Large-time behavior of solutions to the equations of a one-dimensional viscous polytropic ideal gas in unbounded domains *

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Abstract

The large-time behavior of solutions to the initial and initial boundary value problems for a one-dimensional viscous polytropic ideal gas in unbounded domains is investigated. Using a special cut-off function to localize the problem, we derive a local representation for the specific volume. With the help of the local representation, and certain new estimates for the temperature and the stress, and the weighted energy estimates, we prove that in any bounded interval, the specific volume is pointwise bounded from below and above for all $t \geq 0$ and a generalized solution is convergent as time goes to infinity.

1 Introduction

The present paper is concerned with the large-time behavior of solutions to the equations of motion of a one-dimensional viscous polytropic ideal gas in unbounded domains.

The motion of a viscous polytropic ideal gas in $\mathbb{R}$ is described by the following equations in Lagrangian coordinates (see [4, 22]):

$$u_t = v_x, \tag{1.1}$$

$$v_t = \sigma_x \qquad \left(\sigma := \mu\frac{v_x}{u} - R\frac{\theta}{u}\right), \tag{1.2}$$

$$c_V \theta_t = \left[\lambda\frac{\theta_x}{u}\right]_x + \sigma v_x, \tag{1.3}$$

where u, v, and θ are the specific volume, the velocity, and the absolute temperature respectively, σ is the stress, μ, R, c_V and λ are positive constants.

We shall consider the initial and initial boundary value problems for (1.1)-(1.3) in the region $\{t > 0, x \in \Omega\}$ with the initial conditions

$$(u(x,0), v(x,0), \theta(x,0)) = (u_0(x), v_0(x), \theta_0(x)), \quad x \in \Omega, \tag{1.4}$$

and the boundary conditions

$$v|_{\partial\Omega} = 0, \;\; \theta_x|_{\partial\Omega} = 0, \tag{1.5}$$

or

$$v|_{\partial\Omega} = 0, \;\; \theta|_{\partial\Omega} = 1, \tag{1.6}$$

where $\Omega = \mathbb{R}$ for the Cauchy problem (1.1)-(1.4) and $\Omega = (0, \infty)$ for the initial boundary value problems (1.1)-(1.5), and (1.1)-(1.4), (1.6).

Since the first work of Kazhikhov and Shelukhin [14] on the global existence in the dynamics of a one-dimensional viscous polytropic ideal gas for large initial data, significant progress has been made on the mathematical aspect of the initial and initial boundary value problems for (1.1)-(1.3). For initial boundary value problems in *bounded* domains the existence and uniqueness of global (generalized) solutions and

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the regularity have been known. Moreover, the global solution is asymptotically stable as time tends to infinity; see [1, 19, 16, 17, 18, 2, 3, 5], among others.

For the Cauchy problem (1.1)-(1.4) and the initial boundary value problems for (1.1)-(1.5) and (1.1)-(1.4), (1.6) (in *unbounded* domains), Kazhikhov, Shelukhin [13, 14] (also cf [1, 21, 7, 8]) proved that *if*

$$u_0 - 1,\ v_0,\ \theta_0 - 1 \in H^1; \quad u_0,\ \theta_0 > 0 \ \ on\ \ \bar{\Omega}, \tag{1.7}$$

and u_0, v_0, θ_0 are compatible with (1.5), (1.6), then there exists a unique (generalized) solution of (u, v, θ) with $u, \theta > 0$ such that for any $T > 0$

$$\begin{aligned} &u - 1,\ v,\ \theta - 1 \in L^\infty((0,T), H^1), \quad u_t \in L^\infty((0,T), L^2), \\ &v_t,\ \theta_t,\ u_{xt},\ v_{xx},\ \theta_{xx} \in L^2((0,T), L^2), \end{aligned} \tag{1.8}$$

and the regularity holds. The asymptotic behavior as $t \to \infty$ of the solution has been studied under some smallness conditions on the initial data; see [10, 12, 20, 11, 6, 15, 9]. However, nothing has been known on the large-time behavior of the solution in the case of large data.

The aim of the present work is to investigate the large-time behavior of the generalized solution of (1.1)-(1.4), (1.1)-(1.5), and (1.1)-(1.4), (1.6). To study the large-time behavior of the solution some difficulties arise from the unboundedness of the domains. In the case of *bounded* domains we can derive a useful representation of $u(x,t)$, with which and the imbedding $L^2 \hookrightarrow L^1$ one can obtain pointwise bounds of $u(x,t)$. For *unbounded* domains, however, the useful representation and the imbedding $L^2 \hookrightarrow L^1$ in bounded domains do not remain valid (two of the difficulties). To overcome such difficulties we have to introduce new techniques. In this paper we use a special cut-off function to localize the problem, and then by modifying Kazhikhov's argument for bounded domains [1, 19], we derive a local representation of $u(x,t)$. Because of the choice of the cut-off function the error term induced by the cut-off function has the right sign and can be controlled using the local representation and new estimates for $\theta(x,t)$ and $\sigma(x,t)$ which implicitly show the lower pointwise boundedness of θ for all $t \geq 0$ (cf. (2.15), (2.17), (2.22)), such that in any bounded interval $u(x,t)$ is pointwise bounded from below and above for all $t \geq 0$. With the help of the local pointwise bounds of $u(x,t)$ we apply the weighted energy estimates with different weights to prove that in any bounded interval the solution converges to a (constant) state as time goes to infinity. More precisely, denote

$$u_1(t) := \frac{1}{k}\int_0^k u(x,t)dx, \quad u_2(t) := \frac{1}{2k}\int_{-k}^k u(x,t)dx, \tag{1.9}$$

the main result of this paper reads:

Theorem 1.1 *Let $k > 0$ be an arbitrary but fixed integer. Let (1.7) be satisfied and let (u, v, θ) be a generalized solution of (1.1)-(1.4), or (1.1)-(1.5), or (1.1)-(1.4), (1.6). Then we have the following:*
i) For the initial boundary value problem (1.1)-(1.5) there are positive constants $\underline{u}$ and $\overline{u}$, independent of t, such that $\underline{u} \leq u(x,t) \leq \overline{u}$ for all $x \in [0,k]$ and $t \geq 0$. Moreover, as $t \to \infty$ we have

$$\begin{aligned} &\|u(t) - u_1(t)\|_{L^2(0,k)} \to 0; \\ &for\ an\ arbitrary\ but\ fixed\ integer\ l \geq 2, \quad \|v(t)\|_{L^{2l}(0,k)} \to 0. \end{aligned} \tag{1.10}$$

ii) For the initial boundary value problem (1.1)-(1.4), (1.6) we have

$$\|(u(t) - u_1(t), v(t), \theta(t) - 1)\|_{H^1(0,k)} \to 0 \quad as\ t \to \infty. \tag{1.11}$$

iii) For the Cauchy problem (1.1)-(1.4) we have $\underline{u} \leq u(x,t) \leq \overline{u}$ for all $x \in [-k,k]$ and $t \geq 0$, where $\underline{u}$, $\overline{u}$ are positive constants independent of t. If, in addition, u_0, θ_0 are even functions and v_0 is a odd function, then as $t \to \infty$,

$$\begin{aligned} &\|u(t) - u_2(t)\|_{L^2(-k,k)} \to 0; \\ &for\ an\ arbitrary\ but\ fixed\ integer\ l \geq 2, \quad \|v(t)\|_{L^{2l}(-k,k)} \to 0. \end{aligned} \tag{1.12}$$

We shall prove Theorem 1.1 in Section 3. In Section 2 we derive local pointwise bounds of $u(x,t)$.

Notation *used throughout this paper:* Let G be a domain in $\mathbb{R}$. Let $m \geq 0$ be an non-negative integer and let $1 \leq p \leq \infty$. By $W^{m,p}(G)$ we denote the usual Sobolev space defined over G with norm $\|\cdot\|_{W^{m,p}(G)}$; $W^{m,2}(G) \equiv H^m(G)$ with norm $\|\cdot\|_{H^m(G)}$, $W^{0,p}(G) \equiv L^p(G)$ with norm $\|\cdot\|_{L^p(G)}$. For simplicity we also use the following abbreviations:

$$L^p \equiv L^p(\Omega), \quad H^m \equiv H^m(\Omega); \quad \|\cdot\|_{L^p} \equiv \|\cdot\|_{L^p(\Omega)}, \quad \|\cdot\|_{H^m} \equiv \|\cdot\|_{H^m(\Omega)}.$$

$\|\cdot\|$ stands for the norm in $L^2(\Omega)$. $L^p(I,B)$ resp. $\|\cdot\|_{L^p(I,B)}$ denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p=\infty$) functions from I to B resp. its norm, $I \subset \mathbb{R}$ an interval, B a Banach space. For a vector valued function $f = (f_1, \cdots, f_m)$ and a normed space X with the norm $|||\cdot|||$, $f \in X$ means that each component of f is in X; we put $|||f||| := |||f_1||| + \cdots + |||f_m|||$.

The same letter C (sometimes used as $C(k)$ to emphasize the dependence of C on k) will denote various positive constants which do not depend on the time t.

2 Local pointwise estimates of u

In this section we prove that in any bounded interval $u(x,t)$ is bounded from below and above for all $t \geq 0$. We start with the following lemma, which is motivated by the second law of thermodynamcs and embodies the dissipative effects of viscosity and thermal diffusion.

Lemma 2.1 *There is a positive constant e_0, independent of t, such that*

$$\int_\Omega U(x,t)dx + \int_0^t \int_\Omega \left(\lambda \frac{\theta_x^2}{u\theta^2} + \mu \frac{v_x^2}{u\theta} \right) dxds \leq e_0 \qquad \forall\, t \geq 0, \tag{2.1}$$

where

$$U(x,t) := \left\{ v^2/2 + R(u - \log u - 1) + c_V(\theta - \log\theta - 1) \right\}(x,t). \tag{2.2}$$

Proof. Using Eqs. (1.1)-(1.3), we obtain after a straightforward calculation that

$$U_t + \mu \frac{v_x^2}{u\theta} + \lambda \frac{\theta_x^2}{u\theta^2} = [v\sigma]_x + Rv_x + \lambda \left[\left(1 - \frac{1}{\theta}\right) \frac{\theta_x}{u} \right]_x .$$

By virtue of Taylor's theorem, (1.7), and Sobolev's imbedding theorem ($H^1 \hookrightarrow L^\infty$), $\int_\Omega U(x,0)dx \leq C(1 + \|(u_0 - 1, v_0, \theta_0 - 1)\|^2)$. So, if we integrate the above identity over $\Omega \times [0,t]$ ($t \geq 0$) and use the boundary conditions (1.5) or (1.6), we obtain (2.1). The proof is complete. $\Box$

