Stability of $\mathbf{L}^\infty$ Solutions of Temple Class Systems

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Abstract. Let $u_t + f(u)_x = 0$ be a strictly hyperbolic, genuinely nonlinear system of conservation laws of Temple class. In this paper, a continuous semigroup of solutions is constructed on a domain of $\mathbf{L}^\infty$ functions, with possibly unbounded variation. Trajectories depend Lipschitz continuously on the initial data, in the $\mathbf{L}^1$ distance. Moreover, we show that a weak solution of the Cauchy problem coincides with the corresponding semigroup trajectory if and only if it satisfies an entropy condition of Oleinik type, concerning the decay of positive waves.

1 - Introduction

Consider the Cauchy problem for a strictly hyperbolic system of conservation laws in one space dimension

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0,x) = \bar u(x). \tag{1.2}$$

For initial data with small total variation, the global existence of weak solutions is well known [15]. More recently, various papers [1, 4, 5, 7, 8, 9, 10, 11, 19, 20] have established the uniqueness and Lipschitz continuous dependence on the initial data for entropy admissible BV solutions. We recall that, for a scalar conservation law, the classical work of Kruzhkov [17] proved the stability of solutions of (1.1) within a domain of $\mathbf{L}^\infty$ functions. On the other hand, in the case of $n\times n$ systems, the available stability results apply only to domains $\mathcal{D}$ of functions with uniformly bounded total variation.

It is thus natural to ask whether the Lipschitz continuous semigroups of solutions constructed in [1, 5, 7, 11] can be extended to domains of $\mathbf{L}^\infty$ functions with possibly unbounded variation. The present paper provides an answer to this question in the case of Temple class systems [16, 21]. Namely, we show that, if all characteristic fields are genuinely nonlinear, then for a wide class of initial data $\bar u \in \mathbf{L}^\infty$ the Cauchy problem (1.1)-(1.2) has a unique entropy weak solution, depending Lipschitz continuously on $\bar u$. Any two such solutions thus satisfy

$$\big\|u(t,\cdot) - v(t,\cdot)\big\|_{\mathbf{L}^1} \le L \cdot \big\|u(0,\cdot) - v(0,\cdot)\big\|_{\mathbf{L}^1} \tag{1.3}$$

for some Lipschitz constant L independent of t. By a counterexample, we show that the assumption of genuine nonlinearity cannot be dropped. Indeed, for a particular 2×2 Temple class system with a linearly degenerate characteristic field, the $\mathbf{L}^\infty$ entropic solutions depend continuously but not Lipschitz continuously on the initial data.

The construction of the semigroup is achieved by the same technique as in [1, 5]. We consider a family of piecewise constant approximate solutions obtained by a front tracking algorithm [1, 3], which in this case represents a natural extension of [13]. The distance between two approximate solutions u, v is then estimated by constructing a path of solutions connecting u with v and keeping track of how the length of this path changes in time.

As in [1, 5], the heart of the matter is to provide a priori bounds on the norm of a shift-differential. Let a piecewise constant initial data $u(0,\cdot)$ be perturbed by shifting the location of one of its jumps. Call σ the strength of this jump and ξ its shift rate. At time $t > 0$, the corresponding solution $u(t,\cdot)$ will contain jumps, say of strength $\sigma_1, \ldots, \sigma_m$, shifting at rates $\xi_1, \ldots, \xi_m$. To prove stability, one has to establish the key estimate

$$\sum_{j=1}^{m} |\sigma_j \xi_j| \le L \cdot |\sigma\xi| \tag{1.4}$$

with a constant L independent of t. In general, an estimate of the form (1.4) holds only within a class of solutions with uniformly small total variation. For genuinely nonlinear systems of Temple class, however, we show that (1.4) can be satisfied with a constant L independent of the total variation of u. This remarkable property is based on two special features of such systems:

1. By genuine nonlinearity and finite propagation speed, the total amount of waves in $u(t,\cdot)$ which can be influenced by shifting the wave-front σ remains uniformly bounded for all t. Indeed, call $\bar{x}$ the initial location of σ and let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. Then an infinitesimal shift of the position of σ can affect the values of $u(t,\cdot)$ only within the interval of dependency $I(t) \doteq [\bar{x} - \hat{\lambda}t,\ \bar{x} + \hat{\lambda}t]$. On the other hand, by the decay of positive waves due to genuine nonlinearity, the total amount of waves in $u(t,\cdot)$ contained in an interval $[a,b]$ can be estimated as $\mathcal{O}(1) \cdot \big[1 + (b-a)/t\big]$. In particular, the amount of waves contained in $I(t)$ is uniformly bounded.

2. For solutions of Temple class systems, the support of perturbations satisfies a special localization property. Namely, let u be a solution of (1.1) and fix an interval $[a,b]$. For $i = 1,\ldots,n$, call $t \mapsto x_i^a(t)$, $t \mapsto x_i^b(t)$ respectively the i-characteristics originating from the points a, b (for simplicity, in this informal discussion we are assuming that such characteristics are unique). Now consider a slightly perturbed solution v with $v(0,x) = u(0,x)$ for $x \notin [a,b]$. In general, at any time $t > 0$, the two solutions $u(t,\cdot)$, $v(t,\cdot)$ may have different values throughout the interval of dependency $I(t) \doteq \big[x_1^a(t),\ x_n^b(t)\big]$. However, introducing the integrated functions

$$U(t,x) \doteq \int_0^x u(t,y)\,dy, \qquad\qquad V(t,x) \doteq \int_0^x v(t,y)\,dy,$$

if $U(0,\cdot)$ and $V(0,\cdot)$ coincide outside $[a,b]$ then the solutions $u(t,\cdot)$, $v(t,\cdot)$ can be different only on a small neighborhood of the set

$$J(t) \doteq \bigcup_{i=1}^{n} \big[x_i^a(t),\ x_i^b(t)\big].$$

This property is closely related to the representation formula for a solution of the nonlinear equation $U_t + f(U_x) = 0$ in terms of the envelopes of n families of hyperplanes [21]. We shall repeatedly take advantage of this property in our calculations. Indeed, it will allow us to replace an initial data $\bar{u}$ with a new data $\bar{v}$ having uniformly bounded variation, without affecting the values of the solution $u(t,\cdot)$ in a neighborhood of a given point.

As soon as the basic estimate (1.4) is obtained, a standard argument yields the Lipschitz continuous dependence of front tracking approximations from the initial data. Since the Lipschitz constant is independent of the total variation, taking limits we obtain a semigroup of solutions defined on a domain of $\mathbf{L}^\infty$ functions. Using techniques developed in [8], we then prove that each semigroup trajectory $u(t,\cdot) = S_t\bar u$ provides the unique weak solution of the corresponding Cauchy problem (1.1)-(1.2) which satisfies a suitable "entropy condition" of Oleinik type [6, 8, 16].

Concerning the existence and uniqueness of solutions to Temple class systems with bounded initial data, some earlier results can be found in [14, 16, 21]. For some special 2×2 systems, the continuous dependence of solutions within a domain of $\mathbf{L}^\infty$ initial data has been analyzed in [2, 12].

The paper is organized as follows. Section 2 contains basic definitions and the statement of the main results. In Section 3 we describe the construction of front tracking approximate solutions, while Sections 4 and 5 contain the basic a priori estimates. A proof of the main theorems is given in Section 6, while the last section contains a simple example showing that the Lipschitz continuous dependence may fail if one of the characteristic fields is linearly degenerate.

2 - Statement of the main results

Let (1.1) be a strictly hyperbolic system of conservation laws, where $f:\Omega\mapsto I\!R^n$ is a smooth vector field defined on some open set $\Omega\subseteq I\!R^n$. Call $A(u)\doteq Df(u)$ the Jacobian matrix of f and denote by $\lambda_1(u)<\cdots<\lambda_n(u)$ its eigenvalues. The right and left eigenvectors of $A(u)$ will be written as $r_i(u)$, $l_i(u)$ respectively, and normalized so that

$$\big|r_i(u)\big| = 1, \qquad\qquad l_i(u)\cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i=j,\\ 0 & \text{if} \quad i\neq j.\end{cases} \tag{2.1}$$

We recall that the i-th field is *genuinely nonlinear* if, by a suitable orientation of the eigenvectors r_i, at every point $u\in\Omega$ one has $D\lambda_i\cdot r_i>0$.

The system (1.1) is of *Temple class* if there exists a system of coordinates $w=(w_1,\ldots,w_n)$ consisting of Riemann invariants, and such that the level sets $\big\{u\in\Omega;\ w_i(u)=\text{constant}\big\}$ are hyperplanes [21]. For a Temple class system, the integral curve of the vector field r_i through a point u_0 is the straight line described by the $n-1$ equations

$$w_j(u)=w_j(u_0) \qquad\qquad j\neq i.$$

In particular, shock and rarefaction curves coincide [23].

