Vanishing Viscosity Solutions of Hyperbolic Systems on Manifolds

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Abstract. The paper is concerned with the Cauchy problem for a nonlinear, strictly hyperbolic system with small viscosity:

$$u_t + A(u)u_x = \varepsilon\, u_{xx}, \qquad u(0,x) = \bar u(x). \tag{$*$}$$

We assume that the integral curves of the eigenvectors r_i of the matrix A are straight lines. On the other hand, do not require the system $(*)$ to be in conservation form, nor we make any assumption on genuine linearity or linear degeneracy of the characteristic fields.

In this setting we prove that, for some small constant $\eta_0 > 0$ the following holds. For every initial data $\bar u \in \mathbf{L}^1$ with Tot.Var.$\{\bar u\} < \eta_0$, the solution u^ε of (*) is well defined for all $t > 0$. The total variation of $u^\varepsilon(t,\cdot)$ satisfies a uniform bound, independent of t, ε. Moreover, as $\varepsilon \to 0+$, the solutions $u^\varepsilon(t,\cdot)$ converge to a unique limit $u(t,\cdot)$. The map $(t,\bar u) \mapsto S_t\bar u \doteq u(t,\cdot)$ is a Lipschitz continuous semigroup on a closed domain $\mathcal{D} \subset \mathbf{L}^1$ of functions with small total variation. This semigroup is generated by a particular Riemann Solver, which we explicitly determine.

The above results can also be applied to strictly hyperbolic systems on a Riemann manifold. Although these equations cannot be written in conservation form, we show that the Riemann structure uniquely determines a Lipschitz semigroup of "entropic" solutions, within a class of (possibly discontinuous) functions with small total variation. The semigroup trajectories can be obtained as the unique limits of solutions to a particular parabolic system, as the viscosity coefficient approaches zero.

The proofs rely on some new a priori estimates on the total variation of solutions for a parabolic system whose components drift with strictly different speeds.

1 - Introduction

Consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0. \tag{1.1}$$

For initial data with small total variation, the global existence of weak solutions was proved in [8]. Moreover, the uniqueness and stability of entropy admissible BV solutions was recently established in a series of papers [3,4,5,6]. A long standing open question is whether these discontinuous solutions can be obtained as vanishing viscosity limits. More precisely, given a smooth initial data $\bar{u} : \mathbb{R} \mapsto \mathbb{R}^n$ with small total variation, consider the parabolic Cauchy problem

$$u(0,x) = \bar{u}(x). \tag{1.2}$$

$$u_t + A(u)u_x = \varepsilon\, u_{xx}. \tag{1.3}$$

Here $A(u) \doteq Df(u)$ is the Jacobian matrix of f and $\varepsilon > 0$. It is then natural to expect that, as $\varepsilon \to 0$, the solution u^ε of (1.2)-(1.3) converges to the unique entropy weak solution u of (1.1)-(1.2). Unfortunately, no general theorem in this direction is yet known. Some of the main results available in the literature are listed below.

1) In the case of a scalar conservation law, the entropic solutions of (1.1) determine a semigroup which is contractive w.r.t. the $\mathbf{L}^1$-distance. In this case, a general convergence theorem for vanishing viscosity approximations was proved in the classical work of Kruzhkov [12].

2) For various 2×2 systems, if a uniform $\mathbf{L}^\infty$–bound on all functions u^ε is available, one can consider a weak limit $u^\varepsilon \rightharpoonup u$. By a compensated compactness argument introduced by DiPerna [7], it then follows that u is actually a weak solution of the nonlinear system (1.1). For a comprehensive discussion of the compensated compactness method and its applications to conservation laws, see [18].

3) For $n \times n$ Temple class systems, a proof of the convergence of the viscous solutions u^ε to a solution of (1.1) can be found in [17,18].

4) Assume that all characteristic fields of the system (1.1) are linearly degenerate. Then every solution with small total variation which is initially smooth remains smooth for all positive times [2]. Clearly such solution can be obtained as limit of vanishing viscosity approximations. By a density argument it follows that every weak solution of (1.1) with sufficiently small total variation is a limit of viscous approximations.

5) For a general $n \times n$ strictly hyperbolic system, let u be a piecewise smooth entropic solution of (1.1) with jumps along a finite number of smooth curves in the t-x plane. Thanks to this

additional regularity assumptions on u, it was proved in [10] that there exists a family of viscous solutions u^ε converging to u in $\mathbf{L}^1_{\rm loc}$ as $\varepsilon \to 0$.

From our point of view, the major difficulty toward a general proof of the convergence $u^\varepsilon \to u$ lies in deriving an priori estimate on the total variation of the solution of (1.2)-(1.3), uniformly valid as $\varepsilon \to 0$. To fix the ideas, assume $\bar u \in \mathcal{C}^\infty_c$ with Tot.Var.$(\bar u)$ sufficiently small. Performing the rescalings $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, the Cauchy problem becomes

$$u_t + A(u)u_x = u_{xx}, \tag{1.4}$$

$$u(0,x) = \bar u(\varepsilon x). \tag{1.5}$$

Observe that, as $\varepsilon \to 0$, the initial data $u(0,\cdot)$ has constant total variation, all of its derivatives approach zero, but its $\mathbf{L}^1$-norm approaches infinity. We thus need estimates on the total variation of a solution $u(t,\cdot)$ of (1.4) which are independent of the $\mathbf{L}^1$-norm of the initial data.

To illustrate the heart of the matter, let us denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the $n \times n$ Jacobian matrix $A(u) \doteq Df(u)$, and call $l^1,\ldots,l^n$, $r_1,\ldots,r_n$, its left and right eigenvectors, normalized so that

$$\big|r_i(u)\big| \equiv 1, \qquad\qquad l^i(u)\cdot r_j(u) = \begin{cases} 1 & \text{if} \quad i=j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.6}$$

The directional derivative of a function $\phi = \phi(u)$ in the direction of the eigenvector r_i is written

$$r_i \bullet \phi(u) \doteq \lim_{h\to 0} \frac{\phi\big(u + hr_i(u)\big) - \phi(u)}{h}.$$

Moreover, by $u^i_x \doteq l^i(u)\cdot u_x$ we denote the i-th component of the gradient u_x w.r.t. the basis of right eigenvectors $\{r_1,\ldots,r_n\}$. Recalling (1.6), this implies

$$u_x = \sum_i u^i_x r_i. \tag{1.7}$$

With the above notations, (1.4) becomes

$$u_t + \sum_i \lambda_i u^i_x r_i = \sum_i (u^i_x r_i)_x = \sum_i (u^i_x)_x r_i + \sum_{i,j} u^i_x u^j_x (r_j \bullet r_i). \tag{1.8}$$

Differentiating (1.7) w.r.t. t and (1.8) w.r.t. x and equating the results one obtains

$$\begin{aligned} u_{xt} = & \sum_i (u^i_x)_t r_i - \sum_{i,j} u^i_x u^j_x \lambda_j (r_j \bullet r_i) \\ & + \sum_{i,j} u^i_x (u^j_x)_x (r_j \bullet r_i) + \sum_{i,j,k} u^i_x u^j_x u^k_x \big((r_j \bullet r_k)\bullet r_i\big), \end{aligned}$$

$$
\begin{aligned}
u_{tx}+&\sum_i(\lambda_i u_x^i)_x r_i+\sum_{i,j}\lambda_i u_x^i u_x^j(r_j\bullet r_i)\\
&=\sum_i (u_x^i)_{xx}r_i+\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)+\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)\\
&\qquad+\sum_{i,j}u_x^i(u_x^j)_x(r_j\bullet r_i)+\sum_{i,j,k}u_x^i u_x^j u_x^k\big(r_k\bullet(r_j\bullet r_i)\big),
\end{aligned}
$$

$$
\begin{aligned}
\sum_i(u_x^i)_t r_i&+\sum_i(\lambda_i u_x^i)_x r_i+\sum_{j\neq k}\lambda_j u_x^j u_x^k[r_k,r_j]\\
&=\sum_i(u_x^i)_{xx}r_i+2\sum_{i,j}(u_x^i)_x u_x^j(r_j\bullet r_i)+\sum_{i,j,k}u_x^i u_x^j u_x^k\Big(r_k\bullet(r_j\bullet r_i)-(r_j\bullet r_k)\bullet r_i\Big).
\end{aligned}
\tag{1.9}
$$

Here $[r_k,r_j]\doteq r_k\bullet r_j-r_j\bullet r_k$ is the usual Lie bracket. Taking the inner product of (1.9) with $l^i(u)$ one obtains

$$
\begin{aligned}
&(u_x^i)_t+(\lambda_i u_x^i)_x-(u_x^i)_{xx}\\
&=l^i\cdot\left\{\sum_{j\neq k}\lambda_k[r_k,r_j]u_x^j u_x^k+2\sum_{j,k}(r_k\bullet r_j)(u_x^j)_x u_x^k+\sum_{j,k,\ell}\Big(r_\ell\bullet(r_k\bullet r_j)-(r_\ell\bullet r_k)\bullet r_j\Big)u_x^j u_x^k u_x^\ell\right\}\\
&=\sum_{j\neq k}G^i_{jk}(u)u_x^j u_x^k+\sum_{j,k}H^i_{jk}(u)(u_x^j)_x u_x^k+\sum_{j,k,\ell}K^i_{jk\ell}(u)u_x^j u_x^k u_x^\ell.
\end{aligned}
\tag{1.10}
$$

Setting $v^i\doteq u_x^i$, we thus seek an estimate on the $\mathbf{L}^1$-norm of solutions to

$$
\begin{aligned}
v_t^i+\big(\lambda_i(u)v^i\big)_x-v_{xx}^i&=\sum_{j\neq k}G^i_{jk}(u)v^j v^k+\sum_{j,k}H^i_{jk}(u)v_x^j v^k+\sum_{j,k,\ell}K^i_{jk\ell}(u)v^j v^k v^\ell\\
&\doteq\phi^i(u,v^1,\ldots,v^n).
\end{aligned}
\tag{1.11}
$$

We regard (1.11) as a parabolic system of n scalar equations, coupled through the terms G,H,K. These coupling terms can be split in two groups:

- *Transversal terms* involving at least two distinct components, such as $v^j v^k$, $v_x^j v^k$, $v^j v^k v^\ell$ with $j\neq k$,
- *Non-transversal terms* involving one single component, such as $v^j v_x^j$, $v^j v^j v^j$.

In the present paper we perform a careful study of transversal terms, and show that their total contribution is of quadratic order. Hence, for small initial data, they cannot produce a substantial amplification of the solution of (1.11). As a consequence, if the geometry of the system is such that the diffusion operator yields only transversal terms, then the total variation of solutions to (1.3) remains uniformly bounded as $\varepsilon\to 0$. This is the case of systems, not necessarily of Temple class or not even in conservation form, where the integral curves of the eigenvectors r_i are straight lines, namely:

$$
r_i\bullet r_i(u)=0 \qquad\qquad \text{for all } i,u. \tag{1.12}
$$

We conjecture that uniform bounds on the total variation remain valid also in the presence of non-transversal terms. The analysis of these terms however seems to require substantially different techniques, and will be taken up elsewhere.

Our main results can be stated as follows. We first consider the Cauchy problem for the parabolic system (1.4) Here $u \mapsto A(u)$ is a smooth map defined on an open set $\Omega \subseteq \mathbb{R}^n$, with values in the set of $n \times n$ matrices. We assume that each $A(u)$ is *strictly hyperbolic*, i.e. it has n real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. One can thus choose bases of right and left eigenvectors of $A(u)$, normalized as in (1.6). For a BV function $\bar{u} : \mathbb{R} \mapsto \mathbb{R}^n$ we write $\bar{u}(-\infty) \doteq \lim_{x\to-\infty} \bar{u}(x)$.

Theorem 1. *Assume that all matrices $A(u)$ in (1.4) are strictly hyperbolic and that their normalized eigenvectors r_i satisfy (1.12). Then, for every compact set $K_0 \subset \Omega$ there exist constants $0 < \delta_0 < \delta_1$ so that the following holds. For every initial data $\bar{u}$ such that*

$$\text{Tot.Var.}\{\bar{u}\} \leq \delta_0, \qquad\qquad \bar{u}(-\infty) \in K_0, \tag{1.13}$$

the Cauchy problem (1.2), (1.4) has a unique, global solution u, which satisfies

$$\text{Tot.Var.}\big\{u(t,\cdot)\big\} \leq \delta_1 \qquad\qquad \textit{for all } t \geq 0. \tag{1.14}$$

Moreover there exist constants L, L' such that, for every $\bar{u}, \bar{w}$ satisfying (1.13), the corresponding solutions satisfy

$$\big\|u(t) - w(s)\big\|_{\mathbf{L}^1} \leq L'\big(|t-s|^{1/2} + |t-s|\big) + L\,\big\|\bar{u} - \bar{w}\big\|_{\mathbf{L}^1}, \qquad \text{for all } t, s \geq 0. \tag{1.15}$$

We remark that the bound (1.14) depends only on the total variation of the initial data $\bar{u}$, not on its $\mathbf{L}^1$-norm. The above result thus yields an a priori bound on the total variation of solutions of (1.2)-(1.3), independent of the parameter ε. Our second main result shows that, as $\varepsilon \to 0+$, these solutions u^ε converge to a unique limit u, depending continuously on the initial data $\bar{u}$. Namely, the map $(t, \bar{u}) \mapsto S_t\bar{u} \doteq u(t,\cdot)$ is a Lipschitz continuous semigroup on a closed domain $\mathcal{D}$ of functions with small total variation.

By the analysis in [3], a Lipschitz semigroup S on a domain of BV functions is uniquely determined by its local behavior on piecewise constant initial data. In other words, if we assign a procedure for solving each Riemann problem, then the entire semigroup is completely determined. In the present case, our semigroup of "vanishing viscosity solutions" is generated by the following

Riemann Solver. Consider the Riemann problem

$$u_t + A(u)u_x = 0, \tag{1.16}$$

$$u(0,x) = \begin{cases} u^+ & \text{if} \quad x > 0, \\ u^- & \text{if} \quad x < 0, \end{cases} \tag{1.17}$$

with $|u^+ - u^-|$ suitably small. For $u \in \Omega$, $i = 1, \ldots, n$, define the i-rarefaction curve $\sigma \mapsto R_i(\sigma)(u)$ as the integral curve of r_i through u, parametrized so that

$$R_i(0)(u) = u, \qquad \frac{d}{d\sigma} R_i(\sigma)(u) = r_i\big(R_i(\sigma)(u)\big). \tag{1.18}$$

By the implicit function theorem there exist unique states $\omega_0 = u^-, \omega_1, \ldots, \omega_n = u^+$ and wave sizes σ_i such that

$$\omega_i = R_i(\sigma_i)(\omega_{i-1}) \qquad i = 1, \ldots, n. \tag{1.19}$$

Moreover, by strict hyperbolicity, there exist constants $\bar\lambda_1 < \ldots < \bar\lambda_{n-1}$ such that

$$\lambda_i\big(R_i(\theta\sigma_i)(\omega_{i-1})\big) \in]\bar\lambda_{i-1},\ \bar\lambda_i[\qquad \theta \in [0,1], \quad i = 1, \ldots, n.$$

with the convention $\bar\lambda_0 \doteq -\infty$, $\bar\lambda_n \doteq \infty$. For each i, consider the scalar function

$$F_i(\sigma) \doteq \int_0^\sigma \lambda_i\big(R_i(s)(\omega_{i-1})\big)\ ds, \tag{1.20}$$

and let $z_i(t,x)$ be the unique entropic solution to the Riemann problem for the scalar conservation law

$$z_t + F_i(z)_x = 0, \qquad z(0,x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \sigma_i & \text{if} \quad x > 0. \end{cases} \tag{1.21}$$

A "solution" of the Riemann problem (1.16)-(1.17) is now defined by the assignment

$$u(t,x) \doteq R_i\big(z_i(t,x)\big)(\omega_{i-1}) \qquad \text{if} \quad \frac{x}{t} \in [\bar\lambda_{i-1},\ \bar\lambda_i]. \tag{1.22}$$

We remark that, in general, the function in (1.22) is not a classical solution of (1.16). If the system is not in conservation form, this function cannot even be regarded as a solution in distributional sense. Yet, motivated by our next result it is appropriate to regard (1.22) as the unique solution of (1.16)-(1.17) in the vanishing viscosity sense.

