STABILITY OF A RESONANT SYSTEM OF CONSERVATION LAWS MODELING POLYMER FLOW WITH GRAVITATION

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ABSTRACT. We prove L^1 uniqueness and stability for a resonant 2×2 system of conservation laws that arise as a model for two phase polymer flow in porous media. The analysis uses the equivalence of the Eulerian and Lagrangian formulation of this system, and the results are first established for an auxiliary scalar equation. Our methods are based on front tracking approximations for the auxiliary equation, and the Kružkov entropy condition for scalar conservation laws.

1. Introduction

We study the initial value problem for scalar conservation laws of type

$$u_t + f(a(x), u)_x = 0, \tag{1.1}$$

where the coefficient a is a suitable function of x. Using the transformation given by Wagner in [23], for $u \neq 0$, one finds the equivalent 2×2 system:

$$\begin{aligned} \left(\frac{1}{u}\right)_t - \left(uf\left(a, \frac{1}{u}\right)\right)_x &= 0 \\ \left(\frac{a}{u}\right)_t - \left(auf\left(a, \frac{1}{u}\right)\right)_x &= 0. \end{aligned} \tag{1.2}$$

If $1/u$ is interpreted as the saturation of water, and a the concentration of polymer dissolved in the water, then (1.2) is a model of polymer flow in a porous two phase environment. This model is one of the prime motivations for the present paper. Furthermore, conservation laws of the type (1.1) are related to models of transonic flow of gas in a variable area duct.

If the coefficient a is smooth, and a' is of bounded variation, one can use finite differences to show existence of a weak solution to (1.1). This method was first used by Oleinik in [19], where she used the Lax-Friedrichs scheme.

As long as a is smooth, we can also show uniqueness using the Kružkov entropy condition, and the classical "doubling of the variables" technique, see Kružkov [15] or Kuznetsov [16].

However, if a is allowed to be discontinuous, none of the above techniques work. In this case, it is more profitable to view (1.1) as a system, by adding the trivial conservation law

$$a_t = 0.$$

This system has characteristic speeds $\lambda^a = 0$ and $\lambda^u = f_u$. If these two speeds coincide, the system is non-strictly hyperbolic, and is called *resonant*. The solution of the Riemann problem for resonant conservation laws is more complicated and interesting than the solution in the strictly hyperbolic case, see Isaacson and Temple [8]. This paper ([8]) studies a general system of resonant conservation laws, and solves the Riemann problem locally around a state where two wave speeds coincide.

Date: November 1, 1999.

1991 *Mathematics Subject Classification.* 65M10,35L65,35L67.

Key words and phrases. conservation laws,stability, non-strictly hyperbolic.

Nils Henrik Risebro is grateful to the University of Heidelberg and to SFB359 for support during the completion of this work. The authors are grateful to John Towers for spotting a mistake in an earlier version of this work.

For the 2×2 systems (1.2) or (1.1), the global Riemann solution was studied by Gimse and Risebro in [4], where it was shown that for a large class of flux functions f, a unique solution exists, subject to an additional entropy condition. However, for certain fs, some Riemann problems do not have a solution. Also for the special case of the polymer model without gravitation, the solution of the Riemann problem was reported in Isaacson [7], and the solution of the Riemann problem with gravitation was investigated in [3]

Utilizing the solution of the Riemann problem, Temple [21] showed existence of a weak solution for the polymer model. This was done by generating approximate solutions by the Glimm scheme, and then showing compactness using a nonlinear functional. This functional was subsequently used in [5], to show existence for a model of two phase flow in a heterogeneous porous medium, including gravitational effects. The Temple functional was also used in [14] where uniqueness and existence was shown for a model problem where $f(a, u) = ag(u)$ for a convex g. Here, uniqueness was obtained using a wave entropy condition. This model problem was also studied by Klausen and Risebro [13, 12], where it was shown that there is a unique solution, which is the limit of solutions with smooth coefficients, and in addition a stability estimate was established, both with respect to a and u.

Moreover, the functional introduced in [21] was used to prove convergence of the Godunov scheme for (1.1) by Longwai, Temple and Wang in [17], and for an inhomogeneous balance law by Isaacson and Temple in [9]. In these papers, the coefficient a was assumed to be Lipschitz continuous with the total variation of a' bounded. In this case, the solution generated by Godunov's method satisfies Kružkov's entropy condition, so that one has L^1 stability with respect to u, see [18].

For discontinuous a, L^1 contraction of the solution operator with respect to u can be proved by using front tracking and the semigroup approach by Bressan, [2, 10], and in the special case where $f_u \neq 0$ L^1 stability was shown by Baiti and Jenssen in [1].

Note that by Wagner's fundamental result [23], corresponding existence and stability results as the ones alluded to above, hold for (1.2) as well.

Without using this equivalence, Tveito and Winther [22] proved existence and L^1 stability for the polymer system (1.10), (1.11), in the case without gravitational effects.

In this paper we incorporate gravitational effects (i.e., f in (1.1) need not be convex in u), and then proceed to prove existence and L^1 stability without using smoothness assumptions on either the initial data or on the coefficients.

The rest of this paper is organized as follows: In the remainder of this introductory section we motivate the polymer model. In section 2 we define a "canonical" auxiliary conservation law. When viewed in the proper coordinates, the Riemann solution for the auxiliary model is identical to the Riemann solution for (1.10), (1.11). We therefore carry out the bulk of our analysis for the auxiliary model, and defer the discussion of the corresponding results for the polymer model to the next section. Based on the solution of the Riemann problem, we then define approximate solutions by a front tracking scheme. Using a Temple functional, we proceed to show compactness, and hence existence of a weak solution. For the moment assuming that the coefficient a is smooth, we next show that the limit satisfies Kružkov's entropy condition, and hence in this case, the solution operator is L^1 contractive. Furthermore, in the case that a' is bounded, we obtain a convergence rate for the front tracking approximation. Then, using both the front tracking approximations and the Temple functional, we show a stability estimate for non-smooth coefficients. Finally in section 3 we establish corresponding results for the polymer system with gravitation.

1.1. **The polymer model.** In this section we will motivate the polymer model for for a one dimensional reservoir with constant geological properties, i.e., porosity and permeability. Assume that we have two phases present in the reservoir, oil and water. Let s denote the saturation of the water. The saturation of a phase is defined as the percentage of the available pore volume occupied by that phase. Hence the saturation of the other phase, oil, is $1 - s$. Let x denote the position in the reservoir and let t denote the time. Partial derivatives will be denoted by subscripts.

The velocity of each phase, denoted by subscript i, is assumed to obey (the experimentally verified) Darcy's law

$$v_i = -\lambda_i \left((P_i)_x + \rho_i g D_x \right) \tag{1.3}$$

where the mobility of each phase is defined as

$$\lambda_i = K \frac{k_i}{\mu_i}. \tag{1.4}$$

Here K denotes the absolute permeability of the rock, k_i denotes the relative permeability of phase i, and μ_i the viscosity of the phase. Furthermore, ρ_i denotes the density of the phase, g the gravitational acceleration, and D measures vertical distance in the reservoir. The relative permeabilities are convex functions from $[0,1]$ to $[0,1]$. In subsequent equations the index i will be $\mathbf{o}$(denoting the oleic phase) and $\mathbf{w}$(denoting the aqueous phase). We will ignore the capillary pressure and set $P_{\mathbf{o}} = P_{\mathbf{w}} = P$. Conservation of mass for the aqueous phase now reads

$$-\left(\alpha \rho_{\mathbf{w}} v_{\mathbf{w}}\right)_x + \alpha q_{\mathbf{w}} = \alpha \left(\theta \rho_{\mathbf{w}} s_{\mathbf{w}}\right)_t ,$$

where α denotes the cross section of the reservoir, θ denotes the porosity, i.e., the fraction of the total volume available for one of the phases. The term $q_{\mathbf{w}}$ denotes sources or sinks present in the reservoir. We will assume that the cross section is constant, and that the densities and the porosity are independent of the pressure. Hence, formally, the conservation equation reads

$$-\left(v_{\mathbf{w}}\right)_x + \frac{q_{\mathbf{w}}}{\alpha \rho_{\mathbf{w}}} = \theta \left(s_{\mathbf{w}}\right)_t , \tag{1.5}$$

and similarly for the oleic phase. Adding the conservation equation for oil and for water, and using that $s_{\mathbf{o}} + s_{\mathbf{w}} = 1$, we are left with

$$\begin{aligned} \left(v_{\mathbf{o}} + v_{\mathbf{w}}\right)_x &= -\left(\frac{q_{\mathbf{w}}}{\alpha \rho_{\mathbf{w}}} + \frac{q_{\mathbf{o}}}{\alpha \rho_{\mathbf{o}}} \right) \\ \left(v_{\mathrm{tot}}\right)_x &= Q_{\mathrm{tot}} \end{aligned} \tag{1.6}$$

where the so-called *total velocity* is denoted by v_{tot}, similarly Q_{tot} denotes the total volumetric injection or production rate. Thus we see that in a one-dimensional reservoir, the total velocity is constant if Q_{tot} is zero. Using this in the equation for mass conservation (1.5), we find

$$\theta s_t + \left(f_0(s) \left(v_{\mathrm{tot}} - \left(\rho_{\mathbf{w}} - \rho_{\mathbf{o}}\right) g D_x K \lambda_{\mathbf{o}}(s) \right) \right)_x = 0, \tag{1.7}$$

where we write s for $s_{\mathbf{w}}$ ($s_{\mathbf{o}} = 1 - s$), and the function f_0 is the so called *fractional flow function*, given by

$$f_0(s) = \frac{\lambda_{\mathbf{w}}(s)}{\lambda_{\mathbf{w}}(s) + \lambda_{\mathbf{o}}(s)}. \tag{1.8}$$

In oil reservoir applications, injection of water is done in order to help prevent pressure loss in the reservoir, thereby forcing more oil out. However, since the viscosity of oil is larger than that of water, "fingers" may form, thereby rendering this process less effective than desirable. In order to prevent this fingering, a polymer is sometimes added to the water to increase its viscosity. This polymer is passively transported with the water, yielding a conservation equation

$$(sc)_t + (cf(s,c))_x = 0, \tag{1.9}$$

where $f(s,c)$ is given by

$$f(s,c) = f_0(s,c) \left(v_{\mathrm{tot}} - G \lambda_{\mathbf{o}}(s) \right),$$

and

$$G = \left(\rho_{\mathbf{w}} - \rho_{\mathbf{o}}\right) g D_x K.$$

So summing up, we have the following system of conservation laws

$$s_t + f(s,c)_x = 0 \tag{1.10}$$

$$(sc)_t + (cf(s,c))_x = 0 \tag{1.11}$$

In the figure below we show how f typically looks for two different c values. In the rest of this

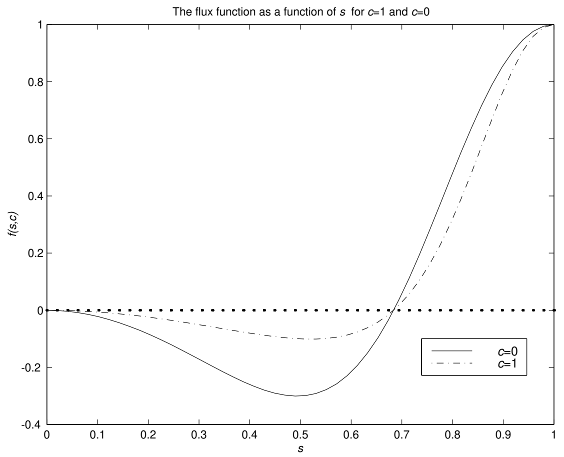


FIGURE 1. The flux function $f(s,c)$ as a function of s for two different c values.