We now consider a convex, compact set $E\subset\Omega$ having the form

$$E=\Big\{u\in\Omega;\quad w_i(u)\in[a_i,\ b_i] \qquad i=1,\ldots,n\Big\}, \tag{2.2}$$

and assume that, as u varies in E, a strengthened version of the strict hyperbolicity condition holds, namely

(SH) For any given $u_1, \ldots, u_n \in E$, the characteristic speeds at these points satisfy $\lambda_1(u_1) < \cdots < \lambda_n(u_n)$. Moreover, the eigenvectors $r_1(u_1), \ldots, r_n(u_n)$ are linearly independent.

Observe that the above assumption is certainly satisfied if the system is strictly hyperbolic and E is contained in a small neighborhood of a given point.

By a translation of coordinates, it is not restrictive to assume that $0 \in E$ and $(w_1, \ldots, w_n)(0) = (0, \ldots, 0)$. We now consider a positively invariant domain of $\mathbf{L}^\infty$ functions, with possibly unbounded variation:

$$\mathcal{D} \doteq \big\{u : I\!\!R \mapsto E; \quad u \in \mathbf{L}^1\big\}. \tag{2.3}$$

Our main result is concerned with the existence of a semigroup generated by the system (1.1) on the domain $\mathcal{D}$. Throughout the following, $w_i(t,x) \doteq w_i\big(u(t,x)\big)$ denotes the i-th Riemann coordinate of u.

Theorem 1. *Let the system (1.1) be of Temple class, let it satisfy the strict hyperbolicity condition (SH) and assume that all characteristic fields are genuinely nonlinear.*

Then there exist constants $L, \kappa > 0$ and a continuous semigroup $S : [0, +\infty[\times\mathcal{D} \to \mathcal{D}$ such that

$$\|S_t\bar u - S_t\bar v\|_{\mathbf{L}^1} \leq L \cdot \|\bar u - \bar v\|_{\mathbf{L}^1} \qquad \text{for all } \bar u, \bar v \in \mathcal{D},\ t \geq 0. \tag{2.4}$$

Moreover, every trajectory $t \mapsto u(t, \cdot) = S_t\bar u$ is a weak solution of (1.1) and satisfies the entropy conditions

$$w_i(t,y) - w_i(t,x) \leq \frac{y-x}{\kappa t} \qquad \text{for all } \ x < y, \quad t > 0,\ i \in \{1, \ldots, n\}. \tag{2.5}$$

Our second result is concerned with the uniqueness of the semigroup. It shows that the Oleinik type estimates (2.5) on the decay of positive waves completely characterize the trajectories of the semigroup.

Theorem 2. *Let the assumptions of Theorem 1 hold. Let $u : [0,T] \mapsto \mathcal{D}$ be continuous as a map with values in $\mathbf{L}^1$, and provide a weak solution to the Cauchy problem (1.1)-(1.2). If the entropy conditions (2.5) hold, then u coincides with the corresponding semigroup trajectory, namely*

$$u(t, \cdot) = S_t\bar u \qquad \text{for all } \ t \geq 0\,. \tag{2.6}$$

Remark 1. For any fixed M, the existence of a Lipschitz semigroup on the domain

$$\mathcal{D}_M \doteq \left\{u : I\!\!R \mapsto E; \quad u \in \mathbf{L}^1, \ \sum_i \text{Tot.Var.}\big\{w_i(u)\big\} \leq M\right\}. \tag{2.7}$$

was proved in [1]. For such solutions, the decay estimates (2.5) can be proved as in [6], while the argument in [8] would show that (2.5) implies (2.6). The key point in the present theorems is that the Lipschitz constant L in (2.4) and the constant κ in (2.4) are both independent of M. Hence the results remain valid for arbitrary solutions within the larger domain $\mathcal{D}$.

3 - Front tracking approximations

We begin the proof of Theorem 1 by describing a front tracking algorithm [1, 3] which constructs piecewise constant approximate solutions of (1.1), continuously depending on the initial data.

Fix an integer $\nu \geq 1$ and consider the discrete set of points in E whose coordinates are integer multiples of $2^{-\nu}$:

$$E^\nu \doteq \big\{u \in E\,; \qquad w_i(u) \in 2^{-\nu}\mathbb{Z},\ i = 1, \ldots, n\big\}.$$

Moreover, consider the domain

$$\mathcal{D}^\nu \doteq \big\{u : \mathbb{R} \mapsto E^\nu; \quad u \in \mathbf{L}^1, \qquad u \text{ is piecewise constant}\big\}\,. \tag{3.1}$$

On $\mathcal{D}^\nu$ we now construct a semigroup S^ν whose trajectories are front tracking approximate solutions of (1.1). To this end, we first describe how to solve a Riemann problem with data $u^-, u^+ \in E^\nu$. In Riemann coordinates, assume that

$$u^- = (w_1^-, \ldots, w_n^-) \qquad u^+ = (w_1^+, \ldots, w_n^+).$$

Consider the intermediate states

$$\omega_0 = u^-, \qquad \ldots, \qquad \omega_i = (w_1^+, \ldots, w_i^+, w_{i+1}^-, \ldots, w_n^-), \qquad \ldots, \qquad \omega_n = u^+. \tag{3.2}$$

If $w_i^+ < w_i^-$, the solution will contain a single i-shock, connecting the states ω_{i-1}, ω_i and travelling with Rankine-Hugoniot speed $\lambda_i(\omega_{i-1}, \omega_i)$. Here and in the sequel, by $\lambda_i(u, u')$ we denote the i-th eigenvalue of the averaged matrix

$$A(u, u') \doteq \int_0^1 A\big(\theta u + (1-\theta)u'\big)\, d\theta. \tag{3.3}$$

If $w_i^+ > w_i^-$, the exact solution of the Riemann problem would contain a centered rarefaction wave. This is approximated by a rarefaction fan as follows. If $w_i^+ = w_i^- + p_i 2^{-\nu}$ we insert the states

$$\omega_{i,\ell} = (w_1^+, \ldots, w_i^- + \ell 2^{-\nu}, w_{i+1}^-, \ldots, w_n^-) \qquad\qquad \ell = 0, \ldots, p_i, \tag{3.4}$$

so that $\omega_{i,0} = \omega_{i-1}$, $\omega_{i,p_i} = \omega_i$. Our front tracking solution will then contain p_i fronts of the i-th family, each connecting a couple of states $\omega_{i,\ell-1}$, $\omega_{i,\ell}$ and travelling with speed $\lambda_i\big(\omega_{i,\ell-1},\ \omega_{i,\ell}\big)$.

For a given initial data $\bar u \in \mathcal{D}^\nu$, the approximate solution $u(t,\cdot) = S_t^\nu \bar u$ is now constructed as follows. At time $t=0$ we solve each of the Riemann problems determined by the jumps in $\bar u$ according to the above procedure. This yields a piecewise constant function with finitely many fronts, travelling with constant speeds. The solution is then prolonged up to the first time where two or more of these fronts interact. At the interaction points, the new Riemann problems are again solved by the above procedure, etc...

As in [1], one checks that these front tracking approximations are well defined for all times $t \geq 0$. Indeed, the following properties hold.

- For each $i = 1, \ldots, n$, the map $t \mapsto \text{Tot.Var.}\big\{w_i(t,\cdot)\big\}$ is non-increasing.
- The total number of wave-fronts in $u(t,\cdot)$ does not increase in time.
- Each trajectory $t \mapsto u(t,\cdot) = S_t^\nu \bar u$ is a weak solution of (1.1) (because all fronts satify the Rankine-Hugoniot conditions), but may not be entropy-admissible (because of the presence of rarefaction fronts).

As $\nu \to \infty$, the domains $\mathcal{D}^\nu$ become dense in $\mathcal{D}$. We will thus define the semigroup S on $\mathcal{D}$ as a suitable limit of the flows S^ν. To ensure the existence of this limit, the key step is to prove the estimate

$$\lim_{\nu\to\infty} \big\|S_t^\nu \bar u - S_t^\nu \bar v\big\|_{\mathbf{L}^1} \leq L \cdot \|\bar u - \bar v\|_{\mathbf{L}^1} \tag{3.5}$$

for some constant L and any $\bar u, \bar v \in \mathcal{D}^\mu$, $\mu \geq 1$. For this purpose, given any two initial data $\bar u, \bar v$, we consider a continuous path $\gamma_0 : \theta \mapsto \bar u^\theta$, with $\gamma_0(0) = \bar u$, $\gamma_0(1) = \bar v$. More precisely, our path will be a *pseudopolygonal*, i. e. a finite concatenation of *elementary paths*, of the form

$$\theta \mapsto u^\theta := \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{]x^\theta_{\alpha-1},\ x^\theta_\alpha]}, \qquad x_\alpha^\theta = x_\alpha + \xi_\alpha \theta, \qquad \theta \in [a,b] \tag{3.6}$$

where χ_I is the characteristic function of the set I, $\omega_0, \ldots, \omega_N \in I\!R^n$ are constant states and ξ_α is the shift rate of the jump at x_α. In (3.6) it is assumed that $x_1^\theta < \ldots < x_N^\theta$ for $a < \theta < b$. If γ is an elementary path of the form (3.6), its $\mathbf{L}^1$-length is computed by

$$\|\gamma\|_{\mathbf{L}^1} = \int_a^b \sum_{\alpha=1}^{N} \big|\Delta u(x_\alpha)\big| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta = \sum_{\alpha=1}^{N} \big|\sigma_\alpha\big|\,|\xi_\alpha|\,(b-a), \tag{3.7}$$

where

$$\sigma_\alpha \doteq \Delta u(x_\alpha) = \omega_{\alpha+1} - \omega_\alpha. \tag{3.8}$$