Theorem 2. *Assume that all matrices $A(u)$ in (1.3) are strictly hyperbolic and that their eigenvectors satisfy (1.12). Then, for every compact set $K_0 \subset \Omega$ there exist constants $L, L', \delta_0 > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}^1_{\text{loc}}$ and a continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$ with the following properties.*

(i) Every function $\bar u$ satisfying (1.13) lies in the domain $\mathcal{D}$ of the semigroup.

(ii) For every $\bar u, \bar w \in \mathcal{D}$ with $\bar u - \bar w \in \mathbf{L}^1$ and every $t, s \geq 0$ one has

$$\|S_t \bar u - S_s \bar w\|_{\mathbf{L}^1} \leq L'|t-s| + L\|\bar u - \bar w\|_{\mathbf{L}^1}\,. \tag{1.23}$$

(iii) For every piecewise constant initial data $\bar u \in \mathcal{D}$, there exists $\tau > 0$ such that the following holds. For $t \in [0, \tau]$, the semigroup trajectory $S_t\bar u$ coincides with the function $u(t,\cdot)$ obtained by piecing together the solutions (1.22) of the Riemann problems determined by the jumps in $\bar u$.

(iv) For every $\bar u \in \mathcal{D}$, the trajectory $u(t,\cdot) = S_t\bar u$ is the unique limit in $\mathbf{L}^1_{\mathrm{loc}}$ of the corresponding solutions $u^\varepsilon(t,\cdot)$ of the viscous Cauchy problem (1.2)-(1.3), as $\varepsilon \to 0+$.

The proofs of the two above theorems are outlined in Section 2. Details are then worked out in Sections 3 to 6. The last section is concerned with a strictly hyperbolic system on a Riemann manifold. By a minor modification of the previous arguments, we show that the Riemann structure uniquely determines a viscosity operator and a continuous semigroup, whose trajectories are obtained as limits of solutions to the corresponding parabolic system, as the viscosity coefficient tends to zero.

2 - Outline of the proofs

In this section we describe the main steps in the proofs of Theorems 1 and 2. The total variation of a solution u will be estimated by deriving an a priori bound on the $\mathbf{L}^1$-norms of the gradient components $v^i \doteq l^i(u)\cdot u_x$, $i = 1,\ldots,n$. Denoting by ϕ^i the right hand side of (1.11), the Cauchy problem (1.4), (1.2) can be rewritten as

$$\begin{cases} v^i_t + [\lambda_i(u)v^i]_x - v^i_{xx} = \phi^i \\ v^i(0,x) = \bar v^i(x) \doteq l^i\big(\bar u(x)\big)\cdot \bar u_x(x), \end{cases} \tag{2.1}$$

where, for each t, the function $u = u(t,x)$ can be recovered from $v = (v^1,\ldots,v^n)$ by solving the system of O.D.E's

$$\begin{cases} u_x = \displaystyle\sum_{i=1}^n v^i(t,x)r_i(u), \\ \displaystyle\lim_{x\to-\infty} u(t,x) = \bar u(-\infty). \end{cases} \tag{2.2}$$

The solution of (2.1)-(2.2) will be obtained as the limit of a sequence of functions $v^{i,p}$, u^p, $p = 0,1,2,\ldots$ defined inductively as follows. We start by setting

$$u^0(t,x) \doteq \bar u(x), \qquad\qquad v^{i,0}(t,x) \doteq l^i\big(\bar u(x)\big)\cdot \bar u_x(x) \qquad\qquad \text{for all } t \geq 0. \tag{2.3}$$

Having defined the functions $v^{i,p}$, $i = 1,\ldots,n$, we then define u^p, $v^{i,p+1}$ to be the solutions of

$$\begin{cases} u^p_x = \displaystyle\sum_{i=1}^n v^{i,p}(t,x)\ r_i(u^p), \\ \displaystyle\lim_{t\to-\infty} u^p(t,x) = \bar u(-\infty), \end{cases} \tag{2.4}$$

$$\begin{cases} v_t^{i,p+1} + \left[\lambda_i(u^p) v^{i,p+1}\right]_x - v_{xx}^{i,p+1} = \phi^{i,p}(t,x), \\ v^{i,p+1}(0,x) = \bar v^i(x). \end{cases} \tag{2.5}$$

Here $\phi^{i,0} \doteq 0$ while, for $p \geq 1$, we set $\phi^{i,p} \doteq \phi^i(u^p, v^{1,p}, \ldots, v^{n,p})$. Since the left hand side of the evolution equation in (2.5) is in conservation form, one clearly has

$$\left\| v^{i,p+1}(t,\cdot)\right\|_{\mathbf{L}^1} \leq \left\|\bar v^i\right\|_{\mathbf{L}^1} + \int_0^t \int_{I\!R} \left|\phi^{i,p}(t,x)\right| \, dxdt. \tag{2.6}$$

Therefore, if the total strength of the source term $\phi^{i,p}$ is bounded, the solution of (2.5) is well defined for all $t \geq 0$. The main goal of this section is to prove an a priori bound on the terms

$$\left\|\phi^{i,p}\right\|_{\mathbf{L}^1([0,\infty[\times I\!R)} = \int_0^\infty \int_{I\!R} \left|\phi^{i,p}(t,x)\right| \, dxdt,$$

uniformly valid for all $i = 1, \ldots, n$ and $p \geq 1$.

We start by choosing constants $c, \delta_1 > 0$ small enough so that the compact set

$$K_1 \doteq \left\{ u \in I\!R^n; \quad dist(u, K_0) \leq \delta_1 \right\}$$

is entirely contained inside Ω, and moreover

$$\lambda_j(u) - \lambda_i(v) \geq c \qquad \text{whenever } i < j, \quad u, v \in K_1, \ |u - v| \leq \delta_1. \tag{2.7}$$

We also choose constants C_0, $\widehat C$ such that

$$\left| r_j \bullet \lambda_i(u)\right| \leq C_0, \qquad \left|G_{j,k}^i(u)\right|, \ \left|H_{j,k}^i(u)\right|, \ \left|K_{j,k,\ell}^i(u)\right| \leq C_0 \qquad \text{for all } u \in K_1 \tag{2.8}$$

$$\widehat C \geq \max\left\{ 2(n+1)^2 C_0, \ \frac{4}{\sqrt{\pi}} \right\}. \tag{2.9}$$

In the first part of our analysis we shall assume that, for all $i = 1, \ldots, n$, the initial data $\bar v^i$ in (2.1) satisfies

$$\int_{I\!R} \left|\bar v^i(t,x)\right| dx \leq \eta_0, \tag{2.10}$$

$$\int_{I\!R} \left|\bar v_x^i(x)\right| dx \leq \frac{4}{\sqrt{\pi}} \widehat C^2 \eta_0^2, \tag{2.11}$$

for some $\eta_0 > 0$. We choose the constant η_0 small enough so that

$$\eta_0 \leq \min\left\{ \frac{c}{4\widehat C}, \ \frac{\delta_1}{2n} \right\}. \tag{2.12}$$

At a later stage, using standard smoothing properties of parabolic equations, we will remove the assumption (2.11).

Bounds on the norms $\|\phi^{i,p}\|_{\mathbf{L}^1}$ will be proved by induction on p. Define the increasing sequence $\{\eta_p\}_{p\geq 0}$ inductively by

$$\eta_{p+1} = \eta_0 + \frac{\widehat{C}}{c}\eta_p^2. \tag{2.13}$$

By (2.12), the above sequence remains uniformly bounded. Indeed,

$$\eta_p \leq 2\eta_0 \qquad\qquad \text{for all } p \geq 0. \tag{2.14}$$

By induction on p, we will show that the $\mathbf{L}^1$-norm of each $v^{i,p}(t,\cdot)$ is bounded by η_p, i.e.

$$\int_{\mathbb{R}} \big|v^{i,p}(t,x)\big|\,dx \leq \eta_p \qquad\qquad \text{for all } t\geq 0. \tag{2.15}$$

Moreover, we will prove the estimates

$$\int_{\mathbb{R}} \big|\phi^{i,p}(t,x)\big|\,dx \leq \frac{8}{\sqrt{\pi}}\widehat{C}^3\eta_p^3 \qquad\qquad t\geq 0, \tag{2.16}$$

$$\int_0^\infty\int_{\mathbb{R}} \big|\phi^{i,p}(t,x)\big|\,dxdt \leq \frac{\widehat{C}}{c}\eta_p^2, \tag{2.17}$$

for every $p \geq 1$.

A detailed proof of (2.16)-(2.17) will be worked out in Section 3, providing a priori bounds on the $\mathbf{L}^1$ norms of the terms $v^j v^k$, $v^j_x v^k$, $v^j v^k v^\ell$ in (1.11), for $j\neq k$. From (2.6), using (2.17) and (2.10) we conclude

$$\begin{aligned}
&\int_{\mathbb{R}} \big|v^{i,p+1}(t,x)\big|\,dx \leq \eta_0 + \frac{\widehat{C}}{c}\eta_p^2 = \eta_{p+1} \leq 2\eta_0,\\
&\int_{\mathbb{R}} \big|v^{i,p+1}(t,x)\big|\,dx \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2 \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2\big(2\eta_0\big)^2.
\end{aligned} \tag{2.18}$$

Recalling that the eigenvectors r_i were chosen with unit length, by (2.4) and the choice of η_0 this yields

$$\begin{aligned}
\big|u^{p+1}(t,x) - \bar{u}(-\infty)\big| &\leq \text{Tot.Var.}\big\{u^{p+1}(t,\cdot)\big\}\\
&\leq \sum_{i=1}^n \big\|v^{i,p+1}(t,x)\big\|_{\mathbf{L}^1} \leq n\cdot 2\eta_0 \leq \delta_1.
\end{aligned} \tag{2.19}$$

Hence all functions u^p take values inside the compact set K_1, and the inductive construction can be iterated for all $p\geq 1$.

In Section 4 we show that, as $p\to\infty$, the sequences $v^{i,p}$, u^p converge to functions v^i, u which provide a solution to (2.1), (2.2), respectively. The bounds (2.19) thus yield an a uniform bound on the total variation of $u(t,\cdot)$. Indeed, for every $t\geq 0$ one has

$$\text{Tot.Var.}\big\{u(t)\big\} \;\leq\; \limsup_{p\to\infty}\Big(\text{Tot.Var.}\big\{u^p(t)\big\}\Big) \;\leq\; \delta_1. \tag{2.20}$$

This establishes global BV bounds on every solution u of (1.2), (1.4) whose initial data lies in the domain

$$\begin{aligned} \mathcal{D}^* \doteq \Big\{ &\bar u : I\!\!R \mapsto I\!\!R^n \ ; \quad u \text{ is absolutely continuous}, \quad \bar u(-\infty) \in K_0, \\ &\big\| l_i(\bar u)\cdot \bar u_x \big\|_{\mathbf{L}^1} \leq \eta_0, \quad \big\| (l_i(\bar u)\cdot \bar u_x)_x \big\|_{\mathbf{L}^1} \leq \frac{4}{\sqrt{\pi}} \widehat C^2 \eta_0^2, \quad i = 1,\ldots,n \Big\}. \end{aligned} \tag{2.21}$$

We then observe that, by the smoothing properties of the parabolic system (1.4), one can choose $t_0, \delta_0 > 0$ so that the following holds. For every initial data $\bar u$ satisfying (1.13), the corresponding solution of (1.2), (1.4) satisfies

$$u(t_0) \in \mathcal{D}^*, \qquad \text{Tot.Var.}\big\{u(t)\big\} \leq \delta_1 \qquad \text{for all } t \in [0, t_0]. \tag{2.22}$$

The previous analysis can thus be applied to the corresponding Cauchy problem on $[t_0, \infty[$. This yields the bounds (1.14), proving the first part of Theorem 1.

Concerning the estimates (1.15), the continuity of a BV solution as a function of time is a well known result for parabolic equations. To prove the Lipschitz continuous dependence w.r.t. the initial data, in Section 5 we first study the linearized equation describing the evolution of an infinitesimal perturbation. Replacing u by $u + \epsilon h$ in (1.4), letting $\epsilon \to 0$ and retaining terms of order ϵ we obtain

$$h_t + \big[DA(u)\cdot h\big] u_x + A(u) h_x = h_{xx}, \tag{2.23}$$

If the total variation of the reference solution u remains small, we show that every solution $h = h(t,x)$ of the linearized system (2.23) satisfies

$$\int_{I\!\!R} \big|h(t,x)\big|\, dx \leq L \cdot \int_{I\!\!R} \big|h(0,x)\big|\, dx \qquad \text{for all } t \geq 0, \tag{2.24}$$

for some uniform constant L. If now two initial data $\bar u, \bar w$ are given, following [2] we construct the smooth path

$$\theta \mapsto \bar u^\theta \doteq \theta \bar u + (1-\theta)\bar w, \qquad\qquad \theta \in [0,1]. \tag{2.25}$$

Calling $t \mapsto u^\theta(t,\cdot)$ the solution of (1.4) with initial data $\bar u^\theta$, we can write

$$\begin{aligned} \big\|u(t) - w(t)\big\|_{\mathbf{L}^1} &\leq \int_0^1 \left\| \frac{du^\theta(t)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\ &\leq L \cdot \int_0^1 \left\| \frac{du^\theta(0)}{d\theta} \right\|_{\mathbf{L}^1} d\theta \\ &= L \cdot \|\bar u - \bar w\|_{\mathbf{L}^1}. \end{aligned} \tag{2.26}$$

Indeed, the tangent vector

$$h^\theta(t,x) \doteq \frac{du^\theta}{d\theta}(t,x)$$

is a solution of the linearized Cauchy problem

$$h^\theta_t + \big[DA(u^\theta)\cdot h^\theta\big]u^\theta_x + A(u^\theta)h^\theta_x = h^\theta_{xx}, \tag{2.27}$$

$$h^\theta(0,x) = \bar h^\theta(x) = \bar u(x) - \bar w(x), \tag{2.28}$$

hence it satisfies (2.24) for every θ. This completes the proof of Theorem 1.

