OPERATOR SPLITTING METHODS FOR GENERALIZED KORTEWEG–DE VRIES EQUATIONS

HELGE HOLDEN, KENNETH HVISTENDAHL KARLSEN, AND NILS HENRIK RISEBRO

ABSTRACT. We apply the method of operator splitting on the generalized Korteweg–de Vries (KdV) equation $u_t+f(u)_x+\varepsilon u_{xxx}=0$, by solving the nonlinear conservation law $u_t+f(u)_x=0$ and the linear dispersive equation $u_t+\varepsilon u_{xxx}=0$ sequentially. We prove that *if* the approximation obtained by operator splitting converges, then the limit function is a weak solution of the generalized KdV equation. Convergence properties are analyzed numerically by studying the effect of combining different numerical methods for each of the simplified problems.

1. INTRODUCTION

There are at least two fundamentally different approaches to computations of numerical solutions of the Korteweg–de Vries (KdV) equation

$$u_t+uu_x+u_{xxx}=0,\quad u(x,0)=u_0(x).$$

One approach is based on appropriate finite difference approximations. See, e.g., [27], [6], [2]. Alternatively, one can use the inverse scattering transform, see, e.g., [23], [4]. This method has had enormous impact on the analysis of the KdV equation and other completely integrable equations, but can also be used numerically, see, e.g., [26, 24, 25]. For an extensive survey of numerical methods associated with completely integrable equations we refer to Taha and Ablowitz, [30, 31, 32]. The inverse scattering transform method can briefly be described as follows: One considers the solution, u, of the nonlinear equation as an entry in an associated linear problem. In the case of the KdV equation one introduces u in the stationary Schrödinger equation, $-\psi_{xx}+u\psi=\lambda\psi$, as a potential parametrized by the variable t. For this equation one first computes the appropriate scattering quantities (reflection and transmission coefficients, bound states, etc.) for the initial data u_0. It turns out that when u solves the the KdV equation, the scattering quantities develop as functions of t in an explicit and trivial manner (e.g., the eigenvalues remain unchanged). Subsequently, one has to solve the inverse problem of computing the potential $u(x,t)$ from the scattering data. The function $u(x,t)$ then solves the KdV equation for any time t. Ingenious as this method is, it is however restricted to the KdV equation and other completely integrable equations. Small perturbations of the equation will yield this method useless.

We will in this paper, in contrast to the methods discussed above, apply the method of operator splitting to generalized KdV equations of the form

$$u_t+f(u)_x+\varepsilon u_{xxx}=0,\quad u(x,0)=u_0(x). \tag{1}$$

In the special case of $f(u)=u^2/2$ and $\varepsilon=1$ it reduces to the KdV equation, while for $f(u)=u^3/3$ and $\varepsilon=1$ we obtain the modified KdV equation, another completely integrable equation. Preliminary qualititative numerical experiments using operator splitting were reported by Tappert [33]. Numerically he investigated equations of the form

$$u_t+f(u)_x+Lu_x=0,\quad u(x,0)=u_0(x), \tag{2}$$

where L is a linear pseudodifferential operator with constant coefficients. By first solving

$$u_t+f(u)_x=0, \tag{3}$$

Date: November 24, 1998.
1991 *Mathematics Subject Classification.* Primary 65M99 35Q53; Secondary 35Q51.
Key words and phrases. Generalized KdV equation, operator splitting, numerical examples.

using an implicit finite difference scheme, and subsequently solving

$$u_t + Lu_x = 0, \tag{4}$$

using discrete fast Fourier transform, he studied perturbations around the KdV equation. The principal result was that the KdV equation is morphologically stable in the sense that a small deviation from KdV equation does not seem to alter qualitative features of the equation, e.g., interaction of solitons.

There are extensive theoretical results on the generalized KdV equation (1). Solutions of (1) are unique within the proper class of function classes. We can summarize the present results as follows: Assume that f is C^1 with $f(0) = 0$ and $|f'(u)| \leq C\,|u|^p$ for $p \in (0,2]$. Consider initial data u_0 such that $(1+\max(x,0))^{\beta/2}u_0 \in L^2(\mathbb{R})$ with $\beta = 1/p - 1/4$. Then there exists, see Ginibre *et al.* [8], [7], a unique solution u such that $u \in L^\infty([0,\infty); L^2(\mathbb{R})) \cap C_w([0,\infty); L^2(\mathbb{R})) \cap L^2_{\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(\mathbb{R}))$. Here C_w denotes the set of weakly differentiable functions. There are also result in the case with $p \in [2,3)$. Briefly, this type of results are obtained by regularizing equation (1) by adding the term $\nu(u_{xxxx} - \mu u_{xx})$ to the left-hand side of (1). The regularized equation has a unique solution, and by a certain compactness argument one obtains convergence of a subsequence in the limit $\nu \to 0$ to a solution of (1). An additional argument proves uniqueness within the proper class of functions.

For initial data u_0 in H^1 the result reads as follows: Assume that f is C^2 with $|f'(u)| \leq C\,|u|^{(3/2)+\gamma}$. Now equation (1) has a unique solution in the class $C(\mathbb{R}; L^2(\mathbb{R})) \cap L^\infty(\mathbb{R}; H^1(\mathbb{R})) \cap L^2(\mathbb{R}; H^1_{\mathrm{loc}}(\mathbb{R})) \cap L^q(\mathbb{R}; L^p_{1+\theta\alpha/2}(\mathbb{R}))$, see Kenig *et al.* [16]. The result holds for any $\theta \in [0,\theta_0]$, $\alpha \in [0,1/2]$ with $\theta_0 = \min(1, 2\gamma)$ and $q = 6/\theta(\alpha+1)$, $p = 2/(1-\theta)$.

For further analytical results see [15], [17], and [38].

We now describe our operator splitting strategy. Let $S_t u_0$ and $A_t u_0$ denote the solution of the initial value problems

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \tag{5}$$

and

$$u_t + \varepsilon u_{xxx} = 0, \quad u(x,0) = u_0(x), \tag{6}$$

respectively.

The idea of operator splitting is to solve equations (5) and (6) sequentially, and approximate the solution of (1) as

$$u_{\Delta t}(x, n\Delta t) = \left[A_{\Delta t} \circ S_{\Delta t}\right]^n u_0(x). \tag{7}$$

The method of operator splitting has successfully been applied to several other problems of the form

$$u_t + f(u)_x = F(x,t,u,u_{xx}), \quad u(x,0) = u_0(x), \tag{8}$$

by solving the equations

$$u_t + f(u)_x = 0$$

and

$$u_t = F(x,t,u,u_{xx})$$

separately. In particular, Karlsen and Risebro, [14], analyzed the case $F = \varepsilon u_{xx}$ using operator splitting. By using a dimensional splitting in addition, their result also covered the multi-dimensional case. A more sophisticated splitting algorithm was presented in [13]. Holden and Risebro, [10], used operator splitting to study the case with $F = g(t,x,u)$ in the more complicated case of a stochastic source. An unconditionally stable splitting scheme for the equation with $F = \varepsilon u_{xx} + g(t,x,u)$ was analyzed in [12]. Finally, Evje and Karlsen, [5], treated the case with a possibly degenerate viscous term $F = (a(u)u_x)_x$, where a may vanish, using operator splitting.

There is an important difference[1] between the diffusive or viscous case

$$u_t + f(u)_x = \varepsilon u_{xx}, \quad u(x,0) = u_0(x), \tag{9}$$

and the dispersive case

$$u_t + f(u)_x = -\varepsilon u_{xxx}, \quad u(x,0) = u_0(x), \tag{10}$$

namely the deep fact that limits as $\varepsilon \to 0$ of solutions of the two equations are distinct, see [18, 19, 20]. Limit solutions of (9) are used as entropy conditions to define the unique weak solution of the conservation law $u_t + f(u)_x = 0$.