paper we will use the following properties of the flux function f. It is a differentiable function of s and c for (s,c) in $[0,1]\times[0,1]$. There is an $\hat{s}$ such that

$$f(\hat{s},c)=0$$

for all c, and $f(s,c)<0$ for c in $(0,\hat{s})$, and $1>f(s,c)>0$ for s in $(\hat{s},1)$. Furthermore $f(0,c)=0$ and $f(1,c)=1$ for all c. Also

$$\begin{aligned}\frac{\partial f}{\partial c}&>0, \quad \text{for } c \text{ in } (0,\hat{s})\\ \frac{\partial f}{\partial c}&<0, \quad \text{for } c \text{ in } (\hat{s},1).\end{aligned}$$

2. The auxiliary model

It turns out that viewed in the proper coordinates, the Riemann problem for (1.10)–(1.11) is identical to the Riemann solution for a scalar equation with x dependent coefficients. This "system" is the following

$$\begin{aligned}\theta_t+(r\sin\theta)_x&=0\\ r_t&=0\end{aligned} \tag{2.1}$$

where the unknown is $\theta(x,t)$. We are interested in the initial value problem for (2.1) and assume that $r(x,t)=r(x)$ and $\theta(x,0)=\theta_0(x)$ are known. Furthermore, we shall assume that $r(x)$ is a positive function of bounded variation, and that the initial function $\theta_0(x)$ takes values in the interval $[-\pi,\pi]$. We let $h(\theta,r)=r\sin\theta$.

2.1. **The Riemann problem.**

The Riemann problem for (2.1) is the initial value problem where

$$\theta_0(x)=\begin{cases}\theta_l, & \text{for } x\le 0,\\ \theta_r, & \text{otherwise,}\end{cases} \qquad r(x)=\begin{cases}r_l, & \text{for } x\le 0,\\ r_r, & \text{otherwise,}\end{cases} \tag{2.2}$$

where $\theta_{l,r}$ and $r_{l,r}$ are constants. Such Riemann problems have been studied and solved by Gimse and Risebro in [4], where it was demonstrated that subject to an extra entropy condition, there existed a unique solution.

The solution Riemann problem for (2.1) consists of two types of waves, r waves, over which h is constant, but r varies, and θ waves, over which r is constant. Note that viewed as a system (2.1), has two characteristic speeds $\lambda_r=0$ and $\lambda_\theta=r\cos\theta$. It is linearly degenerate in the first family, and for $\theta=\pm\pi/2$, the two eigenvalues coincide.

It turns out that the (r, θ) coordinates are awkward to work with when defining the front tracking approximations, and we use instead the following coordinates:

$$\begin{aligned} v &= \operatorname{sign}(\cos(\theta))\, r\,(1 - |\sin(\theta)|) \\ w &= r \sin\theta \end{aligned} \tag{2.3}$$

We denote this coordinate mapping by $\Xi_{\theta r}$, i.e., $(v, w) = \Xi_{\theta r}(\theta, r)$. We see that in the (v, w) plane, the r waves will be lines with constant w, and the θ waves will be on diamonds of constant r. A Riemann problem is said to have a solution θ if the left state is connected with the right state via a θ wave, similarly the solution the two states are connected with an r wave the solution is labeled r. Hence we have essentially three types of solutions to the Riemann problem for (1.10), (1.11): $r\theta$, θr and $\theta r\theta$. In the figure below, we show the solution in each of the cases $-\pi \le \theta < -\pi/2$, $-\pi/2 \le \theta < 0$, $0 \le \theta < \pi/2$ and $\pi/2 \le \theta \le \pi$. These cases appear counterclockwise, starting in the lower left corner. This figure is interpreted by following the dashed lines from, representing

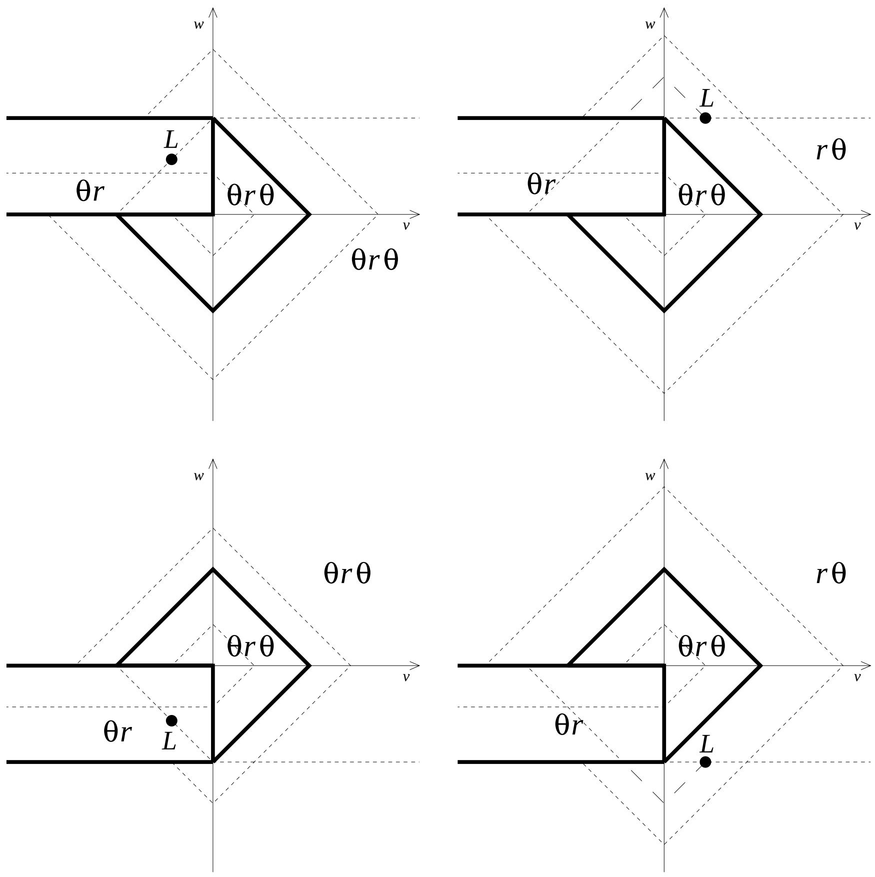
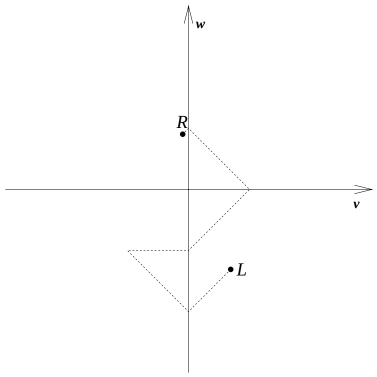


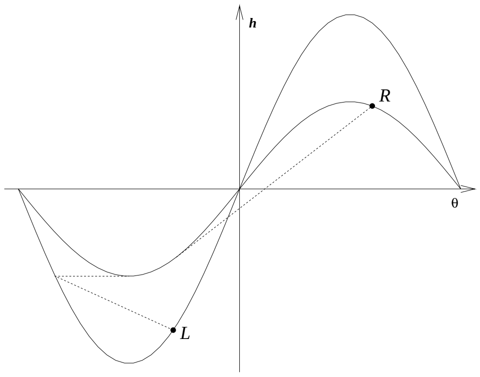
FIGURE 2. The solution of the Riemann problem

either θ waves or r waves, from the left state, denoted L, to any right state. In each case, the $(v.w)$ plane is divided into three regions, separated by thick lines, and in each region the wave configuration is constant, e.g., $\theta r\theta$. Note that r waves cannot cross the lines $\theta = \pm\pi/2$, where the two eigenvalues coincide. Hence to solve a Riemann problem, we first find the wave configuration from this figure 2. If a θ wave appears in the solution, the left and right states have the same r value, and the solution is found by solving a scalar Riemann problem. This is done by finding the lower (if $\theta_l < \theta_r$) or upper (if $\theta_l > \theta_r$) envelope of $h(\theta, r)$. Below we show how this works on a specific example. The diagrams show how the solution will look in (v, w) coordinates, in (θ, h)

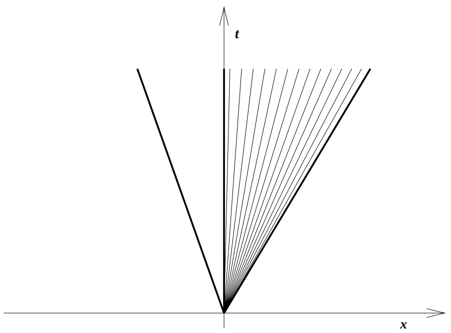
coordinates, and then we show the waves in (x,t) space and finally θ as a function of x/t. For a more detailed explanation of the solution such Riemann solutions, we refer the reader to [4].



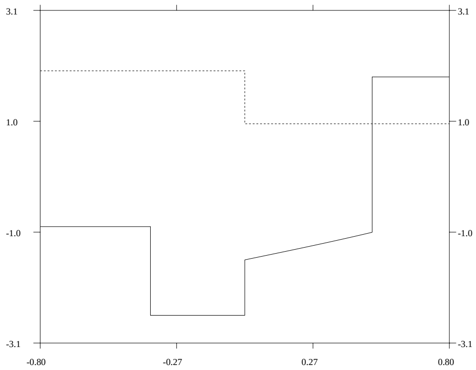
Solution in (v,w) coordinates.



Solution in (θ,h) coordinates.



Solution in (x,t) coordinates.



Solution in $(x/t,\theta)$ coordinates.

2.2. **The front tracking scheme.** Now we define the front tracking approximations which will be our primary tool. The scheme we will use is an adaptation of the schemes used in [5] and in [14].

Fix a (small) positive number δ. For $i \in \mathbb{Z}$, let $v_i = i\delta$ and $w_i = i\delta$ This will be our grid in phase space. For $r_i = |i|\,\delta$, we make a piecewise linear approximation to $h(\theta,r)$, interpolating between the points on the grid, i.e., let

$$\theta_{ij} = \operatorname{sign}(j)\cos^{-1}\left(1 - \frac{|j|}{i}\right), \quad \text{for } -2i \le j \le 2i.$$

Let h^δ be the piecewise linear approximation to h,

$$h^\delta(\theta,r) = h(r_i,\theta_{ij}) + (\theta - \theta_{ij})\,\frac{h(r,\theta_{i,j+1}) - h(r,\theta_{ij})}{\theta_{i,j+1} - \theta_{ij}}, \quad \text{for } r = r_i \text{ and } \theta \in [\theta_{ij}, \theta_{i,j+1}] \tag{2.4}$$

For a fixed i, the initial value problem for

$$\phi_t + h^\delta(\phi, r_i)_x = 0, \quad \phi(x,0) \in \{\theta_{ij}\}_{j=-2i}^{2i}, \tag{2.5}$$

can be solved exactly using Dafermos' method, see [6]. Furthermore, the unique weak solution $\phi(x,t)$ will be constant on a finite number of polygons in the (x,t) plane, and take values in the set $\{\theta_{ij}\}_{j=-2i}^{2i}$.

