{x=x_L} = F_L(v)\,, \qquad r - \epsilon j\,|_{x=x_R} = F_R(v)\,. \tag{3.12}$$

When $\epsilon << 1$, a reasonable approximation of the second equation in (3.3) is

$$j = -v\partial_x r. \tag{3.13}$$

Actually, (3.13) is exact for steady state solutions. Applying this in (3.12) one gets

$$r - \epsilon v \partial_x r|_{x=x_L} = F_L(v)\,, \qquad r + \epsilon v \partial_x r|_{x=x_R} = F_R(v)\,. \tag{3.14}$$

Thus by implementing (3.13) and (3.14) we get the desired boundary conditions also for U and V.

Remark 3 : In the general case the splitting method applied to the diffuse relaxation system gives

$$\begin{aligned} \partial_t r &= -\frac{\sigma_S}{\epsilon^2}\,(r-\rho), \\ \partial_t j &= -\frac{1}{\epsilon^2}\,(\sigma_S j + (1-\epsilon^2\phi) v \partial_x r), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} \partial_t r + v\partial_x j &= -\sigma_A r + Q, \\ \partial_t j + \phi v \partial_x r &= -\sigma_A j. \end{aligned} \tag{3.16}$$

Step (3.15) can be evaluated exactly as before, whereas the source term and the absorption terms in (3.16) are treated in an explicit way. At the boundaries we use the equilibrium approximation $j = -(v/\sigma)\,\partial_x r$. It is easy to show that this splitting preserves the correct diffusion limit as $\epsilon \to 0$.

3.3 The numerical diffusion limit

Now we consider the behavior of the scheme when $\epsilon \to 0$ for a fixed time discretization. The fully space-time discretized splitting scheme can be written as

$$\begin{aligned} r^{k+1/2} &= r^k - v\frac{\Delta t}{\Delta x} D^u j^k, \\ j^{k+1/2} &= j^k - v\frac{\Delta t}{\Delta x} \Phi D^u r^k, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} r^{k+1} &= A r^{k+1/2} + B\rho^{k+1/2}, \\ j^{k+1} &= A j^{k+1/2} - B(1-\epsilon^2\phi) v \frac{D^c}{\Delta x} r^{k+1}, \end{aligned} \tag{3.18}$$

where $A = \epsilon^2/(\epsilon^2 + \Delta t)$, $A + B = 1$, and the operators D^u and D^c denotes respectively the upwind discretization and the centered discretization of the spatial derivatives. Considering the relaxation step (3.18), as $\epsilon \to 0$,

$$r^{k+1} = \rho^{k+1/2}, \quad j^{k+1} = -v\frac{D^c}{\Delta x}\rho^{k+1/2}.$$

Combining the two steps together and using the fact that $\rho^k = \rho^{k-1/2}$ we obtain

$$
\begin{aligned}
r^{k+1} &= \rho^k - \frac{\Delta t}{\Delta x} D^u \left(\int_0^1 j^k \, v \, dv \right) \\
&= \rho^k + \frac{\Delta t}{\Delta x} D^u \left(\frac{D^c}{\Delta x} \rho^{k-1/2} \int_0^1 v^2 \, dv \right) \\
&= \rho^k + \frac{1}{3} \frac{\Delta t}{(\Delta x)^2} D^u D^c \rho^k .
\end{aligned}
$$

Integrating over v we get

$$
\rho^{k+1} = \rho^k + \frac{1}{3} \frac{\Delta t}{(\Delta x)^2} D^u D^c \rho^k . \tag{3.19}
$$

Equation (3.19) is a consistent and stable numerical discretization of the diffusion equation (3.6) under a parabolic CFL condition of the type $\Delta t \approx (\Delta x)^2$.

Remark 4 : In [KL1], Klar also used a similar decomposition to construct the scheme that can work effectively in the diffusive regime. There are two major differences between our approach and his. First, he still performs his computation in the whole velocity space, while we, by taking advantages of the symmetry of the even- and odd-parities, compute only in Ω^+, where all velocity components are non-negative. This is much faster and more efficient, and the difference is more striking in higher dimensions. In fact, Klar's formulation does not fit into our program in dimension higher than one. Secondly, our formulation of the diffusive relaxation system allows us to derive a scheme that works uniformly in ϵ, namely, it works not only in the diffusive regime but also in the rarefied regime, while Klar's splitting, which corresponds to the choice $\phi = 0$ in our method, works primarily in the diffusive regime.

4. 2-D Method

4.1. Reformulation to diffusive relaxation system

Consider the two-dimensional transport equation

$$
\epsilon \, \partial_t f + \mathbf{v} \cdot \nabla_x f = \frac{1}{\epsilon} \left(\frac{1}{2\pi} \int_{|\mathbf{v}|=1} f d\mathbf{v}' - f \right), \qquad \mathbf{x} \in \Gamma \subset \mathbb{R}^2, \quad |\mathbf{v}| = 1, \tag{4.1}
$$

with $\mathbf{v} = (\xi, \eta)$,$-1 \leq \xi, \eta \leq 1$, $\xi^2 + \eta^2 = 1$. At the boundary, we have

$$
f(t, \mathbf{x}, \mathbf{v}) = F_B(\mathbf{x}, \mathbf{v}) \qquad \text{for} \quad \mathbf{n} \cdot \mathbf{v} < 0 \, , \quad \mathbf{x} \in \partial\Gamma,
$$

where $\mathbf{n}$ is the outer normal of the boundary. We rewrite the equation for the positive ξ, η only

$$\begin{aligned}
\epsilon\, \partial_t f(\xi,\eta) + \xi \partial_x f(\xi,\eta) + \eta \partial_y f(\xi,\eta) &= \frac{1}{\epsilon}\left[\rho - f(\xi,\eta)\right], \\
\epsilon\, \partial_t f(-\xi,-\eta) - \xi \partial_x f(-\xi,-\eta) - \eta \partial_y f(-\xi,-\eta) &= \frac{1}{\epsilon}\left[\rho - f(-\xi,-\eta)\right], \\
\epsilon\, \partial_t f(\xi,-\eta) + \xi \partial_x f(\xi,-\eta) - \eta \partial_y f(\xi,-\eta) &= \frac{1}{\epsilon}\left[\rho - f(\xi,-\eta)\right], \\
\epsilon\, \partial_t f(-\xi,\eta) - \xi \partial_x f(-\xi,\eta) + \eta \partial_y f(-\xi,\eta) &= \frac{1}{\epsilon}\left[\rho - f(-\xi,\eta)\right],
\end{aligned}$$

where

$$\rho = \frac{1}{2\pi} \int_{|\mathbf{v}|=1} f\, d\mathbf{v}.$$

Introducing the even and odd parities

$$\begin{aligned}
r_1(\xi,\eta) &= \frac{1}{2}[f(\xi,-\eta) + f(-\xi,\eta)], \\
r_2(\xi,\eta) &= \frac{1}{2}[f(\xi,\eta) + f(-\xi,-\eta)], \\
j_1(\xi,\eta) &= \frac{1}{2\epsilon}[f(\xi,-\eta) - f(-\xi,\eta)], \\
j_2(\xi,\eta) &= \frac{1}{2\epsilon}[f(\xi,\eta) - f(-\xi,-\eta)],
\end{aligned}$$

we obtain the system

$$\begin{aligned}
\partial_t r_1 + \xi \partial_x j_1 - \eta \partial_y j_1 &= -\frac{1}{\epsilon^2}(r_1 - \rho), \\
\partial_t r_2 + \xi \partial_x j_2 + \eta \partial_y j_2 &= -\frac{1}{\epsilon^2}(r_2 - \rho), \\
\partial_t j_1 + \frac{\xi}{\epsilon^2}\partial_x r_1 - \frac{\eta}{\epsilon^2}\partial_y r_1 &= -\frac{1}{\epsilon^2} j_1, \\
\partial_t j_2 + \frac{\xi}{\epsilon^2}\partial_x r_2 + \frac{\eta}{\epsilon^2}\partial_y r_2 &= -\frac{1}{\epsilon^2} j_2.
\end{aligned} \tag{4.2}$$