More precisely, as $\varepsilon \to 0+$, the generalized KdV equation (1) formally reduces to the conservation law

$$u_t + f(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{11}$$

whose solutions may develop shocks. It is well-known that the correct shock (discontinuous) solutions of (11) — interpreted in the sense of distributions — can be obtained as strong L^1 limits of smooth solutions of the dissipative regularization (9) as $\varepsilon \to 0+$. The situation for the dispersive regularization (10) is radically different. More precisely, when t exceeds the shock formation time for the conservation law (11) with $f = u^2$, the solution u^ε of (10) behaves in an oscillatory manner over some x interval. As $\varepsilon \to 0+$, the amplitude of these oscillations remains bounded but does not tend to zero, and its wave length is of order $\mathcal{O}(\varepsilon)$. Thus dispersive solutions u^ε only converge weakly to some limit function $\bar{u}$ as $\varepsilon \to 0+$. Furthermore, it is easy to see that the limit $\bar{u}$ does not satisfy the conservation law (11) in the sense of distributions. On the other hand, it is well-known that when the solution of the conservation law (11) is smooth, then the dispersive solution of (10) converges uniformly to that smooth solution as $\varepsilon \to 0+$. Recall that when $f = u^2$, the weak limit $\bar{u}$ can be calculated explicitly, see Lax and Levermore [18, 19, 20] and Venakides [34, 35]. We also the refer to the recent review papers of Levermore [21] and Venakides [36] (and the references therein) for a detailed overview of the known results of the KdV small dispersion limit problem. We refer to Goodman and Lax [9] and Hou and Lax [11] for numerical investigations of the phenomena of oscillations and weak convergence of solutions of dispersive difference schemes for (11).

The very fact that our operator splitting method uses (approximate) solutions of the conservation law (11) that are consistent with the dissipative equation (9), and not in general the dispersive equation (10), may at first glance generate some doubts about the soundness of the operator splitting method and whether it can produce approximate solutions that will converge to the correct solution of the generalized KdV equation (1) as $\Delta t \to 0+$. But concerning this issue one should of course bear in mind that when Δt becomes sufficiently small, the hyperbolic solution $S_{\Delta t} u_0$ remains smooth (no shocks are formed), at least for smooth initial data, and then, in view of the discussion above, this solution will be consistent with both the dissipative equation (9) and the dispersive equation (10). Due to inconsistencies between the topologies used in the analysis of conservation laws and generalized KdV equations, we have not been able to show that the operator splitting approximations converge. But *if* they converge to some limit function, we show below that this limit must in fact satisfy the generalized KdV equation (1) in the sense of distributions. Furthermore, the numerical results presented here strongly suggest that operator splitting approximations converge to the correct solutions as the discretization parameters tend to zero.

We will now describe the content of the paper more precisely. In Section 2 we first describe fundamental properties of solutions of the conservation law (5) and the linear dispersive equation (10). The incompatibility of properties of solutions of the two equations makes it hard to obtain strong rigorous results. However, we prove a Lax–Wendroff type of result: Assuming that the operator splitting method converges, we establish that the limit indeed is a weak solution of the generalized KdV equation.

In actual computations exact solutions of the conservation law and the linear dispersive equation will have to be replaced by approximations, and in Subsections 2.2 and 2.3 we describe the

[1] The sign in front of the highest derivative is vital in the diffusive case, but only changes the direction of the waves in the dispersive case.

numerical methods employed for the linear dispersive equation and the conservation law, respectively. In the first case we consider a direct difference approximation and a fast Fourier transform. In the latter case we utilize the Godunov method, an ENO scheme as well as a spectral viscosity method.

In Section 3 we present numerical results for an explicit solution of the differential equations, namely one and two-soliton solutions of the KdV equation. The results are summarized in Section 4.

2. Operator splitting

We consider the Cauchy problem for the generalized KdV equation

$$\begin{aligned} u_t + f(u)_x + \varepsilon u_{xxx} &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{12}$$

with a suitable initial function u_0. Our strategy is alternately to solve the conservation law

$$\begin{aligned} u_t + f(u)_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{13}$$

and the linear dispersive equation

$$\begin{aligned} u_t + \varepsilon u_{xxx} &= 0, \\ u(x,0) &= u_0(x). \end{aligned} \tag{14}$$

Let S_t be the solution operator associated with the conservation law (13), i.e., we write the unique weak solution to (13) as $u(x,t) = S_t u_0(x)$. Similarly we denote the solution operator associated with the linear dispersive equation (14) by A_t. Then we approximate the solution of (12) by

$$u_{\Delta t}(x, n\Delta t) = \left[A_{\Delta t} \circ S_{\Delta t}\right]^n u_0(x) \tag{15}$$

for some (small) timestep Δt. Of course, when implementing this approach, both A_t and S_t must be replaced by numerical methods.

2.1. **Analytical results.** Solutions of (1) possess a smoothing property that stems from the dispersive term. See Craig *et al.* [3] for details and precise statements. The linear dispersive equation has strong smoothing properties, while one for the nonlinear hyperbolic conservation law encounters steepening of gradients and formation of discontinuities, i.e., shocks. Thus the two equations are characterized by quite distinct behavior. The solution $u = S_t u_0$ of the hyperbolic conservation law (12) has several important properties:

(i) Maximum principle, $\|u(\,\cdot\,,t)\|_\infty \leq \|u_0\|_\infty$.
(ii) L^p stability, $\|u(\,\cdot\,,t)\|_p \leq \|u_0\|_p$ for $p \geq 1$.
(iii) Total variation dimishing (TVD), $T.V.(u(\,\cdot\,,t)) \leq T.V.(u_0)$, where $T.V.$ denotes the total variation[2].
(iv) L_1 contractive, $\|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_1 \leq \|u_0 - v_0\|_1$, where $v = S_t v_0$ is another solution of (13).
(v) Lipschitz continuous in time, $\|u(\,\cdot\,,t) - u(\,\cdot\,,s)\|_1 \leq \mathcal{O}(1)\, T.V.(u_0)\, |t-s|$.

On the other hand, the linear dispersive equation (14) will in general have small dispersive waves moving rapidly in one direction. More precisely, we have (see [17], [7], and [16]) with initial data u_0 in $L^2(\mathbb{R})$ and $D := \int_0^\infty x^{\varepsilon+1/2}\,|u_0(x)|\,dx < \infty$, that the solution $u = A_t u_0$ satisfies:

(i) u is continuous for any $t > 0$.
(ii) $\sup_{0\leq t\leq T} \|u(\,\cdot\,,t)\|_2 \leq \|u_0\|_2$ for some $T > 0$.
(iii) $\sup_{x_0 \leq x} |u(x,t)| \leq C t^{-1/3}$ for $0 \leq t \leq T$ with the constant C depending on ε, T, x_0, $\|u_0\|_2$, and D.
(iv) $u_x \in L^2(\mathbb{R}; L^2_{\text{loc}}(\mathbb{R}))$.

[2] The total variation may be defined by $T.V.(u) = \sup \sum_i |u(x_{i+1}) - u(x_i)|$ where the supremum is taken over all finite partitions $\{x_i\}$ with $x_i < x_{i+1}$.