Let $r^\delta(x)$ be an approximation to $r(x)$, such that r^δ takes values in the set $\{r_i\}$, and

$$\left\| r^\delta - r \right\|_1 \to 0 \tag{2.6}$$

as $\delta \to 0$. Similarly $\theta_0^\delta(x)$ be an approximation to θ_0 taking values in the set $\{\theta_{ij}\}$, and

$$\left\| \theta_0^\delta - \theta_0 \right\|_1 \to 0 \tag{2.7}$$

as $\delta \to 0$. We shall use front tracking to define a weak solution to

$$\begin{aligned} \theta_t^\delta + h^\delta\left(\theta^\delta, r^\delta\right)_x &= 0. \\ r_t^\delta &= 0 \end{aligned} \tag{2.8}$$

Firstly, note that each initial Riemann problem defined by r^δ and θ_0^d will have solution which are piecewise constant, and that takes values on the grid (v_i, w_i). Furthermore, the discontinuities emanating from each initial discontinuity, all have finite speed. Therefore, by solving the initial Riemann problems, we have defined a weak solution initial two discontinuities collide. At the collision point, we solve the Riemann problem defined by the states to the left and right of the collision point. Since these states are on our grid, also the resulting states will be on the grid. Now we have a solution defined until the next collision point. Therefore, we have a weak solution up to any collision point. We call this construction *front tracking.*

It remains, of course, to show that this process is well defined, we have to check whether we can reach any predetermined time, and whether the number of discontinuities stays finite for any time. To accomplish this we define a functional on $\left(\theta^\delta, r^\delta\right)$. This functional is chosen so that it dominates the total variation in (v, w), and such that it is non-increasing in t.

Let now $\mathbf{u}^\delta = \left(v^\delta, w^\delta\right)$, we have that $\mathbf{u}^\delta$ defines a path on the (v, w) grid. This path consists of θ waves and r waves. We call any finite connected sequence of θ and r waves an I curve, and say that I connects $\mathbf{u}_L$ to $\mathbf{u}_R$ if the left state of the first segment is $\mathbf{u}_L$ and the right state of the last segment is $\mathbf{u}_R$. An I curve can then be written $r_1\theta_1 r_2\theta_2 \ldots r_N$. This terminology and the subsequent techniques are ultimately borrowed from Temple [21], but see also [5] and [14].

We first define F on simple wave segments, for a θ wave, let

$$F(\theta) = T.V.\left(w(\theta)\right), \tag{2.9}$$

i.e., $F(\theta)$ is δ times the number of horizontal grid lines crossed by the θ wave. For an r wave connecting points (v_L, w) and (v_R, w), let $\Delta r = v_R - v_L$. Then

$$F(r) = \left(3 - \operatorname{sign}(\Delta r)\right)\left|\Delta r\right|. \tag{2.10}$$

For general I curves, $I = b_1 b_2 \cdots b_N$, F is defined additively, i.e.,

$$F(I) = \sum_{i=1}^{N} F\left(b_i\right).$$

For later use, note that $F\left(\mathbf{u}^\delta\right)$ can also be written

$$F\left(\mathbf{u}^\delta\right) = \int_{\mathbb{R}} \left(3 - \operatorname{sign}\left(v_x^\delta\right)\right)\left|v_x^\delta\right| + \left|w_x^\delta\right| \, dx, \tag{2.11}$$

and we then clearly have that

$$T.V.\left(\mathbf{u}^\delta\right) \le F\left(\mathbf{u}^\delta\right) \le 4T.V.\left(\mathbf{u}^\delta\right).$$

Now we can show that

Lemma 2.1. *Let I be any I curve connecting $\mathbf{u}_L$ to $\mathbf{u}_R$, and let $[\mathbf{u}_L, \mathbf{u}_R]$ be the I curve defined by the solution of the Riemann problem with left state $\mathbf{u}_L$ and right state $\mathbf{u}_R$. Then*

$$F\left([\mathbf{u}_L, \mathbf{u}_R]\right) \le F(I). \tag{2.12}$$

Proof. First we show the lemma in case where $\operatorname{sign}(I)$ is constant, i.e, I does not cross the v axis. Then it suffices to observe that $\theta \in [-\pi, 0]$ and $\theta \in [0, \pi]$ are both invariant regions with respect to the solution of the Riemann problem, cf. figure 2. In this case the proof of this lemma is identical to the proof of the corresponding lemma in [14] or [21].

Next assume that $w_L = w_R = 0$,and $v_L, v_R > 0$, in this case the $[\mathbf{u}_L, \mathbf{u}_R] = r$, and by the above remarks $F\left([\mathbf{u}_L, \mathbf{u}_R]\right) \le F(J)$ for any J curve connecting the states and not crossing the v axis.

Assume next that $w_L \ge 0$ and $w_R \ge 0$, and I crosses the v axis. Let J be the I curve from $\mathbf{u}_L$ to $\mathbf{u}_R$ consisting of the part of I with non-negative w coordinate, together with parts on the v axis. Then

$$J = J_1 J_1^0 J_2 J_2^0 \ldots J_\ell^0 J_{\ell+1}$$

where J_k denotes the parts of I with non-negative w and J_k^0 denotes the parts of J with zero w. Corresponding to J_k^0 we let I_k denote the part of I connecting the same points (on the v axis) as J_k^0. Then we have

$$\begin{aligned} F\left([\mathbf{u}_L, \mathbf{u}_R]\right) &\leq F(J) \\ &= \sum_{k=1}^{\ell} F\left(J_k^0\right) + \sum_{k=1}^{\ell+1} F\left(J_k\right) \\ &\leq \sum_{k=1}^{\ell} F\left(I_k\right) + \sum_{k=1}^{\ell+1} F\left(J_k\right) \\ &= F(I). \end{aligned}$$

An identical argument covers the case where $w_L, w_R < 0$.

Assume now that $\operatorname{sign}(w_L) \neq \operatorname{sign}(w_R)$. From figure 2 we see that

$$[\mathbf{u}_L, \mathbf{u}_R] = [\mathbf{u}_L, \mathbf{u}_M]\,[\mathbf{u}_M, \mathbf{u}_R],$$

where $w_M = 0$. Furthermore

$$F\left([\mathbf{u}_L, \mathbf{u}_R]\right) = F\left([\mathbf{u}_L, \mathbf{u}_M]\right) + F\left([\mathbf{u}_M, \mathbf{u}_R]\right).$$

First we show the lemma in case I crosses the v axis once at $\mathbf{u}_1$, the general case will then follow by induction. Now we have that

$$[\mathbf{u}_1, \mathbf{u}_R] = r\theta = [\mathbf{u}_1, \mathbf{u}_M]\,[\mathbf{u}_M, \mathbf{u}_R].$$

Let I_1 denote the part of I connecting $\mathbf{u}_L$ and $\mathbf{u}_1$ and let I_2 denote the part connecting $\mathbf{u}_1$ and $\mathbf{u}_R$. Then

$$\begin{aligned} F(I) &= F\left(I_1\right) + F\left(I_2\right) \\ &\geq F\left(I_1\right) + F\left([\mathbf{u}_1, \mathbf{u}_M]\right) + F\left([\mathbf{u}_M, \mathbf{u}_R]\right) \\ &= F\left(I_1\,[\mathbf{u}_1, \mathbf{u}_M]\right) + F\left([\mathbf{u}_M, \mathbf{u}_R]\right) \\ &\geq F\left([\mathbf{u}_L, \mathbf{u}_M]\right) + F\left([\mathbf{u}_M, \mathbf{u}_R]\right) \\ &= F\left([\mathbf{u}_L, \mathbf{u}_R]\right). \end{aligned}$$

Regarding the general case, assuming that I crosses the v axis k times. Using the above arguments, we can find another I curve $\tilde{I}$, with $F(I) \geq F\left(\tilde{I}\right)$ connecting $\mathbf{u}_L$ and $\mathbf{u}_R$ and crossing the v axis $k-1$ times. This concludes the proof of Lemma 2.1. □

From this lemma the following is immediate:

Lemma 2.2. *$F\left(\mathbf{u}^\delta\right)$ is non-increasing in time.*

Proof. It is clear that F only changes value when two discontinuities in $\mathbf{u}^\delta$ collide. At collision points, a section of the I curve traced out by $\mathbf{u}^\delta$ connecting $\mathbf{u}_L$ and $\mathbf{u}_R$ is replaced by $[\mathbf{u}_L, \mathbf{u}_R]$. Hence, F is non-increasing. □

Now we can also use this to show that front tracking is well defined. We have that the following types of collisions can occur

$$\begin{aligned} &\theta\,\theta \to \theta' \\ &r\,\theta \to \theta'\,r' \\ &\theta\,r \to r'\,\theta' \\ &r\,\theta \to \theta_1'\,r'\,\theta_2' \\ &\theta\,r \to \theta_1'\,r'\,\theta_2' \end{aligned}$$

meaning that the two waves to the left of the arrow collides, producing a solution indicated by the waves to the right of the arrows. Each of the θ' waves to the right may consist of several discontinuities. This will be the case if the θ' wave is approximating a rarefaction. To aid the further analysis we use the next lemma

Lemma 2.3. *If we have a collision of type $r\,\theta \to \theta_1'\,r'\,\theta_2'$ or $\theta\,r \to \theta_1'\,r'\,\theta_2'$, then $F - F' \geq \delta$.*

Proof. The proof of this is a study of cases. We prove the lemma in the case where we have a collision of type $\theta\, r$, the other case being entirely similar.

Let θ separate $\mathbf{u}_L$ and $\mathbf{u}_M$, and r separate $\mathbf{u}_M$ and $\mathbf{u}_R$. The point $\mathbf{u}_L$ has coordinates (θ_L, r_L). Similarly for the points $\mathbf{u}_M$ and $\mathbf{u}_R$. To show the lemma we consider four cases, depending on which quadrant in the (v, w) plane $\mathbf{u}_L$ is in. With a slight abuse of notation, let θ_L be the θ coordinate of the left state. The reader is urged to consult the Riemann solution, fig. 2 as a tool to understand our arguments.

Case 1: $-\pi \leq \theta_L < -\pi/2$:
In this case $-\pi/2 \leq \theta_R \leq \pi$, and $r_R \leq r_L$, since this is the set where $[\mathbf{u}_L, \mathbf{u}_R] = \theta\, r\, \theta$. Now $r_M = r_L$ since $\mathbf{u}_L$ and $\mathbf{u}_M$ are connected via a θ wave. Furthermore, this θ wave has positive speed. Now there are no single θ waves of positive speed having θ_L as its left state, which follows from the Rankine-Hugoniot condition. Hence in case $-\pi \leq \theta_L < -\pi/2$ there are no such collisions.

Case 2 $-\pi/2 \leq \theta_L < -0$:
In this case the set of states where $[\mathbf{u}_L, \mathbf{u}_R] = \theta\, r\, \theta$ is given by $-\pi/2 \leq \theta_R \leq \pi$, and $r_R \leq |w_L| - \delta$, where w_L is the w coordinate of $\mathbf{u}_L$. Since $\mathbf{u}_M$ and $\mathbf{u}_R$ are connected by an r wave, $\mathbf{u}_M$ must be such that $|w_M| \leq |w_L| - \delta$, and since $\mathbf{u}_L$ and $\mathbf{u}_M$ are connected by a θ wave, $r_M = r_L$. Now it is a simple exercise to control that $F' - F \geq \delta$.

Case 3 $0 \leq \theta_L < \pi/2$:
Now $-\pi \leq \theta_R \leq \pi/2$ and $|w_R| \leq w_L - \delta$. Also in this case it is straightforward to check that F decreases by at least δ.