From Lemma 2.1 we see that

$$\int_i^{i+1} (u - \log u - 1)(x,t)dx, \ \int_i^{i+1} (\theta - \log\theta - 1)(x,t)dx \leq \frac{e_0}{\min\{R, c_V\}}, \tag{2.3}$$

where $i = 0, 1, 2, \cdots$ for the initial boundary value problems and $i = 0, \pm 1, \pm 2, \cdots$ for the Cauchy problem. Hence by terms of the mean value theorem, for each $t \geq 0$ there are points $a_i(t), b_i(t) \in [i, i+1]$ such that

$$0 < \alpha_1 \leq u(a_i(t), t), \ \theta(b_i(t), t) \leq \alpha_2, \quad t \geq 0, \tag{2.4}$$

where α_1, α_2 are two (positive) roots of the equation $y - \log y - 1 = e_0/\min\{R, c_V\}$, and $i = 0, 1, 2, \cdots$ for the initial boundary value problems and $i = 0, \pm 1, \pm 2, \cdots$ for the Cauchy problem. Moreover, if we utilize (2.3) and apply Jensen's inequality to the convex function $y - \log y - 1$, we obtain:

$$\begin{aligned} &\int_i^{i+1} u(x,t)dx - \log \int_i^{i+1} u(x,t)dx - 1, \\ &\int_i^{i+1} \theta(x,t)dx - \log \int_i^{i+1} \theta(x,t)dx - 1 \leq e_0/\min\{R, c_V\}, \end{aligned}$$

which gives $\alpha_1 \leq \int_i^{i+1} u(x,t)dx$, $\int_i^{i+1} \theta(x,t)dx \leq \alpha_2$. In view of (2.4), we thus have proved

Lemma 2.2 *Let α_1, α_2 be two (positive) roots of the equation $y - \log y - 1 = e_0/\min\{R, c_V\}$ and the constant e_0 be the same as in Lemma 2.1. Then*

$$\alpha_1 \le \int_i^{i+1} u(x,t)dx, \; \int_i^{i+1} \theta(x,t)dx \le \alpha_2 \qquad t \ge 0, \tag{2.5}$$

and for each $t \ge 0$ there are points $a_i(t), b_i(t) \in [i, i+1]$ such that

$$\alpha_1 \le u(a_i(t),t), \;\; \theta(b_i(t),t) \le \alpha_2, \qquad t \ge 0, \tag{2.6}$$

where $i = 0, 1, 2, \cdots$ for the initial boundary value problems and $i = 0, \pm 1, \pm 2, \cdots$ for the Cauchy problem.

Our next aim is to give a local representation of u by using some cut-off function.

Let $\phi \in W^{1,\infty}(\mathbb{R})$ be defined by

$$\phi(x) := \begin{cases} 1, & x \le k+1 \\ k+2-x, & k+1 \le x \le k+2 \\ 0, & x \ge k+2 \end{cases} \tag{2.7}$$

For simplicity we denote $\Omega_k := (0, k+1)$ for the initial boundary value problems and $\Omega_k := (-k-1, k+1)$ for the Cauchy problem. We have the following local representation:

Lemma 2.3 *We have*

$$u(x,t) = B(x,t)Y(t) + \frac{R}{\mu}\int_0^t \frac{B(x,t)Y(t)}{B(x,s)Y(s)}\theta(x,s)ds, \quad x \in \bar{\Omega}_k, t \ge 0, \tag{2.8}$$

where

$$\begin{aligned} B(x,t) &:= u_0(x) \exp\left\{\frac{1}{\mu}\int_x^\infty (v_0(y) - v(y,t))\phi(y)dy\right\}, \\ Y(t) &:= \exp\left\{\frac{1}{\mu}\int_0^t \int_{k+1}^{k+2} \sigma(y,s)dyds\right\}. \end{aligned} \tag{2.9}$$

Proof. We multiply Eq. (1.2) by ϕ to obtain: $[\phi v]_t = [\sigma\phi]_x - \phi_x\sigma$. Integrating this over (x, ∞) ($x \in \bar{\Omega}_k$) with respect to x and recalling (1.1) and the definition of ϕ, σ, we arrive at

$$\begin{aligned} -\int_x^\infty [v\phi]_t dy &= \sigma + \int_x^\infty \sigma\phi_x dy \\ &= \mu[\log u]_t - R\frac{\theta}{u} - \int_{k+1}^{k+2} \sigma(y,t)dy, \quad x \in \bar{\Omega}_k. \end{aligned} \tag{2.10}$$

Recalling the definition of $B(x,t)$ and $Y(t)$, we integrate (2.10) over (0,t) with respect to t and then take the exponential on both sides of the resulting equation to deduce that

$$\frac{1}{B(x,t)Y(t)} = \frac{1}{u(x,t)} \exp\left\{\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}ds\right\}, \quad x \in \bar{\Omega}_k, \, t \ge 0. \tag{2.11}$$

Multiplying (2.11) by $R\theta(x,t)/\mu$ and integrating over $(0,t)$, we infer

$$\exp\left\{\frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{u(x,s)}ds\right\} = 1 + \frac{R}{\mu}\int_0^t \frac{\theta(x,s)}{B(x,s)Y(s)}ds.$$