Incompabilities between the topologies used in the analysis of the conservation law (13) and the linear dispersive equation (14) makes it very difficult to establish rigorously convergence of the operator splitting method. However, it is possible to prove that *if* the operator splitting approximations converge to some limit function as Δt tends to zero, then this limit must in fact be a weak solution of the Cauchy problem (12). Suppose that $u_0 \in L^2(\mathbb{R})$. We then say that u is a weak solution on $\mathbb{R} \times [0,T]$, $T>0$, of the Cauchy problem (12) if

$$u \in L^\infty([0,T]; L^2(\mathbb{R})) \tag{16}$$

and u satisfies the the Cauchy problem (12) in the sense of distributions; that is, for any test function $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$,

$$\int_0^T \int_{-\infty}^{\infty} \Big(\phi_t u + \phi_x f(u) + \varepsilon \phi_{xxx} u\Big)\, dxdt + \int_{-\infty}^{\infty} \phi(x,0) u_0(x)\, dx = 0. \tag{17}$$

It has recently been proved that weak solutions are uniquely determined by their data in the special case $f = u^2$, see Zhou [38]. In what follows, we assume that the functions $S_t u_0$ and $A_t u_0$ are (exact) weak solutions of their respective equations. Inspired by [13], let us introduce the auxiliary function $u_{\Delta t}$,

$$u_{\Delta t}(t) = \begin{cases} S_{2(t-t_n)} u^n, & t \in [t_n, t_{n+1/2}], \\ \big[A_{2(t-t_{n+1/2})} \circ S_{\Delta t}\big] u^n, & t \in [t_{n+1/2}, t_{n+1}]. \end{cases} \tag{18}$$

where we for brevity write (cf. (15))

$$u^n(x) = u_{\Delta t}(x, n\Delta t) = \left[A_{\Delta t} \circ S_{\Delta t}\right]^n u_0(x) \tag{19}$$

Clearly

$$u_{\Delta t}(t_n) = u^n = [A_{\Delta t} \circ S_{\Delta t}]^n u_0, \qquad \text{for all } n.$$

Note also that

$$\|u_{\Delta t}(\cdot,t)\|_2 \le \|u_{\Delta t}(\cdot,0)\|_2, \qquad \forall t > 0,$$

at least when $u_0 \in L^2(\mathbb{R})$, and thus

$$u_{\Delta t} \in L^\infty([0,T]; L^2(\mathbb{R})). \tag{20}$$

In what follows, we assume that

$$u_{\Delta t} \to u \text{ strongly in } L^2_{\mathrm{loc}}(\mathbb{R} \times [0,T]). \tag{21}$$

More generally, we could also have assumed that $u_{\Delta t} \to u$ almost everywhere in $\mathbb{R} \times [0,T]$. In view of (20), it certainly follows that

$$u \in L^\infty([0,T]; L^2(\mathbb{R})). \tag{22}$$

We would like to show that limit in (21) is a weak solution of (12). To this end, fix a test function $\phi \in C_0^\infty(\mathbb{R} \times [0,T])$ and define a new test function by

$$\varphi(x,t) = \phi\Big(x, \frac{t}{2}\Big).$$

Let

$$v^n(t) = S_t u^n, \qquad t \in [0, \Delta t].$$

Then since $v^n(t)$ satisfies the conservation law (11) in the sense of distributions on $\mathbb{R} \times [0,\Delta t]$ with initial data u^n, the following integral equality holds

$$\begin{aligned} &\int_{t_n}^{t_{n+1/2}} \int_{-\infty}^{\infty} \Big(\frac{1}{2}\phi_t u_{\Delta t} + \phi_x f(u_{\Delta t})\Big)\, dxdt \\ &\quad = \frac{1}{2}\int_0^{\Delta t} \int_{-\infty}^{\infty} \Big(\varphi_\tau(x,\tau+2t_n) v^n(\tau) + \varphi_x(x,\tau+2t_n) f(v^n(\tau))\Big)\, dxd\tau \\ &\quad = \frac{1}{2}\int_{-\infty}^{\infty} \phi(x,t_{n+1/2}) u^{n+1/2}\, dx - \frac{1}{2}\int_{-\infty}^{\infty} \phi(x,t_n) u^n\, dx, \end{aligned} \tag{23}$$

where we have used the substitution $\tau = 2(t-t_n)$ and introduced the short-hand notation $u^{n+1/2} = S_{\Delta t}u^n$. Similarly, let

$$w^n(t) = A_t u^{n+1/2}, \qquad t \in [0, \Delta t].$$

Then since $w^n(t)$ satisfies the the linear dispersive equation (14) in the sense of distributions on $\mathbb{R} \times [0, \Delta t]$ with initial data $u^{n+1/2}$, the following integral equality holds

$$\begin{aligned}
&\int_{t_{n+1/2}}^{t_{n+1}} \int_{-\infty}^{\infty} \Big(\frac{1}{2}\phi_t u_{\Delta t} + \varepsilon \phi_{xxx} u_{\Delta t}\Big)\, dxdt \\
&\quad = \frac{1}{2}\int_0^{\Delta t} \int_{-\infty}^{\infty} \Big(\varphi_\tau(x, \tau + 2t_{n+1/2}) w^n(\tau) + \varepsilon \varphi_{xxx}(x, \tau + 2t_{n+1/2}) w^n(\tau)\Big)\, dxd\tau \\
&\quad = \frac{1}{2}\int_{-\infty}^{\infty} \phi(x, t_{n+1}) u^{n+1}\, dx - \frac{1}{2}\int_{-\infty}^{\infty} \phi(x, t_{n+1/2}) u^{n+1/2}\, dx,
\end{aligned} \tag{24}$$

where we have used the substitution $\tau = 2(t - t_{n+1/2})$ and let u^{n+1} denote $[A_{\Delta t} \circ S_{\Delta t}]u^n$. Adding together (23) and (24), multiplying with 2, and summing the result over all $n = 0, \dots, N-1$, where $N\Delta t = T$, yields the global integral equality

$$\int_0^T \int_{-\infty}^{\infty} \Big(u_{\Delta t}\phi_t u + 2\chi_n \phi_x f(u_{\Delta t}) + \varepsilon 2(1 - \chi_n)\phi_{xxx} u_{\Delta t}\Big)\, dxdt + \int_{-\infty}^{\infty} \phi(x, 0) u_0(x)\, dx = 0, \tag{25}$$

where $\chi_n = \chi_n(x, t)$ is the characteristic function of the set $\mathbb{R} \times \big[t_n, t_{n+1/2}\big]$. Then because of (21) and since $\chi_n(x, t) \rightharpoonup \frac{1}{2}$ in $L^2(\mathbb{R} \times [0, T])$, it follows that

$$\begin{aligned}
&\int_0^T \int_{-\infty}^{\infty} \chi_n f(u_{\Delta t})\phi_x\, dxdt \to \frac{1}{2}\int_0^T \int_{-\infty}^{\infty} f(u)\phi_x\, dxdt \\
&\int_0^T \int_{-\infty}^{\infty} (1 - \chi_n) u_{\Delta t}\phi_{xxx}\, dxdt \to \frac{1}{2}\int_0^T \int_{-\infty}^{\infty} u\phi_{xxx}\, dxdt.
\end{aligned}$$

Hence, passing to the limit in (25), we obtain (17). Since ϕ was arbitrary and because of (22), it follows that the limit u is a weak solution of the Cauchy problem (12).

Summing up, we have thus proved the following Lax–Wendroff type theorem for operator splitting for generalized KdV equations:

Theorem 2.1. *Suppose that $u_0 \in L^2(\mathbb{R})$. Consider the semi-discrete sequence of operator splitting approximations $\{u_{\Delta t}\}$ given by* (18) *and* (19)*. Suppose that $u_{\Delta t}$ converges strongly in $L^2_{\mathrm{loc}}(\mathbb{R} \times [0, T])$ to u as $\Delta t \to 0+$. Then $u \in L^\infty([0, T]; L^2(\mathbb{R}))$ is a weak solution of the Cauchy problem* (12)*, that is, it satisfies* (17)*.*

To prove this theorem we did *not* require any smoothness of the operator splitting approximation $u_{\Delta t}$. From the point of view of rigorous analysis and the fact the hyperbolic solution operator S_t in general maps its (even smooth) data into BV, we stress that this is indeeed the sort result that we are interested in. But we mention again that it is an open problem to establish the strong convergence (21) presupposed in the above theorem.

