Granular avalanche flow down a smoothly varying slope: the existence of entropy solutions

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Abstract

We consider a model developed by Savage and Hutter [SH91] which describes the flow of granular avalanches down a smoothly varying slope. The system of two conservation equations for height and momentum of the avalanche resembles the shallow water equations.

The existence of entropy solutions to this model is proved using the vanishing viscosity method. Existence can generally only be guaranteed for a finite time depending on the initial data and the curvature of the slope.

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1 Introduction

We shall investigate the existence of entropy solutions to a one-dimensional continuum model for the flow of a granular avalanche down a smoothly varying slope. In [SH89], Savage and Hutter derived a model for a one-dimensional avalanche flowing down a slowly varying slope. The same authors then improved this model in [SH91] to allow a smoothly varying slope. Other works ([GKH94],[KGH94], [GT97],[TG97]) describe the derivation of some other related models, as well as some experimental and numerical results.

Denoting by (X, Y) the usual Cartesian coordinates, we define a new system of orthogonal curvilinear coordinates (x, y) such that the y is constant on lines parallel to the slope ($y = 0$) and x is constant on lines perpendicular to the slope. At a point on the slope $(x, 0)$, the angle to the horizontal is denoted by $\zeta(x)$. Given the initial height $y = h_0(x)$ and (height-averaged) velocity $u_0(x)$ of the avalanche we wish to determine the height $h(t, x)$, and velocity, $u(t, x)$ at later times (see Figure 1).

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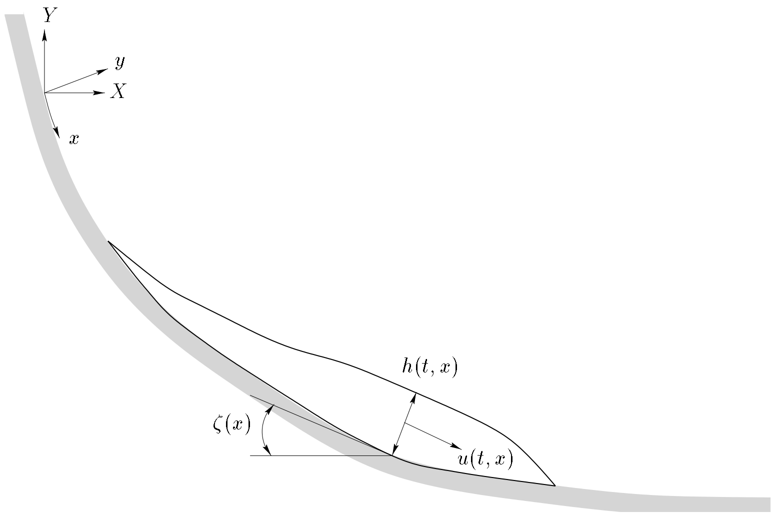

Figure 1: A finite mass granular avalanche with depth-averaged velocity $u(t,x)$ and perpendicular height $y = h(t,x)$ above a curved base with variable slope angle $\zeta(x)$.

Using a Mohr-Coulomb yield criterion to model the pressure as well as scalings similar to the shallow water scalings (see [SH91]), Savage and Hutter obtain a nonstrictly hyperbolic system of two equations for h and hu corresponding to the conservation of mass and momentum respectively.

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (hu)_t + (hu^2 + \tfrac{1}{2}\beta h^2)_x &= h\,g(x,u), \end{aligned} \tag{1.1}$$

with initial conditions:

$$\begin{aligned} h(0,x) &= h_0(x) \geq 0 \\ u(0,x) &= u_0(x), \end{aligned} \tag{1.2}$$

where the function β is strictly positive and takes the form

$$\beta(x, u_x) = \epsilon k_{\mathrm{act,pass}}(u_x) \cos\zeta(x),$$

and $k_{\mathrm{act,pass}}(u_x)$ is the earth pressure coefficent. The dependence on u_x only exhibits itself as a switching between two positive states depending on whether u_x is positive or negative. In this paper we shall concentrate on the simpler system obtained by supposing that these two states are equal and thus assuming that $\beta = \beta(x)$. We shall make the further assumption that β is differentiable and that $\beta'/\beta \in L^\infty(\mathbb{R})$.

In the Savage-Hutter model, the source term g takes the form

$$g(x,u) = \sin\zeta(x) - \frac{u}{|u|}\tan\delta_F(x)\cos\zeta(x). \tag{1.3}$$

where $\delta_F(x)$ is the dynamic friction angle at the position x. Unfortunately g is discontinuous as a function of u. The existence result we shall prove only applies to source terms g which are at least locally Lipschitz continuous with respect to u, though the dependence in x may be less smooth. As a result we can only apply our results to a modified system in which g is replaced by an approximation which is locally Lipschitz with respect to u. For example

$$g(x,u) = \sin\zeta(x) - \frac{u}{|u|+\xi}\tan\delta_F(x)\cos\zeta(x). \tag{1.4}$$

for some small parameter ξ. We remark that the only point where the Lipschitz continuity of g with respect to u is necessary, is in proving that a sequence of approximate solutions we construct converges to an entropy solution of the original system. Without the assumption, we still have strong convergence of the approximations, though we may no longer have convergence to a solution.

It is helpful to transform the system so that the x-dependence in the flux term is transferred to the source term. To this aim we introduce the new unknowns, $\rho = \beta h$, $m = \beta h u$ and define the function $\gamma = \beta'/\beta$. We then derive the following system:

$$v_t + \nabla F(v) v_x = G(v) \quad \text{in } Q_T, \tag{1.5}$$

$$v(0,x) = v_0(x) \tag{1.6}$$

where $v = (\rho, m)$, $v_0 = (\beta h_0, \beta h_0 u_0) =: (\rho_0, m_0)$, $F(v) = (m, m^2/\rho + \frac{1}{2}\rho^2)$ and $G(v,x) = (\gamma m, \gamma\,(m^2/\rho + \frac{1}{2}\rho^2) + \rho\, g)$.

This is a nonstrictly hyperbolic system with a parabolic degeneracy at $\rho = 0$. In general, we expect shocks to develop in finite time regardless of how smooth we choose the initial data. To allow nonsmooth solutions we shall seek weak solutions of (1.5)-(1.6) which also satisy an entropy condition.

Definition 1.1
Suppose that $\eta = \eta(v)$, $q = q(v)$ are scalar C^1 functions satisfying,

$$\nabla\eta\,\nabla F = \nabla q.$$

Such functions η and q are called Entropy-Flux Pairs. *If $\eta(\rho{=}0){=}0$ then (η, q) is a* Weak Entropy-Flux Pair. *If η is convex then (η, q) is a* Convex Entropy-Flux Pair.