System (4.2) can be written as the diffusive relaxation system

$$\begin{aligned}
&\partial_t r_1 + \xi \partial_x j_1 - \eta \partial_y j_1 = -\frac{1}{\epsilon^2}(r_1 - \rho), \\
&\partial_t j_1 + \phi \xi \partial_x r_1 - \phi \eta \partial_y r_1 = -\frac{1}{\epsilon^2}\left[j_1 + (1-\epsilon^2\phi)\xi \partial_x r_1 - (1-\epsilon^2\phi)\eta \partial_y r_1\right], \\
&\partial_t r_2 + \xi \partial_x j_2 + \eta \partial_y j_2 = -\frac{1}{\epsilon^2}(r_2 - \rho), \\
&\partial_t j_2 + \phi \xi \partial_x r_2 + \phi \eta \partial_y r_2 = -\frac{1}{\epsilon^2}\left[j_2 + (1-\epsilon^2\phi)\xi \partial_x r_2 + (1-\epsilon^2\phi)\eta \partial_y r_2\right].
\end{aligned} \tag{4.3}$$

As $\epsilon \to 0$, one has

$$r_1 = \rho, \quad r_2 = \rho, \quad j_1 = -\xi\partial_x r_1 + \eta\partial_y r_1\,, \quad j_2 = -\xi\partial_x r_2 - \eta\partial_y r_2. \tag{4.4}$$

Applying (4.4) in (4.2) gives

$$\begin{aligned} \partial_t\rho - \xi^2\partial_{xx}\rho + 2\xi\eta\partial_{xy}\rho - \eta^2\partial_{yy}\rho = 0, \\ \partial_t\rho - \xi^2\partial_{xx}\rho - 2\xi\eta\partial_{xy}\rho - \eta^2\partial_{yy}\rho = 0. \end{aligned} \tag{4.5}$$

Adding the two equations in (4.5) we get

$$\partial_t\rho - \xi^2\partial_{xx}\rho - \eta^2\partial_{yy}\rho = 0. \tag{4.6}$$

Now, integrating over $\xi^2 + \eta^2 = 1$ we obtain the diffusion equation

$$\partial_t\rho = \frac{1}{2}\left(\partial_{xx}\rho + \partial_{yy}\rho\right). \tag{4.7}$$

4.2. The diffusive relaxation scheme

The previous splitting applied to (4.3) gives the relaxation step

$$\begin{aligned} \partial_t r_1 &= -\frac{1}{\epsilon^2}(r_1 - \rho), \\ \partial_t j_1 &= -\frac{1}{\epsilon^2}\left[j_1 + (1-\epsilon^2\phi)\xi\partial_x r_1 - (1-\epsilon^2\phi)\eta\partial_y r_1\right], \\ \partial_t r_2 &= -\frac{1}{\epsilon^2}(r_2 - \rho), \\ \partial_t j_2 &= -\frac{1}{\epsilon^2}\left[j_2 + (1-\epsilon^2\phi)\xi\partial_x r_2 + (1-\epsilon^2\phi)\eta\partial_y r_2\right], \end{aligned} \tag{4.8}$$

and the transport step

$$\begin{aligned} \partial_t r_1 + \xi\partial_x j_1 - \eta\partial_y j_1 = 0, \\ \partial_t j_1 + \phi\xi\partial_x r_1 - \phi\eta\partial_y r_1 = 0, \\ \partial_t r_2 + \xi\partial_x j_2 + \eta\partial_y j_2 = 0, \\ \partial_t j_2 + \phi\xi\partial_x r_2 + \phi\eta\partial_y r_2 = 0. \end{aligned} \tag{4.9}$$

Now, for (4.8) we can use the backward Euler method

$$\begin{aligned} \frac{r_1^* - r_1^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[r_1^* - \rho^*\right], \\ \frac{j_1^* - j_1^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[j_1^* + (1-\epsilon^2\phi)\xi\partial_x r_1^* - (1-\epsilon^2\phi)\eta\partial_y r_1^*\right], \\ \frac{r_2^* - r_2^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[r_2^* - \rho^*\right], \\ \frac{j_2^* - j_2^n}{\Delta t} &= -\frac{1}{\epsilon^2}\left[j_2^* + (1-\epsilon^2\phi)\xi\partial_x r_2^* + (1-\epsilon^2\phi)\eta\partial_y r_2^*\right]. \end{aligned} \tag{4.10}$$

As in the one-dimensional case, although we have an implicit relaxation step, this step can be implemented explicitly, due to the observation that ρ is preserved in (4.10), i.e., $\rho^* = \rho^n$. This can be seen by adding the first and third equations in (4.10) and integrating over $\xi^2 + \eta^2 = 1$. Thus these steps can be replaced by

$$\frac{r_1^* - r_1^n}{\Delta t} = -\frac{1}{\epsilon^2}(r_1^* - \rho^n), \qquad \frac{r_2^* - r_2^n}{\Delta t} = -\frac{1}{\epsilon^2}(r_2^* - \rho^n). \tag{4.11}$$

Now, one can solve (4.11) for r_1^* and r_2^* explicitly, and then get j_1^* and j_2^* from (4.11) also explicitly, provided the spatial derivatives are discretized using linear schemes.

Setting, for $0 \le \lambda \le 1$,

$$\xi = \cos(\lambda\pi/2), \qquad \eta = \sin(\lambda\pi/2), \tag{4.12}$$

we have

$$\rho = \frac{1}{2}\int_0^1 [r_1(\xi,\eta) + r_2(\xi,\eta)]\, d\,\lambda. \tag{4.13}$$

Thus to evaluate ρ, we can use the Gaussian quadrature rule, simply mapping the quadrature points for $\lambda \in [0,1]$ to ξ and η using relations (4.12).

We now illustrate how the boundary conditions will be defined for the diffusive relaxation system. Assume the domain in a rectangle $[x_L, x_R] \times [y_B, y_T]$, and the incoming boundary conditions are

$$\begin{aligned} f(x_L, y, \xi, \eta) &= F_{LB}(y,\xi,\eta)\,, \quad f(x, y_B, \xi, \eta) = F_{BL}(x,\xi,\eta)\,, \\ f(x_L, y, \xi, -\eta) &= F_{LT}(y,\xi,\eta)\,, \quad f(x, y_T, \xi, -\eta) = F_{TL}(x,\xi,\eta)\,, \\ f(x_R, y, -\xi, \eta) &= F_{RB}(y,\xi,\eta)\,, \quad f(x, y_B, -\xi, \eta) = F_{BR}(x,\xi,\eta)\,, \\ f(x_R, y, -\xi, -\eta) &= F_{RT}(y,\xi,\eta)\,, \quad f(x, y_T, -\xi, -\eta) = F_{TR}(x,\xi,\eta)\,. \end{aligned} \tag{4.14}$$

For example, to get the boundary condition on x-direction for r_2 and j_2, we use relations

$$r_2 + \epsilon j_2|_{x=x_L} = F_{LB}(y,\xi,\eta)\,, \qquad r_2 - \epsilon j_2|_{x=x_R} = F_{RB}(y,\xi,\eta)\,. \tag{4.15}$$

For $\epsilon << 1$, a reasonable approximation of the second equation in (4.2) is

$$j_2 = -\xi\partial_x r_2 - \eta\partial_y r_2. \tag{4.16}$$

For steady state solution, (4.16) is actually exact. Applying this in (3.12) one gets

$$\begin{aligned} r_2 - \epsilon\xi\partial_x r_2 - \epsilon\eta\partial_y r_2|_{x=x_L} &= F_{LB}(y,\xi,\eta)\,, \\ r_2 - \epsilon\xi\partial_x r_2 - \epsilon\eta\partial_y r_2|_{x=x_R} &= F_{RB}(y,\xi,\eta)\,. \end{aligned} \tag{4.17}$$

Thus if we implement (4.16) and (4.17) we get the desired boundary conditions. Other parts of the boundary can be treated similarly.