One should note that the above theorem is valid if the exact solutions operators S_t and A_t are replaced by numerical methods which produce approximate solutions which satisfy their respective partial differential equations in the sense of distributions with error terms that tend to zero (slightly) faster than $\mathcal{O}(\Delta t)$. Finally, the above theorem can easily be extended to more general equations of the form [22]

$$u_t + f(u)_x + \varepsilon g(u)_{xxx} = 0.$$

We leave the details to the interested reader.

2.2. **Methods for the linear dispersive equation.** In this section we describe the methods that we use to approximate the linear dispersive equation. For simplicity we drop the ε, so that the equation reads

$$u_t + u_{xxx} = 0, \quad u|_{t=0} = u_0. \tag{26}$$

The solution can be expressed explicitly (see, e.g., [17]),

$$u(x,t) = \frac{1}{\sqrt[3]{3t}} \int_{-\infty}^{\infty} \mathrm{Ai}\,\big(\frac{x-y}{\sqrt[3]{3t}}\big) u_0(y)\, dy, \tag{27}$$

where Ai is the Airy function (see, e.g., [1], p. 446ff),

$$\mathrm{Ai}(x) = \frac{1}{\pi} \int_0^{\infty} \cos\big(\frac{y^3}{3} + yx\big)\, dy. \tag{28}$$

Most of the technical problems in treating the generalized KdV equation are intrinsically connected with the oscillatory behavior of the Airy function. Indeed we have

$$\mathrm{Ai}(x) = \begin{cases} c_- \left|x\right|^{-1/4} \sin(\frac{2}{3}\left|x\right|^{3/2} + \tilde{c}) & \text{for } x \to -\infty, \\ c_+ x^{-1/4} \exp\big(-2x^{3/2}/3\big) & \text{for } x \to \infty, \end{cases} \tag{29}$$

for constants $c_\pm$ and $\tilde{c}$. For numerical computations the explicit formula (27) is of little help.

2.2.1. *A direct difference method.* Writing $u_i = u(ih,t)$, we approximate u_{xxx} by the difference quotient

$$u_{xxx}(ih,t) \approx \frac{1}{2h^3}\left(-u_{i-2} + 2u_{i-1} - 2u_{i+1} + u_{i+2}\right), \tag{30}$$

for $i = -N, \dots, N$. Writing $\mathbf{u} = (u_{-N}, \dots, u_N)'$, we obtain the linear system of ordinary differential equations

$$\mathbf{u}_t = \mathbf{B}\mathbf{u}, \tag{31}$$

where the matrix $\mathbf{B}$ is defined by (30) and the type of boundary condition we use. (In our examples we use periodic boundary conditions.) To solve (31), we use the midpoint rule,

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} = \frac{1}{2}\mathbf{B}\left(\mathbf{u}^{n+1} + \mathbf{u}^n\right), \tag{32}$$

where $\mathbf{u}^n$ denotes the approximate solution at $n\Delta t$. The advantage of using the midpoint rule, rather than an explicit method, is that the midpoint rule does not have the prohibiting timestep restriction $\Delta t = \mathcal{O}\left(h^3\right)$. Of course, we pay for this by having to solve a system of linear equations, but the matrix $\mathbf{B}$ is banded and solving this system is not expensive in terms of CPU-time or memory. Our shorthand for this method is "`Diff`".

2.2.2. *Methods based on the fast Fourier transform.* Assuming periodic boundary conditions in the interval $[-\pi, \pi]$, we write the solution of (26) as

$$u(x,t) = \sum_n c_n e^{i\left(nx + n^3 t\right)}, \tag{33}$$

where the constants c_n are Fourier coefficients of the initial function.

Numerically, we can effectively implement this via the fast Fourier transform. Let

$$\hat{u}_k = \sum_{j=-N/2}^{N/2} u_j e^{2\pi ijk/(N+1)} \tag{34}$$

where $u_j = u(jh, \cdot)$. The inverse transform is given by

$$u_j = \frac{1}{N+1} \sum_k \hat{u}_k e^{-2\pi ijk/(N+1)}. \tag{35}$$

The discrete version of the dispersive equation (26) now reads with $h = 1/N$

$$\left(\hat{u}_k\right)_t - i\left(2\pi k\right)^3 \hat{u}_k = 0, \tag{36}$$

for each $k = -N/2, \dots, N/2$. This ordinary differential equation can be solved to give

$$\hat{u}(t) = \hat{u}(0)\, e^{i(2\pi k)^3 t}, \tag{37}$$

and one can then use the inverse transform (35) to find u_j. We label this method "`FFT`". However, we found that it is unstable when used in conjunction with some of the numerical methods for the conservation law. Therefore we also solve (36) by using a Crank-Nicholson scheme,

$$\left[1 - \frac{\Delta t}{2} w(k)\right] \hat{u}_k^{n+1} = \left[1 + \frac{\Delta t}{2} w(k)\right] \hat{u}_k^n, \tag{38}$$

where $\hat{u}_k^n = \hat{u}_k(n\Delta t)$ and $w(k) = i\,(2\pi k)^3$. Our shorthand for this method if "`FFT-CN`".

2.3. **Methods for the conservation law.** There is a great variety of methods to choose from for the conservation law. Since we have concentrated on smooth solutions of the KdV equation, we have chosen two methods that give a high order approximation to smooth solutions, and for comparison, one first order method that is simple and easy to implement. The first order method is Godunov's method, and the higher order methods are ENO schemes, and the spectral viscosity method. We now give a short description of these methods.

2.3.1. *Godunov's method.* Again, we use the notation $u_j^n = u(jh, n\Delta t)$. Let $\tilde{u}(x,0)$ be defined by

$$\tilde{u}(x,0) = u_j^n \quad \text{for } (j-1/2)h \le x < (j+1/2)h,$$

for $j = -N-1, \dots, N$, and where $u_{-N-1}^n = u_N^n$. Godunov's method is based on using $\tilde{u}(x,0)$ as initial data for the conservation law (13), and solving for a timestep Δt, and subsequently defining

$$u_j^{n+1} = \frac{1}{h} \int_{(j-1/2)h}^{(j+1/2)h} \tilde{u}(x, \Delta t)\, dx, \tag{39}$$

where $\tilde{u}(x, \Delta t)$ is the solution with initial data $\tilde{u}(x,0)$. Using the conservation law, we find that

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{h}\left(F\left(u_j^n, u_{j+1}^n\right) - F\left(u_{j-1}^n, u_j^n\right)\right),$$

provided $\Delta t \max f' < h$, and where the so-called numerical flux function F is defined by

$$F(a,b) = \begin{cases} \min_{a \le u \le b} f(u) & \text{if } a \le b, \\ \max_{b \le u \le a} f(u) & \text{if } a > b. \end{cases} \tag{40}$$

In the KdV case, where $f(u) = u^2/2$, this simplifies to

$$F(a,b) = \frac{1}{2}\begin{cases} 0 & \text{if } a \le 0 \le b, \\ \min\left[a^2, b^2\right] & \text{if } ab > 0 \text{ and } a < b, \\ \max\left[a^2, b^2\right] & \text{if } a > b. \end{cases}$$

We label this method "`Godunov`".

2.3.2. *ENO-schemes.* ENO (Essentially non-oscillatory) schemes are finite difference schemes based on interpolation of discrete data using polynomials. As long as the data are smooth inside the approximation stencil, the order of the method is high. To circumvent the problem of discontinuouities arising in the solution of conservation laws, a variable stencil is used.