Definition 1.2
We call $v \in L^\infty_+(Q_T) \times L^\infty(Q_T)$ a Weak Entropy Solution *to the system* (1.5) *with initial conditions* (1.6) *if it is a weak solution of* (1.5),(1.6), *i.e.:*

$$\iint_{t>0} (\, v \cdot \psi_t + F(v) \cdot \psi_x \, + G \cdot \psi \,) \; dx \, dt = \int_{t=0} v_0 \cdot \psi \, dx$$

for all test functions $\psi \in (C_0^1([0,T) \times \mathbb{R}))^2$, *and the entropy inequality*

$$\iint_{t>0} (\, \eta \phi_t + q \phi_x \, + \nabla \eta \, G \, \phi \,) \; dx \, dt \geq \int_{t=0} \eta \phi \, dx$$

holds for all test functions $\phi \in C_0^1([0,T) \times \mathbb{R})$ *and all convex weak entropy flux pairs* (η, q).

In this paper we shall prove the following result:

Theorem 1.3
Let $\beta : \mathbb{R} \to \mathbb{R}_+$ *be a strictly positive differentiable function with* $\gamma := \beta'/\beta \in L^\infty(\mathbb{R})$. *Let* $\rho_0, m_0 \in L^\infty(\mathbb{R})$ *with* $\rho_0 \geq 0$ *and* $\rho_0, m_0 \to 0$ *as* $|x| \to \infty$. *Let* $\gamma \in L^\infty(\mathbb{R})$. *Let* $g : \mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto g(x,y)$ *be a uniformly bounded function which is locally Lipschitz with respect to* y. *Let* $P = \frac{5}{8}||\gamma||_\infty$, $Q = ||g||_\infty$ *and* $M_0 = \max(2\|\sqrt{\rho_0}\|_\infty, \|m_0/\rho_0\|_\infty)$.

1. *If* $P = Q = 0$ *then the hyperbolic system* (1.5) *has a global in time entropy solution in the sense of Definition 1.2 which is uniformly bounded with respect to time. This bound depends only on the quantity* M_0.

2. *If* $P = 0$ *and* $Q > 0$ *then* (1.5) *has an entropy solution which is bounded for all finite times. The bound depends only on* M_0, Q *and* t.

3. *If* $P > 0$ *there exists a time* T_{max} *so that* (1.5) *has an entropy solution for times* $t < T_{max}$. *In the case* $Q = 0$ *it is sufficient to take* $T_{max} = 1/M_0 P$, *and when* $Q > 0$ *it is sufficient to take*

$$T_{max} = \frac{1}{PQ} \left(\frac{\pi}{2} - \tan^{-1} M_0 \sqrt{P/Q} \; \right).$$

We remark that there is a one-to-one correspondence between weak entropy solutions of (1.5)-(1.6) and weak entropy solutions of (1.1)-(1.2) obtained via the transformations $\rho = \beta h$, $m = \beta h u$ and $\gamma = \beta'/\beta$ (the Rankine-Hugoniot conditions are the same for both systems). Also, if v is a C^1 weak solution of (1.5)-(1.6), then v is a classical solution of (1.5)-(1.6).

Our strategy for finding weak entropy solutions is the vanishing viscosity method and compensated compactness. First of all we add small second order terms to the system in order to make it strictly parabolic. Standard local existence results for parabolic systems furnish us with a classical local solution of the approximate problem. Next we extend this local solution to a finite time solution, where the existence time can be controlled explicitly in terms of initial data and the curvature. In our case, the source terms grow too rapidly to allow us to prove global existence results. Next we require some estimates which allow us to control the entropy and which allow us to apply the div-curl lemma. Results of Lions, Perthame and Souganidis [LPS96] then allow us to show the existence of an entropy solution of the original system.

2 Approximation by a parabolic system

Let $\rho_0^\varepsilon, m_0^\varepsilon$ be smooth functions such that $\rho_0^\varepsilon \to \rho_0$ and $m_0^\varepsilon \to m_0$ pointwise a.e. as $\varepsilon \to 0$, and that $\rho_0^\varepsilon \geq \delta(\varepsilon) > 0$. We consider the parabolic system:

$$\begin{gathered} v_t^\varepsilon + \nabla F(v^\varepsilon) v_x^\varepsilon = G^\varepsilon(v^\varepsilon) + \varepsilon v_{xx}^\varepsilon \quad \text{in } Q_T, \\ v^\varepsilon(0,x) = (\rho_0^\varepsilon(x), m_0^\varepsilon(x)),\ x \in \mathbb{R} \\ \rho^\varepsilon \to \delta,\ m^\varepsilon \to 0 \text{ as } |x| \to \infty \end{gathered} \tag{2.1}$$

where $v^\varepsilon = (\rho^\varepsilon, m^\varepsilon)$ and $G^\varepsilon(v,x) = \chi_{(-1/\varepsilon, 1/\varepsilon)}(x) G(v,x)$.

In order to prove existence results for the above problem, we first consider a finite domain problem for each fixed $\varepsilon > 0$. We approximate ρ^ε, m^ε by smooth functions $\rho_0^{\varepsilon,L} \geq \delta(\varepsilon)$ and $m_0^{\varepsilon,L}$, such that $\operatorname{supp}(\rho_0^{\varepsilon,L} - \delta), \operatorname{supp}(m_0^{\varepsilon,L}) \subset \Omega_L := (-L, L)$. We suppose $\rho_0^{\varepsilon,L} \to \rho_0^\varepsilon$ and $m_0^{\varepsilon,L} \to m_0^\varepsilon$ in $\mathbb{R}$, pointwise almost everywhere as $L \to \infty$. We study the following parabolic system:

$$\begin{gathered} v_t^{\varepsilon,L} + \nabla F(v^{\varepsilon,L}) v_x^{\varepsilon,L} = G^\varepsilon(v^{\varepsilon,L}) + \varepsilon v_{xx}^{\varepsilon,L} \quad \text{in } (0,T) \times \Omega_L, \\ v^{\varepsilon,L}(0,x) = (\rho_0^{\varepsilon,L}(x), m_0^{\varepsilon,L}(x)),\ x \in \Omega_L \\ \rho^{\varepsilon,L}(t, \pm L) = \delta(\varepsilon),\ m^{\varepsilon,L}(t, \pm L) = 0,\ t \in (0,t). \end{gathered} \tag{2.2}$$

In the following section we shall show that a classical solution to this problem will exist for some small time. The existence time can be extended to the maximum interval in which $||\rho^{\varepsilon,L}||_\infty$, $||m^{\varepsilon,L}||_\infty$ and $||1/\rho^{\varepsilon,L}||_\infty$ remain bounded.