Remark 3 : In the general case we have the splitting

$$\begin{aligned}
\partial_t r_1 &= -\frac{\sigma_S}{\epsilon^2}(r_1-\rho),\\
\partial_t j_1 &= -\frac{1}{\epsilon^2}\left[\sigma_S j_1 + (1-\epsilon^2\phi)\xi\partial_x r_1 - (1-\epsilon^2\phi)\eta\partial_y r_1\right],\\
\partial_t r_2 &= -\frac{\sigma_S}{\epsilon^2}(r_2-\rho),\\
\partial_t j_2 &= -\frac{1}{\epsilon^2}\left[\sigma_S j_2 + (1-\epsilon^2\phi)\xi\partial_x r_2 + (1-\epsilon^2\phi)\eta\partial_y r_2\right],
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
&\partial_t r_1 + \xi\partial_x j_1 - \eta\partial_y j_1 = -\sigma_A r_1 + Q,\\
&\partial_t j_1 + \phi\xi\partial_x r_1 - \phi\eta\partial_y r_1 = -\sigma_A j_1,\\
&\partial_t r_2 + \xi\partial_x j_2 + \eta\partial_y j_2 = -\sigma_A r_2 + Q,\\
&\partial_t j_2 + \phi\xi\partial_x r_2 + \phi\eta\partial_y r_2 = -\sigma_A j_2.
\end{aligned} \tag{4.19}$$

As in the one-dimensional case, step (4.18) is discretized implicitly, but can be implemented explicitly, exactly in the same way as before, whereas (4.19) is discretized explicitly.

Similarly, one can easily verify that the numerical diffusion limit of the fully discretized equations corresponding to (4.8) - (4.9) is a consistent explicit discretization of the diffusion equation (4.13) under a parabolic CFL condition of the type $\Delta t \approx (\Delta x)^2 + (\Delta y)^2$.

5. A Uniform Bound on Consistency Error

The goal of this section is to show that the splitting approximation we introduced is uniformly accurate in ϵ, provided the solution to the transport equation is smooth.

For the sake of notation simplicity, our analysis will be restricted to one-dimensional situations of the form (3.1). The extension to the general case with a source term and different scattering and absorption cross sections will not present any difficulty and we leave it to the interested readers.

The analysis that follows is concerned with the study of the consistency error, which will be shown to be of size $O(\Delta t + (\Delta x)^\alpha)$ uniformly in ϵ, where the exponent $\alpha \geq 1$ is related to the order of approximation of the spatial derivatives. A complete stability analysis is not yet available. We do not discuss the accuracy of the velocity discretization either. For issues on the accuracy of the velocity discretization, we refer the reader to [PS, GJL].

The key idea as in [CJR] is to consider the difference between the deviation from the equilibrium (i.e. $r-\rho$) and its numerical counterpart, instead of the difference between the pure kinetic variable r and j and their numerical approximations.

To this aim, integrating the first equation in (3.4) over $\Omega=[x_L,x_R]$ we otain the continuity equation

$$\partial_t\rho+\partial_x(\rho u)=0. \tag{5.1}$$

Subtracting (5.1) from the first equation in (3.4) gives, for the new variable $w=r-\rho$, the equation

$$\partial_t w+\frac{1}{\epsilon^2}w=-\partial_x(vj-\rho u)\,, \tag{5.2}$$

where w measures the deviation of r from its local equilibrium ρ. By introducing the new variable $q=j+\lambda v\partial_x r$ with $\lambda=(1-\epsilon^2\phi)$, the second equation in (3.4) gives

$$\partial_t q+\frac{1}{\epsilon^2}q=-v\phi\partial_x r+\lambda v\partial_{tx}r. \tag{5.3}$$

The previous equations (5.2), (5.3) can be written in integral form for each x as

$$\begin{aligned} w(t)&=\exp\left\{-\frac{t}{\epsilon^2}\right\}w(0)+\int_0^t\exp\left\{-\frac{t-s}{\epsilon^2}\right\}g(s)\,ds,\\ q(t)&=\exp\left\{-\frac{t}{\epsilon^2}\right\}q(0)+\int_0^t\exp\left\{-\frac{t-s}{\epsilon^2}\right\}h(s)\,ds \end{aligned} \tag{5.4}$$

where $g=-\partial_x(vj-\rho u)$ and $h=-v\phi\partial_x r+\lambda v\partial_{tx}r$.

Now, we can consider the fully space-time discretized equations (3.17) - (3.18). By combining the two steps together we obtain for the new variables w and q the discrete equations

$$\begin{aligned} w^{k+1}&=A\left(w^k+\Delta t g^k\right),\\ q^{k+1}&=A\left(q^k+\Delta t h^k\right), \end{aligned} \tag{5.5}$$

with

$$g^k=\frac{D^u}{\Delta x}\left[vj^k-(\rho u)^k\right],\quad h^k=-v\phi\frac{D^u}{\Delta x}r^k+\frac{\lambda}{\Delta t}\left(\frac{D^c}{\Delta x}r^{k+1}-\frac{D^c}{\Delta x}r^k\right), \tag{5.6}$$

and $A=\epsilon^2/(\epsilon^2+\Delta t)$.

Let $t=n\Delta t$. Then at the end of the iterations we obtain for w^n and q^n the recursion formula

$$\begin{aligned} w^n&=A^nw^0+\Delta t\sum_{k=0}^{n-1}A^{n-k}g^k,\\ q^n&=A^nq^0+\Delta t\sum_{k=0}^{n-1}A^{n-k}h^k. \end{aligned} \tag{5.7}$$

Next, approximating the integral equations (5.4) with integrals over step functions defined in each interval of lenght Δt one gets

$$\begin{aligned}\bar{w}(t) &= \exp\left\{-\frac{t}{\epsilon^2}\right\} w(0) + \Delta t \sum_{k=0}^{n-1} \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k} g(k\Delta t),\\ \bar{q}(t) &= \exp\left\{-\frac{t}{\epsilon^2}\right\} q(0) + \Delta t \sum_{k=0}^{n-1} \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k} h(k\Delta t).\end{aligned} \tag{5.8}$$

In our hypotheses of smoothness it is easy to check that

$$|\bar{w}(t) - w(t)| \leq c\Delta t, \quad |\bar{q}(t) - q(t)| \leq c\Delta t, \tag{5.9}$$

where c is a constant which depends on t but independent of ϵ.