Let $u^\theta(t,\cdot) = S_t^\nu \bar u^\theta$ be the corresponding solutions. Since the number of wave-fronts in these solutions is a-priori bounded and the locations of the interaction points in the t-x plane are determined by a linear system of equations, it is clear that, at any time $\tau > 0$, the corresponding path $\gamma_\tau : \theta \mapsto u^\theta(\tau,\cdot)$ is still a pseudopolygonal. Moreover, there exist finitely many parameter values $0 = \theta_0 < \theta_1 < \ldots < \theta_m = 1$ such that the wave-front configuration of u^θ remains the same as θ ranges on each of the open intervals $I_j \doteq]\theta_{j-1}, \theta_j[$. In this case, the lengths of the paths γ_0 and γ_τ are measured by an expression of the form

$$\|\gamma\|_{\mathbf{L}^1} = \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\alpha \left|\Delta u^\theta(x_\alpha^\theta)\right| \left|\frac{\partial x_\alpha^\theta}{\partial \theta}\right| d\theta. \tag{3.9}$$

By deriving a priori estimates on the integrand in (3.9), we will prove that $\|\gamma_\tau\|_{\mathbf{L}^1} \leq L\|\gamma_0\|_{\mathbf{L}^1}$, for some constant L independent of total variation.

4 - Estimates on front tracking solutions

Throughout this section we fix $\nu \geq 1$ and consider a piecewise constant solution u constructed by the front tracking algorithm, so that $u(t,\cdot) = S_t^\nu \bar u$ for some $\bar u \in \mathcal{D}^\nu$. We then perturb this solution, shifting the locations x_α of the jumps at rates ξ_α. In other words, for θ suitably close to zero, the perturbation $u^\theta(t,\cdot)$ will be a function with jumps at the points $x_\alpha^\theta = x_\alpha + \theta\xi_\alpha$. As long as the wave-front configuration of the functions u, u^θ is the same, the shifts $\xi_\beta(t)$ are uniquely determined as linear functions of the shifts $\xi_\alpha(0)$. Some properties of these shift differentials are investigated in the next lemmas.

Lemma 1. *Consider a bounded, open region Γ in the t-x plane. Call σ_α, $\alpha = 1,\ldots,N$ the fronts entering Γ and let ξ_α be their shifts. Assume that the fronts leaving Γ, say σ_β, $\beta = 1,\ldots,N'$, are linearly independent. Then the shifts ξ_β are uniquely determined by the linear relation*

$$\sum_{\alpha=1}^N \xi_\alpha \sigma_\alpha = \sum_{\beta=1}^{N'} \xi_\beta \sigma_\beta\,. \tag{4.1}$$

Proof. Let u, u^θ be the original and the perturbed solution. By possibly modifying the region Γ, we can assume that its boundary consists only of horizontal or vertical segments, and that all fronts of u cross the boundary $\partial\Gamma$ within segments parallel to the x-axis (fig. 1). We now integrate the conservation equation over the region Γ, using the divergence theorem. Observing that $u = u^\theta$ along all vertical segments of the boundary and calling $\partial^-\Gamma$, $\partial^+\Gamma$ respectively the portions of the

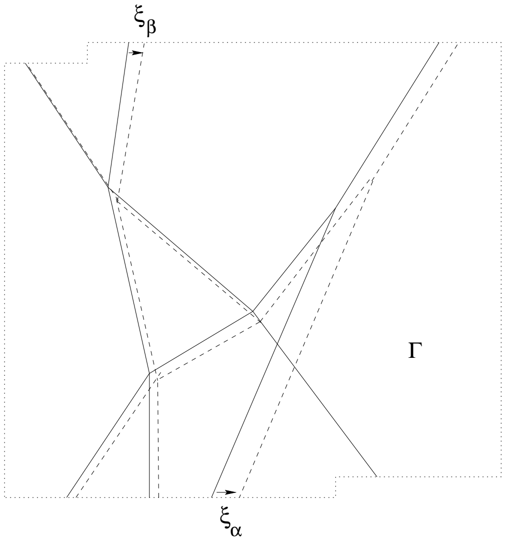

figure 1

boundary containing incoming and outgoing fronts, we have

$$
\begin{aligned}
0 &= \frac{1}{\theta}\int_\Gamma [u^\theta - u]_t + \big[f(u^\theta) - f(u)\big]_x \; dxdt \\
&= \frac{1}{\theta}\int_{\partial^+\Gamma} [u^\theta(t,x) - u(t,x)]dx - \frac{1}{\theta}\int_{\partial^-\Gamma} [u^\theta(t,x) - u(t,x)]dx \\
&= -\sum_{\beta=1}^{N'} \xi_\beta \sigma_\beta + \sum_{\alpha=1}^{N} \xi_\alpha \sigma_\alpha .
\end{aligned}
$$

Since the vectors $\sigma_1, \ldots, \sigma'_N$ are linearly independent, the coefficients ξ_β are uniquely determined by (4.1).

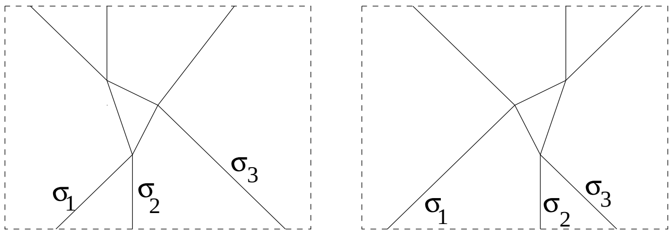

figure 2

Remark 2. According to Lemma 1, the shift rates of the outgoing fronts depend only on the shift rates of the incoming ones, and not on the order in which these wave-fronts interact inside Γ. More precisely, one can perform the following two operations, without changing the shift rates of the outgoing fronts:

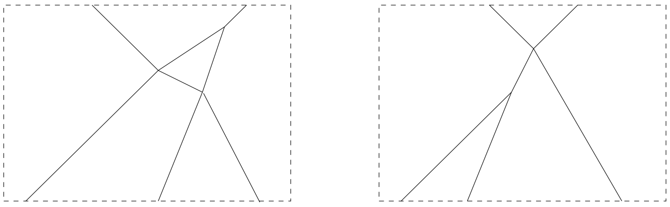

figure 3

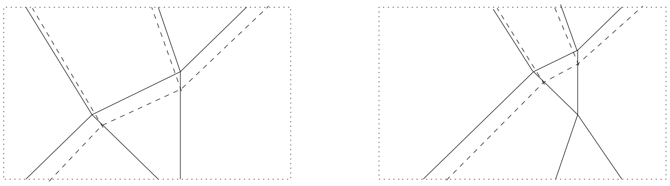

figure 4

(O1) Switch the order in which three fronts interact (fig. 2, fig. 3).
(O2) Invert the order of two fronts at time $t = 0$, provided that both fronts have zero shift rate (fig. 4).
This property will be repeatedly used in our future estimates. Indeed, in the calculation of a shift rate, we can suitably alter the order of wave interactions and thus reduce the problem to a case where the wave-front configuration is particularly simple.

Lemma 2. *Assume that a front tracking solution u contains two wave-fronts of the same k-th family, originating at distinct points and located at $x_{\alpha'}(t) \leq x_{\alpha''}(t)$, $t \in [0, T]$. Then it is possible to assign shift rates $\xi_\alpha(0)$ to all fronts in $u(0, \cdot)$ so that $\xi_{\alpha'}(0) = 1$ and moreover, in the corresponding solution u, all fronts $x_\beta(t)$ outside the strip $\Gamma \doteq \big\{(t, x);\quad t \in [0, T],\ x_{\alpha'}(t) \leq x \leq x_{\alpha''}(t)\big\}$ have zero shift rate.*

In other words (fig. 5), the perturbation of the initial data can be chosen so that one particular front shifts at unit rate, but the corresponding solution remains unaffected outside the region Γ.
Proof. The shift rates $\xi_\alpha(0)$ will be chosen so that, for all t, x,

$$-\frac{d}{d\theta} \int_{-\infty}^{x} u^\theta(t, y)\ dy = \sum_{x_\alpha(t) < x} \xi_\alpha(t) \sigma_\alpha(t) = c(t, x) r_k\big(u(t, x)\big)\,, \tag{4.2}$$

where $c = c(t, x)$ is a scalar function identically equal to zero outside Γ. To achieve (4.2), we proceed by induction. We denote by $x_1(0) < \cdots < x_N(0)$ the locations of the jumps in $u(0, \cdot)$. Moreover, we consider the intermediate states and the jumps

$$u_\alpha \doteq u(0, x) \qquad x \in]x_\alpha,\ x_{\alpha+1}[\,, \qquad\qquad \sigma_\alpha \doteq u_\alpha - u_{\alpha-1}.$$