3 Local existence for the parabolic system (2.2)

We now prove that the parabolic system (2.2) has a unique classical solution for some small time interval. This is essentially standard.

We write (2.2) in the form

$$\frac{dw(t)}{dt} = Aw(t) + B(w(t))$$

where $w(t) = (\rho^{\varepsilon,L}(t) - \delta, m^{\varepsilon,L}(t))$, $A = -\varepsilon\Delta$, $\mathcal{D}(A) = H^2(\Omega_L) \cap H_0^1(\Omega_L)$ and $B(w) = G^\varepsilon(v^{\varepsilon,L}) - \nabla F(v^{\varepsilon,L})v_x^{\varepsilon,L}$. The operator A with the domain $\mathcal{D}(A)$ generates an analytic semigroup and we can define its fractional powers. For $\alpha \in [0,1]$, we define X_α to be the Banach space $\mathcal{D}(A^\alpha)$ equipped with the graph norm. We define the open set

$$U = \{w = (\rho^{\varepsilon,L} - \delta, m^{\varepsilon,L}) \in \mathcal{D}(A^\alpha) \cap L^\infty(\Omega_L); \rho^{\varepsilon,L} > 0\}.$$

Let $w_1, w_2 \in U_\alpha$, $w_i = (\rho_i^{\varepsilon,L} - \delta, m_i^{\varepsilon,L})$, $i = 1,2$. If we suppose that $\alpha \in (\frac{1}{2}, 1]$ then we have the continuous embeddings $X_\alpha \hookrightarrow H^1(\Omega_L)$ and $X_\alpha \hookrightarrow L^\infty(\Omega_L)$. Using these embeddings, one can show that B satisfies a local Lipschitz condition of the form

$$||B(w_1) - B(w_2)||_{L^2(\Omega\varepsilon)} \leq C||A^\alpha(w_1 - w_2)||_{L^2(\Omega_L)}.$$

where

$$C = C(||\rho_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}, ||m_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}, ||1/\rho_i^{\varepsilon,L}||_{L^\infty(\Omega_L)}).$$

Hence for data in U and some small enough $t_0 > 0$, there exists a unique strong local solution of (2.2) (e.g. [Paz83]),

$$(\rho^{\varepsilon,L}, m^{\varepsilon,L}) \in C^0([0,t_0), L^2(\Omega_L)) \cap C^1((0,t_0), L^2(\Omega_L)).$$

Regularity arguments allow us to conlude *a posteriori* that $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ is in fact a classical solution of (2.2).

To extend this local solution to a global solution, we need to ensure that solutions cannot leave the set U (*n.b.* the size of the interval of existence does not depend on the initial data). This means that we require estimates for the quantities $\rho^{\varepsilon,L}$, $m^{\varepsilon,L}$ and $1/\rho^{\varepsilon,L}$ in $L^\infty(\Omega_L)$. In order to be able to pass to the limit as $L \to \infty$ these estimates should be independent of L. Furthermore, to pass to the limit as $\varepsilon \to 0$, the estimates for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$ should be independent of both L and ε.

4 An *a priori* estimate for $\rho^{\varepsilon,L}$

Before we can prove that $\rho^{\varepsilon,L}$ is strictly positive for all $\varepsilon > 0$, we shall first need to prove a slightly weaker estimate. Namely, in this section we show that $\rho^{\varepsilon,L} \geq 0$. This result will be required in the next section, where we prove L^∞ bounds for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$.

Suppose that ρ is a classical solution of the equation

$$\begin{aligned} \rho_t + (\rho u)_x &= \gamma \rho u + \varepsilon \rho_{xx} \\ \rho(0,x) &= \rho_0(x) \geq 0, \end{aligned} \tag{4.1}$$

where u is a given function in $C^1((0,T)\times\Omega_L)\cap L^\infty((0,T)\times\Omega_L)$. Notice that if $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ is a classical solution of (2.2) and $m^{\varepsilon,L}/\rho^{\varepsilon,L}$ is bounded, then $\rho = \rho^{\varepsilon,L}$ satisfies the above conditions with with $u = m^{\varepsilon,L}/\rho^{\varepsilon,L}$.

Multiply the equation by $\phi = \min(0,\rho)$ and integrate over the region $(0,T)\times\Omega_L$ to get

$$\begin{aligned} &\tfrac{1}{2}\int_{\Omega_L} \phi(T)^2\,dx + \varepsilon \int_0^T\!\!\int_{\Omega_L} \phi_x^2\,dxdt \\ &= \int_0^T\!\!\int_{\Omega_L} u\phi\phi_x\,dxdt + \int_0^T\!\!\int_{\Omega_L} \gamma u \phi^2\,dxdt \\ &\leq ||u||_\infty \int_0^T\!\!\int_{\Omega_L} |\phi|\,|\phi_x|\,dxdt + ||\gamma u||_\infty \int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \\ &\leq \varepsilon \int_0^T\!\!\int_{\Omega_L} \phi_x^2\,dxdt + \left(\frac{||u||_\infty^2}{4\varepsilon} + ||\gamma u||_\infty\right) \int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \end{aligned}$$

Hence

$$\tfrac{1}{2}\int_{\Omega_L} \phi(T)^2\,dx \leq \left(\frac{||u||_\infty^2}{4\varepsilon} + ||\gamma u||_\infty\right) \int_0^T\!\!\int_{\Omega_L} \phi^2\,dxdt \tag{4.2}$$

and so $\phi = 0$ by Gronwall's lemma. In other words, $\rho \geq 0$.

5 An invariant region theorem

In this section we modify the well-known invariant region theorem of Chueh, Conley and Smoller [CCS77], and use it to prove L^∞-estimates for $\rho^{\varepsilon,L}$ and $m^{\varepsilon,L}$, at least for some finite time which can be explicitly calculated in terms of the initial data and the curvature of the slope.