If cell averages of a function $u(x)$ are given by

$$\bar{u}_j = \frac{1}{h} \int_{(j-1/2)h}^{(j+1/2)h} u(x)\, dx,$$

we find a polynomial $p_j(x)$ of degree at most $k-1$ such that it is a kth order accurate approximation of u inside the cell $I_j = [(j-1/2)h, (j+1/2)h]$,

$$p_j(x) = u(x) + \mathcal{O}\left(h^k\right) \quad \text{for } x \in I_j.$$

Let u_j^- and u_j^+ be defined as

$$u_j^{\pm} = p_j\left((j \pm 1/2)h\right).$$

If we choose a stencil based on r cells to the left, and s cells to the right, $r+s+1=k$, then there are constants c_{rj} and $\tilde{c}_{rj}$, depending only on r, s and k, such that

$$u_j^- = \sum_{i=0}^{k-1} c_{ri}\bar{u}_{j-r+i}, \qquad u_j^+ = \sum_{i=0}^{k-1} \tilde{c}_{ri}\bar{u}_{j-r+i}.$$

For the actual values of the constants c_{ri} and $\tilde{c}_{ri}$, see [28].

The basic idea of ENO methods is to avoid including discontinuouities in the stencil. Therefore, for a fixed j, all possible stencils are compared, and the one where with the "smoothest" data is used.

The conservation law is then approximated by a system of ordinary differential equations,

$$\begin{aligned} \frac{du_j}{dt} &= -\frac{1}{h}\left[F\left(u_{j+1}^+, u_{j+1}^-\right) - F\left(u_j^+, u_j^-\right)\right], \\ &=: L_j(u) \end{aligned} \tag{41}$$

where F is the numerical flux function. We have used a Godunov numerical flux (40). To integrate (41) we have used a third order TVD Runge-Kutta method,

$$\begin{aligned} v_j^1 &= u_j^n - \Delta t L_j\left(u^n\right) \\ v_j^2 &= \frac{3}{4}u_j^n + \frac{1}{4}\left[v_j^1 - \Delta t L_j\left(v^1\right)\right] \\ u_j^{n+1} &= \frac{1}{3}u_j^n + \frac{2}{3}\left[v_j^2 - \Delta t L_j\left(v^2\right)\right] \end{aligned} \tag{42}$$

for $j=-N,\dots,N$ and $n=0,1,2,\dots$. This method is TVD if $\Delta t/h<1$. The reason for choosing a third order method rather than a fourth or higher order method, is that there are no higher (>3) order methods that are TVD. Also good higher order methods are more complicated to implement and demand significantly more storage and computer time.

Since we have used a third order integration in time, we also use a third order ENO interpolation in space.

2.3.3. *The spectral viscosity method.* The conservation law is not a linear equation, and therefore not obviously suited for spectral methods. However, the spectral viscosity method, developed by Tadmor [29], has proven to be a good method for conservation laws.

Consider a periodic function u with period 1. Let P_N be defined as the truncated Fourier expansion, that is,

$$P_N u(x) = \sum_{k=-N}^{N} \hat{u}_k\, e^{2\pi i k x}. \tag{43}$$

Now, instead of trying to solve the conservation law, (13), we modify this as follows

$$\left[P_N u\right]_t + \left[P_N f\left(P_N u\right)\right]_x = \varepsilon_N\left(Q_N(x,t) * \left(P_N u\right)_x\right)_x, \tag{44}$$

where Q_N augments high frequency oscillations, i.e., $\widehat{Q_N}_k \approx 0$ for $|k| \lll N$ and $\widehat{Q_N}_k \approx 1$ for $|k|$ close to N. Furthermore, the numerical viscosity coefficient behaves like $\varepsilon_N = \mathcal{O}\left(1/\sqrt{N}\right)$. In Fourier space, (44) is a system of ordinary differential equations

$$\left(\hat{u}_k\right)_t + 2\pi i k\, \hat{f}_k = -\varepsilon_N (2\pi k)^2\, \widehat{Q_N}_k\, \hat{u}_k, \tag{45}$$

where $\hat{f} = \widehat{f\left(P_N u\right)}$. This system of equations can be solved by the Euler method, i.e.,

$$\hat{u}_k^{n+1} = \hat{u}_k^n - \Delta t\left(2\pi i k\, \hat{f}_k^n + \varepsilon_N (2\pi k)^2\, \hat{Q}_k\, \hat{u}_k\right),$$

for $n=0,1,\dots$ and $k=-N,\dots,N$. For brevity we write $\hat{Q}_k = \widehat{Q_N}_k$.

We found that this method to integrate (45) was not suitable, due to severe restrictions on the timestep Δt. Instead we integrated (45) by a Crank-Nicholson scheme, at the cost of having to solve a nonlinear system of equations;

$$\hat{u}_k^{n+1} = \left[1 - \varepsilon_N \hat{Q}_k \frac{(2\pi k)^2}{2} \Delta t\right]^{-1} \left\{ \left[1 + \varepsilon_N \hat{Q}_k \frac{(2\pi k)^2}{2} \Delta t\right] \hat{u}_k^n - \frac{\Delta t}{2} (2\pi i k) \left(\hat{f}_k^n + \hat{f}_k^{n+1}\right) \right\}, \tag{46}$$

at each timestep. This system can be solved by fixpoint iteration, we iterated until $\max_k \left(\hat{u}_k^n - \hat{u}_k^{n+1}\right)$ was less than Δt^2. Usually, this required two or three iterations when the initial guess was $\hat{u}^n$. For the parameters Q and ε_N we used

$$\varepsilon_N = \frac{1}{\sqrt{N}}, \qquad \hat{Q}_k = \begin{cases} 0 & \text{for } k \leq \sqrt{N}, \\ \frac{1}{2}\left(1 + \tanh\left(\frac{1}{2}\left(k - \sqrt{N}\right)\right)\right) & \text{otherwise.} \end{cases}$$

This method is labeled "`SpVi`".

2.3.4. *A comparison method.* For comparison with other tested methods for solving the KdV equation, we also implemented the implicit spectral method of Wineberg *et al.* [37]. This method is based the Fourier transform of (12)

$$\left(\hat{u}_k\right)_t + (2\pi i k) \hat{f}_k - \varepsilon i (2\pi k)^3 \hat{u}_k = 0. \tag{47}$$

Again, the numerical integration is done with a Crank-Nicholson method, iterating to find the solution of the system of equations:

$$\hat{u}_k^{n+1} = \left[1 - \frac{\Delta t}{2} \varepsilon i (2\pi k)^3\right]^{-1} \left\{ \left[1 + \frac{\Delta t}{2} \varepsilon i (2\pi k)^3\right] \hat{u}_k^n - \frac{\Delta t}{2} \left(\hat{f}_k^n + \hat{f}_k^{n+1}\right) \right\}. \tag{48}$$

In [37] it is shown that iterating twice gives a second order accurate scheme, consequently we also iterate twice. We label this method "`Spectral`".

Note that integrating (47) by splitting the linear term and the nonlinear term, is not good, since solving the nonlinear term amounts to using a spectral method for the conservation law *without* spectral viscosity, and this leads to spurious oscillations, which in turn create instabilities in the numerical solution of the linear part of (47).

3. Numerical results

In practice, we do not use the Godunov splitting (15), but rather the Strang splitting

$$u^{n+1} = \left[S_{\Delta t/2} \circ A_{\Delta t} \circ S_{\Delta t/2}\right] u^n, \tag{49}$$

where u^n denotes the numerical approximation at $t = n\Delta t$ and A and S are the numerical methods for the linear dispersive equation (14) and the conservation law (13), respectively.

In order to test our methods, we use exact one-soliton and two-soliton solutions for the KdV equation

$$u_t + \frac{1}{2} u_x^2 + \varepsilon u_{xxx} = 0. \tag{50}$$

The one-soliton solution is given by

$$u(x,t) = 3c \operatorname{sech}^2\left(\sqrt{\frac{c}{4\varepsilon}}\,(x - ct)\right). \tag{51}$$

We used $\varepsilon = 0.0013020833$ and $c = 1/3$.