Now, let

$$\begin{aligned}w^*(t) &= \exp\left\{-\frac{t}{\epsilon^2}\right\} w(0) + \Delta t \sum_{k=0}^{n-1} \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k} g^k,\\ q^*(t) &= \exp\left\{-\frac{t}{\epsilon^2}\right\} q(0) + \Delta t \sum_{k=0}^{n-1} \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k} h^k,\end{aligned} \tag{5.10}$$

be the sums obtained by replacing $g(k\Delta t)$, $h(k\Delta t)$ in (5.8) with g^k and h^k given by (5.6) . Since from (3.5) both ϕ and λ are positive quantities bounded by one independently on ϵ, we have

$$|g^k - g(k\Delta t)| \leq c(\Delta x)^\alpha, \quad |h^k - h(k\Delta t)| \leq c[\Delta t + (\Delta x)^\alpha], \tag{5.11}$$

that, along with (5.9), yields

$$|w(t) - w^*(t)| \leq c\left[\Delta t + (\Delta x)^\alpha\right], \quad |q(t) - q^*(t)| \leq c\left[\Delta t + (\Delta x)^\alpha\right]. \tag{5.12}$$

Finally, let us compare w^n and q^n given by (5.7) with $w^*(t)$ and $q^*(t)$ in (5.10). Considering that

$$A \geq \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\},$$

we have for $|w^n - w^*(t)|$ the bound

$$\begin{aligned}|w^n - w^*(t)| \leq& \left[A^n - \exp\left\{-\frac{t}{\epsilon^2}\right\}\right] |w(0)|\\ &+\Delta t \sum_{k=0}^{n-1} \left[A^{n-k} - \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k}\right] |g^k|.\end{aligned} \tag{5.13}$$

An identical bound holds for $|q^n - q^*(t)|$.

For the first term in (5.13) one can use the following estimate [CJR]

$$A^n - \exp\left\{-\frac{t}{\epsilon^2}\right\} \leq c\Delta t, \tag{5.14}$$

with c independent of ϵ.

Moreover we have

$$\sum_{k=0}^{n-1}\left[A^{n-k} - \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{n-k}\right] = \sum_{k=1}^{n}\left[A^{k} - \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{k}\right]$$

$$\leq \sum_{k=0}^{\infty}\left[A^{k} - \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}^{k}\right] = \frac{1}{1-A} - \left(1 - \exp\left\{-\frac{\Delta t}{\epsilon^2}\right\}\right)^{-1} \leq \frac{1}{2}. \tag{5.15}$$

Hence, if $c_1 = \sup_k\{|g^k|, |h^k|\}$,

$$|w^n - w^*(t)| \leq (c + c_1)\Delta t, \quad |q^n - q^*(t)| \leq (c + c_1)\Delta t. \tag{5.16}$$

Collecting bounds (5.9), (5.12) and (5.16) we conclude with the following

Proposition : *Suppose that the solution to system (3.4) is smooth. Then the consistency error for the discretized transport equations (3.17), (3.18) satisfies*

$$|w(t) - w^n| \leq C\left[\Delta t + (\Delta x)^\alpha\right], \quad |q(t) - q^n| \leq C\left[\Delta t + (\Delta x)^\alpha\right], \tag{5.17}$$

where the constant C does not depend on ϵ.

6. Numerical Examples

In this section, we present numerical results for several problems chosen to verify the performance of the schemes described above. To evaluate the spatial derivatives on the right hand side of the relaxation step of our schemes a central difference discretization as in [JPT] has been used. This choice, as observed in [JPT], allows one to explicitly implement the scheme, and provides second order accuracy without spurious oscillations. As usual, the transport step has been solved by a second order shock capturing upwind scheme using slope limiters. To obtain uniformly accurate second order schemes both in space and time the two steps are then combined using the second order Runge-Kutta splitting developed in [CJR]. The boundary conditions has been applied using a second order implementation of equations (3.12) and (4.15) based on central differences.

In all the test problems we have used the function ϕ given by (3.5) and the S_{16} standard Gaussian quadrature set for the velocity space. As usual, the 'exact' solutions were obtained using a much finer grid in space and the same S_{16} standard Gaussian quadrature set.

6.1. 1-D problems

We shall consider five transport problems in slab-geometry. Both the transient and the steady state solutions will be presented.

Problem I :

$$x \in [0,1], \quad F_L(v) = 1, \quad F_R(v) = 0,$$
$$\sigma_S = 1, \quad \sigma_A = 0, \quad Q = 0, \quad \epsilon = 10^{-8}.$$

The initial distribution is a Riemann problem defined by

$$f(x,v,t=0) = 0, \quad 0 < x < 0.5,$$
$$f(x,v,t=0) = 1, \quad 0.5 < x < 1.$$

In the small mean free path limit this problem leads to the well-known Riemann problem for the heat equation. The numerical results at $t = 0.05$ and the steady state are reported in figure 1 and 2. In both cases, the results clearly show that the scheme is able to capture the correct diffusion limit even when $\Delta t, \Delta x >> \epsilon$.

Problem II :

$$x \in [0,1], \quad F_L(v) = 0, \quad F_R(v) = 0,$$
$$\sigma_S = 1 + (10x)^2, \quad \sigma_A = 0, \quad Q = 1, \quad \epsilon = 0.01.$$

The initial distribution is $f(x,v,t=0) = 0$. In figure 4 we report the numerical solution at time $t = 0.4$. In this problem we have a source term and the scattering cross section depends on x, so the scaling term σ_S/ϵ ranges from $1/\epsilon$ to $O(1)$. This example shows that the solution is approximated with good accuracy for different ranges of relaxation rates.

Problem III :

$$x \in [0,20], \quad F_L(v) = 0, \quad F_R(v) = 0,$$
$$\sigma_S = 0, \quad \sigma_A = 10, \quad Q = 10, \quad \epsilon = 1, \quad x \in [0,10[,$$
$$\sigma_S = 1, \quad \sigma_A = 0, \quad Q = 0, \quad \epsilon = 0.01, \quad x \in [10,20].$$

The initial distribution is $f(x,v,t=0) = 0$. The systems consists of a purely absorption slab with a flat interior source adjoining a purely scattering slab with no interior source. The exact solution is nearly constant in the source region and it varies linearly in the scattering region. Although, the purely absorbing region is not diffusive, the approximation to the steady state solution is highly accurate. See figure 5.

Problem IV :

$$x \in [0,11], \quad F_L(v) = 5\sin(v), \quad F_R(v) = 0,$$

$$\sigma_S = 1, \quad \sigma_A = 0, \quad Q = 1, \quad \epsilon = 0.05.$$

The initial distribution is $f(x, v, t = 0) = 0$. The asymptotic solution of this problem is a kinetic boundary layer at $x = 0$ and is shown in figure 6. One can clearly observe that even for coarse diffusive discretization the description of the boundary layer is found with good accuracy.

Problem V :

$$x \in [0, 11], \quad F_L(v) = 5, \quad F_R(v) = 0,$$

$$\sigma_S = 0, \quad \sigma_A = 1, \quad Q = 0, \quad \epsilon = 1, \quad x \in [0, 1],$$
$$\sigma_S = 1, \quad \sigma_A = 0, \quad Q = 0, \quad \epsilon = 0.01, \quad x \in]1, 20].$$

The initial distribution is $f(x, v, t = 0) = 0$. In the purely absorbing region the solution decays exponentially whereas in the purely scattering region the solution is diffusive. A internal layer is present at the interface $x = 0.1$ between the two materials. The numerical scheme clearly gives a good description of the internal layer and of the solution in the absorption and diffusive regions.