Case 4 $0 \leq \theta_L < \pi/2$:
This case is similar to the first case.

This concludes the proof of the lemma. ☐

A consequence of this lemma is that for some fixed δ, "reflections" of θ waves can only occur a finite number of times. Hence after some finite time, all θ waves of nonzero speed will have passed into the region to the left or right where r is constant. Recall that the initial data, and in particular r, is constant outside some bounded interval.

Regarding $\theta\,\theta$ collisions, these always result in a single θ discontinuity if F is constant, and if more than one discontinuity results, then $F - F' \geq \delta$, see [6] for details. Hence after some finite time, any $\theta\,\theta$ collisions will result in a single θ discontinuity. Also the speed of this discontinuity will be between the speeds of the colliding discontinuities. In particular, two colliding θ discontinuities of negative speed, will result in a single discontinuity moving with negative speed.

Combining the above remarks, we see that after some finite time T_δ, there will be no further collision of fronts. Hence the front tracking method is well defined, and for a fixed δ, requires only a finite number of operations.

2.3. **Compactness and convergence.** Now we are in a position to use standard arguments to show that there is a subsequence of $\mathbf{u}^\delta$ that converges to some $\mathbf{u}$, such that $(\theta, r) \stackrel{\text{def}}{=} \Xi_{\theta r}^{-1}\mathbf{u}$ is a weak solution to (2.1).

For the auxiliary system the relevant theorem is:

Theorem 2.1. *Let* $\mathbf{u}_0 = \Xi_{\theta r}(\theta_0, r)$ *be such that* $T.V.(\mathbf{u})$ *is finite, and assume that* $\mathbf{u}_0$ *is in* L^1. *Let* $\mathbf{u}^\delta$ *denote the function defined by the front tracking construction. Then for any sequence* $\{\delta\}$ *such that* $\delta \to 0$, *there exists a subsequence* $\{\delta_j\}$, *such that for any finite time* $t \geq 0$, $\theta_{\delta_j}(\cdot, t)$ *converges uniformly in* L^1_{loc}, *where* $(\theta^\delta, r^\delta) = \Xi_{\theta r}^{-1}\mathbf{u}^\delta$. *Furthermore the limit of* θ^{δ_j} *is a weak solution of* (2.1).

Proof. The compactness of the sequence $\{\mathbf{u}\}^\delta$ follows by standard arguments, see e.g., [20], for $\mathbf{u}$ satisfies

$$
\begin{aligned}
\left\|\mathbf{u}^\delta\right\|_\infty &\leq M,\\
\left\|\mathbf{u}^\delta(\cdot,t)\right\|_1 &\leq M,\\
T.V.(\mathbf{u}) &\leq M,\\
\left\|\mathbf{u}^\delta(\cdot,s)-\mathbf{u}^\delta(\cdot,t)\right\|_1 &\leq M\left|t-s\right|,
\end{aligned}
\tag{2.13}
$$

where M is a generic constant. The proof of the the fourth of these inequalities follows from finite speed of propagation and the third inequality. Then the convergence of a subsequence of $\{\mathbf{u}\}$ in L^1_{loc} follows by applying Helly's theorem and using a further diagonal argument, see e.g., [20]. Since $\Xi_{\theta r}$ is injective, the corresponding convergence for a subsequence of $\left\{\theta^\delta\right\}$ follows.

For simplicity we now denote the subsequence $\{\delta_j\}$ by $\{\delta\}$. For a test function φ we define the functional

$$
W(\theta)=\int_{\mathbb{R}}\int_0^\infty \theta\varphi_t+h(\theta,r)\varphi_x\,dxdt+\int_{\mathbb{R}}\theta(x,0)\varphi(x,0)\,dx.
\tag{2.14}
$$

By the front tracking construction, $\left(\theta^\delta,r^\delta\right)$ is a weak solution to the approximate problem

$$
\left(\theta^\delta\right)_t+h^\delta\left(\theta^\delta,r^\delta\right)_x=0.
$$

Let $\theta=\lim\theta^\delta$, then we find

$$
\begin{aligned}
\left|W\left(\theta\right)\right| &= \left|\iint\left(\theta-\theta^\delta\right)\varphi_t+\left(h(\theta,r)-h^\delta\left(\theta^\delta,r^\delta\right)\right)\varphi_x\,dxdt+\int\left(\theta_0-\theta_0^\delta\right)\varphi(x,0)\,dx\right|\\
&\leq M\left(\left\|\theta-\theta_\delta\right\|_1+\left\|h(\theta,r)-h\left(r^\delta,\theta^\delta\right)\right\|_1+\left\|\theta_0-\theta_0^\delta\right\|_1\right),
\end{aligned}
$$

where M is a bound on $|\varphi|$, $|\varphi_x|$ and $|\varphi_t|$, and as $\left(\theta^\delta,r^\delta\right)$ takes values at precisely those points where $h^\delta=h$, we have replaced h^δ by h in the middle term. Since θ^δ converges to θ and r^δ to r in L^1, the terms on the right hand side can be made arbitrarily small by making δ small. Hence θ is a weak solution to (2.1). ☐

2.4. **Stability for the scalar equation.** We first study stability for the scalar auxiliary equation with respect to perturbations in the initial value θ. If r is twice differentiable, we can use the classical results of Oleinik [19] and Kružkov [15] to show that

$$
\left\|\theta^1(\cdot,t)-\theta^2(\cdot,t)\right\|_1\leq\left\|\theta_0^1-\theta_0^2\right\|_1,
\tag{2.15}
$$

where θ^i is the entropy solution of (2.1) with initial data θ_0^i for $i=1,2$. Now, our first goal is to show that this also holds for the solutions produced by front tracking.

To do this, we must first show that for smooth r, the front tracking method produces the correct entropy solution.

Lemma 2.4. *Assume that $r(x)$ is uniformly Lipschitz continuous and constant outside some bounded interval. Let $\theta=\lim_{\delta\to 0}\theta^\delta$. Then we have that*

$$
\begin{aligned}
\int_0^T\int_{\mathbb{R}}&\left|\theta-k\right|\varphi_t+\operatorname{sign}\left(\theta-k\right)\left[\left(h(\theta,r)-h(k,r)\right)\varphi_x-\left(h(k,r)_x\right)\varphi\right]\,dxdt\\
&-\int_{\mathbb{R}}\left|\theta(x,T)-k\right|\varphi(x,T)\,dx+\int_R\left|\theta_0-k\right|\varphi(x,0)\,dx\geq 0,
\end{aligned}
\tag{2.16}
$$

for all non-negative test functions φ and all constants k, i.e., θ satisfies the Kružkov entropy condition.

Proof. Let the discontinuities of r^δ be located at x_i, $i=1,\ldots,N$. Since θ^δ is a weak solution in each interval $\langle x_i, x_{i+1}\rangle$, we have that

(2.17)
$$\begin{aligned}\int_0^T\int_{x_i}^{x_{i+1}} &\left|\theta^\delta-k\right|\varphi_t+\operatorname{sign}\left(\theta^\delta-k\right)\left[h^\delta\left(\theta^\delta,r^\delta\right)-h^\delta\left(k,r^\delta\right)\right]\varphi_x\,dxdt\\ &+\int_0^T\varphi\left(x_{i+1},t\right)\operatorname{sign}\left(\theta^\delta\left(x_{i+1},t\right)-k\right)\left[h^\delta\left(\theta^\delta\left(x_{i+1},t\right),r^\delta\left(x_{i+1}\right)\right)-h^\delta\left(k,r^\delta\left(x_{i+1}\right)\right)\right]\\ &\qquad-\varphi\left(x_i,t\right)\operatorname{sign}\left(\theta^\delta\left(x_i,t\right)-k\right)\left[h^\delta\left(\theta^\delta\left(x_i,t\right),r^\delta\left(x_i\right)\right)-h^\delta\left(k,r^\delta\left(x_i\right)\right)\right]\,dt\\ &\qquad\quad-\int_{x_i}^{x_{i+1}}\left|\theta^\delta(x,T)-k\right|\varphi(x,T)\,dx+\int_{x_i}^{x_{i+1}}\left|\theta_0^\delta-k\right|\varphi(x,0)\,dx\geq 0.\end{aligned}$$

Thus

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}&\left|\theta^\delta-k\right|\varphi_t+\operatorname{sign}\left(\theta^\delta-k\right)\left[h^\delta\left(\theta^\delta,r^\delta\right)-h^\delta\left(k,r^\delta\right)\right]\varphi_x\,dxdt\\ &-\int_{\mathbb{R}}\left|\theta^\delta(x,T)-k\right|\varphi(x,T)\,dx+\int_{\mathbb{R}}\left|\theta_0^\delta-k\right|\varphi(x,0)\,dx\\ &-\int_0^T\sum_{i=1}^N\varphi_{i+1}\operatorname{sign}\left(\theta_{i+1,-}^\delta-k\right)\left[h^\delta\left(\theta_{i+1,-}^\delta,r_{i+1,-}^\delta\right)-h^\delta\left(k,r_{i+1,-}^\delta\right)\right]\\ &\qquad-\varphi_i\operatorname{sign}\left(\theta_{i,+}^\delta-k\right)\left[h^\delta\left(\theta_{i,+}^\delta,r_{i,+}^\delta\right)-h^\delta\left(k,r_{i,+}^\delta\right)\right]\;dt\geq 0\end{aligned}\tag{2.18}$$

where $r_{i,\pm}^\delta=\lim_{y\to x_i\pm}r^\delta(y)$, similarly for $\theta_{i,\pm}^\delta$, and $\varphi_i=\varphi\left(x_i,t\right)$. Consider now the last term in (2.18), since r is smooth, we can define

$$r_{i,+}^\delta=r\left(x_i\right)=:r_i.$$

Furthermore, by the solution of the Riemann problem,

$$h^\delta\left(\theta_{i+1,-}^\delta,r_i\right)=h^\delta\left(\theta_{i+1,+}^\delta,r_{i+1}\right)=:h_i.$$

Let now

$$\mathcal{P}=\left\{x_j\;\mid\;\max\left\{\left|\theta_i^-\right|,\left|\theta_i^+\right|\right\}>\pi/2\right\},\quad\text{and}\quad\mathcal{S}=\left\{x_j\;\mid\;\max\left\{\left|\theta_i^-\right|,\left|\theta_i^+\right|\right\}\leq\pi/2\right\}.$$

Then we write the sum under the integral in (2.18) as

$$\sum_{\mathcal{S}}\varphi_i\left[h_i-h^\delta\left(k,r_{i-1}\right)\right]\left[\operatorname{sign}\left(\theta_{i,-}^\delta-k\right)-\operatorname{sign}\left(\theta_{i,+}^\delta-k\right)\right]\tag{2.19}$$

$$+\,\varphi_i\operatorname{sign}\left(\theta_{i,+}^\delta-k\right)\left[h\left(k,r_i\right)-h\left(k,r_{i-1}\right)\right]\tag{2.20}$$

$$+\sum_{\mathcal{P}}\varphi_i\left[h_i-h^\delta\left(k,r_i\right)\right]\left[\operatorname{sign}\left(\theta_{i,-}^\delta-k\right)-\operatorname{sign}\left(\theta_{i,+}^\delta-k\right)\right]\tag{2.21}$$

$$+\,\varphi_i\operatorname{sign}\left(\theta_{i,-}^\delta-k\right)\left[h\left(k,r_i\right)-h\left(k,r_{i-1}\right)\right]\tag{2.22}$$

As $\delta\to 0$, clearly the second and fourth terms, (2.20) and (2.22) tend to

$$\int_{\mathbb{R}}\varphi\operatorname{sign}\left(\theta-k\right)h(k,r)_x\,dx.$$

Regarding the first and the third term, (2.19) and (2.21), it is a simple exercise to verify that solution of the Riemann problem implies that (2.19) and (2.22) are non-negative. Hence the limit θ satisfies the Kružkov entropy condition. ☐

Thus, for smooth r, the front tracking construction will give a limit which is L^1 contractive, i.e., (2.15) holds. We now show that also if r is not continuous, there is a weak solution to (2.1) such that (2.15) holds .