We now give a proof of Theorem 2. For a fixed $\varepsilon > 0$, thanks to the coordinate rescaling $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, the solution u^ε of (1.2)-(1.3) can be written in the form

$$u^\varepsilon(t,x) = U^\varepsilon(t/\varepsilon,\ x/\varepsilon),$$

where U^ε is the solution of (1.4)-(1.5). Clearly, for all $t \geq 0$ one has

$$\text{Tot.Var.}\big\{u^\varepsilon(t,\cdot)\big\} = \text{Tot.Var.}\big\{U^\varepsilon(t,\cdot)\big\}.$$

Therefore, by Theorem 1, for each initial condition $\bar u$ in the set

$$\mathcal{D}_0 \doteq \Big\{\bar u \in \mathbf{L}^1_{\rm loc}\,;\quad \text{Tot.Var.}\{\bar u\} \leq \delta_0, \quad \bar u(-\infty) \in K_0\Big\} \tag{2.29}$$

the corresponding solution of (1.3) satisfies (1.14). Moreover, given two initial data $\bar u, \bar w$, from (1.15) we deduce

$$\big\|u(t) - w(s)\big\|_{\mathbf{L}^1} \leq L'\big(\sqrt{\varepsilon|t-s|} + |t-s|\big) + L\,\big\|\bar u - \bar w\big\|_{\mathbf{L}^1} \qquad \text{for all } t,s \geq 0. \tag{2.30}$$

For each $\bar u \in \mathcal{D}_0$ we can thus use Helly's compactness theorem and deduce the existence of a subsequence $\varepsilon_\nu \to 0$ such that the corresponding solutions u^{ε_ν} converge to some function u, namely

$$\lim_{\nu\to\infty} u^{\varepsilon_\nu}(t) = u(t) \qquad \text{in } \mathbf{L}^1_{\rm loc}\,, \quad \text{for all } t \geq 0. \tag{2.31}$$

By a diagonalization argument, we can assume that, with the same sequence ε_ν, the convergence (2.31) holds for all solutions starting from a countable dense set of initial data in $\mathcal{D}_0$. In order to characterize this limit and show that the whole sequence u^ε converges as $\varepsilon \to 0$, we first derive a bound on the propagation speed of perturbations. By studying again the linear variational equation (2.23), in Section 6 we show that, for any interval $[a,b]$ and any two initial data $\bar u, \bar w$, the corresponding vanishing viscosity limit solutions satisfy

$$\int_{a+\tilde\lambda t}^{b-\tilde\lambda t} \big|u(t,x) - w(t,x)\big|\,dx \leq L\cdot\int_a^b \big|\bar u(x) - \bar w(x)\big|\,dx\,. \tag{2.32}$$

Here $\tilde\lambda$ is a suitably large constant providing an upper bound for all wave speeds, so that $\big|\lambda_i(\omega)\big| \leq \tilde\lambda$ for all $\omega \in K_1$, $i = 1, \ldots, n$. We can now define

$$\lim_{\nu\to\infty} u^{\varepsilon_\nu}(t,\cdot) = u(t,\cdot) \doteq S_t\bar u \qquad\qquad \bar u \in \mathcal{D}_0. \tag{2.33}$$

Indeed, since the limit (2.31) exists for a dense set of initial data $\bar u$, the uniform continuity property (2.32) implies the existence of the limit in (2.33) for all initial data in $\mathcal{D}_0$. By restricting this definition to a smaller, positively invariant domain $\mathcal{D} \subset \mathcal{D}_0$ we obtain a continuous semigroup $S : \mathcal{D} \times [0,\infty[\mapsto \mathcal{D}$. In the case where $\|\bar u - \bar w\|_{\mathbf{L}^1} < \infty$, from (2.30) and (2.32) it follows (1.23).

It remains to show that the flow of S is compatible with the Riemann Solver defined in Section 1. By the property (2.32), it suffices to consider the case where the initial data has a single jump as in (1.17), and show that the limit of viscous approximations converge to the solution defined at (1.22).

Consider first the special case where the states u^-, u^+ lie on a single i-rarefaction curve, say

$$u^+ = u^- + \sigma_i r_i(u^-).$$

In this case, the intermediate states in (1.19) are

$$\omega_0 = \cdots = \omega_{i-1} = u^-, \qquad \omega_i = \cdots = \omega_n = u^+.$$

Define the scalar function F_i as in (1.20). By the standard theory of scalar conservation laws [12], it is well known that, as $\varepsilon \to 0$, the solution v^ε of the viscous Cauchy problem

$$z_t^\varepsilon + F_i(z^\varepsilon)_x = \varepsilon\, z_{xx}^\varepsilon, \qquad\qquad v^\varepsilon(0,x) = \begin{cases} 0 & \text{if} \quad x < 0, \\ \sigma_i & \text{if} \quad x > 0. \end{cases} \tag{2.34}$$

converges to the unique entropic solution of (1.21). Since, by assumptions, all rarefaction curves are straight lines, the solution of (1.3) with initial data (1.17) is given by

$$u^\varepsilon(t,x) = u^- + z^\varepsilon(t,x) r_i(u^-),$$

and thus converges to the function u in (1.22), as $\varepsilon \to 0$.

To handle the general case, we consider a perturbed initial data of the form

$$\bar u^\delta(x) \doteq \begin{cases} u^- & \text{if} \quad x < \delta, \\ \omega_i & \text{if} \quad i\delta < x < (i+1)\delta,\ i = 1, \ldots, n-1, \\ u^+ & \text{if} \quad x > n\delta. \end{cases} \tag{2.35}$$

Keeping $\delta > 0$ fixed and letting $\varepsilon \to 0$, by the property (2.32) and the analysis of the previous case, the corresponding viscous solutions $u^{\delta,\varepsilon}$ converge to the function

$$u^\delta(t,x) \doteq R_i\big(z_i(t,\ x - i\delta)\big)(\omega_{i-1}) \qquad \text{if} \ \ i\delta + t\bar\lambda_{i-1} \ \leq\ x\ \leq\ (i+1)\delta + t\bar\lambda_i,$$

with $z_i, \bar{\lambda}_i$ as in (1.22). Letting $\delta \to 0$ we obtained the desired result.

Since a Lipschitz continuous semigroup is entirely determined by its local behavior on piecewise constant initial data, the previous analysis uniquely characterizes the limit of viscous approximations. In particular, this limit does not depend on the choice of the particular sequence ε_ν in (2.33). This completes the proof of Theorem 2.

3 - Estimation of the interaction terms

In this section we prove the key estimates (2.16)-(2.17) on the interaction terms $\phi^{i,p}$. More precisely, we assume that these bounds hold for some integer p and prove that they remain valid for $p+1$.

We shall denote by $\Gamma^i(t,x;s,y)$ the Green kernel for the linear equation

$$z^i_t + [\lambda_i(t,x) z^i]_x - z^i_{xx} = 0, \tag{3.1}$$

where $\lambda_i(t,x)$ is a bounded, sufficiently smooth function. In other words, $(t,x) \mapsto \Gamma^i(t,x;s,y)$ is the distributional solution of (3.1) such that

$$\lim_{t \to s+} \int_{\mathbb{R}} \Gamma^i(t,x;s,y) f(y) = f(x),$$

for every continuous functions $f \in \mathbf{L}^1$, see [1,9]. Since (3.1) is in conservation form, it is well known that $\Gamma^i(t,x;s,y)$ is positive and

$$\int_{\mathbb{R}} \Gamma^i(t,x;s,y)\, dx = 1 \qquad\qquad \text{for all } t > s, \quad y \in \mathbb{R}.$$

Taking $\lambda_i(t,x) = \lambda_i\big(u^p(t,x)\big)$ and calling Γ^i the corresponding Green kernel, the solution of the linear non-homogeneous Cauchy problem (2.5) can be represented as

$$v^{i,p+1}(t,x) = \int_{\mathbb{R}} \Gamma^i(t,x;0,y) \bar{v}^i(y)\, dy + \int_0^t \int_{\mathbb{R}} \Gamma^i(t,x;s,y) \phi^{i,p}(s,y)\, dyds. \tag{3.2}$$

The proof of our inductive estimates will be given in three steps.

STEP 1: Estimation of the integral

$$\int_0^\infty \int_{\mathbb{R}} \big|v^{i,p+1}(t,x)\big| \, \big|v^{j,p+1}(t,x)\big|\, dxdt. \tag{3.3}$$

As a preliminary, we prove

Lemma 1. *Consider two scalar parabolic equations in conservation form:*

$$\begin{cases} z^i_t + \left[\lambda^i(t,x)z^i\right]_x - z^i_{xx} = 0, \\ z^j_t + \left[\lambda^j(t,x)z^j\right]_x - z^j_{xx} = 0. \end{cases} \tag{3.4}$$

Assume that there exists $c > 0$ such that

$$\inf_{t,x} \lambda_j(t,x) - \sup_{t,x} \lambda_i(t,x) \geq c.$$

Then for every initial data $\bar z^i$, $\bar z^j \in \mathbf{L}^1$, the corresponding solutions z^i, z^j satisfy

$$\int_0^\infty \int_{I\!R} \left|z^i(t,x)\right| \cdot \left|z^j(t,x)\right| dx dt \leq \frac{1}{c} \int_{I\!R} \left|\bar z^i(y_1)\right| dy_1 \int_{I\!R} \left|\bar z^j(y_2)\right| dy_2. \tag{3.5}$$

Proof. We first establish the bound

$$\sup_{\substack{(y_1,y_2)\in I\!R^2 \\ y_1\neq y_2}} \left\{ \int_0^\infty \int_{I\!R} \Gamma^i(t,x;0,y_1)\cdot\Gamma^j(t,x;0,y_2)\,dxdt \right\} \leq \frac{1}{c}, \tag{3.6}$$

where Γ^i, Γ^j denote the Green kernels corresponding to the two equations in (3.4). By possibly performing a change of variable of the form $x \mapsto x - \bar\lambda t$, which does not affect the value of the integral (3.6), we can assume

$$\lambda_i(t,x) \leq -\frac{c}{2} < 0 < \frac{c}{2} \leq \lambda_j(t,x).$$

This implies

$$\tilde\lambda_i(t,x) \doteq \lambda_i(t,x) + \frac{c}{2} \leq 0, \qquad \tilde\lambda_j(t,x) \doteq \lambda_j(t,x) - \frac{c}{2} \geq 0. \tag{3.7}$$

We now observe that the product function $K(t,x_1,x_2;\, s,y_1,y_2) \doteq \Gamma^i(t,x_1;\, s,y_1)\cdot\Gamma^j(t,x_2;\, s,y_2)$ provides the Green kernel for the linear equation in two space variables:

$$Z_t + \left[\lambda_i(t,x_1)Z\right]_{x_1} + \left[\lambda_j(t,x_2)Z\right]_{x_2} - \Delta Z = 0. \tag{3.8}$$

In the following we denote by $G^\lambda(t,x)$ the standard Gaussian kernel with constant drift λ and by $G^\lambda_x(t,x)$ its derivative w.r.t. x, i.e.

$$G^\lambda(t,x) \doteq \frac{1}{2\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda t)^2}{4t}\right\}, \qquad G^\lambda_x(t,x) \doteq \frac{-(x-\lambda t)}{4t\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda t)^2}{4t}\right\}.$$

Writing (3.8) in the form

$$Z_t - \frac{c}{2}Z_{x_1} + \frac{c}{2}Z_{x_2} - \Delta Z = -\left[\tilde\lambda_i(t,x_1)Z\right]_{x_1} - \left[\tilde\lambda_j(t,x_2)Z\right]_{x_2},$$

the corresponding Green kernel can be represented as

$$
\begin{aligned}
K(t,x_1,x_2;0,y_1,y_2) &= G^{-c/2}(t,x_1-y_1)G^{c/2}(t,x_2-y_2) \\
&\quad -\int_0^t \iint_{I\!R^2} \Big\{ G_x^{-c/2}(t-s,x_1-z_1)G^{c/2}(t-s,x_2-z_2)\times \\
&\qquad\qquad \times K(s,z_1,z_2;0,y_1,y_2)\tilde{\lambda}_i(s,y_1) \Big\} \, dz_1 dz_2 \, ds \\
&\quad -\int_0^t \iint_{I\!R^2} \Big\{ G^{-c/2}(t-s,x_1-z_1)G_x^{c/2}(t-s,x_2-z_2)\times \\
&\qquad\qquad \times K(s,z_1,z_2;0,y_1,y_2)\tilde{\lambda}_j(s,y_2) \Big\} \, dz_1 dz_2 \, ds.
\end{aligned}
$$

We now compute

$$
\begin{aligned}
-\int_{I\!R^+}\int_{I\!R} G^{-c/2}(t,x-z_1)G_x^{c/2}(t,x-z_2)\,dxdt &= \int_{I\!R^+}\int_{I\!R} G_x^{-c/2}(t,x-z_1)G^{c/2}(t,x-z_2)\,dxdt \\
&= \begin{cases} -\frac{1}{2}\, e^{(z_1-z_2)c/2} & \text{if} \quad z_1<z_2, \\ 0 & \text{if} \quad z_1>z_2. \end{cases}
\end{aligned}
$$

Using the above formula and assuming that $y_1 \neq y_2$ (in this case we can change the order of integration), recalling (3.7) and the fact that the kernel K is positive, we conclude

$$
\begin{aligned}
\int_{I\!R^+}\int_{I\!R} \Gamma^i(t,x;0,y_1)\Gamma^j(t,x;0,y_2)\,dxdt &= \int_{I\!R^+}\int_{I\!R} K(t,x,x;0,y_1,y_2)\,dxdt \\
&\le \frac{1}{c} + \int_{I\!R^+}\iint_{z_1<z_2} e^{(z_1-z_2)c/2} K(s,z_1,z_2;0,y_1,y_2)\Big(\tilde{\lambda}_i(s,z_1)-\tilde{\lambda}_j(s,z_2)\Big)\,dz_1dz_2\,ds \\
&\le \frac{1}{c}\,.
\end{aligned}
$$

This establishes (3.6).

Given any two initial conditions $\bar{z}^i, \bar{z}^j$, the corresponding solutions of (3.4) satisfy

$$
\begin{aligned}
&\int_{I\!R^+}\int_{I\!R} \big|z^i(t,x)\big|\,\big|z^j(t,x)\big|\,dxdt \\
&\quad \le \int_{I\!R^+}\int_{I\!R}\iint_{I\!R^2} \big|\bar{z}^i(y_1)\big|\Gamma^i(t,x;0,y_1)\cdot\big|\bar{z}^j(y_2)\big|\Gamma^j(t,x;0,y_2)\;dy_1dy_2\,dxdt \\
&\quad \le \int_{I\!R}\big|\bar{z}^i(y_1)\big|\,dy_1\cdot\int_{I\!R}\big|\bar{z}^j(y_2)\big|\,dy_2 \;\cdot\; \sup_{\substack{(y_1,y_2)\in I\!R^2\\ y_1\neq y_2}} \left\{\int_{I\!R^+}\int_{I\!R}\Gamma^i(t,x;0,y_1)\cdot\Gamma^j(t,x;0,y_2)\,dxdt\right\}.
\end{aligned}
$$

By (3.6) this yields (3.5), proving the lemma.

Consider now two equations of the form (2.5). The solutions $v^{i,p+1}(t,x)$, $v^{j,p+1}(t,x)$ can be written in the form

$$\begin{aligned} v^{i,p+1}(t,x) &= \int_{I\!R} \Gamma^i(t,x;0,y)\bar{v}^i(y)\,dy + \int_0^t \int_{I\!R} \Gamma^i(t,x;s,y)\ \phi^{i,p}(s,y)\,dyds, \\ v^{j,p+1}(t,x) &= \int_{I\!R} \Gamma^j(t,x;0,y)\bar{v}^j(y)\,dy + \int_0^t \int_{I\!R} \Gamma^j(t,x;s,y)\ \phi^{j,p}(s,y)\,dyds. \end{aligned} \tag{3.9}$$

Using (2.10) and the inductive assumptions (2.17), by (3.5) and (3.9) with easy calculations we obtain

$$\int_{I\!R^+}\int_{I\!R} \left|v^{i,p+1}(t,x)\right|\,\left|v^{j,p+1}(t,x)\right|\,dxdt \le \frac{1}{c}\left(\eta_0 + \frac{\widehat{C}}{c}\eta_p^2\right)^2 = \frac{1}{c}\eta_{p+1}^2. \tag{3.10}$$

STEP 2: Estimation of the integral

$$\left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1} = \int_{I\!R} \left|v_x^{i,p+1}(t,x)\right|\,dx. \tag{3.11}$$

For convenience, for $i = 1,\ldots,n$, introduce the quantities

$$\lambda_i^* \doteq \lambda_i\big(\bar{u}(-\infty)\big), \qquad \|\lambda_i\|_\infty \doteq \max_{u\in K_1}\left|\lambda_i(u) - \lambda_i^*\right|, \qquad \|\lambda_i'\|_\infty \doteq \max_{1\le j\le n,\ u\in K_1}\left|r_j \bullet \lambda_i(u)\right|, \tag{3.12}$$

$$G^i(t,x) \doteq G^{\lambda_i^*}(t,x) = \frac{1}{2\sqrt{\pi t}}\ \exp\left\{-\frac{(x-\lambda_i^* t)^2}{4t}\right\}. \tag{3.13}$$

By (2.8), $\|\lambda_i'\|_\infty \le C_0$. We also observe that the heat kernel $G^i(t,x)$ satisfies

$$\int_{I\!R}\left|G_x^i(t,x)\right|dx = \frac{1}{\sqrt{\pi t}}, \qquad \int_{I\!R}\left|G_{xx}^i(t,x)\right|dx = \sqrt{\frac{2}{\pi e}}\cdot\frac{1}{t}. \tag{3.14}$$

Defining the time $\hat{t}$ by

$$\sqrt{\hat{t}} \doteq \min\left\{\frac{\sqrt{\pi}}{16n\|\lambda_i'\|_\infty\eta_{p+1}},\ \frac{\sqrt{\pi}}{8\widehat{C}\eta_{p+1}}\right\} = \frac{\sqrt{\pi}}{8\widehat{C}\eta_{p+1}}, \tag{3.15}$$

we claim that

$$\left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1} \le \widehat{C}\frac{\eta_{p+1}}{\sqrt{t}} \tag{3.16}$$

for all $t\in[0,\hat{t}]$. Clearly (3.16) is satisfied as a strict inequality for t sufficiently close to zero. To prove our claim, we shall assume that the two sides of (3.16) become equal for some $t<\hat{t}$ and derive a contradiction.