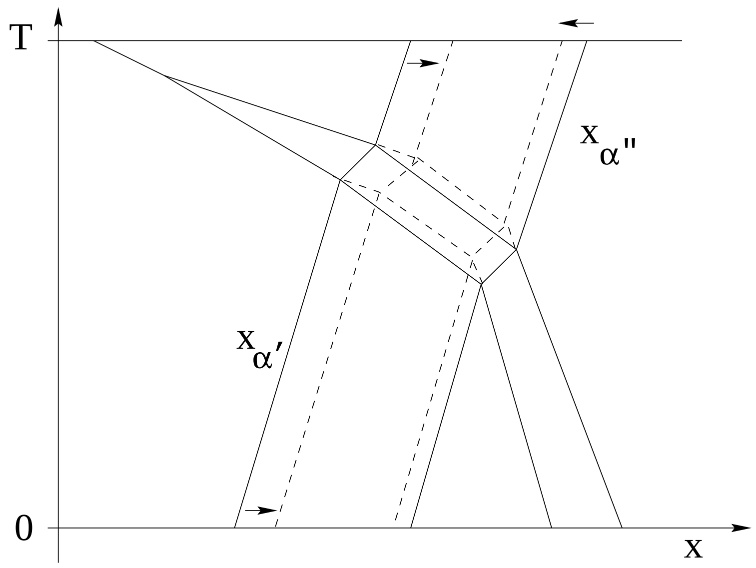

figure 5

The initial shift rates are defined by setting

$$\xi_\alpha = 0 \qquad \text{if} \ \ 1 \le \alpha < \alpha', \qquad \xi_{\alpha'} = 1.$$

If $\alpha' < \alpha \le \alpha''$, assume by induction that $\xi_{\alpha-1}$ has already been defined, so that

$$\sum_{x_\beta < x_\alpha} \sigma_\beta \xi_\beta = c_{\alpha-1} r_k(u_{\alpha-1}). \tag{4.3}$$

for some constant $c_{\alpha-1}$. Two cases will be considered.
Case 1. If the jump at x_α belongs to the k-th family, then the vectors $\sigma_\alpha, r_k(u_{\alpha-1}), r_k(u_\alpha)$ are all parallel, and any choice of the shift ξ_α will automatically satisfy

$$\sum_{x_\beta \le x_\alpha} \sigma_\beta \xi_\beta = c_\alpha r_k(u_\alpha). \tag{4.4}$$

Case 2. If the jump at x_α belongs to the j-th family with $j \neq k$, since the system is of Temple class we have

$$\text{span}\big\{\sigma_\alpha,\ r_k(u_\alpha)\big\} = \text{span}\big\{\sigma_\alpha,\ r_k(u_{\alpha-1})\big\}.$$

Hence there exists a unique shift rate ξ_α and a constant c_α such that

$$c_\alpha r_k(u_\alpha) = \xi_\alpha \sigma_\alpha + c_{\alpha-1} r_k(u_{\alpha-1}). \tag{4.5}$$

Observe that, when $\alpha = \alpha''$, since the front belongs to the k-th family, we can choose the shift rate $\xi_{\alpha''}$ so that

$$\sum_{x_\beta \le x_{\alpha''}} \sigma_\beta \xi_\beta = 0. \tag{4.6}$$

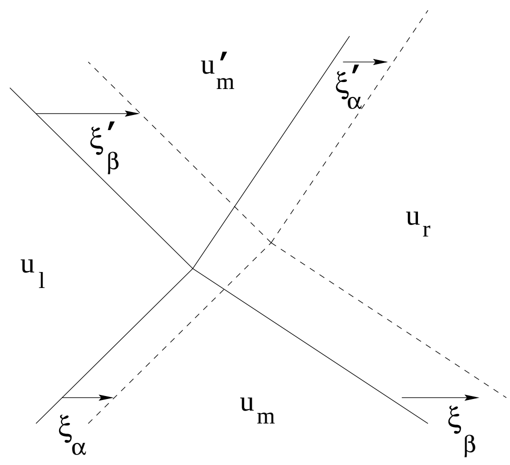

figure 6

We then define $\xi_\alpha = 0$ for all $\alpha > \alpha''$. This achieves (4.2) for all x, at time $t = 0$.

We claim that (4.2) remains valid at all later times. For this purpose, it is enough to study an arbitrary interaction between two wave-fronts, say occurring at a point $(\bar t, \bar x)$. We consider the case of two incoming fronts (fig. 6) of families $k_\alpha \neq k_\beta$, with $k_\alpha, k_\beta \neq k$. The other cases are similar, or easier. Call u_l, u_m, u_r respectively the left, middle and right states before interaction, and let u'_m be the middle state after interaction. We denote by $\sigma_\alpha, \sigma_\beta$ and ξ_α, ξ_β the jumps and shift rates of the incoming fronts, while $\sigma'_\alpha, \sigma'_\beta, \xi'_\alpha, \xi'_\beta$ refer to the outgoing fronts. The inductive assumptions imply

$$\sum_{x_\gamma(\bar t) < \bar x} \sigma_\gamma(\bar t)\xi_\gamma(\bar t) = c_l r_k(u_l) \tag{4.7}$$

for some constant c_l. Moreover

$$c_l r_k(u_l) + \xi_\alpha \sigma_\alpha = c_m r_k(u_m), \tag{4.8}$$

$$c_m r_k(u_m) + \xi_\beta \sigma_\beta = c_r r_k(u_r), \tag{4.9}$$

while the conservation equations yield

$$\xi'_\alpha \sigma'_\alpha + \xi'_\beta \sigma'_\beta = \xi_\alpha \sigma_\alpha + \xi_\beta \sigma_\beta. \tag{4.10}$$

From (4.7)–(4.10) it is clear that

$$\begin{aligned} \sum_{x_\gamma(\bar t) < \bar x} \sigma_\gamma(\bar t)\xi_\gamma(\bar t) + \xi'_\alpha \sigma'_\alpha + \xi'_\beta \sigma'_\beta &= c_l r_k(u_l) + \xi_\alpha \sigma_\alpha + \xi_\beta \sigma_\beta \\ &= c_r r_k(u_r). \end{aligned}$$

It remains to show that, for some constant c'_m, one has

$$c_l r_k(u_l) + \xi'_\beta \sigma'_\beta = c'_m r_k(u'_m). \tag{4.11}$$

This can be easily seen by writing

$$c_l r_k(u_l) + \xi'_\beta \sigma'_\beta = c_r r_k(u_r) - \xi'_\alpha \sigma'_\alpha \tag{4.12}$$

and observing that the vector on the left hand side belongs to the plane $\operatorname{span}\{r_k(u_r),\ r_{k_\beta}(u'_m)\}$, the vector on the right hand side lies in the plane $\operatorname{span}\{r_k(u_l),\ r_{k_\alpha}(u'_m)\}$, and the intersection of these planes is the straight line spanned by $r_k(u'_m)$.

To check that all shifts vanish outside Γ, consider the case where the families of the incoming fronts are $k_\alpha \neq k$, $k_\beta = k$ and the front $t \mapsto x_\beta(t)$ marks the boundary of the region Γ. In this case, the inductive assumption implies $c_r = 0$. From the relation

$$c'_m r_k(u'_m) + \xi'_\alpha \sigma'_\alpha = c_r r_k(u_r) = 0,$$

by the linear independence of the vectors $r_k(u'_m)$, σ'_α it follows $\xi'_\alpha = 0$ as required. This completes the proof of Lemma 2.

Lemma 3. *Let u be a front-tracking solution of (1.1), and consider two wave-fronts, say $t \mapsto x(t)$, $t \mapsto y(t)$, defined for $t \in [0,T]$. Then there exists a second front tracking solution $\tilde u$ with two fronts $\tilde x$, $\tilde y$ such that the following holds.*

(i) $\tilde x(0) = x(0)$, $\tilde y(0) = y(0)$, $\tilde x(T) = x(T)$, $\tilde y(T) = y(T)$.

(ii) $\tilde u = u$ *in a neighborhood of the points* $\big(0, x(0)\big)$, $\big(0, y(0)\big)$, $\big(T, x(T)\big)$, $\big(T, y(T)\big)$.

(iii) Tot.Var.$\big\{\tilde u(0,\cdot)\big\} \leq C_0$, *for some constant C_0 depending only on the system (1.1) and on the set E.*

Proof. To fix the ideas, assume $x(0) < y(0)$. The set $I\!R \setminus \big\{x(0),\ y(0)\big\}$ consists of three connected components which we call J_1, J_2, J_3. Similarly, $I\!R \setminus \big\{x(T),\ y(T)\big\}$ consists of three components J'_1, J'_2, J'_3. Assume that u contains two wave-fronts $t \mapsto z'(t)$, $t \mapsto z''(t)$, of the same k-th family, such that $z'(0)$ and $z''(0)$ lie in the same open set J_i and moreover $z'(T)$ and $z''(T)$ lie in the same J'_j. We now apply Lemma 2, with $x_{\alpha'} = z'$, $x_{\alpha''} = z''$. We claim that, if the k-waves at $z'(0)$ and $z''(0)$ have the same sign, we can choose the initial shift rates $\xi_\alpha(0)$ satisfying the additional conditions

(i) $\xi_{\alpha'}(0) = 1$, $\xi_{\alpha'}(t) \geq 0$ for all $t \in [0,T]$,

(ii) $\xi_\alpha(0) = 0$ if $\alpha' < \alpha < \alpha''$ and the front at x_α is of the k-th family,

(iii) $\xi_{\alpha''}(t) \leq 0$ for all $t \in [0,T]$.