Definition 5.1
Let $R_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1,\ldots,k$, and $M : [0,T] \to \mathbb{R}$ be smooth functions. We define the region

$$\Sigma(t) = \bigcap_{i=1}^{k} \{v \in \mathbb{R}^n : R_i(v) \leq M(t)\}. \tag{5.1}$$

If $v : \mathbb{R}^+\times\mathbb{R} \to \mathbb{R}^n$, we call Σ an invariant region *for v if $v(0,x) \in \Sigma(0)$ implies that $v(t,x) \in \Sigma(t)$ for $t > 0$.*

We now present the modified invariant region theorem (see also *c.f.* [CCS77] and [FI97] which are the cases $M(t) = M_0$ and $M(t) = M_0 e^{\omega t}$ respectively):

THEOREM 5.2
Let $v : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^n$ be a classical solution of the $n \times n$ system:

$$v_t + A(v,x)v_x = B(v,x,t) + \varepsilon C(v,x)v_{xx} \tag{5.2}$$

and Σ be defined by (5.1)*. Suppose that whenever (t_0, v_0) satisfies $R_i(v_0) = M(t_0)$ for some i, the following conditions hold:*

1. *∇R_i at v_0 is a left eigenvector of $C(v_0, x)$ and $A(v_0, x)$ for all $x \in \mathbb{R}$.*
2. *If $\nabla R_i\, C(v_0, x) = \mu \nabla R_i$ and $\mu \neq 0$, then R_i is quasiconvex at v_0 (i.e. $\nabla^2 R_i(r,r) \geq 0$ whenever $\nabla R_i\, r = 0$).*
3. *$\nabla R_i\, B < M'(t_0)$ at $(t,v) = (t_0, v_0)$, for all $x \in \mathbb{R}$.*

Then Σ is an invariant region for v.

PROOF

The proof is *mutatis mutandis* the same as the proofs in [CCS77] and [FI97], though we include it for completeness.

Let us suppose that for some i, $R_i(v(t,x)) < M(t)$ for $0 \leq t < t_0$ and $R_i(v(t_0, x_0)) = M(t_0)$ for some $x_0 \in \mathbb{R}$. It suffices to show that

$$\frac{\partial}{\partial t} R_i(v(t_0, x_0)) < M'(t_0).$$

Now under the assumptions of the theorem, at $v_0 = v(t_0, x_0)$ we have

$$\nabla R_i\, A(v_0, x) v_x = \lambda(x) \nabla R_i\, v_x = \lambda(x)(R_i)_x = 0.$$

(To see this define $h(x) = R_i(v(t_0, x))$, so $h(x_0) = M(t_0)$. If $h'(x_0) > 0$ then by smoothness in x, $h(x) > M(t_0)$ for $|x - x_0| < \delta_0$, and δ_0 sufficiently small. Hence $R_i(v(t_0, x)) > M(t_0)$ for $|x - x_0| < \delta_0$, so by smoothness in t, $R_i(v(t,x)) > M(t)$ for $|x - x_0| < \delta_0$ and $|t - t_0| < \delta_1$, δ_1 sufficiently small. But this is a contradiction. Similarly if we assume $h'(x_0) < 0$. Hence $h'(x_0) = 0$.)

We also have:

$$\nabla R_i\, v_{xx} = (R_i)_{xx} - \nabla^2 R_i(v_x, v_x) \leq 0.$$

(To see this, with h as above, suppose that $h''(x_0) > 0$, then $h(x) > h(x_0) = M(t_0)$, for $|x - x_0| < \delta$, $x \neq x_0$ and δ sufficiently small, so we have a contradiction as before. Hence $h''(x_0) \leq 0$. This accounts for the first term above, and the second is nonpositive due to the quasiconvexity assumption.)

Hence

$$\begin{aligned}
\frac{\partial}{\partial t} R_i(v_0) &= \nabla R_i(v_0)\, v_t \\
&= \nabla R_i(v_0)\, (\varepsilon C v_{xx} - A v_x + B) \\
&= \varepsilon \mu \nabla R_i(v_0)\, v_{xx} - \lambda R_i(v_0)\, v_x + R(v_0)\, B \\
&\leq \nabla R(v_0)\, B(v_0, x_0, t_0) \\
&< M'(t_0).
\end{aligned}$$

□

We now apply this result to the system (2.2). For ease of notation we shall drop the ε, L superscripts. We choose the R_i to be the Riemann invariants of the system so that the first condition of the theorem is satisfied:

$$R_i = \pm \frac{m}{\rho} + 2\sqrt{\rho}$$

It is also easy to verify that R_1 and R_2 are quasiconvex. For the final condition of the theorem, we notice that in our case $B(\rho, m, x) = (\gamma m, \gamma m^2/\rho + \frac{1}{2}\gamma\rho^2 + \rho g)$. Hence:

$$\nabla R_i \cdot B = \frac{\gamma m}{\sqrt{\rho}} \pm \tfrac{1}{2}\gamma\rho + g$$

Suppose that for $t \leq t_0$, $R_i \leq M(t)$, for some function $M(t)$ which is to be chosen. Assume also that $\rho \geq 0$ for $t \leq t_0$. From these conditions we derive:

$$0 \leq \rho \leq \frac{M(t)^2}{4} \tag{5.3}$$

$$\left| \frac{m}{\rho} \right| \leq M(t) \tag{5.4}$$

Substituting these inequalities into the expressions for $\nabla R_1 \cdot B$ and $\nabla R_2 \cdot B$, we see that

$$\nabla R_i \cdot B \leq \tfrac{5}{8} ||\gamma||_\infty M(t)^2 + ||g||_\infty \tag{5.5}$$

Hence condition 3 of Theorem 5.2 will be satisfied if we choose a function M satisfying

$$M'(t) > P M(t)^2 + Q, \tag{5.6}$$

where $P = \frac{5}{8}||\gamma||_\infty$ and $Q = ||g||_\infty$. We also require that the solution lies in the invariant region at time 0. In other words,

$$M(0) \geq M_0 := \max(2\|\sqrt{\rho_0}\|_\infty, \|m_0/\rho_0\|_\infty).$$

Case 1: $P = 0$, $Q = 0$

In this case we can choose, for example, $M(t) = M_0(2 - \exp(-t))$. Since then $M(t) \leq 2M_0$ for all $t \geq (0)$, we can find L^∞ bounds on ρ and m that are independent of t. This case corresponds to a flat slope with no external forces (friction=gravity).

Case 2: $P = 0$, $Q > 0$

This is the case of a flat slope. In this case we can choose $M(t) = M_0 + \tilde{Q}t$, for any $\tilde{Q} > Q$. This time we have a bounded region for any bounded time, hence $\|\rho\|_\infty$ and $\|m\|_\infty$ will be bounded for any bounded time.