The x-derivative of the solution $v^{i,p+1}$ of (2.5) can be written in the form

$$\begin{aligned} v_x^{i,p+1}(t,x) = &\int_{I\!R} G_x^i(t,x-y)\bar{v}^i(y)\,dy \\ &- \int_0^t\int_{I\!R} G_x^i(t-s,x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u^p)-\lambda_i^*\big)v^{i,p+1}(s,y)\Big)\,dy\,ds \\ &+ \int_0^t\int_{I\!R} G_x^i(t-s,x-y)\phi^{i,p}(s,y)\,dy\,ds. \end{aligned} \tag{3.17}$$

Using (3.12), (3.14) and the inductive assumptions (2.15), the $\mathbf{L}^1$-norm of the second integral on the right hand side of (3.17) can be estimated as

$$
\begin{aligned}
&\int_{I\!R}\left|\int_0^t\int_{I\!R} G_x^i(t-s,x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u^p)-\lambda_i^*\big)v^{i,p+1}(s,y)\Big)\,dy\,ds\right|\,dx\\
&\leq \int_0^t \big\|G_x^i(t-s,\cdot)\big\|_{\mathbf{L}^1}\cdot\left\|\frac{\partial}{\partial y}\Big(\big(\lambda_i(u^p)-\lambda_i^*\big)v^{i,p+1}(s,\cdot)\Big)\right\|_{\mathbf{L}^1} ds\\
&\leq \int_0^t \frac{1}{\sqrt{\pi(t-s)}}\cdot\left[\|\lambda_i\|_\infty \widehat{C}\frac{\eta_{p+1}}{\sqrt{s}}+\|\lambda_i'\|_\infty\left(\sum_k \big\|v^{k,p}(s)\big\|_{\mathbf{L}^1}\right)\big\|v^{i,p+1}(s)\big\|_{\mathbf{L}^\infty}\right] ds\\
&\leq 2\widehat{C}n\|\lambda_i'\|_\infty\eta_{p+1}^2\int_0^t\frac{1}{\sqrt{\pi s(t-s)}}\,ds\\
&< \frac{8}{\sqrt{\pi}}\widehat{C}n\|\lambda_i'\|_\infty\eta_{p+1}^2.
\end{aligned}
$$

Using the above estimate together with (2.10) and the inductive assumption (2.16), from (3.17) we obtain

$$
\begin{aligned}
\|v_x^{i,p+1}(t)\|_{L^1} <& \frac{\|\bar v^i\|_{\mathbf{L}^1}}{\sqrt{\pi t}}+\frac{8}{\sqrt{\pi}}\widehat{C}n\|\lambda_i'\|_\infty\eta_{p+1}^2+\frac{2}{\sqrt{\pi}}\sqrt{t}\frac{8}{\sqrt{\pi}}\widehat{C}^3\eta_p^3\\
\leq& \frac{\eta_{p+1}}{\sqrt{t}}\left\{\frac{1}{\sqrt{\pi}}+\frac{8n}{\sqrt{\pi}}\widehat{C}\sqrt{t}\|\lambda_i'\|_\infty\eta_{p+1}+\frac{16}{\pi}t\widehat{C}^3\eta_{p+1}^2\right\}\\
\leq& \frac{\eta_{p+1}}{\sqrt{t}}\left\{\frac{1}{\sqrt{\pi}}+\frac{\widehat{C}}{2}+\frac{\widehat{C}}{4}\right\}\\
\leq& \widehat{C}\frac{\eta_{p+1}}{\sqrt{t}},
\end{aligned}
$$

because of the choice of the constants $C_0, \widehat{C}$ in (2.8)-(2.9). This contradiction proves that (3.16) must hold for all $t\leq\hat t$.

For every $t>\hat t$ we can now repeat the same argument on the time interval $[t-\hat t,\ t]$. This yields

$$
\big\|v_x^{i,p+1}(t)\big\|_{\mathbf{L}^1} \;\leq\; \widehat{C}\frac{\eta_{p+1}}{\sqrt{\hat t}} \;=\; \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2. \tag{3.18}
$$

For $0\leq t\leq\hat t$ we can also write the solution as

$$
\begin{aligned}
v_x^{i,p+1}(t,x) =& \int_{I\!R} G^i(t,x-y)\bar v_y^i(y)\,dy\\
&-\int_0^t\int_{I\!R} G_x^i(t-s,x-y)\frac{\partial}{\partial y}\Big(\big(\lambda_i(u^p)-\lambda_i^*\big)v^{i,p+1}(s,y)\Big)\,dyds\\
&+\int_0^t\int_{I\!R} G_x^i(t-s,x-y)\phi^{i,p}(s,y)\,dyds.
\end{aligned}
$$

With a computation similar to the one above and using (2.13), (2.16), for $t \in [0, \hat{t}]$ we obtain

$$\begin{aligned}\sup_{t\in[0,\hat{t}\,]} \left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1} &\leq \|\bar{v}_x^i\|_{\mathbf{L}^1} + \frac{1}{4}\sup_{t\in[0,\hat{t}\,]} \left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1} + \frac{2}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2 \\ &\leq \frac{6}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2 + \frac{1}{4}\sup_{t\in[0,\hat{t}\,]} \left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1},\end{aligned}$$

hence

$$\sup_{t\in[0,\hat{t}\,]} \left\|v_x^{i,p+1}(t)\right\|_{\mathbf{L}^1} \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2. \tag{3.19}$$

Using (3.18) and (3.19) we conclude that, for all $t \geq 0$,

$$\|v_x^{i,p+1}(t)\|_{\mathbf{L}^1} \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2. \tag{3.20}$$

STEP 3: Evaluation of the double integral

$$\int_0^\infty \int_{I\!R} \left|v_x^{i,p+1}(t,x)\right| \left|v^{j,p+1}(t,x)\right| dxdt. \tag{3.21}$$

The idea is to use (3.19) to show that the quantity in (3.21) can be bounded in terms of the product of $\mathbf{L}^1$-norms of $v_x^{i,p+1}$ and $v^{j,p+1}$. Since (3.17) contains a convolution of $v^{i,p+1}$ with the heat kernel G, we consider the family of functions $v^{j,p+1}(t+\tau, x+z)$, with $(\tau, z) \in I\!R^+ \times I\!R$ and we estimate the quantity

$$\mathcal{I}_\alpha \doteq \sup_{(\tau,z)\in I\!R^+\times I\!R} \int_0^\alpha \int_{I\!R} \left|v_x^{i,p+1}(t,x)\right| \left|v^{j,p+1}(t+\tau, x+z)\right| dxdt, \tag{3.22}$$

where $\alpha \geq 0$. We first study the case $\alpha \leq \hat{t}$, with $\hat{t}$ defined at (3.12). We write the solution of (2.5) as

$$\begin{aligned}v_x^{i,p+1}(t,x) = &\int_{I\!R} G^i(t, x-y)\bar{v}_y^i(y)\, dy - \int_0^t \int_{I\!R} G_x^i(s,y)\big(\lambda_i(u^p) - \lambda_i^*\big)v_x^{i,p+1}(t-s, x-y)\, dyds \\ &- \int_0^t \int_{I\!R} G_x^i(s,y) \left(\sum_{k=1}^n r_k \bullet \lambda_i(u^p)v^{k,p}\right) v^{i,p+1}(t-s, x-y)\, dyds \\ &+ \int_0^t \int_{I\!R} G_x^i(s,y)\phi^{i,p}(t-s, x-y)\, dyds.\end{aligned}$$

We first estimate the following integrals, whose computation is carried out in appendix A:

$$\int_0^\alpha \iint_{I\!R^2} \left|G^i(t, x-y)\bar{v}_x^i(y)v^{j,p+1}(t+\tau, x+z)\right| \leq \frac{4}{c\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^3, \tag{3.23}$$

$$\int_0^\alpha \int_0^t \iint_{I\!R^2} \left|G_x^i(s,y)\big(\lambda_i(u^p) - \lambda_i^*\big)v_x^{i,p+1}(t-s, x-y)v^{j,p+1}(t+\tau, x+z)\right| dxdy\, dsdt \leq \frac{1}{8}\mathcal{I}_\alpha, \tag{3.24}$$

$$\int_0^\alpha \int_0^t \iint_{I\!R^2} \left| G_x^i(s,y)(r_k \bullet \lambda_i) v^{k,p} v^{i,p+1}(t-s,x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\, dsdt \\ \leq \frac{16}{c\pi} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \widehat{C}^2 \eta_{p+1}^4, \tag{3.25}$$

$$\int_0^\alpha \int_0^t \iint_{I\!R^2} \left| G_x^i(s,y) \phi^{i,p}(t-s,x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\ dsdt \leq \frac{2}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3. \tag{3.26}$$

Using the above estimates we obtain

$$\mathcal{I}_\alpha \leq \frac{6}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3 + \frac{1}{8} \mathcal{I}_\alpha + \frac{16n}{c\pi} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \widehat{C}^2 \eta_{p+1}^4 \leq \frac{7}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3 + \frac{1}{8} \mathcal{I}_\alpha. \tag{3.27}$$

This shows that $\mathcal{I}_\alpha$ is uniformly bounded, in the case $\alpha \leq \hat{t}$.

If $\alpha > \hat{t}$ we use (3.17), splitting the integrals in (3.22) in two parts:

$$\int_0^\alpha \int_{I\!R} \left| v_x^{i,p+1}(t,x) \right| \left| v^{j,p+1}(t+\tau,x+z) \right| dxdt \\ = \left\{ \int_0^{\hat{t}} + \int_{\hat{t}}^\alpha \right\} \int_{I\!R} \left| v_x^{i,p+1}(t,x) \right| \left| v^{j,p+1}(t+\tau,x+z) \right| dxdt, \tag{3.28}$$

and write $v_x^{i,p+1}(t,x)$, $t \geq \hat{t}$, using the integral representation

$$\begin{aligned} v_x^{i,p+1}(t,x) = & \int_{I\!R} G_x^i(\hat{t},y) v^{i,p+1}(t-\hat{t},x-y)\, dy \\ & - \int_0^{\hat{t}} \int_{I\!R} G_x^i(s,y) \big(\lambda_i(u^p) - \lambda_i^*\big) v_x^{i,p+1}(t-s,x-y)\, dyds \\ & - \int_0^{\hat{t}} \int_{I\!R} G_x^i(s,y) \left(\sum_{k=1}^n r_k \bullet \lambda_i(u^p) v^{k,p} \right) v^{i,p+1}(t-s,x-y)\, dyds \\ & + \int_0^{\hat{t}} \int_{I\!R} G_x^i(s,y) \phi^{i,p}(t-s,x-y)\, dyds. \end{aligned}$$

We now use the further estimates proved in Appendix A:

$$\int_{\hat{t}}^\alpha \int \int_{I\!R^2} \left| G_x^i(\hat{t},y) v^{i,p+1}(t-\hat{t},x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\ dt \leq \frac{8}{c\pi} \widehat{C} \eta_{p+1}^3, \tag{3.29}$$

$$\int_{\hat{t}}^\alpha \int_0^{\bar{t}} \iint_{I\!R^2} \left| G_x^i(s,y) \left[\lambda_i(u^p) - \lambda_i^* \right] v_x^{i,p+1}(t-s,x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\, dsdt \leq \frac{1}{8} \mathcal{I}_\alpha, \tag{3.30}$$

$$\int_{\hat{t}}^\alpha \int_0^{\hat{t}} \iint_{I\!R^2} \left| G_x^i(s,y) \big((r_k \bullet \lambda_i) v^{k,p} \big) v^{i,p+1}(t-s,x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\, dsdt \\ \leq \frac{16}{c\pi} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \widehat{C}^2 \eta_{p+1}^4, \tag{3.31}$$

$$\int_{\hat{t}}^\alpha \int_0^{\hat{t}} \iint_{I\!R^2} \left| G_x^i(s,y) \phi^{i,p}(t-s,x-y) v^{j,p+1}(t+\tau,x+z) \right| dxdy\, dsdt \leq \frac{2}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3. \tag{3.32}$$

Using the above estimates, for all $(\tau,z)\in I\!R^+\times I\!R$ we obtain

$$\int_{\hat t}^{\alpha}\int_{I\!R}\left|v_x^{1,p+1}(t,x)\right|\,\left|v^{j,p+1}(t+\tau,x+z)\right|dxdt\le\frac{8}{\pi}\widehat{C}\frac{\eta_{p+1}^3}{c}+\frac{3}{c\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^3+\frac{1}{8}\mathcal{I}_\alpha. \tag{3.33}$$

Adding the two expressions (3.27), (3.33), and recalling (2.13), we obtain

$$\mathcal{I}_\alpha\ \le\ \frac{1}{c}\left(\frac{8}{\pi}\widehat{C}+\frac{10}{\sqrt{\pi}}\widehat{C}^2\right)\eta_{p+1}^3+\frac{\mathcal{I}_\alpha}{4}\ \le\ \frac{12}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta_{p+1}^3+\frac{\mathcal{I}_\alpha}{4},$$

from which we deduce

$$\mathcal{I}_\alpha\le\frac{20}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta_{p+1}^3. \tag{3.34}$$

Note that (3.34) holds for every $\alpha\in I\!R^+$. This yields the desired bound for (3.22), namely

$$\int_{I\!R^+}\int_{I\!R}\left|v_x^{i,p+1}(t,x)\right|\left|v^{j,p+1}(t,x)\right|dxdt\ \le\ \limsup_{\alpha\to\infty}\ \mathcal{I}_\alpha\ \le\ \frac{20}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\eta_{p+1}^3. \tag{3.35}$$

By possibly reducing the value of η_0 we can assume that, in addition to (2.12), this constant satisfies

$$\left(1+2\frac{20}{\sqrt{\pi}}\widehat{C}\,2\eta_0+\frac{8n}{\sqrt{\pi}}\widehat{C}^2\,(2\eta_0)^2\right)\le 2. \tag{3.36}$$

Using (3.12), (3.20) and (3.35), we now prove the inductive estimates (2.16)-(2.17), with p replaced by $p+1$.

$$\begin{aligned}\int_{I\!R^+}\int_{I\!R}\left|\phi^{i,p+1}(t,x)\right|dxdt&\le\frac{n^2C_0}{c}\eta_{p+1}^2\left(1+2\frac{20}{\sqrt{\pi}}\widehat{C}\eta_{p+1}+\frac{8n}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2\right)\\&\le\frac{n^2C_0}{c}\eta_{p+1}^2\cdot 2\ \le\ \frac{\widehat{C}}{c}\eta_{p+1}^2,\end{aligned} \tag{3.37}$$

$$\begin{aligned}\int_{I\!R}\left|\phi^{i,p+1}(t,x)\right|dx&\le n^2C_0\frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^3\left(1+2\frac{8}{\sqrt{\pi}}\widehat{C}\eta_{p+1}+\frac{8n}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2\right)\\&\le n^2C_0\frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^3\cdot 2\ \le\ \frac{8}{\sqrt{\pi}}\widehat{C}^3\eta_{p+1}^3.\end{aligned} \tag{3.38}$$

4 - Convergence of the approximating sequence

In this section we prove that, if the total variation of the initial data $\bar u$ is small enough, then the sequence of approximations u^p defined at (2.4)-(2.5) converges to the unique solution u of (1.4), (1.2) for all $t\ge 0$. The uniform bounds obtained on the total variation of each u^p thus yield the desired a priori bound on the total variation of $u(t,\cdot)$, for all $t\ge 0$. This will provide a proof of Theorem 1.