To fix the ideas, let both waves be positive. Since the choice of $\xi_\alpha(0)$ is arbitrary for waves of the k-th family, it is clear that (ii) can be satisfied. To prove (i) and (iii), we first show that the constants $c(t,x)$ vanish outside the strip Γ bounded by z', z'' and are positive inside Γ. At time $t = 0$, by construction we have $c_{\alpha'} > 0$. Moreover, if $c_{\alpha-1} > 0$ for some $\alpha' < \alpha < \alpha''$, then also

$c_\alpha > 0$. This follows from (4.5) and the linear independence assumption (SH). Similarly, at any interaction between two fronts in the interior of the strip Γ, from (4.11)-(4.12) we deduce that, if $c_l, c_m, c_r > 0$, then also $c'_m > 0$. By induction on all finitely many interaction points, we conclude $c > 0$ inside Γ.

Calling $\xi'(t)$, $\sigma'(t)$ respectively the shift rate of the jump at $z'(t)$ and its amplitude, from the relation

$$\xi'(t)\sigma'(t) = c(t, z'(t)+) r_k(u(t, z'(t)+)),$$

since σ' and r_k are parallel and have the same sign, recalling that $c > 0$ inside Γ we conclude $\xi'(t) > 0$. A similar argument yields that the shift of the front z'' satisfies $\xi''(t) < 0$.

By (i) and (iii), we can thus construct a one-parameter family of solutions u^θ shifting forward the front at $z'(\cdot)$ and shifting backwards the front at $z''(\cdot)$. The parameter θ can be raised up to a value $\bar\theta$ where two k-fronts in $u^\theta(0,\cdot)$ coincide. We thus obtain a second solution $u^{\bar\theta}$ which coincides with u outside the strip Γ bounded by the fronts $z'(\cdot)$ and $z''(\cdot)$, but with a smaller number of k-fronts. This construction can be repeated as long as the solution contains fronts of the same family and with the same sign, starting at different points within the same set J_i and ending within the same J'_j. In a finite number of steps, we obtain a new solution $\tilde u$ with the property that, for each $k = 1, \ldots, n$ and $i, j = 1, 2, 3$, there exists at most one point $\bar x \in J_i$ where a positive k-wave originates, terminating within J'_j, and similarly for negative k-waves. Of course, this implies that the total variation of $\tilde u$ is uniformly bounded, with a bound C_0 depending only on n and on the diameter of the set E.

Next we seek an estimate on the amount of positive k-waves in a front tracking solution u at time $\tau > 0$, contained in a bounded interval $[a,\ b]$. As a preliminary, we prove

Lemma 4. *For a fixed $\nu \geq 1$, let $u(t,\cdot) = S_t^\nu \bar u$ be a front tracking solution of the system (1.1). For $t \in [0,T]$, let $x(t) \leq y(t)$ be the positions of two adjacent k-rarefaction fronts. Then for some constant $\kappa > 0$ one has*

$$y(\tau) - x(\tau) \geq \kappa\tau \cdot 2^{-\nu}. \tag{4.13}$$

Proof. By Lemma 3, by possibly replacing u by another solution $\tilde u$, it is not restrictive to assume that $\text{Tot.Var.}\{u(0,\cdot)\} \leq C_0$. Observe that, by construction, each k-rarefaction front has strength $\Delta w_k = 2^{-\nu}$. Let $k_\alpha \in \{1, \ldots, n\}$ be the family of the jump at x_α. Call $\sigma_\alpha \doteq \Delta u(t, x_\alpha) = u(t, x_\alpha+) - u(t, x_\alpha-)$ the jump at x_α and let $\|\sigma_\alpha\| \doteq |\Delta w_{k_\alpha}|$ the strength of the jump, measured in Riemann coordinates. Defining $m(t) \doteq y(t) - x(t)$, by the genuine nonlinearity of the k-th characteristic field we have an estimate of the form

$$\dot m(t) \geq c\cdot 2^{-\nu} - C_1 \cdot \sum_{x(t)<x_\alpha(t)<y(t)} \big\|\sigma_\alpha(t)\big\| \tag{4.14}$$

for some constants $C_1, c > 0$. As usual, we denote by σ_α the value of the jump at x_α, occurring in the k_α-th family. To estimate the contribution of the last term in (4.14), following [6] we introduce the function

$$\Phi(t) \doteq \sum_{k_\alpha\neq k} \phi_{k_\alpha}\big(t, x_\alpha(t)\big)\cdot \big\|\sigma_\alpha(t)\big\|$$

where

$$\phi_i(t,x) \doteq \begin{cases} 1 & \text{if } x < x(t) \\ \frac{y(t)-x}{m(t)} & \text{if } x\in [x(t),\ y(t)] \\ 0 & \text{if } x > y(t) \end{cases} \qquad \text{or} \qquad \phi_i(t,x) \doteq \begin{cases} 0 & \text{if } x < x(t) \\ \frac{x-x(t)}{m(t)} & \text{if } x\in [x(t),\ y(t)] \\ 1 & \text{if } x > y(t) \end{cases}$$

in the cases $i > k$ or $i < k$ respectively. Observe that the map $t\mapsto \Phi(t)$ is piecewise Lipschitz continuous, possibly with a finite number of downward jumps, occurring at interaction times. Because of the strict separation of the characteristic speeds, outside interactions we have

$$\begin{aligned} \dot\Phi(t) &= \sum_{k_\alpha\neq k} \big\|\sigma_\alpha\big\|\cdot \frac{d}{dt}\phi_{k_\alpha}\big(t, x_\alpha(t)\big) \\ &\leq -\sum_{x(t)<x_\alpha(t)<y(t)} \big\|\sigma_\alpha(t)\big\|\cdot \frac{C_2}{m(t)} \end{aligned} \tag{4.15}$$

for some constant $C_2 > 0$. Together, (4.14)-(4.15) yield

$$\dot m(t) - \frac{C_1}{C_2}\cdot \dot\Phi_i(t) m(t) \geq c\cdot 2^{-\nu}, \tag{4.16}$$

$$m(\tau) \geq e^{-(C_1/C_2)\Phi(0)}\cdot c\tau\cdot 2^{-\nu}. \tag{4.17}$$

The uniform bound on the total variation implies $\Phi(0) \leq C_0'$ for some constant C_0'. Hence the lemma holds with $\kappa = ce^{-C_0' C_1/C_2}$.

Lemma 5. *Consider a front tracking solution $u(t,\cdot) = S_t^\nu \bar u$, with $\bar u$ containing N shock fronts of the k-th family. Then, for each $\tau > 0$ and every bounded interval $[a,\ b]$, one has*

$$\textit{Tot.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \leq \frac{2(b-a)}{\kappa\tau} + \|w_k\|_{L^\infty} + (N+1)2^{1-\nu}. \tag{4.18}$$

Proof. By Lemma 4, at time $\tau > 0$ any two adiacent k-rarefaction fronts of u are separated by a distance $\geq \kappa\tau\cdot 2^{-\nu}$. Therefore, the number of rarefaction fronts inside any interval $[a, b]$ is bounded by

$$1 + N + \frac{b-a}{\kappa\tau\cdot 2^{-\nu}}.$$

The positive variation of $w_k(\tau,\cdot)$ on $[a,b]$, i. e. the total amount of upward jumps, thus satisfies

$$\text{Pos.Var.}\big\{w_k(\tau,\cdot);\ [a,\ b]\big\} \leq (1+N)2^{-\nu} + \frac{b-a}{\kappa\tau}. \tag{4.19}$$

In turn, the total variation of $w_k(\tau,\cdot)$ on $[a,b]$ is bounded by $\|w_k\|_{\mathbf{L}^\infty}$ plus twice the positive variation of w_k. Hence (4.18) holds.

5 - Estimates on shift differentials

This section is devoted to the proof of the key estimate on shift differentials.

Lemma 6. *Let $u(t,\cdot) = S_t^\nu \bar u$ be a front tracking solution, with $\bar u$ containing N shocks. Assume that at time $t=0$ one single front is shifted, say of the k-th family, located at $\bar x$ with shift rate $\bar\xi$ and amplitude $\bar\sigma$. Calling $\xi_\alpha(\tau)$, $\sigma_\alpha(\tau)$ the shift rates and the amplitudes of the fronts in $u(\tau,\cdot)$, for some constant C_3 depending only on the system (1.1) and on the domain E we then have*

$$\sum_\alpha \big|\xi_\alpha(\tau)\sigma_\alpha(\tau)\big| \leq C_3(1+N2^{-\nu})|\bar\xi\bar\sigma|. \tag{5.1}$$

Proof. Consider one particular front, say x_{α^*}, of the j-th family, and call $\bar y \doteq x_{\alpha^*}(\tau)$ its terminal point. In the first part of the proof we shall establish the existence of constants C_4, C_5, depending only on the system (1.1) and on the domain E, such that the following properties hold.