For a comparison of the present results with previous results we refer the readers to [LM, JL1, JL2, Kl1]. We remark that for problems III and V no special treatment at the interface has been done except the fact of setting a discretization point at the interface. This simple choice has given a remarkable improvement in the quality of the results.

6.2. 2-D problems

The last numerical experiment refers to a two-dimensional problem defined by *Problem VI :*

$$\Gamma = [0, 1] \times [0, 1], \quad F_B(\mathbf{x}, \mathbf{v}) = 0, \quad \mathbf{n} \cdot \mathbf{v} < 0\,, \quad \mathbf{x} \in \partial\Gamma,$$

$$\sigma_S = 1, \quad \sigma_A = 0, \quad Q = 1, \quad \epsilon = 10^{-8}.$$

Here $\mathbf{n}$ denotes the exterior unit normal vector on $\partial\Gamma$. The initial distribution is $f(\mathbf{x}, \mathbf{v}, t = 0) = 0$. Here, the solution to the problem is the solution of a diffusion equation with a source term. Figure 7 shows the numerical solution at time $t = 0.5$. The results confirm the correct numerical behavior of the scheme in the diffusion limit and are in good agreement with the ones found in [BLA].

7. Concluding Remarks

In this paper we derived a class of diffusive relaxation schemes for time-dependent multiscale transport equations in any space dimension. By using the even- and odd-parity formulation of the transport equation, we are able to formulate a general transport equation into the framework of discrete-velocity Goldstein-Taylor model, which can be treated effectively as in our earlier work [JPT]. This class of diffusive relaxation schemes has a small-parameter-independent stability condition, yet can be implemented explicitly. It works with a uniform accuracy with respect to the relaxation parameter, as verified by both our uniform consistency error analysis and our numerical experiments. It allows the capturing of the macroscopic diffusive behavior with an underresolved numerical discretization, a property highly desirable when the relaxation parameter becomes small and the full numerical resolution of the small parameter becomes impractically expensive, especially in higher dimensions.

Although our examples concentrate on transport equations with isotropic scatterings, and with diffusion approximations, such a formulation has the applicability to be extended to more general transport operators. This will be pursued in our future research.

Appendix: 2-D in Polar Coordinates

In this section we will derive the diffusive relaxation scheme for the the two-dimensional transport equation in polar coordinates.

Consider the two-dimensional transport equation,

$$\epsilon\partial_t f + \mathbf{v}\cdot\nabla_x f = \frac{1}{\epsilon}\left(\frac{1}{2\pi}\int_{|\mathbf{v}|=1} f\,d\mathbf{v}' - f\right), \qquad \mathbf{x}\in\Gamma\subset\mathbb{R}^2, \quad |\mathbf{v}|=1. \tag{A.1}$$

To this aim, let

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \begin{aligned} x &= r\cos\alpha, \\ y &= r\sin\alpha, \end{aligned} \qquad 0\le\alpha\le 2\pi, \qquad 0\le r\le R,$$

then

$$\nabla_{\mathbf{x}} = \frac{\mathbf{x}}{r}\frac{\partial}{\partial r} + \frac{\mathbf{x}^\perp}{r^2}\frac{\partial}{\partial\alpha},$$

where $\mathbf{x}^\perp = (-y, x)$ and hence

$$\mathbf{v}\cdot\nabla_{\mathbf{x}} = \frac{\mathbf{v}\cdot\mathbf{x}}{r}\frac{\partial}{\partial r} + \frac{\mathbf{v}\cdot\mathbf{x}^\perp}{r^2}\frac{\partial}{\partial\alpha}.$$

Now, let $-\pi\le\theta\le\pi$ be the angle between $\mathbf{v}$ and $\mathbf{x}$. Then we have

$$\begin{aligned} \mathbf{v}\cdot\mathbf{x} &= r\cos\theta, \\ \mathbf{v}\cdot\mathbf{x}^\perp &= r\cos\left(\frac{\pi}{2}-\theta\right) = r\sin\theta, \end{aligned}$$

and

$$\mathbf{v} \cdot \nabla_{\mathbf{x}} = \cos\theta \frac{\partial}{\partial r} + \frac{\sin\theta}{r} \frac{\partial}{\partial \alpha}.$$

Finally, since

$$\int_{|\mathbf{v}'|=1} f(\mathbf{x}, \mathbf{v}', t) d\mathbf{v}' = \int_{-\pi}^{\pi} f(r, \alpha, \theta', t) d\theta',$$

under the coordinate (r, α, θ, t), equation (A.1) becomes

$$\epsilon \partial_t f + \cos\theta \partial_r f + \frac{\sin\theta}{r} \partial_\alpha f = \frac{1}{\epsilon} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f d\theta' - f \right), \tag{A.2}$$

or equivalentely

$$\epsilon \partial_t f + \xi \partial_r f + \eta \partial_\alpha = \frac{1}{\epsilon} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f d\theta' - f \right), \tag{A.3}$$

where $\xi = \cos\theta$ and $\eta = (\sin\theta)/r$, $-\pi \leq \theta \leq \pi$, and with $0 \leq \alpha \leq 2\pi$, $0 \leq r \leq R$. Clearly (A.3) has the same structure of (4.1) and thus we can consider the even-parity formulation for positive ξ and η only, i.e. for $0 \leq \theta \leq \pi/2$, using the variables

$$\begin{aligned}
r_1(t, r, \alpha, \theta) &= \frac{1}{2}[f(t, r, \alpha, \theta) + f(t, r, \alpha, \theta - \pi)], \\
r_2(t, r, \alpha, \theta) &= \frac{1}{2}[f(t, r, \alpha, \theta - \pi/2) + f(t, r, \alpha, \theta + \pi/2)], \\
j_1(t, r, \alpha, \theta) &= \frac{1}{2\epsilon}[f(t, r, \alpha, \theta) - f(t, r, \alpha, \theta - \pi)], \\
j_2(t, r, \alpha, \theta) &= \frac{1}{2\epsilon}[f(t, r, \alpha, \theta - \pi/2) - f(t, r, \alpha, \theta + \pi/2)].
\end{aligned}$$

The even-parity equations are exactly (4.2) except that we have derivatives with respect to r and α instead of x and y. Thus the derivation of the diffuse relaxation scheme in section 4 can be applied directly without any modification.

On θ one can use the same symmetric Gaussian quadrature as in section 4, whereas on r and α a uniform grid can be used.

To illustrate the boundary conditions we assume the domain is a disk centered in the origin with radius R. As for the boundary conditions we observe that no boundary condition are needed on θ, whereas for α the boundary condition are assumed to be periodic. Finally for r, we use the following conditions.

For $r = R$ and $\cos\theta < 0$ we have the incoming flux

$$f(R, \alpha, \theta) = F_{RB}(\alpha, \theta).$$

At $r = 0$, we compute the solution directly by extending the equation for $r > 0$ to $r < 0$. In fact, note that Eq. remains unchanged if one replaces r by $-r$, θ by $\pi - \theta$ if $\theta > 0$ and by $-\pi - \theta$ if $\theta < 0$, and α by $2\pi - \alpha$. Thus we use relations

$$f(-r, \alpha, \theta) = \begin{cases} f(r, 2\pi - \alpha, \pi - \theta) & \text{if } \theta > 0, \\ f(r, 2\pi - \alpha, -\pi - \theta) & \text{if } \theta < 0, \end{cases}$$

to obtain the value of f at the point $r = 0$.