Substituting the above identity into (2.11), we obtain (2.8). □

Now in order to bound $u(x,t)$ pointwise, we first show the exponential decay of $Y(t)$ by utilizing Lemmas 2.1, 2.2 and some new estimates for θ, σ (cf. (2.15), (2.17)), then we use the representation (2.8) to obtain the following local uniform bounds on u:

Lemma 2.4 *There are positive constants $\overline{u}$, $\underline{u}$ independent of t, such that*

$$\underline{u} \le u(x,t) \le \overline{u} \qquad \forall\, x \in \bar{\Omega}_k, \; t \ge 0. \tag{2.12}$$

Proof. Recalling (2.7) and (2.9), we get from Cauchy-Schwarz's ineqaulity, (1.7), and (2.1) that

$$C(k)^{-1} \le B(x,t) \le C(k) \qquad \forall\, x \in \bar{\Omega}_k,\ t \ge 0. \tag{2.13}$$

By Cauchy-Schwarz's inequality, (2.1), and (2.5), we see that for $x \in [k+1, k+2]$

$$\begin{aligned}
\left| \int_s^t \int_{b_{k+1}(\tau)}^x \frac{\theta_x(y,\tau)}{\theta(y,\tau)} dy d\tau \right| &\le \int_s^t \left(\int_{k+1}^{k+2} \frac{\theta_x^2}{u\theta^2} dy \right)^{1/2} \left(\int_{k+1}^{k+2} u dy \right)^{1/2} d\tau \\
&\le \frac{\alpha_2}{\log 2} \int_s^t \int_{k+1}^{k+2} \frac{\theta_x^2}{u\theta^2} dx d\tau + (t-s)\log 2 \\
&\le \frac{\alpha_2 e_0}{\lambda \log 2} + (t-s)\log 2,
\end{aligned} \tag{2.14}$$

where $b_{k+1}(t)$ is the same as in Lemma 2.2.

We apply Jensen's inequality to the convex function e^x, and utilize (2.6), (2.14) to obtain that for any $x \in [k+1, k+2]$, $t \ge s \ge 0$

$$\begin{aligned}
&\int_s^t \theta(x,\tau) d\tau = \int_s^t \exp\{\log \theta(x,\tau)\} d\tau \ge (t-s) \exp\left\{ \frac{1}{t-s} \int_s^t \log \theta d\tau \right\} \\
&= (t-s) \exp\left\{ \frac{1}{t-s} \int_s^t \left(\log \frac{\theta(x,\tau)}{\theta(b_{k+1}(\tau),\tau)} + \log \theta(b_{k+1}(\tau),\tau) \right) d\tau \right\} \\
&= (t-s) \exp\left\{ \frac{1}{t-s} \int_s^t \left(\int_{b_{k+1}(\tau)}^x \frac{\theta_x}{\theta} dy + \log \theta(b_{k+1}(\tau),\tau) \right) d\tau \right\} \\
&\ge (t-s) \exp\left\{ \log \alpha_1 - \frac{1}{t-s} \left| \int_s^t \int_{b_{k+1}(\tau)}^x \frac{\theta_x}{\theta} dy d\tau \right| \right\} \\
&\ge \frac{\alpha_1}{2} (t-s) \exp\left\{ \frac{-\alpha_2 e_0}{(t-s)\lambda \log 2} \right\} \ge C(t-s) e^{-1/[C(t-s)]},
\end{aligned} \tag{2.15}$$

which gives

$$-\int_s^t \inf_{[k+1,k+2]} \theta(\cdot,\tau) d\tau \le \begin{cases} 0 & \text{for } 0 \le t-s \le 1 \\ -C(t-s) & \text{for } 1 \le t-s \end{cases}. \tag{2.16}$$

Applying Cauchy-Schwarz's inequality, Jensen's inequality for the convex function $1/x$ $(x>0)$, utilizing (2.1), (2.16), and taking into account that

$$\begin{cases} C & \text{for } 0 \le t-s \le 1 \\ C - (t-s)/C & \text{for } 1 \le t-s \end{cases} \le C - \frac{t-s}{C} \quad \text{for } t \ge s \ge 0,$$

we obtain

$$\begin{aligned}
\int_s^t \int_{k+1}^{k+2} \sigma(x,\tau) dx d\tau &= \int_s^t \int_{k+1}^{k+2} \left(\mu \frac{v_x}{u} - R\frac{\theta}{u} \right) dx d\tau \\
&\le C \int_s^t \int_{k+1}^{k+2} \frac{v_x^2}{u\theta} dx d\tau - \frac{R}{2} \int_s^t \int_{k+1}^{k+2} \frac{\theta}{u} dx d\tau \\
&\le C - \frac{R}{2} \int_s^t \inf_{[k+1,k+2]} \theta(\cdot,\tau) \int_{k+1}^{k+2} \frac{1}{u} dx d\tau \\
&\le C - \frac{R}{2} \int_s^t \inf_{[k+1,k+2]} \theta(\cdot,\tau) \left(\int_{k+1}^{k+2} u dx \right)^{-1} d\tau \\
&\le C - \frac{R}{2\alpha_2} \int_s^t \inf_{[k+1,k+2]} \theta(\cdot,\tau) d\tau \le C - (t-s)/C
\end{aligned} \tag{2.17}$$

for $t \ge s \ge 0$. Recalling the definition of $Y(t)$, it follows from (2.17) that

$$0 \le Y(t) \le Ce^{-t/C}, \qquad \frac{Y(t)}{Y(s)} \le Ce^{-(t-s)/C} \quad \forall\, t \ge 0. \tag{2.18}$$