For later use we now write the entropy inequality satisfied by θ^δ as

$$\begin{aligned}\int_0^T \int_{\mathbb{R}} \left|\theta^\delta - k\right| \varphi_t + \operatorname{sign}\left(\theta^\delta - k\right)\left[h^\delta\left(\theta^\delta, r^\delta\right) - h^\delta\left(k, r^\delta\right)\right] \varphi_x \, dxdt \\ - \int_0^T \sum_{\mathcal{S}} \varphi_i \operatorname{sign}\left(\theta^\delta_{i,+} - k\right)\left[h\left(k, r_i\right) - h\left(k, r_{i-1}\right)\right] \\ + \sum_{\mathcal{P}} \varphi_i \operatorname{sign}\left(\theta^\delta_{i,-} - k\right)\left[h\left(k, r_i\right) - h\left(k, r_{i-1}\right)\right] dt \\ - \int_{\mathbb{R}} \left|\theta^\delta(x,T) - k\right| \varphi(x,T)\, dx + \int_{\mathbb{R}} \left|\theta^\delta_0 - k\right| \varphi(x,0)\, dx \geq 0.\end{aligned} \tag{2.23}$$

If r is discontinuous, let θ^ε_i be entropy solutions to

$$\left(\theta^\varepsilon_i\right)_t + h\left(\theta^\varepsilon_i, r^\varepsilon\right)_x = 0, \tag{2.24}$$

with initial data $\theta_{i,0}$, and $r^\varepsilon = r * \omega_\varepsilon$ for a standard mollifier ω_ε. Using the functional F we can now show that

Theorem 2.2. *For $i = 1,2$, let $(\theta_{i,0}, r)$ be as in Theorem 2.1, and let θ^ε_i be defined by (2.24). Then, for any sequence $\{\varepsilon\}$ such that $\varepsilon \to 0$, there is a subsequence ε_j such $\theta^{\varepsilon_j} \to \theta_i$ in L^1_{loc} as $j \to \infty$. Furthermore θ_i are weak solutions to*

$$\begin{aligned}\left(\theta_i\right)_t + h\left(\theta_i, r\right)_x &= 0,\\ \theta_i(x,0) &= \theta_{i,0}(x).\end{aligned}$$

Proof. We first show the convergence with respect to ε, $\mathbf{u}^\varepsilon = \Xi_{\theta r}\left(\theta^e ps_1, r^\varepsilon\right)$. By Lemma 2.2 and (2.11) we have that

$$T.V.\left(\mathbf{u}^\varepsilon(\cdot,t)\right) \leq 4T.V.\left(\mathbf{u}(\cdot,0)\right).$$

Hence the total variation of $\mathbf{u}$ is bounded independently of ε. Also $\mathbf{u}^\varepsilon$ satisfies the bounds

$$\begin{aligned}\left\|\mathbf{u}^\varepsilon\right\|_\infty &\leq M\\ \left\|\mathbf{u}^\varepsilon(\cdot,s) - \mathbf{u}^\varepsilon(\cdot,t)\right\|_1 &\leq M\left|t-s\right|\\ \left\|\mathbf{u}^\varepsilon(\cdot,t)\right\|_1 &\leq M,\end{aligned}$$

for some constant M independent of ε. So we see that $\mathbf{u}^\varepsilon$ satisfies all the requirements of using the classical technique of Helly's theorem and a further diagonalization to show that there exists a convergent subsequence. Hence θ^ε converges in L^1_{loc} to some function θ. We also have that

$$\begin{aligned}\left|W(\theta)\right| &= \left|\iint \left(\theta - \theta^\varepsilon\right)\varphi_t + \left(h(\theta,r) - h\left(r^\varepsilon, \theta^\varepsilon\right)\right)\varphi_x \, dxdt\right|\\ &\leq M\left(\left\|\theta - \theta^\varepsilon\right\|_1 + \left\|h(\theta,r) - h\left(r^\varepsilon,\theta^\varepsilon\right)\right\|_1\right).\end{aligned}$$

Since θ^ε tends to θ in L^1, the terms on the right hand side can be made arbitrary small, and θ is a weak solution to (2.1). Letting θ^ε_i denote the weak solutions of (2.24) with initial data θ^i_0, we have

$$\left\|\theta_1(\cdot,t) - \theta_2(\cdot,t)\right\|_1 = \lim_{\varepsilon\to 0}\left\|\theta^\varepsilon_1(\cdot,t) - \theta^\varepsilon_2(\cdot,t)\right\|_1 \leq \left\|\theta_{1,0} - \theta_{2,0}\right\|_1,$$

which is (2.15). □

Note that we do not know whether the weak solution constructed directly by front tracking; θ^f, is equal to θ. To show this we must show stability with respect to variations in the coefficient r. Now we show stability with respect also to the r variable, this is done by a "doubling of the variables" approach similar to Kuznetsov [16] original approach, see also [22]. Before stating the lemma, we remind the reader that for r twice differentiable, and $T.V.(r')$ bounded, there exists an entropy solution of (2.1) of bounded variation in any bounded time interval. This follows by showing that the approximate solution generated by the Lax-Friedrichs scheme are of bounded variation, and that they satisfy an approximate entropy inequality, see e.g., [19] or [12].

Lemma 2.5. *Let θ and $\hat\theta$ satisfy*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} |\theta-k|\,\varphi_t + \operatorname{sign}(\theta-k)\left[(h(\theta,r)-h(k,r))\,\varphi_x - (h(k,r)_x)\,\varphi\right]\,dxdt\\ -\int_{\mathbb{R}}|\theta(x,T)-k|\,\varphi(x,T)\,dx + \int_R |\theta_0-k|\,\varphi(x,0)\,dx \ge 0,\end{aligned}\tag{2.25}$$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} \left|\hat\theta-k\right|\varphi_t + \operatorname{sign}\left(\hat\theta-k\right)\left[\left(h(\hat\theta,\hat r)-h(k,\hat r)\right)\varphi_x - (h(k,\hat r)_x)\,\varphi\right]\,dxdt\\ -\int_{\mathbb{R}}\left|\hat\theta(x,T)-k\right|\varphi(x,T)\,dx + \int_R \left|\hat\theta_0-k\right|\varphi(x,0)\,dx \ge 0,\end{aligned}\tag{2.26}$$

for all constants k and all non-negative test functions φ. Assume that either $T.V.(\theta(\cdot,t))$ or $T.V.\left(\hat\theta(\cdot,t)\right)$ is uniformly bounded for $t \le T$, and that either r or $\hat r$ are Lipschitz continuous, and that both $T.V.(r')$, $T.V.(\hat r')$ are bounded. Then

$$\left\|\theta(\cdot,T)-\hat\theta(\cdot,T)\right\|_1 \le M\left(\left\|\theta_0-\hat\theta_0\right\|_1 + \|r-\hat r\|_1 + T.V.(r-\hat r)\right).\tag{2.27}$$

for some finite constant M depending on $\min\left\{T.V._x(\theta), T.V._x\left(\hat\theta\right)\right\}$ and $\min\{\|r'\|_\infty, \|\hat r'\|_\infty\}$, for $t\in[0,T_0]$.

Proof. Let ω_ε be the usual approximate δ function, with $|\omega_\varepsilon| \le M_\omega/\varepsilon$, and set

$$F_1(\theta,r,k) = \operatorname{sign}(\theta-k)\,(h(\theta,r)-h(k,r)), \quad F_2(\theta,r,k) = \operatorname{sign}(\theta-k)\,h(k,r)_x.$$

Then (2.25) and (2.26) read

$$\begin{aligned}\iint |\theta-k|\,\varphi_t + F_1(\theta,r,k)\varphi_x - F_2(\theta,r,k)\varphi\,dxdt\\ +\int |\theta_0-k|\,dx - \int |\theta(T)-k|\,\varphi\,dx \ge 0\end{aligned}\tag{2.28}$$

$$\begin{aligned}\iint \left|\hat\theta-k\right|\varphi_t + F_1(\hat\theta,\hat r,k)\varphi_x - F_2(\hat\theta,\hat r,k)\varphi\,dyds\\ +\int \left|\hat\theta_0-k\right|dy - \int\left|\hat\theta(T)-k\right|\varphi\,dy \ge 0.\end{aligned}\tag{2.29}$$

By using the test function $\varphi(x,y,s,t) = \omega_\varepsilon((x-y))\omega_\varepsilon((t-s))$ and setting $k=\hat\theta(y,s)$ in (2.28) and $k=\theta(x,t)$ in (2.29), then integrating (2.28) with respect to y and s and (2.29) with respect to x and t, we find that

$$\begin{aligned}0 \le \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} &\left|\theta(x,t)-\hat\theta(y,s)\right|\varphi_t\\ &+F_1\left(\theta(x,t),r(x),\hat\theta(y,s)\right)\varphi_x - F_2\left(\theta(x,t),r(x),\hat\theta(y,s)\right)\varphi\,dxdt\,dyds\\ &+\int_0^T \rho_\varepsilon\left(\theta_0,\hat\theta(\cdot,s)\right))\omega_\varepsilon(s)\,ds - \int_0^T\rho_\varepsilon\left(\theta(\cdot,T),\hat\theta(\cdot,s)\right))\omega_\varepsilon(s)\,ds\end{aligned}\tag{2.30}$$

$$\begin{aligned}0 \le \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} &\left|\theta(x,t)-\hat\theta(y,s)\right|\varphi_s\\ &+F_1\left(\hat\theta(y,s),\hat r(y),\theta(x,t)\right)\varphi_y - F_2\left(\hat\theta(y,s),\hat r(y),\theta(x,t)\right)\varphi\,dxdt\,dyds\\ &+\int_0^T \rho_\varepsilon\left(\hat\theta_0,\theta(\cdot,t)\right))\omega_\varepsilon(t)\,dt - \int_0^T\rho_\varepsilon\left(\hat\theta(\cdot,T),\theta(\cdot,t)\right))\omega_\varepsilon(s)\,dt,\end{aligned}\tag{2.31}$$

where ρ_ε is defined by

$$\rho_\varepsilon(w,z) = \int_{\mathbb{R}}\int_{\mathbb{R}} \omega_\varepsilon(x-y)\,|w(x)-z(y)|\,dxdy.$$