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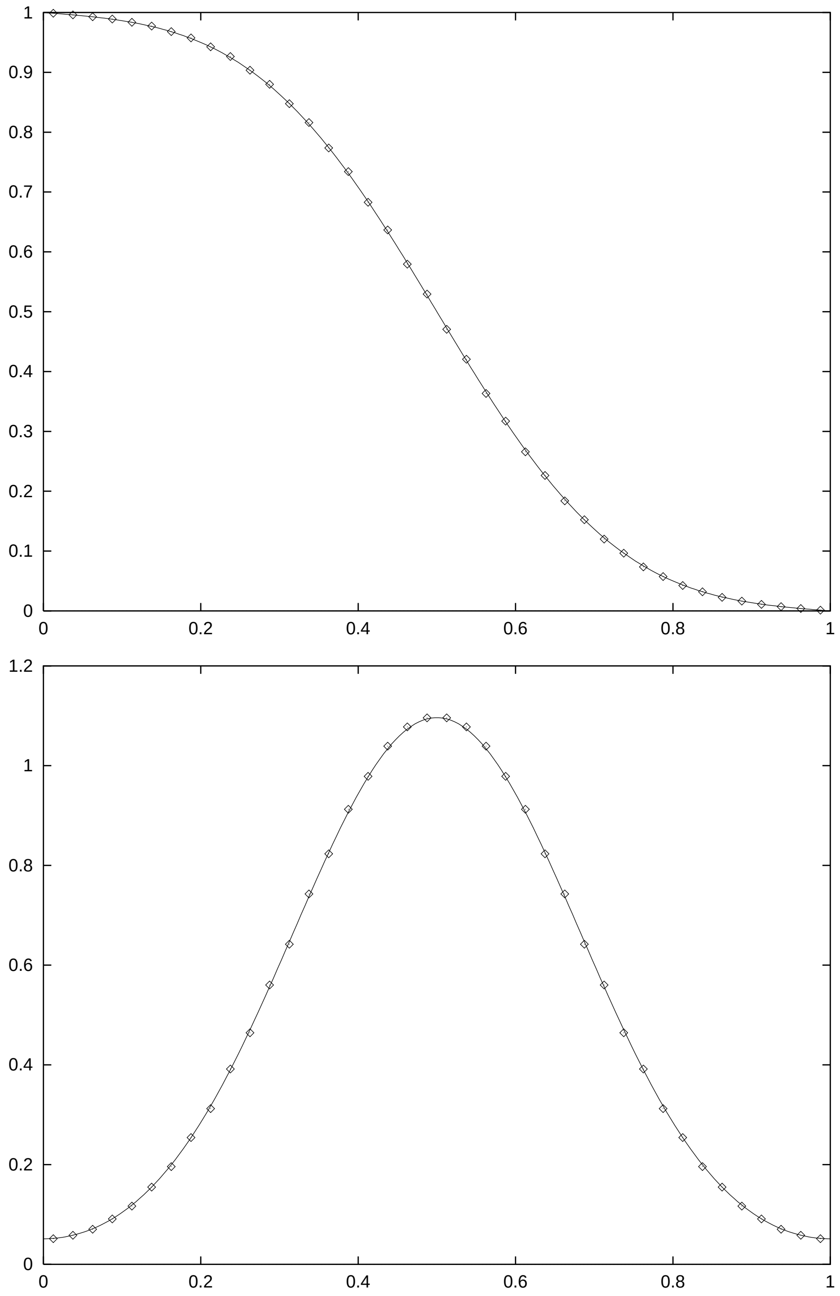


Figure 1 : Solution at $t = 0.05$ of Problem I. Numerical solution is represented by the diamonds, while the exact solution by the solid lines. $\epsilon = 10^{-8}$, $\Delta x = 0.025$ and $\Delta t = 0.0002$. Top: the mass density ρ; bottom: the flux ρu.

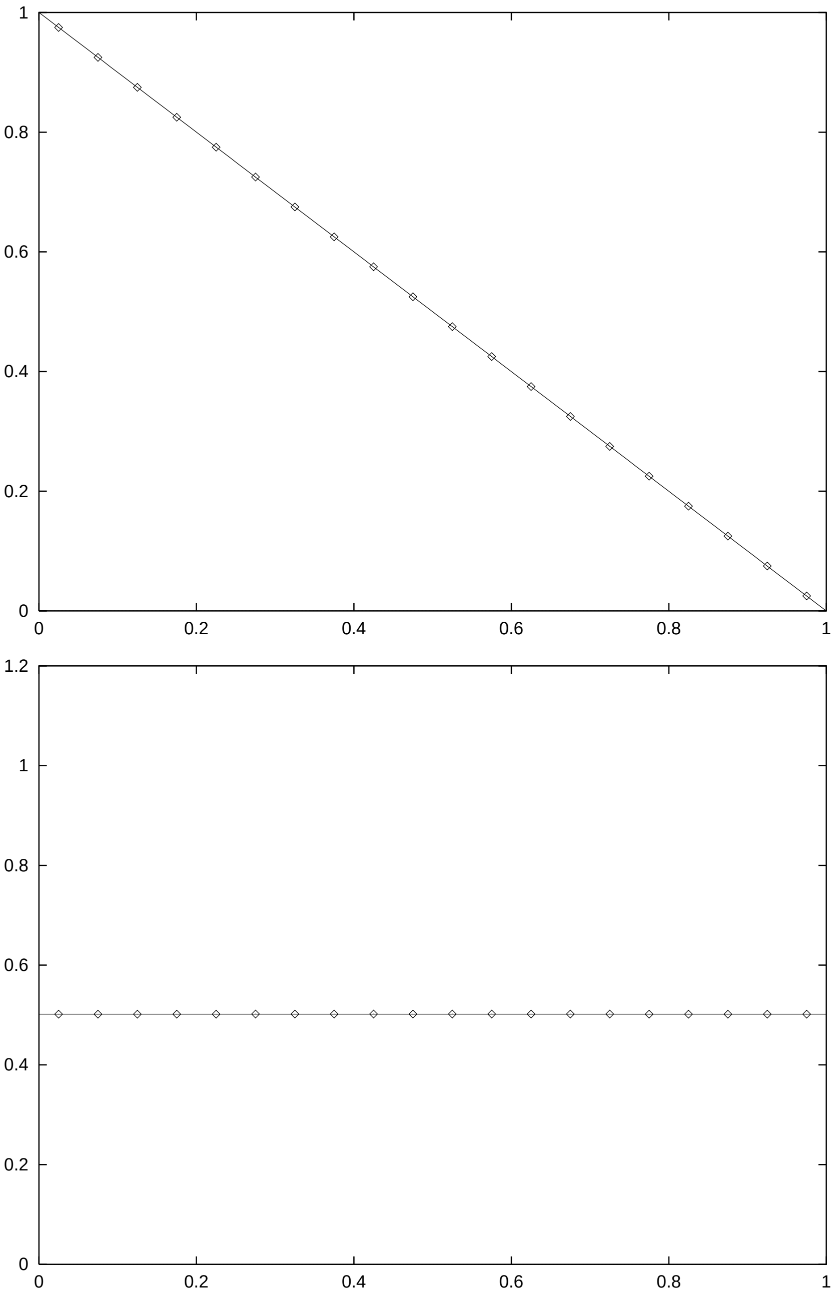


Figure 2 : Steady state solution of Problem I. Numerical solution is represented by the diamonds, while the exact solution by the solid lines. $\epsilon = 10^{-8}$, $\Delta x = 0.025$ and $\Delta t = 0.0002$. Top: the mass density ρ; bottom: the flux ρu.