Inserting (2.13) and (2.18) into (2.8), we infer that

$$u(x,t) \le C + C\int_0^t \theta(x,s)e^{-(t-s)/C}ds \qquad \text{for all } x \in \bar{\Omega}_k,\ t \ge 0. \tag{2.19}$$

On the other hand, by Lemmas 2.1-2.2 we have

$$\begin{aligned}\left|\theta^{1/2}(x,t) - \theta^{1/2}(b_i(t),t)\right| &\le \int_i^{i+1} \theta^{-1/2}|\theta_x|dx \\ &\le \left(\int_i^{i+1} \frac{\theta_x^2}{u\theta^2}dx\right)^{1/2}\left(\int_i^{i+1} u\theta dx\right)^{1/2} \\ &\le \sqrt{\alpha_2}\left(\int_\Omega \frac{\theta_x^2}{u\theta^2}dx\right)^{1/2} \max_{[i,i+1]} u^{1/2}(\cdot,t) \quad \text{for } x \in [i,i+1]\end{aligned}$$

and $i = 0, 1, \cdots$ for the initial boundary value problems and $i = 0, \pm 1, \cdots$ for the Cauchy problem. The above inequality together with (2.6) yields

$$\frac{\alpha_1}{2} - \alpha_2\int_\Omega \frac{\theta_x^2}{u\theta^2}dx\ \max_{\bar{\Omega}_k} u(\cdot,t) \le \theta(x,t) \le 2\alpha_2 + 2\alpha_2\int_\Omega \frac{\theta_x^2}{u\theta^2}dx\ \max_{\bar{\Omega}_k} u(\cdot,t), \quad x \in \bar{\Omega}_k. \tag{2.20}$$

Hence, inserting (2.20) into (2.19), applying Gronwall's inequality, and (2.1), we conclude

$$u(x,t) \le C(k) \qquad \forall\ x \in \bar{\Omega}_k,\ t \ge 0. \tag{2.21}$$

Now, integrating (2.8) over $(0,1)$ with respect to x, using (2.5), (2.13), and (2.18), we obtain

$$\alpha_1 \le Ce^{-t/C} + C\int_0^t \frac{Y(t)}{Y(s)}ds, \qquad t \ge 0. \tag{2.22}$$

It follows from (2.8), (2.13), (2.18), and (2.20)-(2.22), (2.1) that

$$\begin{aligned} u(x,t) &\ge C\int_0^t \frac{Y(t)}{Y(s)}\left(\frac{\alpha_1}{2} - C\int_\Omega \frac{\theta_x^2}{u\theta^2}dx\right)ds \\ &\ge C_0 - C_1e^{-t/C_2} - C\left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{Y(t)}{Y(s)}\int_\Omega \frac{\theta_x^2}{u\theta^2}dxds \\ &\ge C_0 - C_1e^{-t/C_2} - Ce^{-t/(2C)}\int_0^{t/2}\int_\Omega \frac{\theta_x^2}{u\theta^2}dxds - C\int_{t/2}^t\int_\Omega \frac{\theta_x^2}{u\theta^2}dxds \\ &\ge C_0 - C_1e^{-t/C_2} - Ce^{-t/(2C)} - C\int_{t/2}^t\int_\Omega \frac{\theta_x^2}{u\theta^2}dxds \\ &\ge C_2 \qquad \text{for all } x \in \bar{\Omega}_k,\ t \ge T_0, \end{aligned} \tag{2.23}$$

where T_0, C_i $(i = 0, 1, 2)$ are positive constants independent of t. From the proof of the existence in [1, pp.68-75], [13] (also see [8, pp.348-350]) we see that u satisfies

$$0 < m_1(t) \le u(x,t) \qquad \text{for } x \in \bar{\Omega}, t \ge 0, \tag{2.24}$$

where $m_1(t)$ is a continuous function of t (In fact, $m_1(t) = Ce^{-Ct}$). Therefore, Combining (2.21) with (2.23)-(2.24), we get (2.12). The proof is complete. □

3 Proof of Theorem 1.1

We have proved the local pointwise estimates in Theorem 1.1 (i.e. Lemma 2.4). In this section we apply the results obtained in Section 2 to prove the rest of Theorem 1.1. The proof is divided into three cases. *Case* **I.** *for the problem (1.1)-(1.5).* We start with the following lemma:

Lemma 3.1

$$\int_0^t \max_{[0,k+1]} v^2(\cdot,s)ds,\ \int_0^t \max_{[0,k+1]}[\theta(\cdot,s)-\bar\theta(s)]^2 ds \le C \quad \textit{for all } t\ge 0, \tag{3.1}$$

where $\bar\theta(t) := \theta(b_0(t),t)$ and $b_0(t)$ is the same as in Lemma 2.2.