Case 3: $P > 0$, $Q = 0$

We integrate to find that $M(t) > M_0/(1 - M_0 P t)$ We thus see that any region we choose will blow up by $t = T_{\max} := 1/M_0 P$. Since we can choose $M(t) = M_0/(1 - M_0 \tilde{P} t)$ for any $\tilde{P} > P$, we see that $\|\rho^\varepsilon\|_\infty$ and $\|m\|_\infty$ will be bounded for any $t < T_{\max}$.

Case 4: $P > 0$, $Q > 0$

This time we integrate to get

$$M(t) > \sqrt{\frac{Q}{P}} \tan\left(C + t\sqrt{PQ}\right), \quad C = \tan^{-1}(M_0\sqrt{P/Q})$$

As in the previous case any region we choose will blow up in finite time. This time we have that $\|\rho\|_\infty$ and $\|m\|_\infty$ are bounded for any $t < T_{\max}$, where

$$T_{\max} := \frac{1}{PQ}\left(\frac{\pi}{2} - \tan^{-1} M_0\sqrt{P/Q}\right).$$

Notice that in all cases $M(t)$ is independent of both L and ε. In case 1 we have a bound which is independent of time and in case 2 we have a bound which grows with time, but is finite for all finite times. Cases 3 and 4 both provide us with a maximum time $T_{\max}$ for which solutions of the parabolic approximation are guaranteed to be bounded. In both of these cases, we have good control of $T_{\max}$ in terms of the initial data (M_0) and the quantity P, which roughly corresponds to the curvature. $T_{\max}$ will be large if either M_0 or P is small. P is small when $||\gamma||_\infty$ is small. Since $\gamma(x) = \beta'(x)/\beta(x)$, and taking $\beta(x) = k\cos\zeta(x)$, this gives $\gamma(x) = -\zeta'(x)\tan\zeta(x)$. In other words, demanding that $||\gamma||_\infty$ be small is equivalent to demanding that the slope must vary slowly unless the slope angle is small. An example of a curve satisfying $||\gamma||_\infty = 0.001$ is shown in Figure 2.

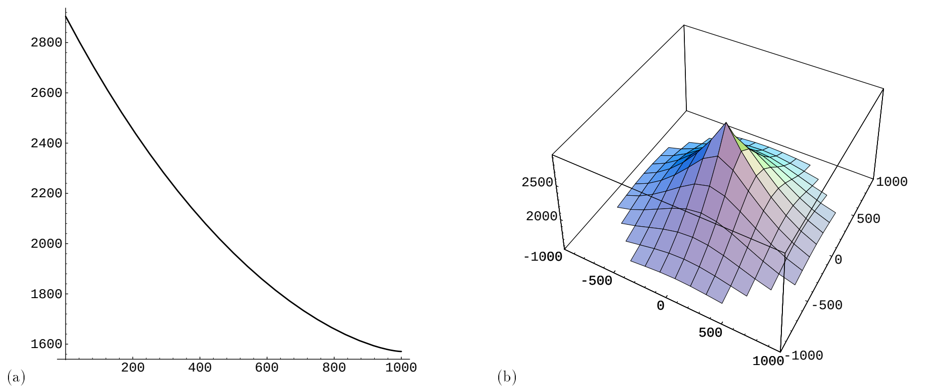

Figure 2: (a) The curve $f(x) = \delta^{-1} e^{1-\delta x}\sqrt{1-e^{2(\delta x-1)}} + \delta^{-1}\sin^{-1}\left(e^{-1+\delta x}\right)$, where $\delta = 0.001$. Setting $\zeta(x) = \tan^{-1} f'(x)$, we find that $\gamma(x) = -\zeta'(x)\tan\zeta(x) = \delta$.
(b) A mountain generated by rotating the curve in (a).

6 An L-independent H^1-estimate

We shall write $w^{\varepsilon,L} = (\rho^{\varepsilon,L} - \delta, m^{\varepsilon,L})$ and $v^{\varepsilon,L} = (\rho^{\varepsilon,L}, m^{\varepsilon,L})$. If $v^{\varepsilon,L}$ is a classical solution of (2.2), then $w^{\varepsilon,L}$ satisfies:

$$w_t^{\varepsilon,L} + (F(v^{\varepsilon,L}))_x = G^\varepsilon(x, v^{\varepsilon,L}) + \varepsilon w_{xx}^{\varepsilon,L}.$$

Multiplying on the left by $w^{\varepsilon,L}$ and integrating over Ω_L, we have,

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_L} |w^{\varepsilon,L}(t)|^2\,dx &+ \varepsilon\int_{\Omega_L} |w_x^{\varepsilon,L}(t)|^2\,dx \\ &= \int_{\Omega_L} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx + \int_{\Omega_L} w_x^{\varepsilon,L}(t)\cdot F(v^{\varepsilon,L}(t))\,dx.\end{aligned}$$

The first term on the right hand side is bounded independent of L, since by Hölder's inequality we have

$$\begin{aligned}\int_{\Omega_L} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx &= \int_{\Omega_L\cap(-1/\varepsilon,1/\varepsilon)} w^{\varepsilon,L}(t)\cdot G^\varepsilon(v^{\varepsilon,L}(t))\,dx \\ &\le \frac{2}{\varepsilon}\|w^{\varepsilon,L}\|_\infty \|G(v^{\varepsilon,L})\|_\infty,\end{aligned}$$

and

$$\begin{aligned}\|w^{\varepsilon,L}\|_\infty &\le \delta + \|\rho^{\varepsilon,L}\|_\infty + \|m^{\varepsilon,L}\|_\infty \\ \|G(v^{\varepsilon,L})\|_\infty &\le ||\gamma||_\infty(||m^{\varepsilon,L}||_\infty + ||m^{\varepsilon,L}/\rho^{\varepsilon,L}||_\infty^2 ||\rho^{\varepsilon,L}||_\infty + \tfrac{1}{2}||\rho^{\varepsilon,L}||_\infty^2) \\ &\quad + ||\rho^{\varepsilon,L}||_\infty ||g||_\infty. \qquad (6.1)\end{aligned}$$

These quantities are all bounded independent of L due to estimates (5.3) and (5.4).