Let the constants $C_0, \widehat{C}, \delta_1, \eta_0$ be as in the previous sections, depending only on the compact set K_0 and on the system (1.4). Using the estimates (3.37)-(3.38), we construct a sequence of functions $v^{i,p}$, u^p, $p \geq 0$, satisfying the inductive relations (2.4)-(2.5) together with

$$\big\|v^{i,p}(t)\big\|_{\mathbf{L}^1} \leq 2\eta_0, \quad \big\|v^{i,p}_x(t)\big\|_{L^1} \leq \frac{8}{\sqrt{\pi}}\widehat{C}^2(2\eta_0)^2, \qquad \text{for all } t \geq 0,\ x \in I\!R. \tag{4.1}$$

$$\text{Tot.Var.}\big\{u^p(t,\cdot)\big\} \leq \delta_1, \qquad u^p(t,x) \in K_1 \qquad \text{for all } \ t \geq 0, \ \ x \in I\!R, \tag{4.2}$$

We first prove that the sequence $v^{i,p}$, u^p, $p \geq 0$, converges in $W^{1,1}$ to the corresponding solutions v^i, u of (2.1)-(2.2). On the vector valued function $v^p = (v^{1,p}, \ldots, v^{n,p})$ we will use the following $\mathbf{L}^1$ and $W^{1,1}$ norms:

$$\|v^p\|_{\mathbf{L}^1} \doteq \max_{i=1,\ldots,n} \|v^{i,p}\|_{\mathbf{L}^1}, \quad \|v^p\|_{W^{1,1}} \doteq \|v^p\|_{\mathbf{L}^1} + \|v^p_x\|_{\mathbf{L}^1}. \tag{4.3}$$

We first estimate the $\mathbf{L}^\infty$-norm of $u^p - u^{p-1}$. One has

$$\begin{aligned}\frac{d}{dx}|u^p - u^{p-1}| &\leq |u^p_x - u^{p-1}_x| \\ &\leq \sum_{i=1}^n \big|v^{i,p} r_i(u^p) - v^{i,p-1} r_i(u^{p-1})\big| \\ &\leq \sum_{i=1}^n |v^{i,p} - v^{i,p-1}| + C\sum_{i=1}^n |v^{i,p-1}|\big|u^p - u^{p-1}\big|,\end{aligned}$$

for some constant C depending only on the system (1.4). Therefore,

$$\begin{aligned}\|u^p - u^{p-1}\|_{\mathbf{L}^\infty} &\leq \int_{-\infty}^x \sum_{i=1}^n |v^{i,p} - v^{i,p-1}| \exp\left\{C\int_y^x \sum_{i=1}^n |v^{i,p-1}|\,dz\right\}\,dy \\ &\leq n\exp\{nC\,2\eta_0\} \cdot \|v^{i,p} - v^{i,p-1}\|_{\mathbf{L}^1}.\end{aligned} \tag{4.4}$$

As usual, for any given $t \geq 0$, we can write the solution of (2.4) in the integral form

$$\begin{aligned}v^{i,p+1}(t+\bar{t},x) &= \int_{I\!R} G^i(\bar{t}, x-y) v^{i,p+1}(t,y)\,dy \\ &\quad - \int_0^{\bar{t}}\int_{I\!R} G^i_x(\bar{t}-s, x-y)\big(\lambda_i(u^p) - \lambda_i^*\big) v^{i,p+1}(t+s,y)\,dyds \\ &\quad + \int_0^{\bar{t}}\int_{I\!R} G^i(\bar{t}-s, x-y)\phi^{i,p}(t+s,y)\,dyds,\end{aligned} \tag{4.5}$$

for every $\bar{t} > 0$. Next, we estimate the $\mathbf{L}^1$-distance between $\phi^{i,p}$ and $\phi^{i,p-1}$. Let L be a Lipschitz constant over K_1 for all functions $G^i_{j,k}$, $H^i_{j,k}$, $K^i_{j,k,\ell}$. Recalling (1.10) and the uniform bounds (2.18) and (4.2), we have (see Appendix B):

$$\begin{aligned}\left\|\sum_{j,k=1}^n G^i_{j,k}(u^p) v^{j,p} v^{k,p} - G^i_{j,k}(u^{p-1}) v^{j,p-1} v^{k,p-1}\right\|_{\mathbf{L}^1} &\leq \frac{8n^2}{\sqrt{\pi}}\widehat{C}^2\big(2\eta_0\big)^3 L\|u^p - u^{p-1}\|_{\mathbf{L}^\infty} \\ &\quad + \frac{8}{\sqrt{\pi}}\widehat{C}^3\big(2\eta_0\big)^2\|v^p - v^{p-1}\|_{\mathbf{L}^1},\end{aligned} \tag{4.6}$$

$$\left\| \sum_{j,k=1}^{n} H^i_{j,k}(u^p) v_x^{j,p} v^{k,p} - H^i_{j,k}(u^{p-1}) v_x^{j,p-1} v^{k,p-1} \right\|_{\mathbf{L}^1} \leq \frac{64n^2}{\pi} \widehat{C}^4 \big(2\eta_0\big)^4 L \|u^p - u^{p-1}\|_{\mathbf{L}^\infty}$$
$$+ \frac{8}{\sqrt{\pi}} \widehat{C}^3 \big(2\eta_0\big)^2 \|v_x^p - v_x^{p-1}\|_{\mathbf{L}^1}, \tag{4.7}$$

$$\left\| \sum_{j,k,\ell=1}^{n} K^i_{j,k,\ell}(u^p) v^{j,p} v^{k,p} v^{\ell,p} - K^i_{j,k,\ell}(u^{p-1}) v^{j,p-1} v^{k,p-1} v^{\ell,p-1} \right\|_{\mathbf{L}^1}$$
$$\leq \frac{64n^3}{\pi} \widehat{C}^4 \big(2\eta_0\big)^5 L \|u^p - u^{p-1}\|_{\mathbf{L}^\infty} + \frac{96n}{\sqrt{\pi}} \widehat{C}^5 \big(2\eta_0\big)^4 \|v^p - v^{p-1}\|_{\mathbf{L}^1}, \tag{4.8}$$

Using (4.4) and (4.6)–(4.8), we finally obtain

$$\|\phi^{i,p} - \phi^{i,p-1}\|_{\mathbf{L}^1} \leq C \|v^p - v^{p-1}\|_{W^{1,1}}, \tag{4.9}$$

for some constant C large enough. A simple calculation now yields

$$\left\| \big(\lambda_i(u^p) - \lambda_i^*\big) v^{i,p+1} - \big(\lambda_i(u^{p-1}) - \lambda_i^*\big) v^{i,p} \right\|_{W^{1,1}} \leq C \|v^{p+1} - v^p\|_{W^{1,1}} + C \|v^p - v^{p-1}\|_{W^{1,1}} \tag{4.10}$$

for a possibly larger constant C. Using (4.9) and (4.10), from (4.5) it now follows

$$\sup_{0 \leq s \leq \bar t} \left\| v^{i,p+1}(t+s) - v^{i,p}(t+s) \right\|_{W^{1,1}} \leq \left\| v^{i,p+1}(t) - v^{i,p}(t) \right\|_{W^{1,1}}$$
$$+ \frac{2}{\sqrt{\pi}} \sqrt{\bar t}\, C \left\{ \sup_{0 \leq s \leq \bar t} \left\| v^{p+1}(t+s) - v^p(t+s) \right\|_{W^{1,1}} + \sup_{0 \leq s \leq \bar t} \left\| v^p(t+s) - v^{p-1}(t+s) \right\|_{W^{1,1}} \right\}.$$

By fixing the time step

$$\sqrt{\bar t} \doteq \frac{\sqrt{\pi}}{3C}$$

we obtain

$$\sup_{0 \leq s \leq \bar t} \left\| v^{i,p+1}(t+s) - v^{i,p}(t+s) \right\|_{W^{1,1}} \leq \frac{3}{2} \left\| v^{i,p+1}(t) - v^{i,p}(t) \right\|_{W^{1,1}}$$
$$+ \frac{1}{2} \sup_{0 \leq s \leq \bar t} \left\| v^{i,p}(t+s) - v^{i,p-1}(t+s) \right\|_{W^{1,1}}. \tag{4.11}$$

Assume now that, at a given time $t = \tau$, the sequence $\left\| v^{i,p+1}(\tau) - v^{i,p}(\tau) \right\|_{W^{1,1}}$ tends to zero as $p \to \infty$. Then by (4.11) we have

$$\limsup_{p \to \infty} \left\{ \sup_{0 \leq s \leq \bar t} \left\| v^{i,p+1}(\tau+s) - v^{i,p}(\tau+s) \right\|_{W^{1,1}} \right\} \leq 3 \limsup_{p \to \infty} \left\| v^{i,p+1}(\tau) - v^{i,p}(\tau) \right\|_{W^{1,1}} = 0.$$

Therefore, convergence occurs also for all times $t \in [\tau,\ \tau + \bar t]$. Iterating the above argument we obtain that, for any fixed interval $[0, T]$, the sequences $v^{i,p}$, u^p tend uniformly in $W^{1,1}$ to some functions v^i, u. Clearly, these limit functions provide a solution to (2.1)-(2.2).

By construction, u solves the Cauchy problem (1.2), (1.4). Moreover, the estimates (4.1)-(4.2) remain valid with $v^{i,p}$, u^p replaced by v^i, u, respectively. Since the functions $\phi^{i,p}$ can be considered as bounded measures on $I\!R^2$, and since by (4.9) they converge to some function ϕ^i in $\mathbf{L}^1([0,T]\times I\!R)$ for any $T<+\infty$, we have

$$\int_0^\infty \int_{I\!R} \big|\phi^i(t,x)\big|\,dxdt \le \liminf_{p\to+\infty} \int_0^\infty \int_{I\!R} \big|\phi^{i,p}(t,x)\big|\,dxdt \le \frac{\widehat{C}}{c}(2\eta_0)^2. \tag{4.12}$$

Consider the set

$$\begin{aligned}\mathcal{D}^* \doteq \Big\{u : I\!R \mapsto I\!R^n\ ; \quad u(-\infty)\in K_0,\ \ \big\|l_i(u)\cdot u_x\big\|_{\mathbf{L}^1} \le \eta_0,& \\ \big\|(l_i(u)\cdot u_x)_x\big\|_{\mathbf{L}^1} \le \frac{4}{\sqrt{\pi}}\widehat{C}^2\eta_0^2, \quad i=1,\dots,n\Big\}.&\end{aligned} \tag{4.13}$$

For every initial data $\bar u\in\mathcal{D}^*$, the above analysis has established the existence of a solution u of the Cauchy problem (1.2), (1.4), whose total variation satisfies the uniform bounds (1.14) for all $t\ge 0$, and hence takes values within the compact set K_1. In the remainder of this section we prove that, by possibly choosing a smaller constant η_0, the result remains valid also without the assumption (2.11). The main argument in this last step of the proof is quite simple: If the total variation of the initial data $\bar u$ is very small, after a short interval of time the corresponding solution of (1.4) will be inside the domain $\mathcal{D}^*$, thanks to the smoothing properties of the parabolic system. Hence our previous analysis can be applied.

With a linear change of coordinates, we can assume that $A^*\doteq A\big(\bar u(-\infty)\big)=\text{diag}(\lambda_1^*,\dots,\lambda_n^*)$. Denoting with $\mathbf{e}_i$ the i-th unit vector and writing $\hat u^i\doteq \mathbf{e}_i\cdot u$, we can write the solutions of (1.4) in the form

$$\hat u^i(t,x) = \int_{I\!R} G^i(t,x-y)\bar u^i(y)\,dy - \int_0^t\int_{I\!R} G^i(t-s,x-y)\ \mathbf{e}_i\cdot\big(A(u)-A^*\big)u_x(s,y)\,dyds, \tag{4.14}$$

$$\hat u^i_x(t,x) = \int_{I\!R} G^i(t,x-y)d\bar u^i(y) - \int_0^t\int_{I\!R} G^i_x(t-s,x-y)\ \mathbf{e}_i\cdot\big(A(u)-A^*\big)u_x(s,y)\,dyds, \tag{4.15}$$

$$\hat u^i_{xx}(t,x) = \int_{I\!R} G^i_x(t,x-y)d\bar u^i(y) - \int_0^t\int_{I\!R} G^i_x(t-s,x-y)\ \mathbf{e}_i\cdot\Big[\big(A(u)-A^*\big)u_x(s,y)\Big]_x\,dyds, \tag{4.16}$$

with obvious meaning of notations. Recall that $G^i\doteq G^{\lambda_i^*}$ is the Green kernel defined at (3.13). We look for an estimate of the form

$$\|u_{xx}(t)\|_{\mathbf{L}^1} \le C\,\frac{\text{Tot.Var.}\{\bar u\}}{\sqrt{t}}. \tag{4.17}$$

Defining the time t_0 and the constant C as

$$\sqrt{t_0}\doteq \frac{\sqrt{\pi}}{32n\|DA\|_\infty\cdot \text{Tot.Var.}\{\bar u\}}, \qquad C\doteq \max\left\{\frac{9}{\sqrt{\pi}},\,\|DA\|_\infty\right\},$$

we claim that (4.17) holds for $t \le t_0$ if the initial datum is smooth enough. Clearly (4.17) is a strict inequality for t sufficiently small. Using the same techniques as in Section 3, step 2, if $0 < t < t_0$ we can estimate the $\mathbf{L}^1$ norms of (4.15) and (4.16) as

$$
\begin{aligned}
\big\|u_x(t)\big\|_{\mathbf{L}^1} &\le \text{Tot.Var.}\{\bar u\} + \frac{2n}{\sqrt{\pi}}\sqrt{t}\big\|A - A^*\big\|_\infty \sup_{0<s\le t}\big\|u_x(s)\big\|_{\mathbf{L}^1} \le 2\,\text{Tot.Var.}\{\bar u\} \\
\big\|u_{xx}(t)\big\|_{L^1} &< \frac{\text{Tot.Var.}\{\bar u\}}{\sqrt{\pi t}} + \frac{8n}{\sqrt{\pi}}C\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar u\} \cdot \sup_{0<s\le t}\big\|u_x(s)\big\|_{\mathbf{L}^1} \\
&\le C\,\frac{\text{Tot.Var.}\{\bar u\}}{\sqrt{t}},
\end{aligned}
\tag{4.18}
$$

where

$$
\|A - A^*\|_\infty \doteq \max_{u\in K_1}\big|A(u) - A^*\big|, \qquad \|DA\|_\infty \doteq \max_{u\in K_1}\left|\frac{d}{du}A(u)\right|.
$$

Here the matrices $A(u)$ are regarded as linear operators in $L(I\!R^n, I\!R^n)$, and $I\!R^n$ has the norm $\|v\| \doteq \sum_{i=1}^n |v_i|$. This concludes the proof if the initial datum is sufficiently smooth. Since the estimates (4.17)-(4.18) depend only on the BV norm of u, by a density argument it follows that (4.17) holds for any enough small BV function. In particular, at time t_0 form (4.17) we have

$$
\big\|u_{xx}(t_0)\big\|_{L^1} \le \frac{32n}{\sqrt{\pi}}C^2\big(\text{Tot.Var.}\{\bar u\}\big)^2. \tag{4.20}
$$

With easy computations, from (1.7) it follows that the components v_x^i are given by

$$
v^i = \sum_{j=1}^n (l^i \cdot \mathbf{e}_j)\,\hat u_x^j, \qquad v_x^i = \sum_{j=1}^n (l^i \cdot \mathbf{e}_j)\,\hat u_{xx}^j + \sum_{j,k=1}^n (\mathbf{e}_k \bullet l^i)\cdot \mathbf{e}_j\ \hat u_x^j \hat u_x^k. \tag{4.21}
$$

Therefore, for a suitable constant C' depending on $\|DA\|_\infty$, by (4.20)-(4.21) we have

$$
\begin{aligned}
\big\|v^i(t_0)\big\|_{\mathbf{L}^1} &\le nC'\big\|u_x(t_0)\big\|_{\mathbf{L}^1} \le 2nC' \cdot \text{Tot.Var.}\{\bar u\} \le \eta_0, \\
\big\|v_x^i(t_0)\big\|_{\mathbf{L}^1} &\le nC'\Big(1 + n\big\|u_x(t_0)\big\|_{\mathbf{L}^1}\Big)\big\|u_{xx}(t_0)\big\|_{\mathbf{L}^1} \\
&\le nC'\Big(1 + 2n\cdot \text{Tot.Var.}\{\bar u\}\Big)\frac{32n}{\sqrt{\pi}}C^2\big(\text{Tot.Var.}\{\bar u\}\big)^2 \\
&\le \frac{4}{\sqrt{\pi}}\widehat{C}^2\eta_0^2,
\end{aligned}
\tag{4.22}
$$

provided that Tot.Var.$\{\bar u\}$ is sufficiently small and $\widehat{C}$ big enough. All previous arguments can now be applied to the solution u on the interval $[t_0,\ \infty[$. This completes the proof of the first part of Theorem 1.