Proof. It follows from (1.5), Cauchy-Schwarz's inequality, (2.1), and Lemma 2.4 that

$$\begin{aligned}
&\int_0^t \max_{[0,k+1]} v^2(\cdot,s)ds \le 2\int_0^t\int_0^{k+1}|vv_x|dxds\\
&\le \int_0^t\int_0^{k+1}\left(C\frac{v_x^2}{u\theta}+\frac{v^2u\theta}{2(k+1)\alpha_2\overline{u}}\right)dxds \le C+\frac{1}{2(k+1)\alpha_2}\int_0^t \max_{[0,k+1]} v^2 \int_0^{k+1}\theta dxds\\
&\le C+\frac12\int_0^t \max_{[0,k+1]} v^2(\cdot,s)ds,
\end{aligned}$$

which implies the first part of (3.1). Similarly, from Cauchy-Schwarz's inequality and Lemma 2.4 we get

$$\begin{aligned}
&\int_0^t \max_{[0,k+1]}[\theta(\cdot,s)-\bar\theta(s)]^2ds \le C\int_0^t\int_0^{k+1}\frac{\theta_x^2}{u\theta^2}dx\int_0^{k+1}\theta^2dxds\\
&\le C+C\int_0^t\int_0^{k+1}\frac{\theta_x^2}{u\theta^2}dx\max_{[0,k+1]}[\theta(\cdot,s)-\bar\theta(s)]^2ds.
\end{aligned}$$

Applying Gronwall's inequality and (2.1) to the above inequality, we obtain the second part of (3.1). The proof is complete. □

Let $\psi\in W^{1,\infty}(\mathbb{R})$, $\psi(x)=1$ for $x\le k$, $\psi(x)=k+1-x$ for $k\le x\le k+1$ and $\psi(x)=0$ for $x\ge k+1$.

Note that Eq. (1.2) can be rewritten as $[\mu u_x/u - v]_t = R[\theta/u]_x$. Multiplying this by $(\mu u_x/u-v)\psi$ in $L^2((0,\infty)\times(0,t))$, applying the inequality $|xy|\le\epsilon x^2+\epsilon^{-1}y^2$, (2.1), (2.12), (2.6), and taking into account $\theta u_x^2=(\bar\theta+\theta-\bar\theta)u_x^2\ge(\alpha_1/2-C(\theta-\bar\theta)^2)u_x^2$, we obtain

$$\begin{aligned}
&\frac{\mu^2}{2}\int_\Omega\frac{u_x^2}{u^2}\psi dx \le C+C\int_\Omega v^2\psi dx - R\mu\int_0^t\int_\Omega\theta\frac{u_x^2}{u^3}\psi\\
&+C\int_0^t\int_\Omega\left(\frac{\theta_x^2}{\epsilon u\theta^2}+\epsilon\theta^2\frac{u_x^2}{u^3}+\epsilon\theta^2\frac{v^2}{u}+\frac{v^2}{\epsilon u}\right)\psi dxds\\
&\le\frac{C}{\epsilon}-R\mu\left(\frac{\alpha_1}{2}-C\epsilon\right)\int_0^t\int_\Omega\frac{u_x^2}{u^3}\psi dxds + C\int_0^t\max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega\frac{u_x^2}{u^3}\psi dxds\\
&+C\epsilon^{-1}\int_0^t\max_{[0,k+1]}[(\theta-\bar\theta)^2+v^2]ds \qquad (0<\epsilon<1),
\end{aligned}$$

which, by choosing ϵ appropriately small, and applying (2.12) and (3.1), gives

$$\int_\Omega u_x^2\psi dx+\int_0^t\int_\Omega u_x^2\psi dxds\le C+C\int_0^t\max_{[0,k+1]}(\theta-\bar\theta)^2\int_\Omega u_x^2\psi dxds. \tag{3.2}$$

Applying Gronwall's inequality to (3.2) and using (3.1), one gets

Lemma 3.2 *We have*

$$\int_\Omega u_x^2(x,t)\psi(x)dx+\int_0^t\int_\Omega u_x^2\psi dxds\le C \qquad \textit{for all } t\ge 0. \tag{3.3}$$

By virtue of Poincaré's inequality and (3.3),

$$\int_0^\infty \|(u-u_1)(t)\|^2_{L^2(0,k)}dt\le C\int_0^\infty\|u_x(t)\|^2_{L^2(0,k)}dt\le C. \tag{3.4}$$

It follows from Eq. (1.1), the mean value theorem (for $u(k,t)-u_1$), integration by parts, (1.5), (3.1), and (3.4) that

$$\int_0^\infty\left|\frac{d}{dt}\|(u-u_1)(t)\|^2_{L^2(0,k)}\right|dt=\int_0^\infty\left|\int_0^k(u-u_1)(v_x-\frac{v(k,t)}{k})dx\right|dt$$

$$\le C\int_0^\infty \{(u(k,t)-u_1)^2+\|u-u_1\|^2_{L^2(0,k)}+v^2(k,t)\}dt+\int_0^\infty\int_0^k |u_x v|dxdt$$
$$\le C\int_0^\infty (\|u_x\|^2_{L^2(0,k)}+\max_{[0,k]} v^2)dt\le C,$$

which combined with (3.4) yields $(1.10)_1$.