For the second term, we have

$$\begin{aligned} w_x^{\varepsilon,L}\cdot F(v^{\varepsilon,L}) &= \rho_x^{\varepsilon,L} m^{\varepsilon,L} + m_x^{\varepsilon,L}\frac{(m^{\varepsilon,L})^2}{\rho^{\varepsilon,L}} + \tfrac{1}{2} m_x^{\varepsilon,L}(\rho^{\varepsilon,L})^2 \\ &= \rho_x^{\varepsilon,L} m^{\varepsilon,L} + m_x^{\varepsilon,L}\frac{(m^{\varepsilon,L})^2}{\rho^{\varepsilon,L}} + \tfrac{1}{2} m_x^{\varepsilon,L}(\rho^{\varepsilon,L}-\delta)^2 \\ &\quad + \delta m_x^{\varepsilon,L}(\rho^{\varepsilon,L}-\delta) - \tfrac{1}{2}\delta^2 m_x^{\varepsilon,L}. \end{aligned}$$

and so (noting that the integral of the final term vanishes):

$$\begin{aligned} \int_{\Omega_L} w_x^{\varepsilon,L}\cdot F(v^{\varepsilon,L})\,dx &\le \int_{\Omega_L} |w_x^{\varepsilon,L}|\,|w^{\varepsilon,L}|(1+\|\frac{m^{\varepsilon,L}}{\rho^{\varepsilon,L}}\|_\infty + \tfrac{1}{2}\|\rho^{\varepsilon,L}\|_\infty + \tfrac{3}{2}\delta)\,dx \\ &\le \tfrac{1}{2}\varepsilon \int_{\Omega_L} |w_x^{\varepsilon,L}(t)|^2\,dx + C(\varepsilon)\int_{\Omega_L} |w^{\varepsilon,L}(t)|^2\,dx, \end{aligned}$$

where the final inequality is due to Young's inequality and estimates (5.3) and (5.4). The constant $C(\varepsilon)$ is independent of L.

Using Gronwall's inequality we can now derive estimates

$$||w^{\varepsilon,L}(t)||_{L^2(\Omega_L)} \le C(\varepsilon) \tag{6.2}$$

$$||w_x^{\varepsilon,L}||_{L^2((0,T)\times\Omega_L)} \le C(\varepsilon) \tag{6.3}$$

for some $C(\varepsilon)$, independent of L. From these estimates, we see that we can extend w by 0 for x outside Ω_L. In other words, we can extend $(\rho^{\varepsilon,L}, m^{\varepsilon,L})$ to the whole of $(0,T)\times\mathbb{R}$ by setting $(\rho^{\varepsilon,L}(t,x), m^{\varepsilon,L}(t,x)) = (\delta, 0)$ for $(t,x)\in(0,T)\times(\mathbb{R}\backslash\Omega_L)$.

A standard compactness and regularity argument now shows that we can pass to the limit $L\to\infty$ and obtain a classical solution $(\rho^\varepsilon, m^\varepsilon)$ to problem (2.1). Notice that all of the estimates we have proved so far are also valid for $(\rho^\varepsilon, m^\varepsilon)$. However, the only estimates we have available which are also independent of ε are (5.3) and (5.4).

We now have another important consequence of (6.2). We define the set:

$$J(t) = \left\{x\in\Omega_L \;\middle|\; \rho^{\varepsilon,L} < \delta/2\right\}. \tag{6.4}$$

We thus have,

$$(\delta/2)^2|J(t)| = \int_{\rho^{\varepsilon,L}<\delta/2}(\delta/2)^2\,dx \le \int_{\rho^{\varepsilon,L}<\delta/2}(\rho^{\varepsilon,L}-\delta)^2\,dx \le \int_{\Omega_L}|w^{\varepsilon,L}|^2\,dx \le C(\varepsilon). \tag{6.5}$$

In other words, the set J has finite measure for each $\varepsilon > 0$. This result will be very useful in the next section since it allows us to do without the assumptions (C2) and (C4) in the paper by Fang and Ito [FI97]. This is crucial since neither of these assumptions hold in our case.

7 Proof that $\rho^\varepsilon \geq C(\varepsilon) > 0$

We now have the existence of a solution $(\rho^\varepsilon, m^\varepsilon)$ to system (2.1) for some small time. In order to extend our solution up to the time $T_{\max}$ we still need to ensure that $\rho^\varepsilon \geq C(\varepsilon) > 0$ for $T < T_{\max}$. We sketch an argument of Fang and Ito [FI97], slightly modified to this situation. Let $r = 1/e^{\mu t}\rho^{\varepsilon,L}$ and $u = m^\varepsilon/\rho^\varepsilon$. Then

$$r_t + ur_x - u_x r + (\mu + \gamma u)r = \varepsilon(r_{xx} - 2r_x^2/r) \tag{7.1}$$

In [FI95] Lemma 2.3, Fang and Ito prove the following result:

LEMMA 7.1
Suppose $\phi \geq 0$ is a nonincreasing function on $[c_0, \infty)$, $\phi(c_0) < \infty$ and there are positive constants K, a and b so that

$$\phi(c_2) \leq K c_1^a (c_2 - c_1)^{-a} \phi(c_1)^{1+b}, \quad \textit{for all } c_2 > c_1 \geq c_0.$$

Then there exists a $c^* < \infty$ such that

$$\phi(c^*) = 0.$$

We apply this lemma to

$$\phi(c) := \sup_{t \in [0,T]} \left| \left\{ x \in \mathbb{R} \;\middle|\; r(t,x) > c \right\} \right|.$$

Because of estimate (6.5), there exists $c_0 < \infty$ so that $\phi(c_0) < \infty$ ($c_0 = \delta$ will do). Now define $\xi = \max(0, r - c_1)$ for some $c_1 > c_0$ (large enough so that $\xi(0,x) = 0$). Multiply (7.1) by ξ^3 and integrate over Q_T to get:

$$\begin{aligned}
\tfrac{1}{4}\int_{\mathbb{R}} \xi^4(T)\,dx + (\mu + \gamma u)\int_0^T\!\!\int_{\mathbb{R}} (\xi^4 + c_1\xi^3)\,dx\,dt& \\
+ \frac{3\varepsilon}{4}\int_0^T\!\!\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt& \\
= -2\varepsilon\int_0^T\!\!\int_{\mathbb{R}} \frac{\xi_x^2}{\xi + c_1}\,dx\,dt - \int_0^T\!\!\int_{\mathbb{R}} (5\xi^3 + 3c_1\xi^2)u\xi_x\,dx\,dt&
\end{aligned} \tag{7.2}$$

Young's inequality yields

$$-\int_0^T\!\!\int_{\mathbb{R}} 5\xi^3 u\xi_x\,dx\,dt \le \frac{\varepsilon}{4}\int_0^T\!\!\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt + \frac{25}{4\varepsilon}\|u\|_\infty^2 \int_0^T\!\!\int_{\mathbb{R}} \xi^4\,dx\,dt$$

and

$$-\int_0^T\!\!\int_{\mathbb{R}} 3c_1\xi^2 u\xi_x\,dx\,dt \le \frac{\varepsilon}{4}\int_0^T\!\!\int_{\mathbb{R}} |(\xi^2)_x|^2\,dx\,dt + \frac{9c_1^2}{4\varepsilon}\|u\|_\infty^2 \phi(c_1)^{2/3}\int_0^T \|\xi(t)\|_6^2\,dt$$