The formula for the two-soliton is more involved

$$u(x,t) = 2\frac{k_1^2\, e^{\theta_1} + k_2^2\, e^{\theta_2} + 2\left(k_2 - k_1\right) e^{\theta_1+\theta_2} + a^2 \left(k_2^2\, e^{\theta_1} + k_1^2\, e^{\theta_2}\right) e^{\theta_1+\theta_2}}{\left(1 + e^{\theta_1} + e^{\theta_2} + a^2\, e^{\theta_1+\theta_2}\right)^2}, \tag{52}$$

where the constants are given by

$$k_1 = \frac{3}{2}, \qquad k_2 = 1, \qquad a^2 = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 = \frac{1}{25},$$

$$\theta_1 = k_1 \frac{x}{6\sqrt{\varepsilon}} - k_1^3 \frac{t}{6^{3/2}\sqrt{\varepsilon}} - 3,$$

$$\theta_2 = k_2 \frac{x}{6\sqrt{\varepsilon}} - k_2^3 \frac{t}{6^{3/2}\sqrt{\varepsilon}} + 3.$$

Experimenting with various combinations of the methods for the linear dispersive equation and the conservation law, we found that not all combinations were equally well suited. After extensive testing, we were left with the combinations: `Diff–Godunov`, `Diff–ENO`, `FFT–ENO` and `FFT-CN–SpVi` as the most stable and accurate. Below we show how these combinations perform on the one-soliton case, i.e., we use (51) with $t = 0$ as initial data. The figure shows the numerical results at $t = 2$, when the peak of the soliton has moved about 0.7 units. In these computations we used 64 gridpoints in the interval $[-1, 2]$ and periodic boundary data. In the figure we see that the

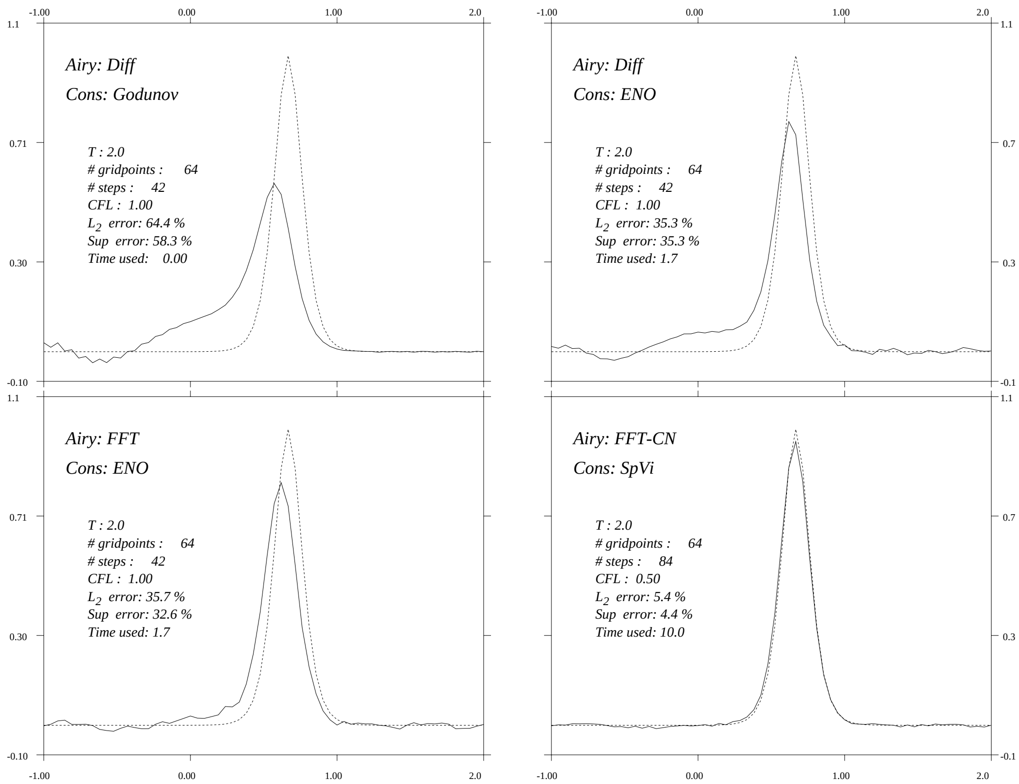

FIGURE 1. Numerical solution of the one-soliton case, the dashed line indicates the exact solution.

combinations seem to rank from `Diff–Godunov` to `FFT-CN–SpVi` in terms of accuracy.

This is also what we found when measuring how the methods compared in terms of CPU-time versus error. The errors reported here are relative percentage errors, i.e.,

$$\text{error} = 100 \frac{\|u_{\text{approx}} - u_{\text{exact}}\|}{\|u_{\text{exact}}\|}$$

where $\|\cdot\|$ is the L_∞ norm or the L_2 norm. In the next figure we show the logarithms of CPU-time [3] and L_2 error respectively for the four combinations and for the spectral method (`Spectral`).

[3] The CPU-time is $100 \cdot$ CPU-time for the calculations, *not* the initial memory allocations and setup of the initial data. The computations were carried out on a Power Macintosh G3/266.

TABLE 1. Convergence rates in Δx

	Rate	χ^2
`Spectral`	0.76	0.367
`Diff-Godunov`	0.52	0.0001
`Diff-ENO`	1.05	0.077
`FFT-ENO`	0.96	0.003
`FFT-CN-SpVi`	0.89	0.024

Again we use the one-soliton case (51), but we have used $2^5 - 2^{11}$ gridpoints. This figure more or

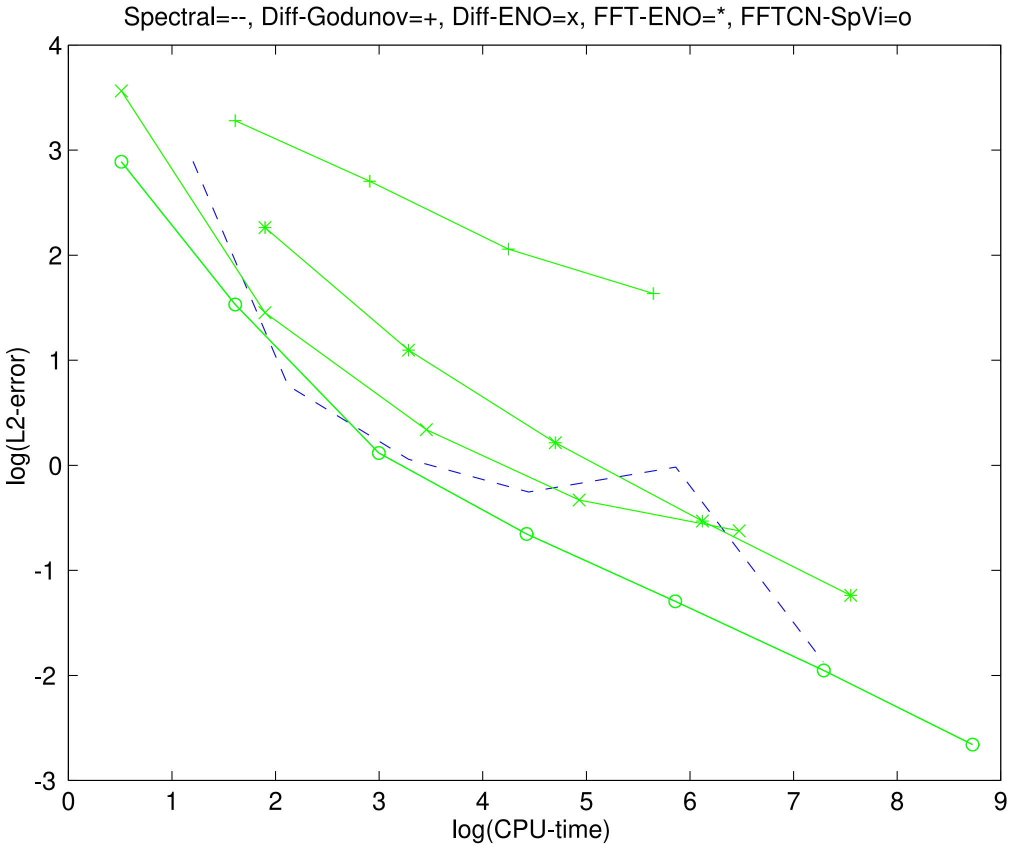

FIGURE 2. L_2 error vs. CPU-time

less confirms the ranking of the methods from the previous example. The methods `Diff–ENO` and `FFT-ENO` seem to perform slightly worse than `Spectral`, and `FFT-CN–SpVi` slightly better. The method `Diff–Godunov`, although it seems to converge, uses significantly more time to reach the same error. Using 2^{10} gridpoints, it has roughly the same error as the other methods using 2^7 gridpoints.