(P1) If x_{α^*} is precisely the k-front starting at $\bar x$, then

$$\big|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\big| \leq C_5|\bar\xi\bar\sigma|. \qquad (5.2).$$

(P2) If x_{α^*} is a j-front, with $j\neq k$, and the backward j-characteristics ending at $\bar y$ include fronts starting from both sides of $\bar x$, then (5.2) again holds.

(P3) If x_{α^*} is a j-front, and the j-fronts ending at $\bar y$ start either all at the left or all at the right of $\bar x$, one then has the sharper estimate

$$\big|\xi_{\alpha^*}(\tau)\big| \leq C_4|\bar\xi\bar\sigma|. \tag{5.3}$$

Toward a proof of (P1)–(P3) we observe that, besides the fronts starting at $\bar x$ and the ones ending at $\bar y$, one can single out four groups of waves:

(1) the waves starting on the left of $\bar x$ and ending on the left of $\bar y$;

(2) the waves starting on the right of $\bar{x}$ and ending on the right of $\bar{y}$;

(3) the waves starting on the right of $\bar{x}$ and ending on the left of $\bar{y}$;

(4) the waves starting on the left of $\bar{x}$ and ending on the right of $\bar{y}$.

According to Remark 2, in our computation of the shift rate $\xi_{\alpha^*}(\tau)$ of the front reaching $\bar{y}$, we can assume that the sets of waves in (1) and (2) are empty. Indeed, we can otherwise shift the locations of all these fronts of type (1) toward the left, until they all lie outside the domain of influence of the initial point $\bar{x}$. Similarly, fronts of type (2) can be shifted toward the right until they lie completely outside this domain of influence.

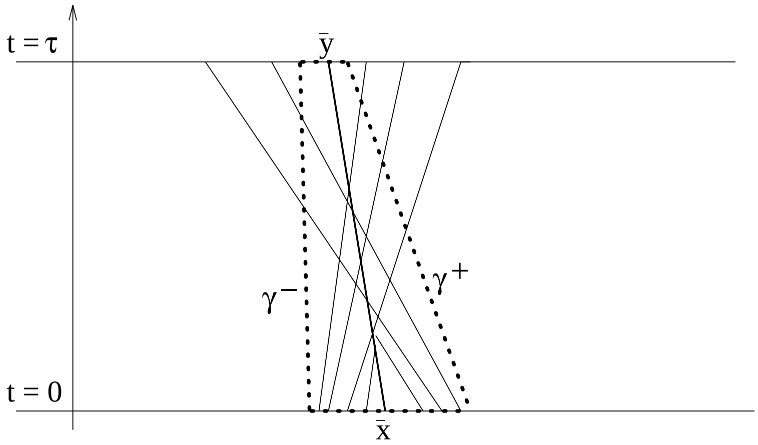

figure 7

Having achieved this simplification, consider the situation described in (P1). We can construct a region of the form

$$\Gamma \doteq \big\{(t,x)\,;\quad t\in[0,\tau],\quad \gamma^-(t)<x<\gamma^+(t)\big\}$$

as in fig. 7, choosing Γ so that all k-fronts starting within $\big[\gamma^-(0),\ \gamma^+(0)\big]$ join together into the single k-front at $\bar{y}$. Again by Remark 2, we can assume that all fronts of type (4) originate from the interval $\big[\gamma^-(0),\ \bar{x}\big]$ and exit from Γ through the side γ^+. Similarly, we can assume that all fronts of type (3) originate from the interval $\big[\bar{x},\ \gamma^+(0)\big]$ and exit from Γ through the side γ^-.

Observe that the fronts of type (3) are those of families $i<k$, while the fronts of type (3) are those of families $i>k$. Applying Lemma 1 to the region Γ, we obtain

$$\bar{\xi}\bar{\sigma}=\sum_{\alpha\in C(\gamma^-)}\xi_\alpha\sigma_\alpha+\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)+\sum_{\alpha\in C(\gamma^+)}\xi_\alpha\sigma_\alpha\,. \tag{5.4}$$

In (5.4) the two summations refer to wave-fronts leaving Γ by crossing the boundary curves γ^- and γ^+ respectively. Calling $u^-=u(\tau,\bar{y}-)$ and $u^+=u(\tau,\bar{y}+)$ the left and right states across the jump at $\bar{y}$, we now observe that the amplitude of each wave σ_α with $\alpha\in C(\gamma^-)$ satisfies

$$\sigma_\alpha\in\mathrm{span}\big\{r_1(u^-),\ldots,r_{k-1}(u^-)\big\}. \tag{5.5}$$

Moreover, the amplitude of each wave σ_α with $\alpha \in C(\gamma^+)$ satisfies

$$\sigma_\alpha \in \mathrm{span}\big\{r_{k+1}(u^-),\ldots,r_n(u^-)\big\}. \tag{5.6}$$

Since $\sigma_{\alpha^*}(\tau)$ is parallel to $r_k(u^+)$, from (5.4)–(5.6) and the assumption (SH) on the linear independence of the eigenvectors, the bound (5.2) follows.

We now establish (P3), first in the case (fig. 8) where no j-wave ending at $\bar y$ crosses the k-wave starting at $\bar x$.

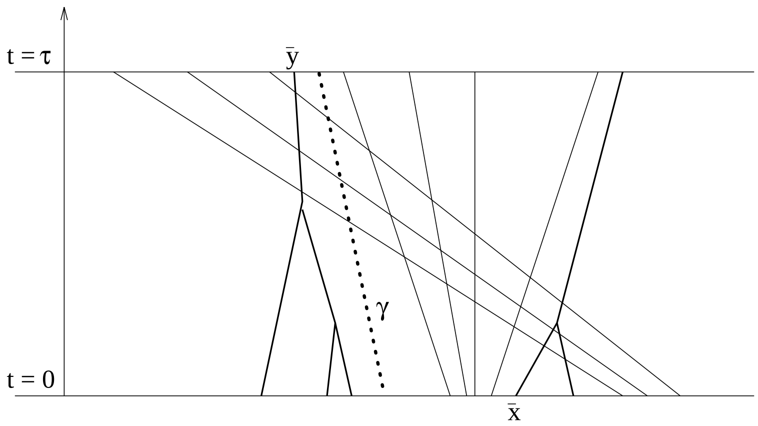

figure 8

Consider a curve γ running slightly to the right of the maximal backward j-front ending at $\bar y$. By Remark 2, after performing the operations (O1)-(O2) a nuber of times, we can consider an equivalent configuration with the following properties:

No front crosses γ from left to right. There exists some index $\ell \le j$ such that only fronts of families $i < \ell$ can cross γ from right to left. Moreover, no i-front with $i < \ell$ terminates at time τ inside the half line $\big[\gamma(\tau),\ \infty\big[$, i. e. all these fronts cross γ before time τ.

Applying Lemma 1 to the region on the right of γ we obtain

$$\sum_{\alpha\in C(\gamma)} \xi_\alpha\sigma_\alpha + \sum_{x_\alpha(\tau)>\gamma(\tau)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar\xi\bar\sigma. \tag{5.7}$$

Here the first summation extends to all fronts crossing the curve γ. As before, call u^- and u^+ the left and right states across the jump at $\bar y$. Observing that the two sums on the right hand side of (5.7) are contained in

$$\mathrm{span}\big\{r_1(u^+),\ldots,r_{\ell-1}(u^+)\big\}, \qquad \mathrm{span}\big\{r_\ell(u^+),\ldots,r_n(u^+)\big\}, \tag{5.8}$$

using the assumption (SH) we conclude

$$\left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_4 |\bar{\xi}\bar{\sigma}|. \tag{5.9}$$

We now apply again Lemma 1 to the region on the left of γ. Observing that the only incoming fronts which carry a nonzero shift rate are those crossing γ from right to left, and that the only outgoing j-front is the one ending at $\bar{y}$, we obtain

$$\sum_{x_\alpha(\tau)<\bar{y}} \xi_\alpha \sigma_\alpha + \xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) = \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{5.10}$$

Recalling the normalization at (2.1), we observe that (5.10) implies

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| = l_j(u^-) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{5.11}$$

On the other hand,

$$l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0. \tag{5.12}$$

Together, (5.11)-(5.12) imply

$$\left|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\right| \leq C' \left|l_j(u^+) - l_j(u^-)\right| \left| \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha \right| \leq C_5 \left|\sigma_{\alpha^*}(\tau)\right| |\bar{\xi}\bar{\sigma}|. \tag{5.13}$$

Hence, in this case, (5.3) holds.

We now prove (P3) in the case where $k > j$ and all j-waves running into $\bar{y}$ cross the k-wave starting from $\bar{x}$, as in figure 9. In this case, we construct a curve γ slightly to the left of the minimal backward j-front ending at $\bar{y}$.