Now choose

$$\mu = \frac{\varepsilon}{4} + \frac{25}{4\varepsilon}\|u\|_\infty^2 + K$$

where $K \ge \|\gamma u\|_\infty$. We thus derive

$$\int_{\Omega_L} \xi^4(T)\,dx + \varepsilon\int_0^T\!\!\int_{\mathbb{R}} (\xi^4 + |(\xi^2)_x|^2)\,dx\,dt \le \frac{9c_1^2}{\varepsilon}\|u\|_\infty^2\phi(c_1)^{2/3}\int_0^T \|\xi(t)\|_6^2\,dt$$

Sobolev's embedding theorem gives $\|\xi(t)\|^2_{L^6(\mathbb{R})} \le C\|\xi(t)^2\|_{H^1(\mathbb{R})}$. Combining this with another application of Young's inequality we thus derive

$$\int_{\mathbb{R}} \xi^4(T)\,dx \le \frac{\alpha c_1^4\phi(c_1)^{4/3}T}{\varepsilon}$$

for some constant α. Furthermore, by definition of ξ, for $c_2 > c_1$ we have

$$\int_{\mathbb{R}} \xi^4(T)\,dx \ge (c_2 - c_1)^4\phi(c_2)$$

and so

$$\phi(c_2) \le Kc_1^4(c_2-c_1)^{-4}\phi(c_1)^{4/3}$$

Hence there exists $c^* < \infty$ such that $\phi(c^*) = 0$, and so $r > c^*$ for all $t < T_{\max}$, and hence

$$\rho^\varepsilon(t,x) \ge 1/c^* e^{\mu t} > 0, \tag{7.3}$$

where the constants c^* and μ depend on ε.

8 Entropy estimates

Estimates (5.3) and (5.4) are independent of ε, hence there exist functions $(\bar h, \bar m)$ satisfying the same estimates, and for a subsequence,

$$(\rho^\varepsilon, m^\varepsilon) \overset{*}{\rightharpoonup} (\bar h, \bar m) \quad \text{in } L^\infty((0,T)\times\mathbb{R}) \text{ as } \varepsilon \to 0.$$

Young's theorem also tells us that this is equivalent to convergence in the sense of measures. To prove that we actually have strong convergence to an entropy solution we shall apply the method of compensated compactness. For this we additionally require some estimates on the entropy. We shall also need to improve the H^1 estimate (6.3). Of course, if it were possible to find an ε-independent H^1 estimate, then by compactness we would immediately have strong convergence. This is too much to expect, though we shall see that we can prove that $\sqrt{\varepsilon} v_x^\varepsilon \in L^2_{\mathrm{loc}}(Q_T)$.

DiPerna [DiP83] showed that every weak entropy-flux pair (η, q) can be expressed in the form of a convolution:

$$\eta(\rho, m) = \int_{\mathbb{R}} f(\xi)\chi(\rho, \xi - m/\rho)\, d\xi$$
$$q(\rho, m) = \tfrac{1}{2} \int_{\mathbb{R}} f(\xi)(\xi + m/\rho)\chi(\rho, \xi - m/\rho)\, d\xi$$

where

$$\chi(\rho, z) = \left(4\rho - z^2\right)_+^{1/2}$$

Furthermore Lions, Perthame and Tadmor [LPT94] showed that η is a convex weak entropy exactly when f is convex. In particular, setting $f(\xi) = \xi^2/2$ yields a strictly convex entropy which represents the mechanical energy:

$$\eta_E = m^2/2\rho + \rho^2/2.$$

The corresponding flux is given by

$$q_E = \frac{\rho}{2}\left(\frac{m}{\rho}\right)^3 + m\rho.$$

Differentiating the convolution integral for η, one can easily derive the following estimates for any C^2 weak entropy:

$$|\nabla\eta| \le C(||f||_{W^{1,\infty}}, ||\rho||_\infty, ||m||_\infty) \tag{8.1}$$
$$|\nabla^2\eta| \le C(||f||_{W^{2,\infty}}, ||\rho||_\infty, ||m||_\infty) \tag{8.2}$$

If (η, q) is an entropy-flux pair, then multiplying (2.1) by $\nabla\eta$ gives:

$$\eta_t + q_x = \varepsilon\big(\, \eta_{xx} - \nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon)\,\big) + \nabla\eta\, G^\varepsilon \tag{8.3}$$

Now multiply by a non-negative test function $\phi \in C_0^\infty(\mathbb{R})$, and integrate over Q_T. We have,

$$\int_{\mathbb{R}} \phi\eta\,\big|_{t=T} dx + \varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon)\, dx\, dt$$
$$= \int_{\mathbb{R}} \phi\eta\,\big|_{t=0} dx + \int_0^T\!\!\int_{\mathbb{R}} (\varepsilon\phi_{xx}\,\eta + \phi_x\, q)\, dx\, dt + \int_0^T\!\!\int_{\mathbb{R}} \phi\nabla\eta\, G^\varepsilon\, dx\, dt.$$

Now suppose that (η, q) is a non-negative, strictly convex entropy-flux pair (e.g. (η_E, q_E)). Then

$$\int_{\mathbb{R}} \phi\eta \,\big|_{t=T} dx + \varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi \nabla^2 \eta(v_x^\varepsilon, v_x^\varepsilon)\, dx\, dt \geq K\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2 \, dx\, dt.$$

and the L^∞ estimates for $\nabla\eta$, ρ^ε and m^ε yield:

$$\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2 \, dx\, dt \leq C \left(1 + \int_0^T\!\!\int_{\mathbb{R}} |\phi\, G| dx\, dt \right).$$

We also have $\|G\|_\infty \leq K(T)$ by (6.1). Hence we have

$$\varepsilon \int_0^T\!\!\int_{\mathbb{R}} \phi |v_x^\varepsilon|^2 \, dx\, dt \leq C < \infty$$

where the constant depends on ϕ and $T < T_{\max}$, but not on $\varepsilon > 0$. In other words, we have shown that $(\sqrt{\varepsilon} v_x^\varepsilon)_{\varepsilon>0}$ is bounded independent of ε in $L^2_{\mathrm{loc}}(Q_T)$.