5 - $\mathbf{L^1}$-stability of viscous solutions

In this section we prove the stability estimate (1.15) for solutions u of (1.4). Toward this goal, we first study the linear variational equation satisfied by infinitesimal perturbations. The evolution equation (2.23) can be more conveniently written as

$$h_t + [A(u)h]_x + (h \bullet A(u))u_x - (u_x \bullet A(u))h = h_{xx}\,. \tag{5.1}$$

We first consider the case in which the initial data $\bar u$ is in D_W. Defining the components of h and u as

$$h = \sum h^j r_j(u), \qquad u_x = \sum v^j r_j(u), \tag{5.2}$$

with computations entirely similar to (1.7)–(1.10) we obtain

$$\begin{aligned} h_t + \sum_{j=1}^n (\lambda_j h^j)_x r_j + \sum_{j,k=1}^n \lambda_j h^j v^k r_k \bullet r_j \\ + \sum_{j,k=1}^n (v^j h^k - h^j v^k)\Big(\lambda_j r_k \bullet r_j + (r_k \bullet \lambda_j) r_j - \sum_{\ell=1}^n \lambda_\ell \big(l^\ell \cdot (r_k \bullet r_j)\big) r_\ell\Big) \\ = \sum_{j=1}^n h^j_{xx} r_j + 2\sum_{j,k=1}^n h^j_x v^k r_k \bullet r_j + \sum_{j,k=1}^n h^j v^k_x r_k \bullet r_j + \sum_{j,k,\ell=1}^n h^j v^k v^\ell r_\ell \bullet (r_k \bullet r_j), \end{aligned}$$

$$\begin{aligned} h_t = \sum_{j=1}^n h^j_t r_j - \sum_{j,k=1}^n \lambda_k h^j v^k (r_k \bullet r_j) + \sum_{j,k=1}^n h^j v^k_x (r_k \bullet r_j) \\ + \sum_{j,k,\ell=1}^n h^j v^k v^\ell (r_\ell \bullet r_k) \bullet r_j, \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n h^j_t r_j + \sum_{j=1}^n (\lambda_j h^j)_x r_j + \sum_{j,k=1}^n (\lambda_j - \lambda_k) h^j v^k (r_k \bullet r_j) \\ + \sum_{j,k=1}^n (v^j h^k - h^j v^k)\Big(\lambda_j r_k \bullet r_j + (r_k \bullet \lambda_j) r_j - \sum_{\ell=1}^n \lambda_\ell \big(l^\ell \cdot (r_k \bullet r_j)\big) r_\ell\Big) \\ = \sum_{j=1}^n h^j_{xx} r_j + 2\sum_{j,k=1}^n h^j_x v^k r_k \bullet r_j + \sum_{j,k,\ell=1}^n h^j v^k v^\ell \Big(r_\ell \bullet (r_k \bullet r_j) - (r_\ell \bullet r_k) \bullet r_j\Big), \end{aligned} \tag{5.3}$$

The inner product of (5.3) with $l_i(u)$ yields

$$
\begin{aligned}
h^i_t + [\lambda_i h^i]_x - h^i_{xx} &= \sum_{j,k=1}^n (\lambda_k - \lambda_i) l^i \cdot [r_k, r_j] h^j v^k + 2 \sum_{j,k=1}^n l^i \cdot (r_k \bullet r_j) h^j_x v^k \\
&\quad + \sum_{j,k,\ell=1}^n l^i \cdot \Big(r_l \bullet (r_k \bullet r_j) - (r_\ell \bullet r_k) \bullet r_j \Big) h^j v^k v^\ell + \sum_{k=1}^n (h^i v^k - h^k v^i)(r_k \bullet \lambda_i) \\
&\doteq \sum_{\substack{j,k=1 \\ j\neq k}}^n G^i_{j,k}(u) h^j v^k + \sum_{\substack{j,k=1 \\ j\neq k}}^n H^i_{j,k}(u) h^j_x v^k + \sum_{\substack{j,k,\ell=1 \\ k\neq \ell}}^n K^i_{j,k,\ell}(u) h^j v^k v^\ell \\
&\quad + \sum_{\substack{j,k=1 \\ j\neq k}}^n J^i_{j,k}(u) h^j (v^k)^2 + \sum_{\substack{k=1 \\ k\neq i}}^n L^i_k(u)(h^i v^k - h^k v^i) \\
&\doteq \psi^i(u,v,h).
\end{aligned}
\tag{5.4}
$$

The following estimates are quite similar to those in Sections 2 - 3.

Let the constant C_0 provide an upper bound for the absolute values of all functions $G^i_{j,k}$, $H^i_{j,k}$, $K^i_{j,k,l}$, $J^i_{j,k}$, L^i_k, as u ranges over the compact set K_1. For a given initial data $\bar h$, consider the sequence of functions $h^{i,p}$, inductively defined by the equations

$$
\begin{cases} h^{i,p+1}_t + \big[\lambda_i(u) h^{i,p+1}\big]_x - h^{i,p+1}_{xx} = \psi^{i,p}(t,x) \\ h^{i,p+1}(0,x) = \bar h^i(x) \doteq l^i\big(u(x)\big) \cdot \bar h(x). \end{cases}
\tag{5.5}
$$

Here $\psi^{i,p} \doteq \psi^i(u,v,h^p)$ is the right hand side of (5.4) with $h = h^p$. We assume that there exists a constant ξ_0 such that

$$
\|\bar h^i\|_{\mathbf{L}^1} \leq \xi_0, \qquad \int_{I\!R} \big|\bar h_x(x)\big|\, dx \leq \frac{4}{\sqrt{\pi}} \widehat C^2 (2\eta_0) \xi_0, \qquad i = 1, \ldots, n,
\tag{5.6}
$$

and we define the increasing sequence ξ_p inductively by

$$
\xi_{p+1} = \xi_0 + \frac{\widehat C}{c} (2\eta_0) \xi_p.
\tag{5.6}
$$

We will show that $h^{i,p}$ and $\psi^{i,p}$ satisfy bounds analogous to (2.15)–(2.17):

$$
\begin{gathered}
\|h^{i,p}\|_{\mathbf{L}^1} \leq \xi_p, \qquad \int_0^\infty \int_{I\!R} \big|\psi^{i,p}(t,x)\big|\, dxdt \leq \frac{1}{c} \widehat C (2\eta_0) \xi_p, \\
\int_{I\!R} \big|\psi^{i,p}(t,x)\big|\, dx \leq \frac{8}{\sqrt{\pi}} \widehat C^3 (2\eta_0)^2 \xi_p.
\end{gathered}
\tag{5.8}
$$

Note that all estimates are linear w.r.t. ξ_p, ξ_0. Since the left hand side of the evolution equation (5.5) is in conservation form, from (5.6) and the second inequality in (5.8) it follows the inductive estimate $\|h^{i,p+1}\|_{\mathbf{L}^1} \leq \xi_{p+1}$.

Assuming that (5.8) holds for some p, we show that all three inequalities are also valid for $p+1$. The proof involves three steps, entirely similar to the ones in Section 3.

1. Writing the solutions $h^{i,p+1}(t,x)$, $v^j(t,x)$ in the integral form

$$\begin{aligned} h^{i,p+1}(t,x) &= \int_{I\!R} \Gamma^i(t,x;0,y)\bar{h}^i(y) + \int_0^t \int_{I\!R} \Gamma^i(t,x;s,y)\psi^{i,p}(s,y)\,dyds, \\ v^j(t,x) &= \int_{I\!R} \Gamma^j(t,x;0,y)\bar{v}^j(y)\,dy + \int_0^t \int_{I\!R} \Gamma^j(t,x;s,y)\phi^j(s,y)\,dyds, \end{aligned}$$

after some calculations we obtain

$$\int_0^\infty \int_{I\!R} \left|v^{i,p+1}(t,x)\right| \left|v^{j,p+1}(t,x)\right| dxdt \le \frac{1}{c}\left(\eta_0 + \frac{\widehat{C}}{c}(2\eta_0)^2\right)\left(\xi_0 + \frac{\widehat{C}}{c}\,2\eta_0\xi_p\right) \ \le\ \frac{1}{c}\,2\eta_0\xi_{p+1}, \tag{5.9}$$

because $\eta_p \le 2\eta_0$ for all p.

2. Concerning $\left\|h_x^{i,p+1}(t)\right\|_{\mathbf{L}^1}$, for every $t \ge 0$ the same calculations as in Section 3, step 2, yield

$$\int_{I\!R} \left|h_x^{i,p+1}(x)\right| dx \le \frac{8}{\sqrt{\pi}}\widehat{C}^2 2\eta_0\xi_{p+1}. \tag{5.10}$$

3. Furthermore, the same computation as in Section 3, step 3 yields

$$\int_0^\infty \int_{I\!R} \left|h_x^{i,p+1}(t,x)\right| \left|v^j(t,x)\right| dxdt \le \frac{20}{\sqrt{\pi}}\frac{\widehat{C}^2}{c}\,(2\eta_0)^2\xi_{p+1}. \tag{5.11}$$

Using the above estimates (5.8)–(5.11) and recalling that η_0 also satisfies (3.36), we obtain

$$\begin{aligned} \int_0^\infty \int_{I\!R} \left|\psi^{i,p+1}(t,x)\right| dxdt &\le \frac{C_0}{c}\,2\eta_0\xi_{p+1}\left[n^2\left(1 + 2\frac{20}{\sqrt{\pi}}\widehat{C}\,(2\eta_0) + \frac{8n}{\sqrt{\pi}}\widehat{C}^2\,(2\eta_0)^2\right) + 2n\right] \\ &\le \frac{\widehat{C}}{c}\,2\eta_0\xi_{p+1}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} \int_{I\!R} \left|\psi^{i,p+1}(t,x)\right| dx &\le C_0\frac{8}{\sqrt{\pi}}\widehat{C}^2\,(2\eta_0)^2\xi_{p+1}\left[n^2\left(1 + 2\frac{8}{\sqrt{\pi}}\widehat{C}^2\,2\eta_0 + \frac{8n}{\sqrt{\pi}}\widehat{C}^2\,(2\eta_0)^2\right) + 2n\right] \\ &\le \frac{8}{\sqrt{\pi}}\widehat{C}^3\,(2\eta_0)^2\xi_{p+1}. \end{aligned} \tag{5.13}$$

By (2.12), the sequence ξ_p is bounded by

$$\lim_{p\to+\infty} \xi_p = \left(1 - \frac{\widehat{C}}{c}\,2\eta_0\right)^{-1}\xi_0 \le 2\xi_0. \tag{5.14}$$

After computations entirely similar to those in Section 4, we conclude that the sequence $h^p = (h^{1,p}, \ldots, h^{n,p})$ tends in $W^{1,1}$ to the classical solution h of (5.1) and

$$\sup_{t\in I\!R^+} \|h(t)\|_{\mathbf{L}^1} \le 2\|h_0\|_{\mathbf{L}^1}. \tag{5.15}$$

We consider now the case $\bar u \in \mathcal{D}$ and $\bar h \in BV$. Denoting the components $\hat h^i \doteq \mathbf{e}_i \cdot h$, we can write the solution of (5.1) in the form

$$
\begin{aligned}
\hat h^i(t,x) = {} & \int_{I\!R} G^i(t,x-y)d\bar h^i(y) - \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(A(u)-A^*\big)h(s,y)\Big]\,dyds \\
& + \int_0^t \int_{I\!R} G^i(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds,
\end{aligned}
\tag{5.16}
$$

$$
\begin{aligned}
\hat h^i_x(t,x) = {} & \int_{I\!R} G^i_x(t,x-y)\,d\bar h^i - \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(A(u)-A^*\big)h(s,y)\Big]_x\,dyds \\
& + \int_0^t \int_{I\!R} G^i_x(t-s,x-y)\mathbf{e}_i \cdot \Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds.
\end{aligned}
\tag{5.17}
$$

Using the same techniques of Section 4, in this case relying on the assumptions

$$
\begin{aligned}
\big\|h_x(t)\big\|_{\mathbf{L}^1} \leq C\frac{\|\bar h\|_{\mathbf{L}^1}}{\sqrt{t}}, \qquad C = \max\left\{\frac{9}{\sqrt{\pi}},\ \|DA\|_\infty\right\}, \\
\sqrt{t} < \sqrt{t_0} \doteq \frac{1}{9nC\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar u\}},
\end{aligned}
\tag{5.18}
$$

for h, and recalling (4.17), we can estimate the $\mathbf{L}^1$-norms of (5.16) and (5.17) as

$$
\begin{aligned}
\big\|h(t)\big\|_{\mathbf{L}^1} &\leq \|\bar h\|_{\mathbf{L}^1} + \left(4C + \frac{4}{\sqrt{\pi}}\right)n\sqrt{t}\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar u\} \cdot \sup_{0<s\leq t}\big\|h(s)\big\|_{\mathbf{L}^1} \leq 2\|\bar h\|_{\mathbf{L}^1}\,, \\
\big\|h_x(t)\big\|_{\mathbf{L}^1} &< \frac{\|\bar h\|_{\mathbf{L}^1}}{\sqrt{\pi t}} + \frac{16n}{\sqrt{\pi}}C\|DA\|_\infty \cdot \text{Tot.Var.}\{\bar u\} \cdot \sup_{0<s\leq t}\big\|h(s)\big\|_{\mathbf{L}^1} \leq C\frac{\|\bar h\|_{\mathbf{L}^1}}{\sqrt{t}}.
\end{aligned}
$$

In particular at time t_0, repeating the arguments at (4.22), for some constant C' we have

$$
\begin{aligned}
\big\|h^i(t_0)\big\|_{\mathbf{L}^1} &\leq nC'\,\big\|h(t_0)\big\|_{\mathbf{L}^1} \leq 2nC' \cdot \|\bar h\|_{\mathbf{L}^1} \leq \xi_0, \\
\big\|h^i_x(t_0)\big\|_{\mathbf{L}^1} &\leq nC'\,\big(1 + 2n\text{Tot.Var.}\{\bar u\}\big)\|h_x(t_0)\|_{\mathbf{L}^1} \\
&\leq nC'\,\big(1 + 2n\text{Tot.Var.}\{\bar u\}\big)9nC^3\text{Tot.Var.}\{\bar u\}\|\bar h\|_{\mathbf{L}^1} \leq \frac{4}{\sqrt{\pi}}\widehat{C}^2\,2\eta_0\xi_0,
\end{aligned}
\tag{5.19}
$$

provided that $\widehat{C}$ is big enough and the total variation of $\bar u$ sufficiently small. The previous estimates on the norms $\big\|h^i(t)\big\|$ can thus be applied for $t \in [t_0,\ \infty[$, proving the estimates (2.24) on tangent vectors. By (2.26), this completes the proof of Theorem 1.