To prove $(1.10)_2$, we multiply (1.2) by $2jv^{2j-1}\psi^{2j}$ $(j\ge 2)$ in $L^2((0,\infty)\times(0,t))$, integrate by parts with respect to x, utilize (1.5), the inequality $|xy|\le x^2+y^2$, and Lemma 2.4, we infer that

$$\begin{aligned}&\int_\Omega v^{2j}(x,t)\psi^{2j}(x)dx+j^2\int_0^t\int_\Omega v^{2j-2}v_x^2\psi^{2j}\,dxds\\ &\quad\le C+Cj^2\int_0^t\int_\Omega\left\{v^{2j}\psi^{2j-2}+\theta|v|^{2j-1}\psi^{2j-1}+\theta^2v^{2j-2}\psi^{2j}\right\}dxds\\ &\quad\equiv C+I_j(t).\end{aligned}\tag{3.5}$$

Define $F_j(t):=\max_{0\le s\le t}\int_\Omega v^{2j}(x,s)\psi^{2j}(x)dx$. Then I_j can be estimated as follows, using the inequality $|xy|\le x^2+y^2$, (2.1) and (3.1).

$$\begin{aligned}I_2(t)&\le C\int_0^t\int_\Omega\left(v^4+(\theta-\bar\theta)^2v^2+v^2\right)\psi^2dxds\\ &\le C\int_0^t\max_{[0,k+1]}\left(v^2+(\theta-\bar\theta)^2\right)(\cdot,s)ds\le C\end{aligned}\tag{3.6}$$

and for $j\ge 3$

$$\begin{aligned}I_j(t)&\le Cj^2\int_0^t\max_{[0,k+1]}\{|\theta-\bar\theta|^2+v^2\}\int_\Omega[(v\psi)^{2j-2}+(v\psi)^{2j-4}]dxds\\ &\le Cj^2(F_{j-1}+F_{j-2})(t).\end{aligned}\tag{3.7}$$

Inserting (3.6)-(3.7) into (3.5), we obtain by induction for j that

$$\|v(t)\psi\|^{2j}_{L^{2j}}+j^2\int_0^t\|v^{j-1}v_x\psi^j\|^2ds\le C_j\qquad\forall\, t\ge 0, j\ge 2,\tag{3.8}$$

where C_j is a positive constant that depends only on j and k.

Using Eq. (1.2), (3.1) and (3.8), following the same arguments as used for (3.5)-(3.7), we get for $j\ge 2$

$$\begin{aligned}\int_0^\infty\left|\frac{d}{dt}\|v(t)\psi\|^{2j}_{L^{2j}}\right|dt&\le C+Cj^2\int_0^\infty\|v^{j-1}v_x\psi^j\|^2dt\\ &+C(j-2)j\max_{t\ge0}\int_\Omega[(v\psi)^{2j-2}+(v\psi)^{2j-4})dx<\infty,\end{aligned}$$

which together with (3.1) and (3.8) yields $(1.10)_2$ in Theorem 1.1.

Case **II.** *for the problem (1.1)-(1.4), (1.6).* First it is easy to see that for the boundary conditions (1.6) Lemmas 3.1, 3.2 still hold. Using (1.6) and applying the same arguments as used for (3.1), we obtain

$$\int_0^\infty\max_{[0,k+1]}[\theta(t)-1]^2dt\le C.\tag{3.9}$$

Now we estimate v_x in L^2-norm. Multiplying (1.2) with $v\psi^2$ in $L^2(\Omega\times(0,t))$, integrating by parts for $[v_x/u]_x$ with respect to x, employing (2.1), (2.12), and Lemma 3.2, we obtain

$$\begin{aligned}&\int_0^t\int_\Omega\frac{v_x^2}{u}\psi^2dxds\le C+C\int_0^t\int_\Omega\left(\frac{|v_x|}{u}|\psi_x|+\frac{|\theta_x|}{u}\psi+\frac{\theta|u_x|}{u^2}\psi\right)|v|\psi dxds\\ &\le C+\int_0^t\int_\Omega\left\{\frac{v_x^2}{2u}+C[\frac{\theta_x^2}{u\theta^2}+\theta^2v^2+u_x^2]\right\}\psi^2dxds+C\int_0^t\max_{[0,k+1]}v^2ds\\ &\le C+\frac12\int_0^t\int_\Omega\frac{v_x^2}{u}\psi^2dxds+C\int_0^t\max_{[0,k+1]}[v^2+(\theta-1)^2]ds,\qquad t\ge 0,\end{aligned}$$

whence (also by Lemma 3.1 and (3.9))

$$\int_0^\infty \|v_x(t)\psi\|^2 dt \leq C. \tag{3.10}$$

The following lemma gives L^2-bounds of v_t.

Lemma 3.3 *For any $t \geq 0$ we have*

$$\|v_x(t)\psi^2\|^2 + \|[\theta(t)-1]\psi^2\|^2 + \int_0^t (\|v_t\psi^2\|^2 + \|\theta_x\psi^2\|^2) ds \leq C. \tag{3.11}$$

Proof. We multiply (1.2) by $uv_t\psi^4$ in $L^2(\Omega \times (0,t))$, integrate by parts for $[v_x/u]_x$ with respect to x, and utilize (2.12), Cauchy-Schwarz's inequality, (3.10) and Lemma 3.2, to get

$$\begin{aligned}
&\|v_x(t)\psi^2\|^2 + \int_0^t \|v_t\psi^2\|^2 ds \leq C + C\int_0^t \int_\Omega \left\{\theta_x^2 + \theta^2 u_x^2 + v_x^2 u_x^2\right\} \psi^4 dxds \\
&\leq C + C\int_0^t \left\{\max_{\bar\Omega} v_x^2\psi^3 \int_\Omega u_x^2\psi dx + \|\theta_x\psi^2\|^2 + \max_{[0,k+1]}(\theta-1)^2\right\} ds. \qquad (3.12)
\end{aligned}$$

Here the term $\max v_x^2\psi^3$ can be bounded as follows, using (2.12), Sobolev's embedding theorem ($W^{1,1} \hookrightarrow L^\infty$), and Eq. (1.2).