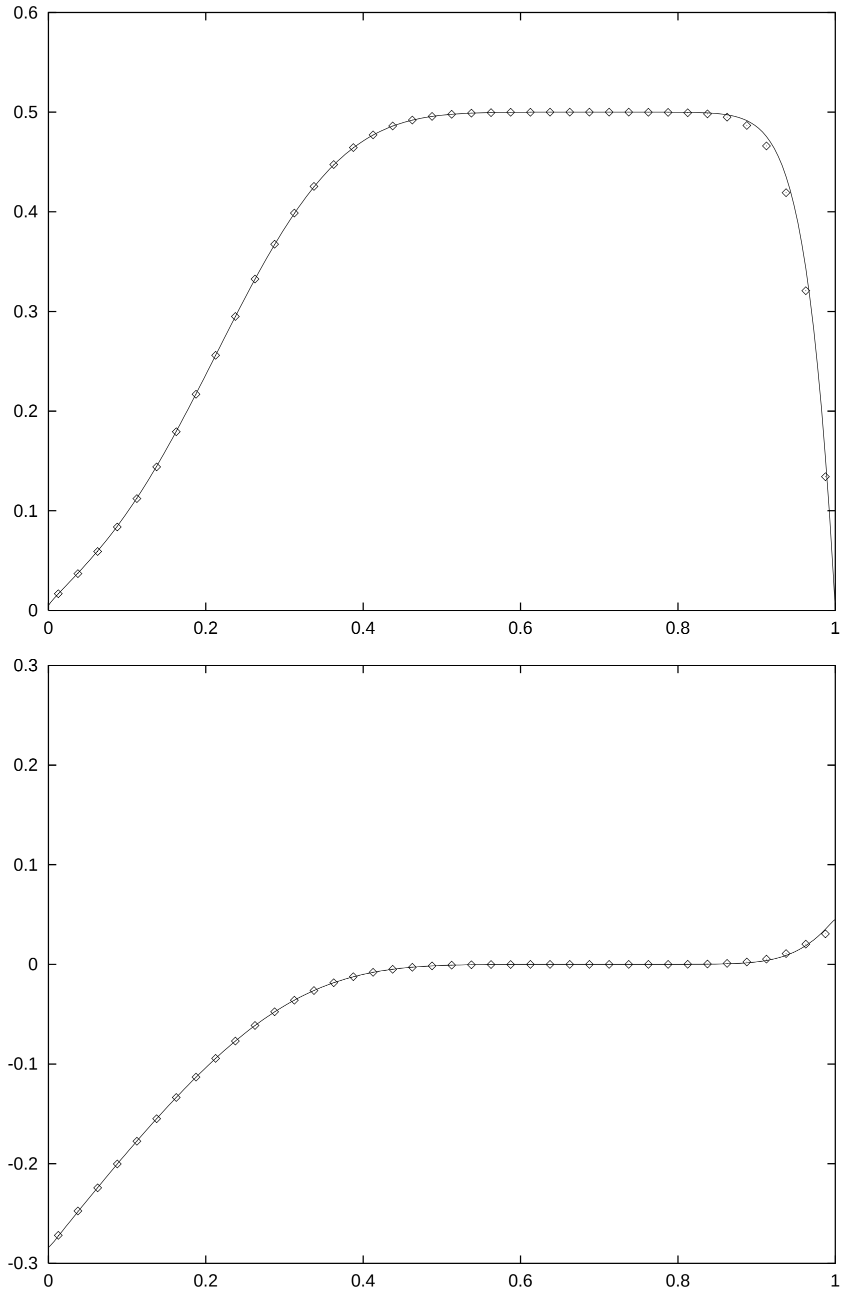


Figure 3 : Solution at $t = 0.4$ of Problem II. Numerical solution is represented by the diamonds, while the exact solution by the solid lines. $\epsilon = 0.01$, $\Delta x = 0.025$ and $\Delta t = 0.0002$. Top: the mass density ρ; bottom: the flux ρu.

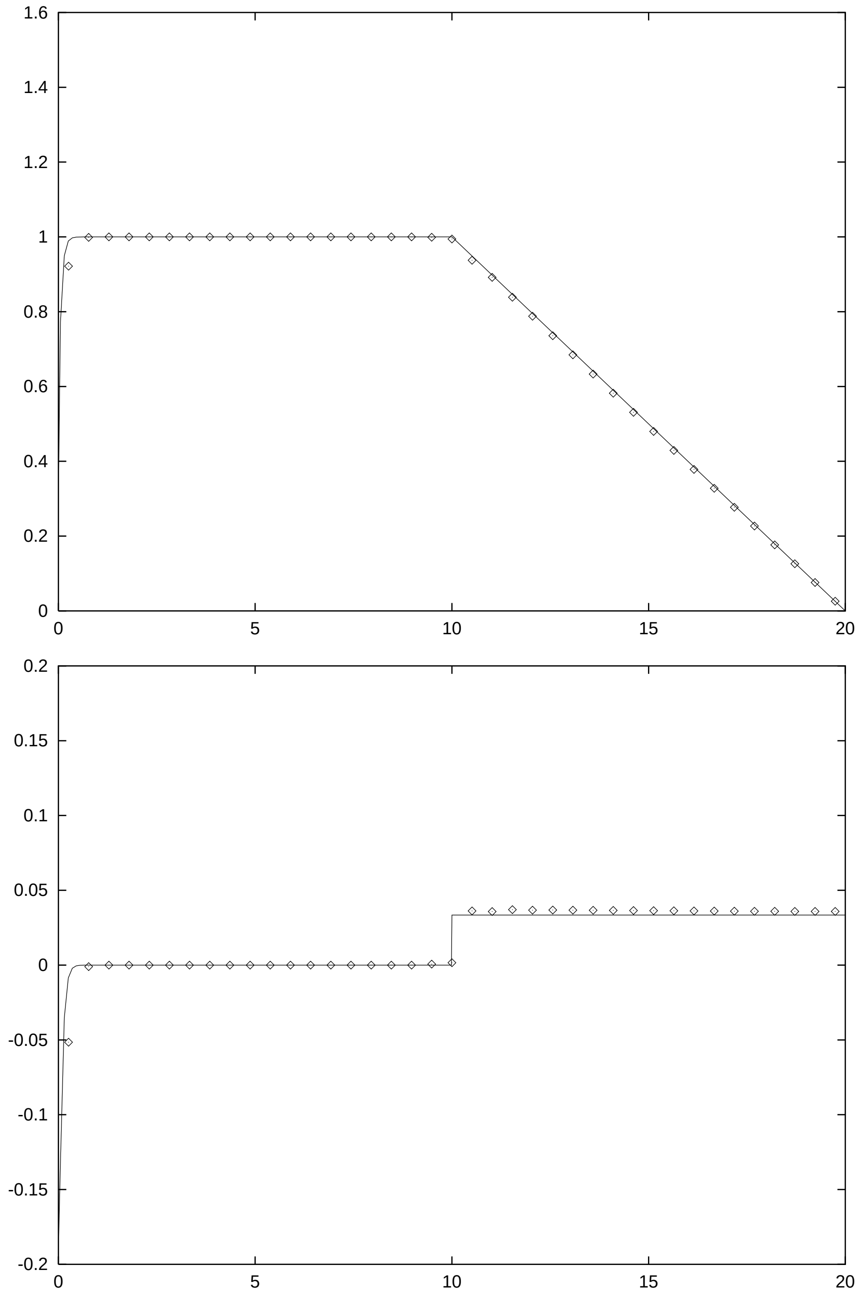


Figure 4 : Steady state solution of Problem III. Numerical solution is represented by the diamonds, while the exact solution by the solid lines. $\Delta x = 0.5$ and $\Delta t = 0.1$. Top: the mass density ρ; bottom: the flux ρu.