6 - The vanishing viscosity limit

To complete the proof of Theorem 2 given in Section 2, it only remains to prove the bound (2.32), for some constants L, $\tilde{\lambda}$. Toward this goal, we first establish an estimates on the size of a solution h to the linear variational equation (5.1). Using the integral representation

$$\begin{aligned} h(t,x) = &\int_{I\!R} G(t,x-y)\bar{h}(y)\,dy - \int_0^t \int_{I\!R} G_x(t-s,x-y)A(u)h(s,y)\,dyds \\ &+ \int_0^t \int_{I\!R} G(t-s,x-y)\Big[\big(u_x \bullet A(u)\big)h(s,y) - \big(h \bullet A(u)\big)u_x(s,y)\Big]\,dyds, \end{aligned} \tag{6.1}$$

we obtain the bound

$$\begin{aligned} \big|h(t,x)\big| \leq &\int_{I\!R} G(t,x-y)\big|\bar{h}(y)\big|\,dy + \|A\|_\infty \int_0^t \int_{I\!R} \big|G_x(t-s,x-y)\big|\,\big|h(s,y)\big|\,dyds \\ &+ 2\|DA\|_\infty \int_0^t \int_{I\!R} \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \cdot G(t-s,x-y)\big|h(s,y)\big|\,dyds\,. \end{aligned} \tag{6.2}$$

Assuming that the initial data satisfies

$$\big|h(0,x)\big| \leq e^{-x} \qquad \text{for all } x \in I\!R, \tag{6.3}$$

we claim that

$$\big|h(t,x)\big| \leq E^+(t,x) \doteq f(t)\exp\left\{4\|DA\|_\infty \cdot \int_0^t \big\|u_x(s)\big\|_{\mathbf{L}^\infty}\,ds + t - x\right\} \tag{6.4}$$

for all $t \geq 0$, $x \in I\!R$, where f is an increasing function such that

$$f(t) \geq 1 + \int_0^t 2\|A\|_\infty f(s)\left(\frac{1}{\sqrt{t-s}} + \sqrt{\pi}\right)ds, \qquad f(0) = 1.$$

It is easy to show that we can take a function $f(t)$ satisfying $f(t) \leq 2\exp\{Ct\}$, if the constant C is big enough. The bound (6.4) follows from the estimates (see appendix C)

$$\int_{I\!R} G(t,x-y)E^+(0,y)\,dy \ \leq\ e^{t-x}, \tag{6.5}$$

$$\|A\|_\infty \cdot \int_0^t \int_{I\!R} \big|G_x(t-s,x-y)\big|E^+(s,y)\,dyds \ \leq\ \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}, \tag{6.6}$$

$$2\|DA\|_\infty \cdot \int_0^t \int_{I\!R} \big\|u_x(s)\big\|_{\mathbf{L}^\infty} \cdot G(t-s,x-y)E^+(s,y)\,dyds \ \leq\ \frac{1}{2}E^+(t,x) - \frac{1}{2}e^{t-x}, \tag{6.7}$$

by a standard comparison argument. An entirely similar computation shows that, if $\big|h(0,x)\big| \leq e^x$, then for all t,x one has

$$\big|h(t,x)\big| \leq E^-(t,x) \doteq f(t)\exp\left\{4\|DA\|_\infty \cdot \int_0^t \big\|u_x(s)\big\|_{L^\infty}\,ds + t + x\right\}. \tag{6.8}$$

Observe that, by (4.19), we can choose a constant $\tilde\lambda$ large enough so that

$$E^+(t,x) \le 2e^{\tilde\lambda(\sqrt{t}+t)-x}, \qquad E^-(t,x) \le 2e^{\tilde\lambda(\sqrt{t}+t)+x}. \tag{6.9}$$

The estimates (6.9) are very rough bounds on the solutions of (6.1): in fact we just need a bound on the speed of propagation of perturbation. Now consider any two initial data $\bar u, \bar w$ satisfying the assumption (1.13) of Theorem 1. For every interval $[a,b]$ and any $\lambda > \tilde\lambda$, defining the path $\theta \mapsto \bar u^\theta$ as in (2.25), we compute

$$\begin{aligned}\int_{a+\lambda t}^{b-\lambda t} \big|u(t,x)-w(t,x)\big|\,dx &\le \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \left|\frac{du^\theta}{d\theta}(t,x)\right|\,dx\,d\theta \\ &= \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \big|h^\theta(t,x)\big|\,dx\,d\theta\,,\end{aligned} \tag{6.10}$$

where h^θ is the solution of the Cauchy problem (2.27)-(2.28). By linearity, we can write this solution as a sum:

$$h^\theta = h_1^\theta + h_2^\theta + h_3^\theta,$$

where h_j^θ, $j = 1,2,3$ denote respectively the solutions of (2.27) with initial data

$$h_1^\theta(0,x) = \begin{cases} \bar u(x) - \bar w(x) & \text{if } x \in [a,b], \\ 0 & \text{otherwise,}\end{cases}$$

$$h_2^\theta(0,x) = \begin{cases} \bar u(x) - \bar w(x) & \text{if } x < a, \\ 0 & \text{otherwise,}\end{cases} \qquad h_3^\theta(0,x) = \begin{cases} \bar u(x) - \bar w(x) & \text{if } x > b, \\ 0 & \text{otherwise.}\end{cases}$$

Applying the estimates (2.24) to h_1^θ, (6.4) to h_2^θ and (6.8) to h_3^θ we obtain

$$\begin{aligned}&\int_{a+\lambda t}^{b-\lambda t} \big|u(t,x)-w(t,x)\big|\,dx \le \int_0^1 \int_{a+\lambda t}^{b-\lambda t} \Big(\big|h_1^\theta(t,x)\big| + \big|h_2^\theta(t,x)\big| + \big|h_3^\theta(t,x)\big|\Big)\,dx d\theta \\ &\le L \cdot \|h_1^\theta(0)\|_{\mathbf{L}^1} + 2(b-a-2\lambda)\big\|h^\theta(0,\cdot)\big\|_{\mathbf{L}^\infty} \cdot \Big(\sup_{x<b-\lambda t} e^{\tilde\lambda(\sqrt{t}+t)-b+x} + \sup_{x>a+\lambda t} e^{a+\tilde\lambda(\sqrt{t}+t)-x}\Big) \\ &\le L \cdot \int_a^b \big|\bar u(x) - \bar w(x)\big|\,dx + 2(b-a)\|\bar u - \bar w\|_{\mathbf{L}^\infty} \cdot e^{(\tilde\lambda-\lambda)t+\tilde\lambda\sqrt{t}}.\end{aligned} \tag{6.11}$$

Given $\varepsilon > 0$, consider now two solutions $u^\varepsilon, w^\varepsilon$ of (1.3), with initial data $\bar u, \bar w$. Applying the previous estimates to the corresponding solutions obtained via the rescalings $t \mapsto t/\varepsilon$, $x \mapsto x/\varepsilon$, and observing that the distance $\big|\bar u(x) - \bar w(x)\big|$ is always bounded by the diameter of the compact set K_1, from (6.11) we deduce

$$\int_{a+\lambda t}^{b-\lambda t} \big|u^\varepsilon(t,x) - w^\varepsilon(t,x)\big|\,dx \le L \cdot \int_a^b \big|\bar u(x) - \bar w(x)\big|\,dx + 2(b-a)\cdot \mathrm{diam}(K_1)\cdot e^{(\tilde\lambda-\lambda)t/\varepsilon+\tilde\lambda\sqrt{t/\varepsilon}}.$$

Letting $\varepsilon \to 0$, the vanishing viscosity limits $u^\varepsilon \to u$, $w^\varepsilon \to w$ thus satisfy

$$\int_{a+\lambda t}^{b-\lambda t} \big|u(t,x) - w(t,x)\big|\,dx \le L \cdot \int_a^b \big|\bar u(x) - \bar w(x)\big|\,dx. \tag{6.12}$$

Since (6.12) is valid for every $\lambda > \tilde\lambda$, it implies (2.32).

7 - Hyperbolic systems on manifolds

Let $\mathcal{M}$ be a smooth n-dimensional Riemann and call T_u the tangent space at $u \in \mathcal{M}$. At each point u, let $A(u) : T_u \mapsto T_u$ be a linear mapping, smoothly depending on $u \in \mathcal{M}$. Assume that each $A(u)$ is *strictly hyperbolic*, i.e. it has n real distinct eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$. If $u : [0, \infty[\times I\!\!R \mapsto \mathcal{M}$ is a smooth map, at any given (t, x) the partial derivatives u_t, u_x are vectors in $T_{u(t,x)}$. It is thus meaningful to consider the system

$$u_t + A(u)u_x = 0 \tag{7.1}$$

In a given set of coordinates, this yields a standard quasilinear hyperbolic system. Smooth solutions are thus well defined. On the other hand. since the equations are not in conservation form, there is no canonical way for defining discontinuous solutions.

Toward this goal, a possible approach is to consider a Riemannian structure on $\mathcal{M}$. In this case, one can choose bases of right and left eigenvectors $r_1(u), \ldots, r_n(u) \in T_u$ and $l^1(u), \ldots, l^n(u) \in T_u^*$ normalized as in (1.6):

$$\langle r_i,\ r_i \rangle = 1, \qquad\qquad l^i \cdot r_j = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{7.2}$$

For any $\varepsilon > 0$ we can now consider the parabolic system

$$u_t + A(u)u_x = \varepsilon \sum_i \big(l^i(u) \cdot u_x\big)_x r_i(u). \tag{7.3}$$

Definition. If there exists a sequence of solutions u^ε of (7.3) which converges to a function $u = u(t, x)$ in $\mathbf{L}^1_{\mathrm{loc}}$, we say that u is a *vanishing viscosity solution* of (7.1).

The same computations as in (1.7)–(1.9) now yield

$$\sum_i (u_x^i)_t r_i + \sum_i (\lambda_i u_x^i)_x r_i + \sum_{j \neq k} \lambda_j u_x^j u_x^k [r_k, r_j] = \varepsilon \left\{ \sum_i (u_x^i)_{xx} r_i + 2 \sum_{j \neq k} (u_x^j)_x u_x^k [r_k, r_j] \right\}. \tag{7.4}$$

Taking the products of (7.4) with $l^i(u)$ and calling $v^i \doteq l^i(u) \cdot u_x^i$, $i = 1, \ldots, n$, we obtain a system similar to (1.11), where the quadratic terms involve only products of waves of distinct families:

$$v_t^i + (\lambda_i v^i)_x - v_{xx}^i = \sum_{j \neq k} G_{ijk} v^j v^k + \sum_{j \neq k} H_{ijk} v_x^j v^k. \tag{7.5}$$

All of our previous analysis can thus be applied. In the following, for simplicity we assume that a compact $K_0 \subset \mathcal{M}$ is given, which lies in the domain of a single chart. In this case, the total variation of a function $\bar{u} : I\!\!R \mapsto \mathcal{M}$ and its $\mathbf{L}^1$-norm can be referred to one particular system of coordinates. The extension to the general case is straightforward.

Theorem 3. *Let (7.1) be a strictly hyperbolic system on a Riemannian manifold $\mathcal{M}$. For every compact set $K_0 \subset \mathcal{M}$, there exist constants $L, L', \eta_0 > 0$, a closed domain $\mathcal{D} \subset \mathbf{L}^1_{\rm loc}$ and a continuous semigroup $S : \mathcal{D} \times [0, \infty[\mapsto \mathcal{D}$ with the following properties.*

(i) Every function $\bar u$ satisfying (1.13) lies in the domain $\mathcal{D}$ of the semigroup.

(ii) For every $\bar u, \bar w \in \mathcal{D}$ with $\bar u - \bar w \in \mathbf{L}^1$ and every $t, s \geq 0$ one has

$$\|S_t \bar u - S_s \bar w\|_{\mathbf{L}^1} \leq L'|t-s| + L\|\bar u - \bar w\|_{\mathbf{L}^1} . \tag{1.23}$$

(iii) For every piecewise constant initial data $\bar u \in \mathcal{D}$, there exists $\tau > 0$ such that the following holds. For $t \in [0, \tau]$, the semigroup trajectory $S_t \bar u$ coincides with the function $u(t, \cdot)$ obtained by piecing together the solutions of the Riemann problems determined by the jumps in $\bar u$, constructed by the Riemann Solver at (1.16)–(1.22).

(iv) Every trajectory of the semigroup is a vanishing viscosity solution of (7.1).

Observe that all steps in the construction of the Riemann Solver at (1.18)–(1.22) remain meaningful also in the case of a Riemann manifold.

To prove the theorem, we observe that all the estimates derived in the previous sections remain valid in this case. In particular, for any smooth initial data $\bar u : I\!R \mapsto \mathcal{M}$ with small total variation, the corresponding solution of (7.3) is well defined for all $t \geq 0$, and its total variation remains uniformly small. Moreover, by the analysis in Sections 5-6, these solutions depend Lipschitz continuously on the initial data, with a Lipschitz constant independent of $\varepsilon > 0$, and converge to a limit as $\varepsilon \to 0$.

On the other hand, by the results in [14], the Riemann Solver described at (1.18)–(1.22) generates a unique Lipschitz semigroup S. To show that he trajectories of this semigroup coincide with our vanishing viscosity limits, by [3] it suffices to check the case of a Riemann initial data. Assume first that both states u^-, u^+ in (1.17) lie on a single i-rarefaction curve, say

$$u^+ = R_i(\sigma_i)(u^-).$$

Then the particular form of the diffusion operator implies that the solution of (7.3) remains on the same i-rarefaction curve for all positive times. We can thus write

$$u(t, x) = R_i\big(z(t, x)\big)(u^-),$$

where z is the solution of the scalar Cauchy problem (1.21).

The general case is proved by considering small perturbations $\bar u^\delta$ of the initial data, as in (2.35), and using the Lipschitz continuous dependence of vanishing viscosity solutions.