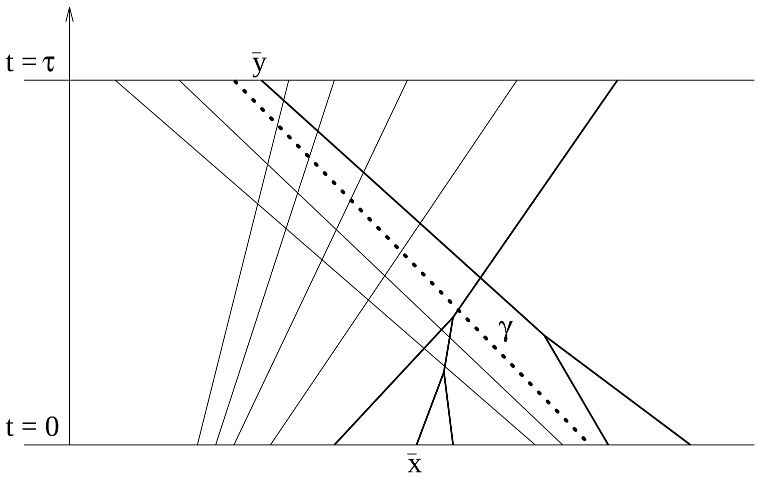

figure 9

Observe that every wave-front crossing γ from left to right must be of a family $i > j$. Moreover, by Remark 2, we can assume that no wave crosses γ from right to left. Applying Lemma 1 to the region on the left of γ we obtain

$$\sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = \bar{\xi}\bar{\sigma}. \tag{5.14}$$

We now apply again Lemma 1 to the region on the right of γ, observing that set of outgoing fronts, crossing the line $t = \tau$, contains the j-front at $\bar{y}$ plus other fronts on the right of $\bar{y}$, of families $i > j$. Together with (5.14), this yields

$$\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau) + \sum_{x_\alpha(\tau) > \bar{y}} \xi_\alpha \sigma_\alpha = \sum_{\alpha \in C(\gamma)} \xi_\alpha(\tau)\sigma_\alpha(\tau) = \bar{\xi}\bar{\sigma}. \tag{5.15}$$

By (5.15),

$$\big|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\big| = l_j(u^+) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha. \tag{5.16}$$

Observing that

$$l_j(u^-) \cdot \sum_{\alpha \in C(\gamma)} \xi_\alpha \sigma_\alpha = 0,$$

we again have an estimate of the form (5.13), hence (P3) holds.

We finally consider the situation described in (P2), where there exist j-waves starting both on the left and on the right of $\bar{x}$ which join into a single front ending at $\bar{y}$, (fig. 10a).

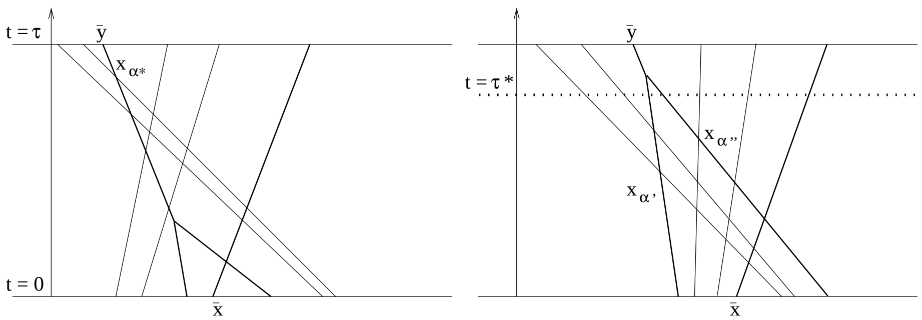

figure 10a figure 10b

According to Remark 2, we can simplify our computations by suitably changing the order of wave-front interactions. In particular, we can perform our estimate in terms of an equivalent wave-front configuration (fig. 10b), where the j-fronts ending at $\bar{y}$ and starting to the left of $\bar{x}$ join together in a single front $x_{\alpha'}$ at a time $\tau^* < \tau$, and all j-fronts ending at $\bar{y}$ and starting to the right of $\bar{x}$ join together in a single front $x_{\alpha''}$ at time τ^*. Applying (P3) to the fronts $x_{\alpha'}$, $x_{\alpha''}$, we obtain

$$\big|\xi_{\alpha'}(\tau^*)\big| \le C_5|\bar{\xi}\bar{\sigma}|, \qquad \big|\xi_{\alpha''}(\tau^*)\big| \le C_5|\bar{\xi}\bar{\sigma}|. \tag{5.16}$$

The final interaction of these two j-fronts determines a single front of strength $\sigma_{\alpha^*} = \sigma_{\alpha'} + \sigma_{\alpha''}$. Using Lemma 1, we compute

$$\begin{aligned}\big|\xi_{\alpha^*}(\tau)\sigma_{\alpha^*}(\tau)\big| &= \big|\xi_{\alpha'}(\tau^*)\sigma_{\alpha'}(\tau^*) + \xi_{\alpha''}(\tau^*)\sigma_{\alpha''}(\tau^*)\big| \\ &\leq C' \cdot \Big(\big|\xi_{\alpha'}(\tau^*)\big| + \big|\xi_{\alpha''}(\tau^*)\big|\Big) \\ &\leq C'C_5|\bar\xi\bar\sigma|.\end{aligned} \tag{5.17}$$

This establishes (P2).

We now complete the proof of Lemma 6. Let $\hat\lambda$ be an upper bound for the absolute values of all characteristic speeds. If the only front shifted at time $t = 0$ is the one at $\bar x$, it follows that at a fixed time $\tau > 0$ the only fronts with nonzero shift rate can be the ones located inside the interval $[\bar x - \hat\lambda\tau,\ \bar x + \hat\lambda\tau]$. Recalling the estimate (4.18) on the total variation, using the properties (P1)–(P3) we thus have

$$\begin{aligned}\sum_\alpha \big|\xi_\alpha(\tau)\sigma_\alpha(\tau)\big| &\leq nC_4|\bar\xi\bar\sigma| + C_5|\bar\xi\bar\sigma| \cdot \text{Tot.Var.}\big\{u(\tau)\,;\ \ [\bar x - \hat\lambda\tau,\ \bar x + \hat\lambda\tau]\big\} \\ &\leq C_3(1 + N2^{-\nu})|\bar\xi\bar\sigma|,\end{aligned}$$

for a suitable constant C_3.

Remark 3. For the front tracking solution u considered in Lemma 6, assume that all fronts in $u(0,\cdot)$ are shifted. More precisely, let $\xi_\alpha(0)$ be the shift rate of the front located at $x_\alpha(0)$, having amplitude $\sigma_\alpha(0)$. Call $\xi_\beta(\tau)$ the corresponding shift rate of the front of $u(\tau,\cdot)$ located at $x_\beta(\tau)$. Observing that the shift differential

$$\big(\xi_1(0), \dots \xi_N(0)\big) \mapsto \big(\xi_1(\tau), \dots \xi_{N'}(\tau)\big)$$

is a linear mapping, from (5.1) it easily follows

$$\sum_\beta \big|\xi_\beta(\tau)\sigma_\beta(\tau)\big| \leq C_3(1 + N2^{-\nu}) \cdot \sum_\alpha \big|\xi_\alpha(0)\sigma_\alpha(0)\big|. \tag{5.14}$$

6 - Proof of the theorems

To construct the semigroup described in Theorem 1, we recall that, for every $M > 0$, by [1] there exists a Lipschitz semigroup S of solutions of (1.1) defined on the domain $\mathcal{D}_M$ defined at (2.7). Moreover, trajectories of this semigroup are the unique limits of front tracking approximations:

$$S_t\bar u = \lim_{\nu\to\infty} S_t^\nu \bar u_\nu$$

for every $\bar{u} \in \mathcal{D}_M$, $\bar{u}_\nu \in \mathcal{D}_M \cap \mathcal{D}^\nu$, $\bar{u}_\nu \to \bar{u}$ in $\mathbf{L}^1$. We need to show that the Lipschitz constant of this semigroup on $\mathcal{D}_M$ does not depend on M. For this purpose, consider any two piecewise constant initial data, say $\bar{u}, \bar{v} \in \mathcal{D}^\mu$. We can now construct a pseudopolygonal path $\gamma_0 : \theta \mapsto \bar{u}^\theta$ connecting $\bar{u}$ with $\bar{v}$ with the following properties. The $\mathbf{L}^1$ length of γ_0 satisfies

$$\|\gamma_0\|_{\mathbf{L}^1} \leq C_6 \|\bar{u} - \bar{v}\|_{\mathbf{L}^1} . \tag{6.1}$$

Moreover, all functions $\bar{u}^\theta$ lie in $\mathcal{D}^\mu$ and have a uniformly bounded number of shocks, say $\leq N$.