Regarding convergence rates in Δx, by using linear regression on the above example, we obtained Table 1. The first column gives the numercal convergence rates, and the second is an estimate on how well the variance in the data is explained by the linear model. A low value for χ^2 indicates good linear fit. The values indicate that `Diff-Godunov` has a convergence rate of $1/2$, while all the other methods seem to converge at a rate of 1. Remarkably, this also holds for the second order spectral method `Spectral`.

We also remark that the L_2 errors are extremely well correlated with the supremum errors. Hence, using supremum errors would give virtually identical results.

As a more complicated case, we use the two-soliton solution (52). The next figure shows how the four splitting methods resolved an interaction between these two solitons. Again `Diff–Godunov` ranks poorest, we see that the the solitons appear somewhat smeared after the interaction, while the three other methods are similar. We used 2^7 gridpoints in the x interval $[-0.5, 3.0]$ and CFL number 1 for all methods but `FFT-CN–SpVi`, where we used CFL number 0.5. This gives 244 timesteps for CFL$= 1$ and 488 for CFL$= 0.5$. The computations were calculated up to $t = 6.0$, well

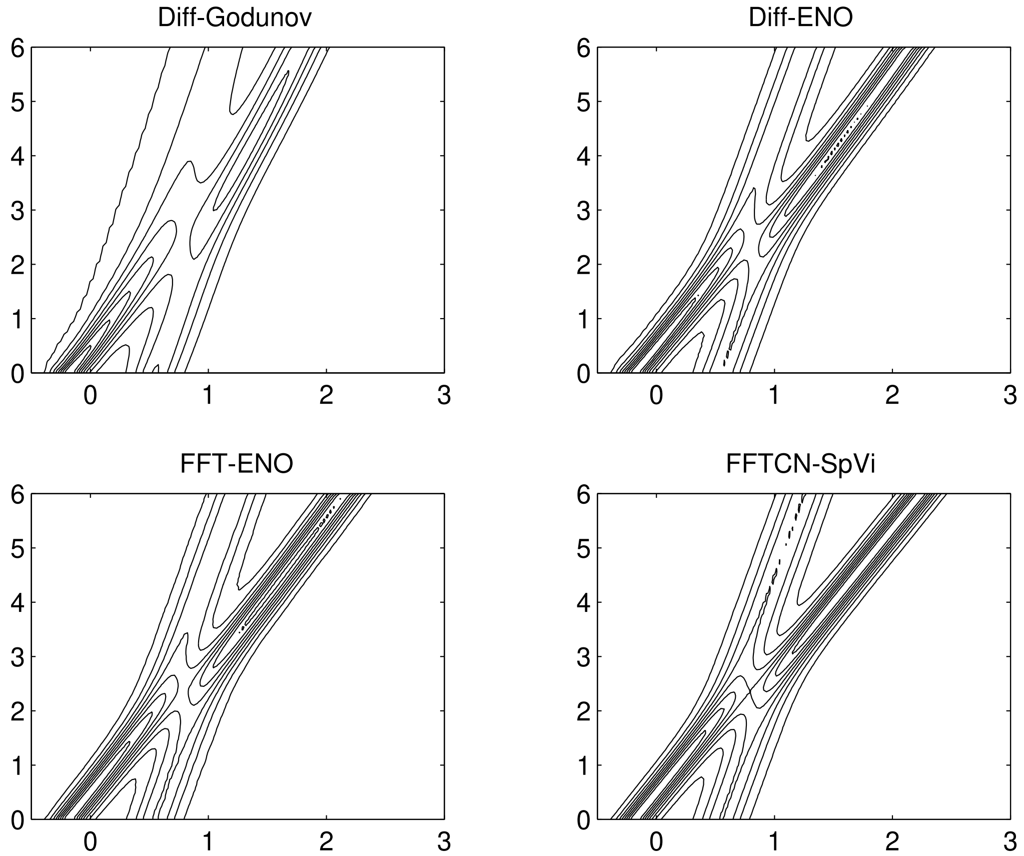

FIGURE 3. A two-soliton interaction

past the interaction.

In the final figure we show the supremum error for these four methods as a function of x and t. In this connection we remark that we can see that the main source of error in `Diff-Godunov` is that the location of the solitons is wrong after the interaction.

4. Conclusions

We have the applied the method of operator splitting to the generalized KdV equation

$$u_t + f(u)_x + \varepsilon u_{xxx} = 0, \quad u(x,0) = u_0(x),$$

by sequentially solving the hyperbolic conservation law,

$$u_t + f(u)_x = 0, \quad u(x,0) = u_0(x), \tag{53}$$

and the linear dispersive equation,

$$u_t + \varepsilon u_{xxx} = 0, \quad u(x,0) = u_0(x). \tag{54}$$

The two simpler equations have quite distict and incompatible properties, making convergence results hard. However, we have proved that if the approximation does converge, then the limit

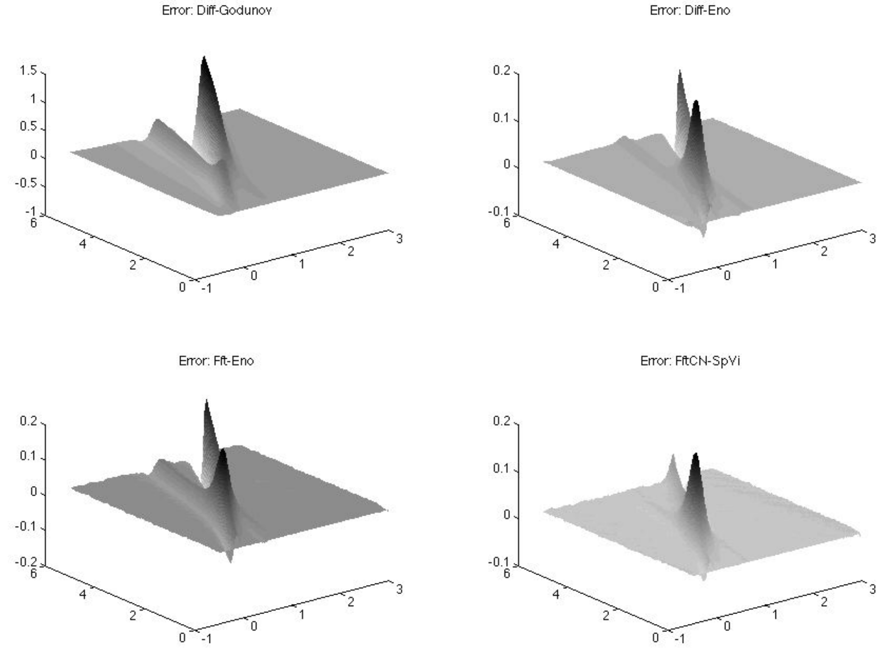

FIGURE 4. Supremum error as a function of x and t.

is a weak solution of the generalized KdV equation. Numerical computatations reveal, with the certainty such computations offer, that the method does indeed converge. The operator splitting method is easy to implement on a computer, and one can combine a variety of methods for each of the equations. We have tested the method numerically by applying selected numerical techniques to each of them. We find that all methods converge numerically, but at different rates. Numerical results are presented for one and two-soliton solutions of the KdV equations. For these and other examples we find that a combination of the fast Fourier transform with a Crank-Nicholsen scheme (for equation (54)) and spectral viscosity method (for equation (53)) is the most accurate.