9 Compensated compactness

Since $(v^\varepsilon)_{\varepsilon>0}$ takes values in a bounded set B, there is a subsequence (v^{ε_j}) and a measurable family $(t, x) \mapsto \nu_{t,x}$ of probability measures (Young measures) with $\operatorname{supp} \nu_{t,x} \subset \bar{B}$ such that

$$f \circ v^{\varepsilon_j} \stackrel{*}{\rightharpoonup} \langle f \rangle \quad \text{in } L^\infty(Q_T)$$

$\langle f \rangle$ is the average with respect to the Young measure:

$$\langle f \rangle(t, x) := \int_{Q_T} f \, d\nu_{t,x}$$

Recall the following result of Murat:

Lemma 9.1 (Murat, [Mur81])
Let U be an open bounded subset of $\mathbb{R}^n$ and $(v^\varepsilon)_{\varepsilon>0}$, a bounded sequence in $W^{-1,\infty}(U)$ which is contained in a set of the form $B+C$ where B is a bounded subset of $\mathcal{M}(U)$ (the space of bounded measures) and C is a compact subset of $H^{-1}(U)$. Then $(v^\varepsilon)_{\varepsilon>0}$ is relatively compact in $H^{-1}(U)$.

Let U be an arbitrary open, bounded subset of Q_T, $0 < T < T_{\max}$. Recall equation (8.3):

$$\eta_t + q_x = \varepsilon \eta_{xx} - \varepsilon \nabla^2 \eta(v_x^\varepsilon, v_x^\varepsilon) + \nabla\eta\, G^\varepsilon$$

Looking at each term in turn, we find:

- $(\eta(v^\varepsilon) + q(v^\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$, and hence also in $W^{-1,\infty}(Q_T)$.
- $\varepsilon\eta_x = \sqrt{\varepsilon}\cdot\nabla\eta\cdot\sqrt{\varepsilon}\,v_x^\varepsilon \longrightarrow 0$ in $L^2(U)$ as $\varepsilon\to 0$. Hence $(\varepsilon\eta_{xx})_{\varepsilon>0}$ is relatively compact in $H^{-1}(U)$.
- $\varepsilon\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon) \le |\nabla^2\eta|\,\varepsilon|v_x^\varepsilon|^2$, so $(\varepsilon\nabla^2\eta(v_x^\varepsilon, v_x^\varepsilon))_{\varepsilon>0}$ is uniformly bounded in $L^1_{\mathrm{loc}}(Q_T)$, and hence in $\mathcal{M}(U)$.
- $(\nabla\eta\, G^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(Q_T)$ and hence in $\mathcal{M}(U)$.

Hence by Murat's lemma, the sequence $(\eta(v^\varepsilon)_t + q(v^\varepsilon)_x)_\varepsilon$ is relatively compact in $H^{-1}_{\mathrm{loc}}(Q_T)$, and so we are able to apply the following lemma due to Tartar.

Lemma 9.2 (Tartar, [Tar79])
If for all weak entropy-flux pairs, the sequence

$$(\eta(v^\varepsilon)_t + q(v^\varepsilon)_x)_{\varepsilon>0}$$

is relatively compact in $H^{-1}_{\mathrm{loc}}(Q_T)$, then for all weak entropy-flux pairs (η_j, q_j), $j = 1, 2$, the following equation holds:

$$\langle \eta_1 q_2 - \eta_2 q_1\rangle = \langle\eta_1\rangle\langle q_2\rangle - \langle\eta_2\rangle\langle q_1\rangle. \tag{9.1}$$

We can then apply results of Lions, Perthame and Souganidis [LPS96] to show that $v^{\varepsilon_j} \to v$ (strong) in $L^p(U)$, for all open, bounded $U \subset Q_T$, $p\in[1,\infty)$.

10 Convergence to an entropy solution

Recall that whenever $T < T_{\max}$, v^ε satisfies:

$$\begin{gathered} v_t^\varepsilon + \nabla F(v^\varepsilon)v_x^\varepsilon = G^\varepsilon(v^\varepsilon) + \varepsilon v_{xx}^\varepsilon \quad \text{ in } Q_T,\\ v^\varepsilon(0,x) = v_0(x) := (\rho_0^\varepsilon(x), m_0^\varepsilon(x)),\ x\in\mathbb{R}\\ \rho^\varepsilon\to\delta,\ m^\varepsilon\to 0, \quad \text{ as } |x|\to\infty. \end{gathered}$$

Multiplying the equation for v^ε by a test function $\psi\in(C_0^1([0,T)\times\mathbb{R}))^2$ and integrating over Q_T, we get:

$$\int_0^T\!\!\int_{\mathbb{R}} (\, v^\varepsilon\cdot\psi_t + F(v^\varepsilon)\cdot\psi_x + G^\varepsilon(v^\varepsilon)\cdot\psi\,)\; dx\,dt + \varepsilon\int_0^T\!\!\int_{\mathbb{R}} v^\varepsilon\cdot\psi_{xx}\,dx\,dt = \int_{\mathbb{R}} v_0^\varepsilon\cdot\psi(0,x)\,dx$$

The second integral converges to 0 as $\varepsilon \to 0$ and the convergence $v^\varepsilon \to v$ strong in L^p, ensures that v is a weak solution of (1.5),(1.6). Similarly, let (η, q) be a convex weak entropy-flux pair.

$$\begin{aligned} \eta_t + q_x &= \varepsilon \eta_{xx} - \varepsilon \nabla^2 \eta(v_x^\varepsilon, v_x^\varepsilon) + \nabla \eta \, G^\varepsilon \\ &\leq \varepsilon \eta_{xx} + \nabla \eta \, G^\varepsilon . \end{aligned}$$

Where $\eta = \eta(v^\varepsilon)$, $q = q(v^\varepsilon)$. Multiply by a test function $\phi \in C_0^1([0,T) \times \mathbb{R})$ and integrate over Q_T to get:

$$\int_0^T \int_{\mathbb{R}} (\, \eta \phi_t + q \phi_x \, + \nabla \eta \, G^\varepsilon \phi \,) \; dx \, dt - \varepsilon \int_0^T \int_{\mathbb{R}} (\nabla \eta \, v_x^\varepsilon) \phi_x \, dx \, dt \geq \int_{\mathbb{R}} \eta(v_0^\varepsilon) \phi(0,x) \, dx$$

Again, as $\varepsilon \to 0$ the second integral vanishes and the strong convergence of the sequence v^ε ensures that the entropy inequality is fulfilled in the limit.

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