Calling $\gamma_\tau : \theta \mapsto u^\theta \doteq S_\tau u^\theta$, we claim that the length of γ_τ remains a bounded multiple of the length of γ_0. Indeed, for every $\nu \geq \mu$, consider the path $\gamma_\tau^\nu : \theta \mapsto u^\theta \doteq S_\tau^\nu u^\theta$. Writing the length of this path in the form (3.9) and using Lemma 6 we obtain

$$\begin{aligned}
\|\gamma_\tau^\nu\|_{\mathbf{L}^1} &= \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} \sum_\beta \left|\Delta u_\nu^\theta(\tau, x_\beta^\theta)\right| \left|\frac{\partial x_\beta^\theta(\tau)}{\partial \theta}\right| d\theta. \\
&\leq \sum_{j=1}^m \int_{\theta_{j-1}}^{\theta_j} C_3(1 + N2^{-\nu}) \cdot \sum_\alpha \left|\Delta u_\nu^\theta(0, x_\alpha^\theta)\right| \left|\frac{\partial x_\alpha^\theta(0)}{\partial \theta}\right| d\theta. \\
&\leq C_3(1 + N2^{-\nu}) \cdot \|\gamma_0^\nu\|_{\mathbf{L}^1} .
\end{aligned} \tag{6.2}$$

Letting $\nu \to \infty$ we obtain (2.4) for all $\bar{u}, \bar{v} \in \mathcal{D}^*$. The semigroup S can thus be extended by continuity to the whole domain $\mathcal{D}$, preserving the property (2.4). Every semigroup trajectory is thus a limit of front tracking approximations, hence it provides a solution of the system (1.1).

Concerning the entropy condition (2.5), fix an interval $[a, b]$ and an initial condition $\bar{u}$. We can now approximate the solution $u(t, \cdot) = S_t \bar{u}$ with a sequence of front tracking solutions $u_\nu(t, \cdot) = S_t^\nu \bar{u}_\nu$, choosing initial data $\bar{u}_\nu$ having a number of shocks $N_\nu \leq \nu$. By (4.19), the total number of positive wave-fronts in $u_\nu(\tau, \cdot) = S_\tau^\nu \bar{u}_\nu$ on a given interval $[a, b]$ satisfies

$$\text{Pos.Var.}\big\{w_k^\nu(\tau, \cdot);\ [a,\ b]\big\} \leq (1 + N_\nu)2^{-\nu} + \frac{b - a}{\kappa\tau}, \tag{6.3}$$

where $w_k^\nu \doteq w_k(u_\nu)$ is the k-th Riemann coordinate of u_ν. Letting $\nu \to \infty$ in (6.3) we obtain

$$\text{Pos.Var.}\big\{w_k\big(u(\tau, \cdot)\big);\ [a,\ b]\big\} \leq \frac{b - a}{\kappa\tau}. \tag{6.4}$$

Hence (2.5) holds. This completes the proof of Theorem 1.

Remark 4. In general the semigroup S_t constructed as above will not be Lipschitz continuous w. r. t. time. Indeed, the map $t \mapsto S_t \bar{u}$ from $[0, \infty[$ to $\mathbf{L}^1(\mathbb{R})$ may not be Lipschitz continuous at time $t = 0$ if $\bar{u}$ has unbounded total variation.

We now give a proof of Theorem 2. Fix any $R > 0$ and let $\hat{\lambda}$ be an upper bound for the absolute values of all characteristic speeds. Let u be a weak solution of (1.1) satisfying the decay estimate (2.5). For every $\delta > 0$ the restriction of $u(t,\cdot)$ to the intervals $I(t) \doteq [-R+\hat{\lambda}t,\ R-\hat{\lambda}t]$, $t \in [\delta,\ R/\hat{\lambda}]$ has uniformly bounded variation. Therefore, the uniqueness theorem in [8] yields

$$u(t,\cdot) = S_{t-\delta}u(\delta) \qquad \text{restricted to } I(t). \tag{6.5}$$

$$\begin{aligned}\int_{I(t)} \big|u(t,x) - (S_t\bar{u})(x)\big|\,dx &\le \int_{I(t)} \Big|\big(S_{t-\delta}u(\delta)\big)(x) - \big(S_{t-\delta}\circ S_\delta u(\delta)\big)(x)\Big|\,dx \\ &\le L\cdot \big\|u(\delta,\cdot) - S_\delta\bar{u}\big\|_{\mathbf{L}^1}\,.\end{aligned} \tag{6.6}$$

Letting $\delta \to 0$ we conclude that the left hand side of (6.6) must be zero. Since R was arbitrary, this proves (2.6).

7 - A counterexample

The following example shows that the Lipschitz continuous dependence on the initial data may not hold if one drops the assumption of genuine nonlinearity of all characteristic fields.

Consider the 2×2 system

$$\begin{aligned}(u_1)_t + \left(\frac{u_1}{1+u_1+u_2}\right)_x &= 0,\\ (u_2)_t + \left(\frac{u_2}{1+u_1+u_2}\right)_x &= 0,\end{aligned} \tag{7.1}$$

where $u_1,\ u_2 > 0$. Here, the first characteristic field is genuinely nonlinear, the second is linearly degenerate. In terms of the Riemann coordinates $w_1 \doteq u_1+u_2$, $w_2 \doteq u_2/u_1$, this system takes the form

$$\begin{aligned}(w_1)_t + \left(\frac{w_1}{1+w_1}\right)_x &= 0,\\ (w_2)_t + \frac{1}{1+w_1}(w_2)_x &= 0.\end{aligned} \tag{7.2}$$

Fix some coordinate values $0 < w_1^- < w_1^+$, $0 < w_2^- < w_2^+$. We now consider an initial data $\bar{u}(x) = \big(\bar{w}_1(x),\ \bar{w}_2(x)\big)$ whose Riemann coordinates are defined as follows. Given three points $a < b < c$ on the real line, we take

$$\bar{w}_1(x) \doteq \begin{cases} w_1^- & \text{if} \quad x < c,\\ w_1^+ & \text{if} \quad x > c.\end{cases}$$

Moreover we set

$$\bar{w}_2(x) \doteq w_2^- \qquad \text{if } \ x \notin [a,b]$$

and let $\bar{w}_2$ oscillate countably many times between the two values w_2^-, w_2^+ as x ranges inside $[a,b]$, so that its total variation is infinite.

The corresponding solution $u = u(t,x)$ will thus contain a 1-shock, travelling with speed λ_1, and countably many 2-contact discontinuities, travelling with speeds λ_2^-, λ_2^+ respectively before and after the interaction with the shock. An explicit computation yields

$$\lambda_1 = \frac{w_1^+/(1+w_1^+) - w_1^-/(1+w_1^-)}{w_1^+ - w_1^-}, \qquad \lambda_2^- = \frac{1}{1+w_1^-} > \lambda_2^+ = \frac{1}{1+w_1^+}. \tag{7.3}$$

Now consider a perturbed initial condition, obtained by shifting the 1-shock from c to $c+\theta$, and fix a time $t > 0$ large enough so that waves of different families will have crossed each other (fig. 11). Then $u(t,\cdot)$ contains a 1-shock located at a point $c' = c + \lambda_1 t$ and a family of 2-waves located within an interval $[a', b']$, while the corresponding perturbed solution $u^\theta(t,\cdot)$ will contain a 1-shock located at $c' + \theta$ and a family of 2-waves, located within the interval $[a' + \xi\theta,\ b' + \xi\theta]$. Here the shift rate is computed by

$$\xi = \frac{\lambda_2^- - \lambda_2^+}{\lambda_2^- - \lambda_1} \neq 0. \tag{7.3}$$

Observing that

$$\frac{1}{\theta} \cdot \left\| \bar u^\theta - \bar u \right\|_{\mathbf{L}^1} = \left| \Delta \bar u(c) \right|,$$

$$\begin{aligned}\limsup_{\theta\to 0} \frac{1}{\theta} \int_{-\infty}^{\infty} \left| w_2^\theta(t,x) - w_2(t,x) \right| \, dx &= \limsup_{\theta\to 0} \frac{1}{\theta} \int_{-\infty}^{\infty} \left| w_2^\theta(t,x-\xi\theta) - w_2(t,x) \right| \, dx \\ &= \xi \cdot \text{Tot.Var.}\left\{ w_2(t,\cdot) \right\} = \infty,\end{aligned}$$

it is clear that the map $\bar u \mapsto S_t \bar u$ is not Lipschitz continuous w. r. t. the $\mathbf{L}^1$ distance.

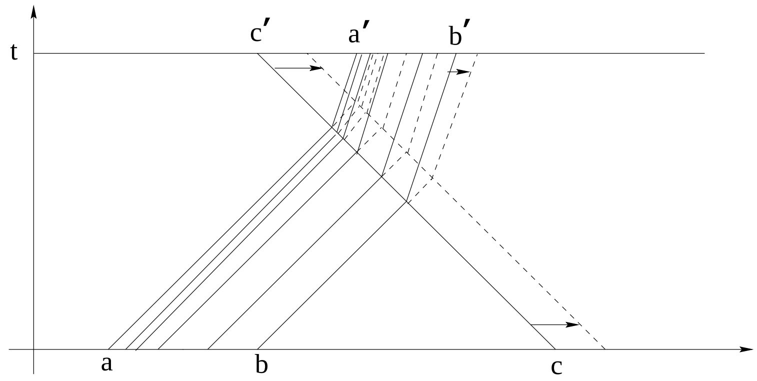

figure 11

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