Using that $\varphi_x = -\varphi_y$ and $\varphi_t = -\varphi_s$, and adding (2.30) and (2.31), we obtain

$$\begin{aligned}
&\int_0^T \rho_\varepsilon\left(\theta(\cdot,T),\hat\theta(\cdot,s)\right)\omega_\varepsilon(T-s)\,ds + \int_0^T \rho_\varepsilon\left(\hat\theta(\cdot,T),\theta(\cdot,t)\right)\omega_\varepsilon(t-T)\,dt \\
&\quad \le \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{F_1\left(\theta(x,t),r(x),\hat\theta(y,s)\right) - F_1\left(\hat\theta(y,s),\hat r(y),\theta(x,t)\right)\right\}\varphi_x\,dxdt\,dyds \\
&\qquad - \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{F_2\left(\theta(x,t),r(x),\hat\theta(y,s)\right) - F_2\left(\hat\theta(y,s),\hat r(y),\theta(x,t)\right)\right\}\varphi\,dxdt\,dyds \\
&\qquad + \int_0^T \rho_\varepsilon\left(\theta_0,\hat\theta(\cdot,s)\right)\,ds + \int_0^T \rho_\varepsilon\left(\hat\theta_0,\theta(\cdot,t)\right)\,dt.
\end{aligned}$$

We write this equation as

$$L^1(\varepsilon) + L_2(\varepsilon) \le R_1(\varepsilon) + R_2(\varepsilon) + R_3(\varepsilon) + R_4(\varepsilon). \tag{2.32}$$

Now standard arguments, see e.g., [15] imply that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left[L_1(\varepsilon) + L_2(\varepsilon)\right] = \left\|\theta(\cdot,T) - \hat\theta(\cdot,T)\right\|_1 \\
&\lim_{\varepsilon\to 0}\left[R_3(\varepsilon) + R_4(\varepsilon)\right] = \left\|\theta_0 - \hat\theta_0\right\|_1.
\end{aligned} \tag{2.33}$$

Regarding R_2, assuming that r' is bounded, we have that

$$\begin{aligned}
R_2(\varepsilon) &\le \int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left|h_r\left(\theta(x,t),\hat r(y)\right)\left(\hat r'(y) - r'(x)\right)\right|\varphi \\
&\qquad\qquad + \left|h_r\left(h_r\left(\theta(x,t),\hat r(y) - \hat\theta(y,s),r(x)\right)\right)\right|\left|r'(x)\right|\varphi\,dxdt\,dyds \\
&\le M^1\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left\{\left|\hat r'(y) - r'(x)\right| + \left|\hat\theta(y,s) - \theta(x,t)\right| + \left|r(x) - \hat r(y)\right|\right\}\varphi\,dxdt\,dyds,
\end{aligned}$$

where M^1 depends linearly on $\|r'\|_\infty$ but not on ε or on $\|\hat r\|_\infty$. Now

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left|r(x) - \hat r(y)\right|\varphi\,dxdt\,dyds \xrightarrow{\varepsilon\to 0} T\int_{\mathbb{R}}\left|r(x) - \hat r(x)\right|\,dx \\
&\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left|r'(x) - \hat r'(y)\right|\varphi\,dxdt\,dyds \xrightarrow{\varepsilon\to 0} T\int_{\mathbb{R}}\left|r'(x) - \hat r'(x)\right|\,dx \\
&\int_0^T\int_{\mathbb{R}}\int_0^T\int_{\mathbb{R}} \left|\theta(x,t) - \hat\theta(y,s)\right|\varphi\,dxdt\,dyds \xrightarrow{\varepsilon\to 0} \int_0^T\int_{\mathbb{R}}\left|\theta(x,t) - \hat\theta(x,t)\right|\,dxdt.
\end{aligned} \tag{2.34}$$

This provides the bound

$$\lim_{\varepsilon\to 0}\left|R_2(\varepsilon)\right| \le M^1\left(T\left(\|r-\hat r\|_1 + T.V.\,(r-\hat r)\right) + \int_0^T \left\|\theta(\cdot,t) - \hat\theta(\cdot,t)\right\|_1\,dt\right). \tag{2.35}$$

Now we estimate R_1. Let

$$\begin{aligned}
Q(x,t,y,s) &= \operatorname{sign}\left(\theta(x,t) - \hat\theta(y,s)\right)\times \\
&\qquad \left[\left(h(\theta(x,t),r(x)) - h(\theta(x,t),\hat r(y))\right) - \left(h(\hat\theta(y,s),r(x)) - h(\hat\theta(y,s),\hat r(y))\right)\right],
\end{aligned}$$

Then

$$R_1(\varepsilon) = \iiiint Q(x,t,y,s)\varphi_x\,dxdt\,dyds.$$

Observe that

$$\iiiint Q(y,s,y,s)\varphi_x\,dxdt\,dyds = 0.$$

Hence we can write R_1 as

$$R_1(\varepsilon) = \iiiint \left\{Q(x,t,y,s) - Q(y,s,y,s)\right\}\varphi_x\,dxdt\,dyds.$$

Estimating the difference $Q(x,t,y,s) - Q(y,s,y,s)$, we find that

$$|Q(x,t,y,s) - Q(y,s,y,s)| \leq M\left(|r(x) - \hat{r}(y)|\ |\theta(x,t) - \theta(y,s)| + |r(x) - r(y)|\ \left|\theta(y,s) - \hat{\theta}(y,s)\right|\right)$$

for some constant M depending on h. Consequently

$$\begin{aligned}
|R_1(\varepsilon)| \leq M\bigg\{&\int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} |r(x) - \hat{r}(y)|\ |\theta(x,t) - \theta(y,s)|\ |\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dxdt\,dyds \\
&+ \int_0^T \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}} |r(x) - r(y)|\ \left|\theta(y,s) - \hat{\theta}(y,s)\right|\ |\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dxdt\,dyds\bigg\} \\
=:&\ I_1(\varepsilon) + I_2(\varepsilon).
\end{aligned}$$

Now

$$\begin{aligned}
I_1(\varepsilon) &\leq M \iiiint \left(|\hat{r}(y) - r(y)| + |r(y) - r(x)|\right) |\theta(x,t) - \theta(y,s)|\ |\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dxdt\,dyds \\
&\leq M \iiiint |r(y) - \hat{r}(y)| \left(|\theta(x,t) - \theta(y,t)| + |\theta(y,t) - \theta(y,s)|\right) |\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dxdt\,dyds \\
&\quad + \frac{M^1 M_\omega^2}{\varepsilon^2} \int_0^T \int_{\mathbb{R}} \int_{t-\varepsilon}^{t+\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} |\theta(x,t) - \theta(y,s)|\ dyds\,dxdt.
\end{aligned}$$

Since θ is bounded and measurable, the second term above tends to zero as ε becomes small. The integral in the first term is bounded by

$$\begin{aligned}
\|r - \hat{r}\|_\infty &\int_0^T \int_0^T \omega_\varepsilon(t-s)\bigg\{\int_{\mathbb{R}} \int_{\mathbb{R}} |\theta(x,t) - \theta(y,t)|\ |\omega_\varepsilon{}'(x-y)|\ dxdy \\
&\qquad + \int_{\mathbb{R}} |\theta(y,t) - \theta(y,s)| \left[\int_{y-\varepsilon}^{y+\varepsilon} |\omega_\varepsilon{}'(x-y)|\ dx\right] dy\bigg\}\,ds\,dt \\
&\leq \|r - \hat{r}\|_\infty \int_0^T \int_0^T \omega_\varepsilon(t-s)\bigg\{\int_{\mathbb{R}} |\omega_\varepsilon{}'(z)| \int_{\mathbb{R}} |\theta(x,t) - \theta(x-z,t)|\ dx\,dz \\
&\qquad + \frac{2M_\omega}{\varepsilon} \|\theta(\cdot,t) - \theta(\cdot,s)\|_1\bigg\}\,ds\ dt \\
&\leq \|r - \hat{r}\|_\infty \int_0^T \left(M^2 T.V.(\theta(\cdot,t)) + M^3\right)\,dt
\end{aligned}$$

where the constant M^2 depends on $\int z\omega_\varepsilon{}'(z)dz$ and M^3 depends linearly on $T.V.(\theta)$. Hence

$$\lim_{\varepsilon\to 0} I_1(\varepsilon) \leq M^4 T\,\|r - \hat{r}\|_\infty\,,$$

for some constant M^4 depending linearly on $T.V._x(\theta)$. We estimate I_2 by using the Lipschitz continuity of r.

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_2(\varepsilon) &\leq M^1 \lim_{\varepsilon\to 0} \iiiint \left|\theta(y,s) - \hat{\theta}(y,s)\right|\ |x-y|\,|\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dxdt\,dyds \\
&\leq M^1 \lim_{\varepsilon\to 0} \iint \left|\theta(y,s) - \hat{\theta}(y,s)\right| \int |x-y|\,|\omega_\varepsilon{}'(x-y)|\,\omega_\varepsilon(t-s)\,dx\,dy\,ds \\
&\leq M^1 M^2 \int_0^T \left\|\theta(\cdot,t) - \hat{\theta}(\cdot,t)\right\|_1\,dt.
\end{aligned}$$

Consequently,

$$\lim_{\varepsilon\to 0} |R_1(\varepsilon)| \leq M^5\left(T\,\|r - \hat{r}\|_\infty + \int_0^T \left\|\theta(\cdot,t) - \hat{\theta}(\cdot,s)\right\|_1\,dt\right), \tag{2.36}$$

for some constant M^5 depending linearly on $\|r'\|_\infty$ and $T.V.(\theta)$. Collecting the bounds on the terms in (2.32), and using the relation $\|r-\hat{r}\|_\infty \leq T.V.(r-\hat{r})$, we find that

$$\left\|\theta(\cdot,T)-\hat{\theta}(\cdot,T)\right\|_1 \leq \left\|\theta_0-\hat{\theta}_0\right\|_1 + M^6\left(T\left(\|r-\hat{r}\|_1 + T.V.(r-\hat{r})\right) + \int_0^T \left\|\theta(\cdot,t)-\hat{\theta}(\cdot,s)\right\|_1 dt\right) \tag{2.37}$$

for some constant M^6 depending linearly on $\|r'\|_\infty$ and $T.V.(\theta)$. The lemma then follows by the Gronwall inequality. □

Note that for $r=\hat{r}$, this proof will yield Kružkov's stability result (2.15).

Remark. By itself, this lemma is not quite enough to show that the weak solution generated by front tracking; θ^f, is the same weak solution obtained by letting $\varepsilon \to 0$ in (2.24). The next section is devoted to obtaining the necessary tools for showing this.

2.5. **An estimate of the convergence rate if r is smooth.** We can actually extract more information from the entropy inequality for the approximate solutions θ^δ, (2.18). To this end, we need some technical results.

For a function $u(x,t)$, assume that there exist continuous functions ν_x and ν_t $\mathbb{R}_+ \mapsto \mathbb{R}_+$, with $\nu_{x,t}(0)=0$, such that

$$\begin{aligned}
&\sup_{|y|\leq\varepsilon}\int_{\mathbb{R}} |u(x+y,t)-u(x,t)|\,dx \leq \nu_x(\varepsilon)\\
&\sup_{|s|\leq\tau}\int_{\mathbb{R}} |u(x,t+s)-u(x,t)|\,dx \leq \nu_t(\tau)
\end{aligned} \tag{2.38}$$

If we can find such functions, we say that u has moduli of continuity in x and t. Note that if u is of bounded variation in x, then ν_x can be chosen as $C\varepsilon$. The next proposition states that functions (θ,r) that have the property that $\Xi_{\theta r}(\theta,r)$ is of bounded variation, have moduli of continuity.