Appendix A

By the same computations as in Section 3, step 1, it follows that (3.10) holds also if we translate $v^{j,p+1}$:

$$\int_0^\infty \int_{I\!R} \left|v^{i,p+1}(t,x)\right| \left|v^{j,p+1}(t+u,x+y)\right| \leq \frac{1}{c}\eta_{p+1}^2. \qquad (A.1)$$

A.1. Estimation of (3.23): from (2.10) and (A.1) it follows

$$\begin{aligned}
&\int_0^\alpha \iint_{I\!R^2} \left|G^i(t,x-y)\,\bar v_x^i(y)\,v^{j,p+1}(t+u,x+z)\right| dx dy\, dt \\
&\quad \leq \int_{I\!R} \left|\bar v_x^i(y)\right| \int_0^\infty \int_{I\!R} G^i(t,x-y)\left|v^{j,p+1}(t+u,x+z)\right| dx\, dt\, dy \\
&\quad \leq \frac{4}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2\frac{\eta_{p+1}}{c}\,.
\end{aligned}$$

A.2. Estimation of (3.24): the definitions of $\mathcal{I}_\alpha$ in (3.22) and of $\hat t$ in (3.20) yield

$$\begin{aligned}
&\int_0^\alpha \int_0^t \iint_{I\!R^2} \left|G_x^i(s,y)\left[\lambda_i(u^p)-\lambda_i^*\right]v_x^{i,p+1}(t-s,x-y)v^{j,p+1}(t+u,x+z)\right| dx dy\, ds dt \\
&\leq \|\lambda_i\|_\infty \int_0^\alpha \int_{I\!R} \left|G_x^i(s,y)\right| \int_s^\alpha \int_{I\!R} \left|v_x^{i,p+1}(t-s,x-y)v^{j,p+1}(t+u,x+z)\right| dx dt\, dy ds \\
&\leq \|\lambda_i\|_\infty \int_0^\alpha \int_{I\!R} \left|G_x^i(s,y)\right| \int_0^{\alpha-s} \int_{I\!R} \left|v_x^{i,p+1}(t,x)v^{j,p+1}(t+u+s,x+z+y)\right| dx dt\, dy ds \\
&\leq \frac{2}{\sqrt{\pi}}\sqrt{\alpha}\|\lambda_i\|_\infty\mathcal{I}_\alpha \;\leq\; \frac{2}{\sqrt{\pi}}\sqrt{\hat t}\|\lambda_i\|_\infty\mathcal{I}_\alpha \;\leq\; \frac{2n}{\sqrt{\pi}}\sqrt{\hat t}\|\lambda_i'\|_\infty\eta_{p+1}\mathcal{I}_\alpha \;\leq\; \frac{1}{8}\mathcal{I}_\alpha.
\end{aligned}$$

A.3. Evaluation of (3.25): by (3.20) and (A.1) we have

$$\begin{aligned}
&\int_0^\alpha \int_0^t \iint_{I\!R^2} \left|G_x^i(s,y)\Big((r_k\bullet\lambda_i)v^{k,p}\Big)v^{i,p+1}(t-s,x-y)v^{j,p+1}(t+u,x+z)\right| dy dx\, ds dt \\
&\leq \|\lambda_i'\|_\infty \sup_{t\geq 0}\left\|v^{k,p}(t)\right\|_{\mathbf{L}^\infty} \\
&\qquad \cdot \int_0^\alpha \int_{I\!R} \left|G_x^i(s,y)\right| \int_s^\alpha \int_{I\!R} \left|v^{i,p+1}(t-s,x-y)\cdot v^{j,p+1}(t+u,x+z)\right| dx dt\, dy ds \\
&\leq \|\lambda_i'\|_\infty \sup_{t\geq 0}\left\|v_x^{k,p}(t)\right\|_{\mathbf{L}^1} \\
&\qquad \cdot \int_0^\alpha \int_{I\!R} \left|G_x^i(s,y)\right| \int_0^{\alpha-s} \int_{I\!R} \left|v^{i,p+1}(t,x)\cdot v^{j,p+1}(t+s+u,x+y+z)\right| dx dt\, dy ds \\
&\leq \|\lambda_i'\|_\infty \frac{8}{\sqrt{\pi}}\widehat{C}^2\eta_{p+1}^2\frac{2}{\sqrt{\pi}}\sqrt{\alpha}\left(\frac{1}{c}\eta_{p+1}^2\right) \\
&\leq \frac{16}{c\pi}\sqrt{\hat t}\|\lambda_i'\|_\infty\widehat{C}^2\eta_{p+1}^4\,.
\end{aligned}$$

A.4. Estimation of (3.26): using (3.20) and (2.17) we obtain

$$\int_0^\alpha \int_0^t \iint_{I\!R^2} \left| G_x^i(s,y) \phi^{i,p}(t-s,x-y) v^{j,p+1}(t+u,x+z) \right| dydx\, dsdt$$
$$\leq \sup_{t\geq 0} \left\| v^{i,p+1}(t) \right\|_{\mathbf{L}^\infty} \int_0^\alpha \int_{I\!R} \left| G_x^i(s,y) \right| \int_s^\alpha \int_{I\!R} \left| \phi^{i,p}(t-s,x-y) \right| dxdt\, dyds$$
$$\leq \frac{8}{\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^2 \frac{2}{\sqrt{\pi}} \sqrt{\alpha} \frac{\widehat{C}}{c} \eta_p^2 \;\leq\; \frac{16}{c\pi} \sqrt{\hat{t}} \widehat{C}^3 \eta_{p+1}^4 \;=\; \frac{2}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3.$$

A.5. Estimation of (3.29): from (2.10) and (A.1) it follows

$$\int_{\hat{t}}^\alpha \iint_{I\!R^2} \left| G_x^i(\hat{t},y) v^{i,p+1}(t-\hat{t},x-y) v^{j,p+1}(t+u,x+z) \right| dydx\, dt$$
$$= \int_{I\!R} \left| G_x^i(\hat{t},y) \right| \int_0^{\alpha-\hat{t}} \int_{I\!R} \left| v^{i,p+1}(t,x-y) \right| \left| v^{j,p+1}(t+u+\hat{t},x+z) \right| dxdt\, dy$$
$$\leq \frac{1}{\sqrt{\pi \hat{t}}} \frac{\eta_{p+1}^2}{c} \;=\; \frac{8}{c\pi} \widehat{C} \eta_{p+1}^3\,.$$

A.6. Estimation of (3.30): the definitions of $\mathcal{I}_\alpha$ in (3.22) and of $\hat{t}$ in (3.15) yield

$$\int_{\hat{t}}^\alpha \int_0^{\hat{t}} \iint_{I\!R^2} dxdy \left| G_x^i(s,y) \left[\lambda_i(u^p) - \lambda_i^* \right] v_x^{i,p+1}(t-s,x-y) v^{j,p+1}(t+u,x+z) \right| dxdy\, dsdt$$
$$\leq \|\lambda_i\|_\infty \int_0^{\hat{t}} \int_{I\!R} \left| G_x^i(s,y) \right| \int_{\hat{t}-s}^{\alpha-s} \int_{I\!R} \left| v_x^{i,p+1}(t,x) \right| \left| v^{j,p+1}(t+u+s,x+z+y) \right| dxdt\, dyds$$
$$\leq \frac{2n}{\sqrt{\pi}} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \eta_{p+1} \mathcal{I}_\alpha \;\leq\; \frac{1}{8} \mathcal{I}_\alpha.$$

A.7. Estimation of (3.31): by (3.20) and (A.1) we have

$$\int_{\hat{t}}^\alpha \int_0^{\hat{t}} \iint_{I\!R^2} \left| G_x^i(s,y) \big((r_k \bullet \lambda_i) v^{k,p} \big) v^{i,p+1}(t-s,x-y) v^{j,p+1}(t+u,x+z) \right| dxdy\, dsdt$$
$$\leq \|\lambda_i'\|_\infty \sup_{t\geq 0} \left\| v^{k,p}(t) \right\|_{\mathbf{L}^\infty}$$
$$\cdot \int_0^{\hat{t}} \int_{I\!R} \left| G_x^i(s,y) \right| \int_{\hat{t}}^\alpha \int_{I\!R} \left| v^{i,p+1}(t-s,x-y) \right| \left| v^{j,p+1}(t+u,x+z) \right| dxdt\, dyds$$
$$\leq \frac{2}{\sqrt{\pi}} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \frac{8}{\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^2 \left(\frac{1}{c} \eta_{p+1}^2 \right) \;=\; \frac{16}{c\pi} \sqrt{\hat{t}} \|\lambda_i'\|_\infty \widehat{C}^2 \eta_{p+1}^4.$$

A.8. Estimation of (3.32): using (3.20) and (2.17) we obtain

$$\int_{\hat{t}}^\alpha \int_0^{\hat{t}} \iint_{I\!R^2} \left| G_x^i(s,y) \phi^{i,p}(t-s,x-y) v^{j,p+1}(t+u,x+z) \right| dxdy\, dsdt$$
$$\leq \sup_{t\geq 0} \left\| v^{i,p+1}(t) \right\|_{\mathbf{L}^\infty} \int_0^{\hat{t}} \int_{I\!R} \left| G_x^i(s,y) \right| \int_{\bar{t}}^\alpha \int_{I\!R} \left| \phi^{i,p}(t-s,x-y) \right| dxdt\, dyds$$
$$\leq \frac{8}{\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^2 \frac{2}{\sqrt{\pi}} \sqrt{\hat{t}}\, \frac{\widehat{C}}{c} \eta_p^2 = \frac{16}{c\pi} \sqrt{\hat{t}}\, \widehat{C}^3 \eta_{p+1}^4 \;=\; \frac{2}{c\sqrt{\pi}} \widehat{C}^2 \eta_{p+1}^3.$$

Appendix B

B.1. Estimation of (4.6):

$$
\begin{aligned}
&\left\| \sum_{j,k=1}^{n} G^i_{j,k}(u^p) v^{j,p} v^{k,p} - G^i_{j,k}(u^{p-1}) v^{j,p-1} v^{k,p-1} \right\|_{\mathbf{L}^1} \\
&\leq \sum_{j,k=1}^{n} \Big\{ \left\| \big(G^i_{j,k}(u^p) - G^i_{j,k}(u^{p-1})\big) v^{j,p} v^{k,p} \right\|_{\mathbf{L}^1} + \left\| G^i_{j,k}(u^{p-1})(v^{j,p} - v^{j,p-1}) v^{k,p} \right\|_{\mathbf{L}^1} \\
&\qquad\qquad + \left\| G^i_{j,k}(u^{p-1}) v^{j,p-1}(v^{k,p} - v^{k,p-1}) \right\|_{\mathbf{L}^1} \Big\} \\
&\leq \frac{8n^2}{\sqrt{\pi}} \widehat{C}^2 \big(2\eta_0\big)^3 L \|u^p - u^{p-1}\|_{\mathbf{L}^\infty} + \frac{8}{\sqrt{\pi}} \widehat{C}^3 \big(2\eta_0\big)^2 \|v^p - v^{p-1}\|_{\mathbf{L}^1}.
\end{aligned}
$$

B.2. Estimation of (4.7):

$$
\begin{aligned}
&\left\| \sum_{j,k=1}^{n} H^i_{j,k}(u^p) v_x^{j,p} v^{k,p} - H^i_{j,k}(u^{p-1}) v_x^{j,p-1} v^{k,p-1} \right\|_{\mathbf{L}^1} \\
&\leq \sum_{j,k=1}^{n} \Big\{ \left\| \big(H^i_{j,k}(u^p) - H^i_{j,k}(u^{p-1})\big) v_x^{j,p} v^{k,p} \right\|_{\mathbf{L}^1} + \left\| H^i_{j,k}(u^{p-1})(v_x^{j,p} - v_x^{j,p-1}) v^{k,p} \right\|_{\mathbf{L}^1} \\
&\qquad\qquad + \left\| H^i_{j,k}(u^{p-1}) v_x^{j,p-1}(v^{k,p} - v^{k,p-1}) \right\|_{\mathbf{L}^1} \Big\} \\
&\leq \frac{64n^2}{\pi} \widehat{C}^4 \big(2\eta_0\big)^4 L \|u^p - u^{p-1}\|_{\mathbf{L}^\infty} + \frac{8}{\sqrt{\pi}} \widehat{C}^3 \big(2\eta_0\big)^2 \|v_x^p - v_x^{p-1}\|_{\mathbf{L}^1}.
\end{aligned}
$$

B.3. Estimation of (4.8):

$$
\begin{aligned}
&\left\| \sum_{j,k,\ell=1}^{n} K^i_{j,k,\ell}(u^p) v^{j,p} v^{k,p} v^{\ell,p} - K^i_{j,k,\ell}(u^{p-1}) v^{j,p-1} v^{k,p-1} v^{\ell,p-1} \right\|_{\mathbf{L}^1} \\
&\leq \sum_{j,k,\ell=1}^{n} \Big\{ \left\| \big(K^i_{j,k,\ell}(u^p) - K^i_{j,k,\ell}(u^{p-1})\big) v^{j,p} v^{k,p} v^{\ell,p} \right\|_{\mathbf{L}^1} + \left\| K^i_{j,k,\ell}(u^{p-1})(v^{j,p} - v^{j,p-1}) v^{k,p} v^{\ell,p} \right\|_{\mathbf{L}^1} \\
&\quad + \left\| K^i_{j,k,\ell}(u^{p-1}) v^{j,p-1}(v^{k,p} - v^{k,p-1}) v^{\ell,p} \right\|_{\mathbf{L}^1} + \left\| K^i_{j,k,\ell}(u^{p-1}) v^{j,p-1} v^{k,p-1}(v^{\ell,p} - v^{\ell,p-1}) \right\|_{\mathbf{L}^1} \Big\} \\
&\leq \frac{64n^3}{\pi} \widehat{C}^4 \big(2\eta_0\big)^5 L \|u^p - u^{p-1}\|_{\mathbf{L}^\infty} + \frac{96n}{\sqrt{\pi}} \widehat{C}^5 \big(2\eta_0\big)^4 \|v^p - v^{p-1}\|_{\mathbf{L}^1}.
\end{aligned}
$$

Appendix C

C.1. Estimation of (6.5):

$$\int_R G(t,x-y)E^+(0,y)\,dy \le \frac{1}{2\sqrt{\pi t}}\int_{I\!R} \exp\left\{\frac{(y+2t-x)^2}{4t}+t-x\right\}\,dy = e^{t-x}\,.$$

C.2. Estimation of (6.6):

$$\begin{aligned}
&\|A\|_\infty\cdot\int_0^t\int_{I\!R}\big|G_x(t-s,x-y)\big|E^+(s,y)\,dy\,ds\\
&=\|A\|_\infty\int_0^t\int_{I\!R}\frac{|x-y|}{4(t-s)\sqrt{\pi(t-s)}}f(s)\\
&\qquad\qquad\cdot\exp\left\{-\frac{(x-y)^2}{4(t-s)}+4\|DA\|_\infty\int_0^s\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+s-y\right\}\\
&\le\|A\|_\infty\exp\left\{4\|DA\|_\infty\int_0^t\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+t-x\right\}\\
&\qquad\qquad\cdot\int_0^t\frac{f(s)}{4(t-s)\sqrt{\pi(t-s)}}\cdot\int_{I\!R}|x-y|\exp\left\{-\frac{(y+2(t-s)-x)^2}{4(t-s)}\right\}\,dyds\\
&=\exp\left\{4\|DA\|_\infty\int_0^t\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+t-x\right\}\cdot\int_0^t\frac{\|A\|_\infty f(s)}{\sqrt{\pi(t-s)}}\int_{I\!R}\big|\zeta-\sqrt{t-s}\big|e^{-\zeta^2}\,d\zeta ds\\
&\le\exp\left\{4\|DA\|_\infty\int_0^t\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+t-x\right\}\cdot\int_0^t\|A\|_\infty f(s)\left(\frac{1}{\sqrt{t-s}}+\sqrt{\pi}\right)ds\\
&\le\exp\left\{4\|DA\|_\infty\int_0^t\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+t-x\right\}\cdot\left(\frac{f(t)}{2}-\frac{1}{2}\right)\\
&=\frac{1}{2}E^+(t,x)-\frac{1}{2}\exp\left\{4\|DA\|_\infty\int_0^t\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma+t-x\right\}\\
&\le\frac{1}{2}E^+(t,x)-\frac{1}{2}e^{t-x}\,.
\end{aligned}$$

C.3. Estimation of (6.7):

$$\begin{aligned}
&2\|DA\|_\infty\cdot\int_0^t\big\|u_x(s)\big\|_{\mathbf{L}^\infty}\int_{I\!R}G(t-s,x-y)E^+(s,y)\,dyds\\
&=2\|DA\|_\infty\cdot\int_0^t f(s)\cdot\exp\left\{4\|DA\|_\infty\int_0^s\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}\,d\varsigma+s\right\}\\
&\qquad\qquad\cdot\frac{\big\|u_x(s)\big\|_{\mathbf{L}^\infty}}{2\sqrt{\pi(t-s)}}\int_{I\!R}\exp\left\{-\frac{(x-y)^2}{4(t-s)}-y\right\}\,dy\,ds\\
&\le f(t)\cdot e^{t-x}\cdot\int_0^t 2\|DA\|_\infty\big\|u_x(s)\big\|_{\mathbf{L}^\infty}\cdot\exp\left\{4\|DA\|_\infty\cdot\int_0^s\big\|u_x(\varsigma)\big\|_{\mathbf{L}^\infty}d\varsigma\right\}\,ds\\
&\le\frac{1}{2}E^+(t,x)-\frac{1}{2}f(t)\cdot e^{t-x}\\
&\le\frac{1}{2}E^+(t,x)-\frac{1}{2}e^{t-x}.
\end{aligned}$$

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