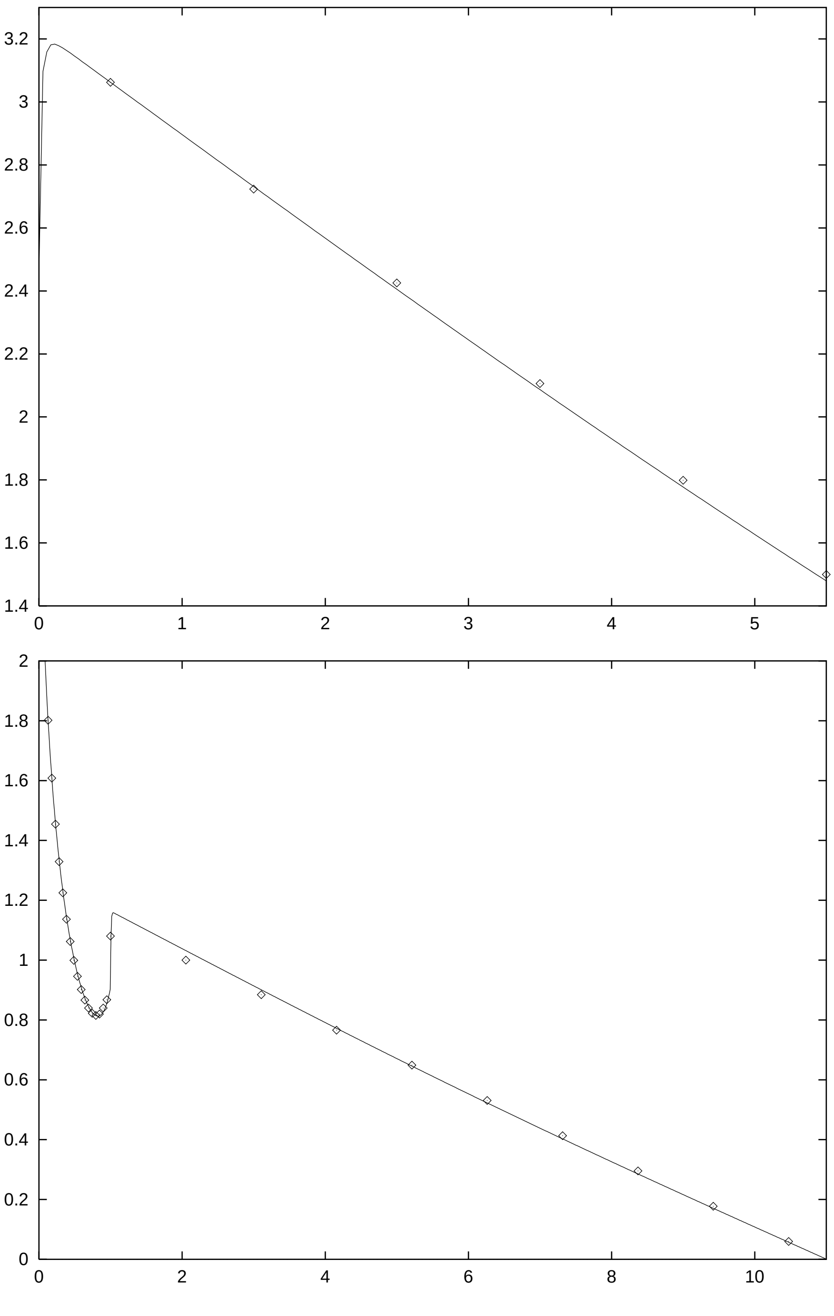
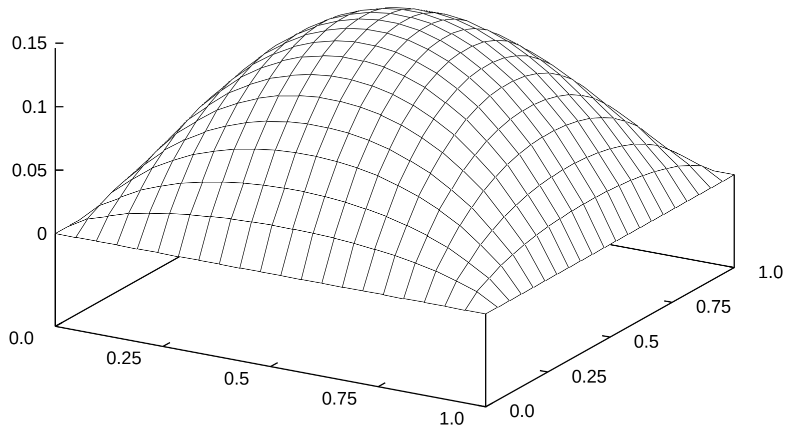


Figure 5 : Steady state solutions of the mass density ρ for Problem IV (top) and Problem V (bottom). Numerical solution is represented by the diamonds, while the exact solution by the solid lines. In Problem IV : $\sigma_A = 0$, $\sigma_S = 1$, $\epsilon = 0.05$, $\Delta x = 1$, $\Delta t = 0.25$. In Problem V : $\sigma_A = 1$, $\sigma_S = 0$, $\epsilon = 1$, $\Delta x = 0.05$ on $]0, 1[$, $\sigma_A = 1$, $\sigma_S = 0$, $\epsilon = 0.01$, $\Delta x = 1$ on $[1, 11[$ and $\Delta t = 0.025$.



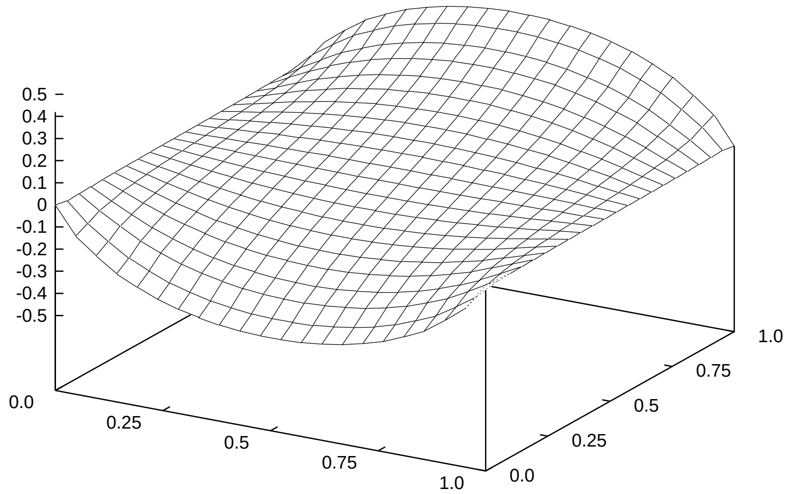


Figure 6 : Numerical solution for Problem VII with $\epsilon = 10^{-8}$, $\Delta x = 0.05$ and $\Delta t = 0.001$. In Problem VI : $Q = 1$, $\sigma_A = 0$ and $\sigma_S = 1$. The mass density ρ (top) and the flux ρu (bottom) are plotted at $t = 0.5$.

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