REFERENCES

[1] M. Abramowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, New York, 1972.
[2] J. L. Bona, V. A. Dougalis, O. A. Karakashian, and W. McKinney. Conservative, high-order numerical schemes for the generalized Korteweg–de Vries equation. *Philos. Trans. Roy. Soc. London Ser. A*, 351:107–164, 1995.
[3] W. Craig, T. Kappeler, and W. Strauss. Gain of regularity for equations of KdV type. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 9:147–186, 1992.
[4] P. G. Drazin and R. S. Johnson. *Solitons: An Introduction*. Cambridge Univ. Press, Cambridge, 1989.
[5] S. Evje and K. Hvistendahl Karlsen. Viscous splitting approximation of mixed hyperbolic-parabolic convection-diffusion equations. *Num. Math.* To appear.
[6] B. Garcia-Archilla. A supraconvergent scheme for the Korteweg–de Vries equation. *Num. Math.*, 61:291–310, 1992.
[7] J. Ginibre and Y. Tsutsumi. Uniqueness of solutions for the generalized Korteweg–de Vries equation. *SIAM J. Math. Anal.*, 20:1388–1425, 1989.
[8] J. Ginibre, Y. Tsutsumi, and G. Velo. Uniqueness and uniqueness of solutions for the generalized Korteweg–de Vries equation. *Math. Z.*, 203:9–36, 1990.
[9] J. Goodman and P. D. Lax. On dispersive difference schemes. I. *Comm. Pure Appl. Math.*, 41:591–613, 1988.
[10] H. Holden and N. H. Risebro. Conservation laws with a random source. *Appl. Math. Optim.*, 36:229–241, 1997.
[11] T. Y. Hou and P. D. Lax. Dispersive approximation in fluid dynamics. *Comm. Pure Appl. Math.*, 44:1–40, 1991.
[12] K. Hvistendahl Karlsen and K.-A. Lie. An unconditionally stable splitting scheme for a class of nonlinear parabolic equations. *IMA Num. Anal.* submitted.
[13] K. Hvistendahl Karlsen and N. H. Risebro. Corrected operator splitting for nonlinear parabolic equations. *SIAM J. Numer. Anal.* To appear.

[14] K. Hvistendahl Karlsen and N. H. Risebro. An operator splitting method for convection-diffusion equations. *Num. Math.*, 77:365–382, 1997.

[15] T. Kato. On the Cauchy problem for the (generalized) Korteweg–de Vries equation. *Stud. Appl. Math.*, 8:93–128, 1983.

[16] C. Kenig, G. Ponce, and L. Vega. On the (generalized) Korteweg–de Vries equation. *Duke Math. J.*, 59:585–610, 1989.

[17] S. N. Kruzhkov and A. V. Faminskii. Generalized solutions of the Cauchy problem for the Korteweg–de Vries equation. *Math. USSR-Sb.*, 120:391–421, 1983.

[18] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg–de Vries equation. I. *Comm. Pure Appl. Math.*, 36:253–290, 1983.

[19] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg–de Vries equation. II. *Comm. Pure Appl. Math.*, 36:571–593, 1983.

[20] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg–de Vries equation. III. *Comm. Pure Appl. Math.*, 36:809–830, 1983.

[21] C. D. Levermore. The KdV zero-dispersion limit and densities of Dirichlet spectra. In *Recent Advances in Partial Differential Equations. Venice 1996*, pages 187–210. Amer. Math. Soc., Providence, RI, 1998.

[22] Yi. A. Li, P. J. Olver, and P. Rosenau. Non-analytic solutions of nonlinear wave models. Technical Report 1591, Institute of Mathematics and Applications, Minnesota, 1998.

[23] S. Novikov, S. V. Manakov, L. P. Pitaevskii, and V. E. Zakharov. *Theory of Solitons*. Consultants Bureau, New York, 1984.

[24] A. R. Osborne. Nonlinear Fourier analysis for the infinite-interval Korteweg–de Vries equation I: An algorithm for the direct scattering transform. *J. Comput. Phys.*, 94:284–313, 1991.

[25] A. R. Osborne and E. Segre. The numerical inverse scattering transform for the periodic Korteweg–de Vries equation. *Phys. Lett. A*, 173:131–142, 1993.

[26] A. Provenzale and A. R. Osborne. Nonlinear Fourier analysis for the infinite-interval Korteweg–de Vries equation II: Numerical tests of the direct scattering transform. *J. Comput. Phys.*, 94:314–351, 1991.

[27] A. B. Shamardan. Central difference schemes for nonlinear dispersive waves. *Comput. Math. Appl.*, 19:9–15, 1990.

[28] C.-W. Shu. Essentially non-oscillatory and weighted essentially non-oscillatory schemes for hyperbolic conservation laws. In A. Quarteroni, editor, *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, Lecture Notes in Mathematics, Berlin. Springer. to appear.

[29] E. Tadmor. Super-viscosity and spectral approximations of nonlinear conservation laws. In K. W. Morton and J. M. Baines, editors, *Numerical Methods for Fluid Dynamics IV*, pages 69–81. Clarendon Press, Oxford, 1993.

[30] T. R. Taha and M. J. Ablowitz. Analytical and numerical aspects of certain nonlinear evolution equations. I. Analytical. *J. Comput. Phys.*, 55:192–202, 1984.

[31] T. R. Taha and M. J. Ablowitz. Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, nonlinear Schrödinger equation. *J. Comput. Phys.*, 55:203–230, 1984.

[32] T. R. Taha and M. J. Ablowitz. Analytical and numerical aspects of certain nonlinear evolution equations. III. Numerical, nonlinear Korteweg–de Vries equation. *J. Comput. Phys.*, 55:231–253, 1984.

[33] F. Tappert. Numerical solutions of the Korteweg–de Vries equation and its generalizations by the split-step Fourier method. In A. C. Newell, editor, *Nonlinear Wave Motion*, volume 15 of *Lectures in Appl. Math.*, pages 215–216. Amer. Math. Soc., Providence, 1974.

[34] S. Venakides. The zero dispersion limit of the Korteweg– de Vries equation with periodic initial data. *Trans. Amer. Math. Soc.*, 301:189–226, 1987.

[35] S. Venakides. The Korteweg-de Vries equation with small dispersion: higher order Lax-Levermore theory. *Comm. Pure Appl. Math.*, 43:335–361, 1990.

[36] S. Venakides. The small dispersion KdV equation with decaying initial data. In *Recent Advances in Partial Differential Equations. Venice 1996*, pages 369–392. Amer. Math. Soc., Providence, RI, 1998.

[37] S. B. Wineberg, J. F. McGrath, E. F. Gamble, and C. E. Southwell. Implicit spectral methods for wave propagation problems. *J. Comput. Phys.*, 97:311–336, 1991.

[38] Yi Zhou. Uniqueness of weak solutions of the KdV equation. *Internat. Math. Res. Notices*, (6):271–283, 1997.

(Holden)
Department of Mathematical Sciences
Norwegian University of Science and Technology
N–7043 Trondheim, Norway
E-mail address: holden@math.ntnu.no
URL: http://www.math.ntnu.no/~holden/

(Karlsen)
Department of Mathematics
University of Bergen
Johs. Bruunsgt. 12
N–5007 Bergen, Norway
E-mail address: kennethk@math.uib.no
URL: http://www.mi.uib.no/~kennethk/

(Risebro)
Department of Mathematics
University of Oslo
P.O. Box 1053, Blindern
N–0316 Oslo, Norway
E-mail address: nilshr@math.uio.no
URL: http://www.math.uio.no/~nilshr/