Proposition 2.1. *Let $\theta:\mathbb{R}\mapsto[-\pi,\pi]$ and $r\mapsto[a,b]$ where $0<a<b$. Assume that $\Xi_{\theta r}(\theta,r)$ has bounded variation. Furthermore assume that both r and θ are constant outside some bounded interval I. Then θ has a module of continuity ν_x which can be chosen as $C\varepsilon^{1/2}$ for some constant C depending on h, a, b, $T.V.(\Xi_{\theta r}(\theta,r))$ and I.*

Proof. Without loss of generality, we can assume that both θ and r are right continuous. Let $\varepsilon>0$, and set $x_i=i\varepsilon$ for $i\in\mathbb{Z}$. Set $\theta_i=\theta(x_i+)$, and $r_i=r(x_i+)$. Define the sets

$$\mathcal{S}_\varepsilon^\pm = \left\{x_i \;\middle|\; |\theta_i \mp \pi/2| \leq \sqrt{\varepsilon} \text{ and } |\theta_{i-1}\mp\pi/2|\leq\sqrt{\varepsilon}\right\} \quad\text{and}\quad \mathcal{B}_\varepsilon = \left\{x_i \;\middle|\; x_i\notin\mathcal{S}_\varepsilon^\pm\right\}.$$

Note that for $|\theta\mp\pi/2|>\sqrt{\varepsilon}$, $|\partial_\theta\Xi_{\theta r}|\geq C_1\sqrt{\varepsilon}$ for some constant C_1 depending on h and a. Since θ and r are constant outside I, $\sum_{\mathcal{S}_\varepsilon} 1 = \mathcal{O}(1/\varepsilon)$. Therefore,

$$C_2 = \sum_{\mathcal{S}_\varepsilon}\left(\sqrt{\varepsilon}\right)^2 \geq \sqrt{\varepsilon}\sum_{\mathcal{S}_\varepsilon}|\theta_i-\theta_{i-1}|.$$

Furthermore since $T.V.(\Xi_{\theta r}(\theta,r))$ is finite

$$\sum_{\mathcal{B}_\varepsilon}\left|(\Xi_{\theta r})_i-(\Xi_{\theta r})_{i-1}\right| \leq C_3,$$

for some constant C_1 not depending on ε. Combining the above we obtain

$$\begin{aligned}
\sqrt{\varepsilon}\sum_i|\theta_i-\theta_{i-1}| &= \sqrt{\varepsilon}\sum_{\mathcal{B}_\varepsilon}|\theta_i-\theta_{i-1}| + \sqrt{\varepsilon}\sum_{\mathcal{S}_\varepsilon}|\theta_i-\theta_{i-1}|\\
&\leq C_4\sum_{\mathcal{B}_\varepsilon}\left|(\Xi_{\theta r})_i-(\Xi_{\theta r})_{i-1}\right| + C_2\\
&\leq C_5.
\end{aligned}$$

Since ε is arbitrary, this implies the conclusion of the proposition. □

Lemma 2.6 (Kružkov's interpolation lemma). *Let $z(x,t)$ be a bounded measurable function defined in $\Pi_T = \mathbb{R} \times [0,T]$. For $t \in [0,T]$ assume that z possesses a spatial modulus of continuity ν_x that does not depend on t. Suppose that for any $\phi \in C_0^\infty(\mathbb{R})$ and any $t_1, t_2 \in [0,T]$,*

$$\left| \int_{\mathbb{R}} \left(z\left(x,t_2\right) - z\left(x,t_1\right) \right) \phi(x)\, dx \right| \leq \mathrm{Const}_T \cdot \left\|\phi'\right\|_\infty \left|t_2 - t_1\right|. \tag{2.39}$$

Then for t and $t+\tau \in [0,T]$ and all $\varepsilon > 0$

$$\int_{\mathbb{R}} \left| z(x,t+\tau) - z(x,t) \right| \, dx \leq \mathrm{Const}_T \cdot \left(\frac{|\tau|}{\varepsilon} + \nu_x(\varepsilon) \right). \tag{2.40}$$

For a proof of this lemma, see [11].

Now it is straightforward to show that θ^δ satisfies (2.39). Hence we can use the lemma and Proposition 2.1 to conclude that θ^δ possesses a modulus of continuity ν_t which can be chosen as

$$\nu_t(\tau) = C\tau^{1/3}.$$

Consequently both the approximate solutions θ^δ and the limit θ possess moduli of continuity

$$\nu_x(\varepsilon) = C\varepsilon^{1/2}, \quad \nu_t(\tau) = C\tau^{1/3}, \tag{2.41}$$

where the constant C does not depend on t for $t \leq T$.

Let $\omega_\varepsilon(x)$ be a standard mollifier as before, and define a test function $\varphi(x,y,t,s)$ by

$$\varphi(x,y,t,s) = \omega_\varepsilon(x-y)\omega_\varepsilon(t-s).$$

For a function $u = u(x,t)$ let $\lambda_\varepsilon(u,c)$ be defined as

(2.42)

$$\lambda_\varepsilon(u,k) = -\iint\limits_{\Pi_T} |u-k|\,\varphi_t + \mathrm{sign}\,(u-c)\left(h(u,r) - h(k,r)\right)\varphi_x - \mathrm{sign}\,(u-c)\, h(k,r)_x \varphi \, dxdt + \int\limits_{\mathbb{R}} |u-k|\,\varphi \,\Big|_{t=0}^{t=T} \; dx.$$

For two functions u and v we define the functional $\Lambda_\varepsilon(u,v)$ as

$$\Lambda_\varepsilon(u,v) = \iint\limits_{\Pi_T} \lambda_\varepsilon(u, v(y,s))\, dyds, \tag{2.43}$$

where $u = u(x,t)$ and $v = v(y,s)$. In passing, we note that if u is an entropy solution of (2.1), then

$$\Lambda_\varepsilon(u,v) \leq 0. \tag{2.44}$$

For two arbitrary functions u and v we have the following result:

Lemma 2.7 (Kuznetsov's lemma). *Assume that that r is in $C(\mathbb{R})$ and r' is in $L^\infty(\mathbb{R})$ and that both r and r' have moduli of continuity. If u and v are in $L^1(\Pi_T)$ and have moduli of continuity in space and time, then*

$$\begin{aligned} \left\|u(\cdot,T) - v(\cdot,T)\right\|_1 \leq{}& \left\|u(\cdot,0) - v(\cdot,0)\right\|_1 + \Lambda_\varepsilon(u,v) + \Lambda_\varepsilon(v,u) \\ &+ \frac{1}{2}\left[\nu_x(u(\cdot,0);\varepsilon) + \nu_x(v(\cdot,0);\varepsilon) + \nu_x(u(\cdot,T);\varepsilon) + \nu_x(v(\cdot,T);\varepsilon)\right] \\ &+ \frac{1}{2}\left[\nu_t(u(\cdot,T);\varepsilon) + \nu_t(v(\cdot,T);\varepsilon) + \nu_t(u(\cdot,0);\varepsilon) + \nu_t(v(\cdot,0);\varepsilon)\right] \\ &+ \left\|r'\right\|_\infty \left\|h_r\right\|_\infty T \sup_{0\leq t\leq T} \left(\nu_x(u(\cdot,t);\varepsilon) + \nu_t(u(\cdot,t);\varepsilon)\right) \\ &+ CT\left(\left\|r'\right\|_\infty \nu_x(r;\varepsilon) + \nu_x(r';\varepsilon)\right), \end{aligned} \tag{2.45}$$

where $\nu_x(\cdot;\cdot)$ and $\nu_t(\cdot;\cdot)$ denote moduli of continuity. The constant C depends on h_r and h_{rr}.

For a proof of this lemma, see Karlsen and Risebro [11]. Applying Kuznetsov's lemma with $u = \theta$ and $v = \theta^\delta$ and using (2.41), we find

$$\left\|\theta(\cdot,T) - \theta^\delta(\cdot,T)\right\|_1 \leq \left\|\theta_0 - \theta_0^\delta\right\|_1 + \Lambda_\varepsilon(\theta^\delta,\theta) + C\cdot\|r'\|_\infty\left(\varepsilon + \varepsilon^{1/2} + \varepsilon^{1/3}\right). \tag{2.46}$$

Here we tacitly assume that $\|r'\|_\infty > 0$, as this will be the case for our primary application of the results following from this.

It remains to estimate $\Lambda_\varepsilon(\theta^\delta,\theta)$. Let $\tilde\lambda$ be defined by the negative of (2.23), then $\tilde\lambda \leq 0$, and

$$\lambda_\varepsilon(\theta^\delta,k) = \left(\lambda_\varepsilon(\theta^\delta,k) - \tilde\lambda\right) + \tilde\lambda \leq \left|\lambda_\varepsilon(\theta^\delta,k) - \tilde\lambda\right|.$$

Now

$$\begin{aligned}\left|\lambda_\varepsilon(\theta^\delta,k) - \tilde\lambda\right| \leq \iint\limits_{\Pi_T} &\left(\left|h^\delta(\theta^\delta,r^\delta) - h(\theta^\delta,r)\right| + \left|h^\delta(k,r^\delta) - h(k,r)\right|\right)|\varphi_x|\,dxdt \\ &+ \int_0^T \sum_i \int_{x_i}^{x_{i+1}} |\varphi(x_i,t) - \varphi(x,t)|\,|h(k,r)_x|\,dx\,dt.\end{aligned}$$

Regarding the first term on the right, we have that $h^\delta(\theta^\delta,r^\delta) = h(\theta^\delta,r^\delta)$, hence

$$\left|h^\delta(\theta^\delta,r^\delta) - h(\theta^\delta,r)\right| \leq C\delta, \tag{2.47}$$

since $\left|r^\delta - r\right| \leq \delta$. Also

$$\left|h^\delta(k,r^\delta) - h(k,r)\right| \leq \left|h^\delta(k,r^\delta) - h^\delta(k,r)\right| + \left|h^\delta(k,r) - h(k,r)\right| \leq C\delta \tag{2.48}$$

by the construction of h^δ. To estimate the second term, we must specify how the points x_i are chosen. By assumption, r is constant outside a finite interval $I = \langle a,b\rangle$. Let $x_0 = a$, if necessary modify δ such that $r(a) = j\delta$ for some $j \in \mathbb{Z}$. Then define

$$x_{i+1} = \inf\left\{x > x_i \mid |r(x) - r(x_i)|) \geq \delta\right\}. \tag{2.49}$$

If this infimum does not exist, then $x_{i+1} = b$. Since r and r' are of bounded variation, for sufficiently small δ, r is monotone in each interval $\langle x_i, x_{i+1}\rangle$. Then

$$\begin{aligned}\int_{x_i}^{x_{i+1}} |\varphi(x_i,t) - \varphi(x,t)|\,|h(k,r)_x|\,dx &\leq C\int_{x_i}^{x_{i+1}}\int_{x_i}^{x} |\varphi_z(z,t)|\,dz\,|r'(x)|\,dx \\ &\leq C\,(x_{i+1} - x_i)\,\delta\int_{x_i}^{x_{i+1}} |\varphi_x(x,t)|\,dx\end{aligned} \tag{2.50}$$

Combining (2.47), (2.48) and (2.50) we then find

$$\lambda_\varepsilon(\theta^\delta,k) \leq C\cdot\delta\int_0^T\left(\int_{\mathbb{R}} + \sum_i (x_{i+1} - x_i)\int_{x_i}^{x_{i+1}} |\varphi_x(x,t)|\,dx\right)dt$$

Recall that we have defined our approximation θ^δ such that

$$\left|\theta_0^\delta(x) - \theta_0(x)\right| \leq \sqrt{\delta}$$

for all x. Since θ_0 has compact support, this implies that

$$\left\|\theta_0^\delta - \theta_0\right\|_1 \leq C\delta^{1/2}.$$

Finally, using our special choice of test function, we find that

$$\left\|\theta(\cdot,T) - \theta^\delta(\cdot,T)\right\|_1 \leq +C\cdot\left[\delta^{1/2} + \frac{\delta}{\varepsilon} + \|r'\|_\infty\left(\varepsilon + \varepsilon^{1/2} + \varepsilon^{1/3}\right)\right], \tag{2.51}$$

where C depends also on T. Minimizing with respect to ε gives

$$\left\|\theta(\cdot,T) - \theta^\delta(\cdot,T)\right\|_1 \leq C\cdot\left(\|r'\|_\infty^3\,\delta\right)^{1/4} \tag{2.52}$$

To conclude this section, we state our results:

Theorem 2.3. *Let θ_0 and r be functions such that the assumptions of Theorem 2.1 hold. Furthermore assume that r is continuous and r' uniformly bounded, and that θ_0 and r' have bounded support. Let θ denote the entropy solution to*

$$\theta_t + h(\theta, r)_x = 0, \quad \theta(x,0) = \theta_0(x),$$

and let θ^δ denote the front tracking approximation to θ. Then the bound (2.52) holds, where the constant C only depends on h, T and the support of θ_0 and r'.