$$\begin{aligned}
\max_{\bar\Omega} v_x^2\psi^3 &\leq C\max(\sigma^2+\theta^2)\psi^3 \leq \epsilon\|\sigma_x\psi^2\|^2 + C\epsilon^{-1}\|\sigma\psi\|^2 + C\max_{[0,k+1]}\theta^2 \\
&\leq C\epsilon^{-1}\{1 + \max_{[0,k+1]}(\theta-1)^2 + \|v_x\psi\|^2\} + \epsilon\|v_t\psi^2\|^2 \quad (0<\epsilon<1).
\end{aligned}$$

Substituting the above estimate into (3.12), choosing ϵ appropriately small, and taking into account (3.3), (3.9)-(3.10), we conclude

$$\|v_x(t)\psi^2\|^2 + \int_0^t \|v_t\psi^2\|^2 ds \leq C + C\int_0^t \|\theta_x\psi^2\|^2 ds. \tag{3.13}$$

If we multiply Eq. (1.3) with $(\theta-1)\psi^4$ in $L^2(\Omega\times(0,t))$, apply Cauchy-Schwarz's inequality, (3.9)-(3.10), we find that

$$\|[\theta(t)-1]\psi^2\|^2 + \int_0^t \|\theta_x\psi^2\|^2 ds \leq C + C\int_0^t \max_{[0,k+1]}(\theta-1)^2\|v_x\psi^2\|^2 ds. \tag{3.14}$$

Multiplying (3.14) with $2C$ and adding the resulting inequality to (3.13), applying then Gronwall's inequality and (3.9), we obtain (3.11). $\square$

As a result of Lemma 3.3 we have

$$\int_0^\infty \int_\Omega (u_{tx}^2\psi^4 + v_{xx}^2\psi^4) dxds \leq C. \tag{3.15}$$

In fact, from Sobolev's imbedding theorem ($W^{1,1} \hookrightarrow L^\infty$), (1.1)-(1.2), and (2.12), (3.3) and (3.9)-(3.10), we get

$$\int_0^t \int_\Omega (u_{tx}^2 + v_{xx}^2)\psi^4 dxds \leq C + C\int_0^t \max_{\bar\Omega} v_x^2\psi^3 ds \leq C + \int_0^t \int_\Omega (Cv_x^2\psi + \frac{1}{2}v_{xx}^2\psi^4) dxds,$$

which yields (3.15).

If we multiply (1.3) by $u\theta_t\psi^8$, integrate and keep in mind that

$$\max_\Omega \theta_x^2\psi^6 \leq C(\epsilon^{-1}\|\theta_x\psi^2\|^2 + \epsilon\|[\theta_x/u]_x\psi^4\|^2),$$

we obtain by the arguments similar to those used in Lemma 3.3 and (3.15) that

$$\|\theta_x(t)\psi^4\|^2 + \int_0^t \int_\Omega (\theta_t^2 + \theta_{xx}^2)\psi^8 dxds \leq C \qquad \forall\, t \geq 0. \tag{3.16}$$

Now we are in a position to prove (1.11). By (3.3), (3.10)-(3.11), (3.15)-(3.16), and the identity $\int_\Omega v_x v_{xt}\psi^8 dx = -\int_\Omega v_{xx} v_t \psi^8 dx - 8\int_\Omega v_x v_t \psi^7 \psi_x dx$, we see that

$$\int_0^\infty \left\{ \left|\frac{d}{dt}\|u_x\psi^8\|^2\right| + \left|\frac{d}{dt}\|v_x\psi^8\|^2\right| + \left|\frac{d}{dt}\|\theta_x\psi^8\|^2\right| \right\} dt \le C,$$

which together with (3.3), (3.10)-(3.11) gives

$$\|(u_x(t), v_x(t), \theta_x(t))\|_{L^2(0,k)} \to 0 \quad \text{as } t \to \infty. \tag{3.17}$$

In view of the boundary conditions (1.6), we apply Poincaré's inequality and (3.17) to obtain $\|(u(t) - u_1(t), v(t), \theta(t) - 1)\|_{H^1(0,k)} \to 0$ as $t \to \infty$. This proves (1.11).

Case **III.** *for the Cauchy problem* Let $(\tilde u, \tilde v, \tilde\theta)$ be the generalized solution of the initial boundary value problem (1.1)-(1.5) with the initial data (u_0, v_0, θ_0) in the half line. Then we define $u(x,t), v(x,t), \theta(x,t)$ in $\mathbb{R} \times (0,\infty)$ by

$$u(x,t) := \begin{cases} \tilde u(x,t), & x \ge 0 \\ \tilde u(-x,t), & x < 0 \end{cases}, \quad v(x,t) := \begin{cases} \tilde v(x,t), & x \ge 0 \\ -\tilde v(-x,t), & x < 0 \end{cases},$$
$$\theta(x,t) := \begin{cases} \tilde\theta(x,t), & x \ge 0 \\ \tilde\theta(-x,t), & x < 0 \end{cases}. \tag{3.18}$$

Then u, v, θ satisfy (1.8) and the Cauchy problem (1.1)-(1.4). Therefore, (u, v, θ) defined by (3.18) is a generalized solution of (1.1)-(1.4). By virtue of the uniqueness (u, v, θ) defined by (3.18) coincides with the generalized solution indicated in (1.8). From the definition (3.18) and i) of Theorem 1.1, (1.12) follows immediately. The proof of Theorem 1.1 is complete.

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