2.6. **Entropy solutions with discontinuous r.** Let now θ^ε denote the solution to (2.24). Since r is bounded we have that

$$\|(r^\varepsilon)'\|_\infty \leq \frac{C}{\varepsilon},$$

for some constant C. Furthermore we have established that there is a subsequence of $\{\varepsilon\}$; $\{\varepsilon_j\}$ such that

$$\left\|\theta^{\varepsilon_j} - \theta^0\right\|_1 \to 0,$$

where θ^0 is a weak solution of (2.53). If $\theta^{\delta,\varepsilon}$ denotes the front tracking approximation to θ^ε then the previous theorem; Theorem 2.3 says that

$$\left\|\theta^{\delta,\varepsilon} - \theta^\varepsilon\right\|_1 \leq C\left(\frac{\delta}{\varepsilon^3}\right)^{1/4}.$$

Let now θ^δ denote the front tracking approximation to

$$\theta_t + h(\theta, r)_x = 0, \tag{2.53}$$

with r bounded but possibly discontinuous. Note that while we must define $r^{\delta,\varepsilon}$ according to (2.49), however we can define r^δ to be any function taking values in the set $\{j\delta\}_{j\in\mathbb{Z}}$, such that

$$\lim_{\delta\to 0} \left\|r^\delta - r\right\|_1 = 0. \tag{2.54}$$

Now set $\delta = \varepsilon^4$, by choosing subsequences if necessary, we have that

$$\left\|\theta^\delta - \theta^f\right\|_1 \to 0, \quad \text{and} \quad \left\|\theta^\varepsilon - \theta^0\right\|_1 \to 0,$$

as δ $(\varepsilon) \to 0$. Now define r^δ as $r^{\delta,\varepsilon}$, when $\varepsilon = \delta^{1/4}$ this choice clearly satisfies (2.54). Since the initial data coincide, also $\theta^{\delta,\varepsilon} = \theta^\delta$, thus

$$\begin{aligned}\left\|\theta^f - \theta^0\right\|_1 &\leq \left\|\theta^f - \theta^\delta\right\|_1 + \left\|\theta^\delta - \theta^{\delta,\varepsilon}\right\|_1 + \left\|\theta^{\delta,\varepsilon} - \theta^\varepsilon\right\|_1 + \left\|\theta^\varepsilon - \theta^0\right\|_1 \\ &\leq \left\|\theta^f - \theta^\delta\right\|_1 + C\cdot\varepsilon^{1/4} + \left\|\theta^\varepsilon - \theta^0\right\|_1.\end{aligned}$$

Now let $\varepsilon \to 0$ and conclude that $\theta^f = \theta^0$. In other words, also the solution obtained by front tracking converges to the solution obtained by smoothing the coefficients and letting the smoothing radius tend to zero. Furthermore by (2.15) this limit is unique. We therefore say that a weak solution to (2.1) is andn *entropy solution* if it is the L^1 limit as $\varepsilon \to 0$ of Kružkov entropy solutions to

$$\theta_t^\varepsilon + h\left(\theta^\varepsilon, r * \omega_\varepsilon\right)_x = 0. \tag{2.55}$$

Summing up, we have shown:

Theorem 2.4. *Let (θ_0, r) be such that $\mathbf{u}_0 = \Xi_{rc}(\theta_0, r)$ is in $B.V. \cap L^1$. Then, there exists a unique entropy solution to*

$$\theta_t + h(\theta, r)_x = 0, \quad \theta(x,0) = \theta_0(x).$$

This entropy solution can be constructed by front tracking. Furthermore, if $\hat{\theta}$ is another entropy solution to the above equation, but with initial data $\hat{\theta}_0$, then

$$\left\|\theta(\cdot,t) - \hat{\theta}(\cdot,t)\right\|_1 \leq \left\|\theta_0 - \hat{\theta}_0\right\|_1.$$

Note that for the specific equation as (2.1), using the methods in [13], we can actually prove a stronger stability estimate

$$\left\|\theta(\cdot,t)-\hat{\theta}(\cdot,t)\right\|_1 \leq \left\|\theta_0-\hat{\theta}_0\right\|_1 + CtT.V.\left(r-\hat{r}\right) \tag{2.56}$$

for some constant C depending only on r, $\hat{r}$, h and the initial data. This estimate is possible only for those systems on the form (2.1) where the transition curves are *not* a function of r. In the proof of Theorem 2.4, we have not used this, and our arguments relies only on the fact that the solution of the Riemann problem can transformed into the diagrams in figure 2. While not attempting to describe precisely which scalar conservation laws that have this property, we will display one such equation in the next section.

3. The polymer system

In this section we shall show analogous results for the polymer system (1.10), (1.11), as we did in the previous section for the auxiliary system (2.1). To do this we adapt Wagner's general results [23], to the polymer case.

For technical reasons, we assume that the relative permeability of the water satisfies

$$\lambda_{\mathbf{w}}(s)=0, \quad \text{for all } s\leq\alpha$$

for some constant $\alpha<1$. This assumption means that a certain amount of the water is bound to the rock. We begin by translating the stability results obtained for smooth coefficients in the previous section, to the corresponding results for the polymer system.

3.1. Stability estimates for smooth c.

Let now $g=f/s$. In [22], Tveito and Winther defined an entropy solution to (1.10), (1.11) as follows: A pair of functions (s,c) is called an *entropy solution* to (1.10), (1.11) in $\mathbb{R}\times[0,T]$ if:

Definition 3.1. 1. *(s,c) take values in $[\alpha,1]\times[0,1]$, (s,c) is $L^1(dx)$-Lipschitz continuous in t, $s(\cdot,t)$ and $c_x(\cdot,t)$ are Lipschitz continuous and of bounded variation, and c is Lipschitz continuous.*

2. *For all non-negative test functions φ, all constants $q\in[\alpha,1]$ if*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} |s-q|\,\varphi_t + \operatorname{sign}(s-q)\left(f(s,c)-f(q,c)\right)\varphi_x - \operatorname{sign}(s-q)\,f(q,c)_x\varphi\,dxdt\\ -\int_{\mathbb{R}}|s(x,T)-q|\,\varphi(x,T)\,dx+\int_{\mathbb{R}}|s(x,0)-q|\,\varphi(x,0)\,dx\geq 0.\end{aligned} \tag{3.1}$$

3. *For almost all (x,t) in $\mathbb{R}\times[0,T]$, c satisfies*

$$c_t+g(s,c)c_x=0. \tag{3.2}$$

By Theorem 2 in [23], there is a one-to-one correspondence between weak solutions of (1.10), (1.11), satisfying $0<\alpha\leq s\leq 1$, and weak solutions of

$$\begin{aligned}\left(\frac{1}{s}\right)_\tau-\left(\frac{f(s,c)}{s}\right)_y&=0\\ c_\tau&=0.\end{aligned} \tag{3.3}$$

The new coordinates (y,τ) are given by

$$\frac{\partial y}{\partial x}=s,\quad \frac{\partial y}{\partial t}=-f(s,c),\quad \frac{\partial \tau}{\partial x}=0,\quad\text{and}\quad \frac{\partial\tau}{\partial t}=1.$$

Then

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial t}+\frac{f}{s}\frac{\partial}{\partial x},\quad \frac{\partial}{\partial y}=\frac{1}{s}\frac{\partial}{\partial x}.$$

For smooth c, a Kružkov entropy solution of (3.3) satisfies

(3.4)
$$\int_0^T \int_{\mathbb{R}} \left|\frac{1}{s} - k\right| \varphi_\tau - \operatorname{sign}\left(\frac{1}{s} - k\right)\left(\frac{f(s,c)}{s} - kf\left(\frac{1}{k}, c\right)\right)\varphi_y$$
$$+ \operatorname{sign}\left(\frac{1}{s} - k\right)\left(kf\left(\frac{1}{k}, c\right)\right)_y \varphi\, dy\, d\tau$$
$$- \int_{\mathbb{R}} \left|\frac{1}{s(T)} - k\right| \varphi(T)\, dy + \int_{\mathbb{R}} \left|\frac{1}{s_0} - k\right| \varphi(0)\, dy \geq 0$$

for all constants $k \in [1, 1/\alpha]$, and all non-negative test functions φ. Changing variables to (x,t) we find that the first integral above equals

$$\int_0^T \int_{\mathbb{R}} \left\{ \left|\frac{1}{s} - k\right| \left(\varphi_t + \frac{f}{s}\varphi_x\right) - \operatorname{sign}\left(\frac{1}{s} - k\right)\left(\frac{f(s,c)}{s} - kf\left(\frac{1}{k}, c\right)\right)\frac{1}{s}\varphi_x \right.$$
$$\left. + \operatorname{sign}\left(\frac{1}{s} - k\right) kf\left(\frac{1}{k}, c\right)_x \frac{1}{s}\varphi \right\} s\, dx dt$$
$$= \int_0^T \int_{\mathbb{R}} k\left|\frac{1}{k} - s\right| \varphi_t + \operatorname{sign}\left(\frac{1}{k} - s\right)\left[\left(\frac{1}{s} - k\right) f - \frac{f}{s} + kf\left(\frac{1}{k}, c\right)\right]\varphi_x$$
$$+ \operatorname{sign}\left(\frac{1}{k} - s\right) kf\left(\frac{1}{k}, c\right)_x \varphi\, dx dt$$
$$= k\int_0^T \int_{\mathbb{R}} \left|s - \frac{1}{k}\right| \varphi_x + \operatorname{sign}\left(s - \frac{1}{k}\right)\left[f(s,c) - f\left(\frac{1}{k}, c\right)\right]\varphi_x - \operatorname{sign}\left(s - \frac{1}{k}\right) f\left(\frac{1}{k}, c\right)_x \varphi\, dx dt.$$

Similarly
$$\int_{\mathbb{R}} \left|\frac{1}{s} - k\right| \varphi\, dy = k\int_{\mathbb{R}} \left|s - \frac{1}{k}\right| \varphi\, dx.$$

Also, if $c(y)$ is smooth, then
$$c_t + \left(\frac{f(s,c)}{s}\right) c_x = 0$$

almost everywhere. Consequently, entropy solutions of (1.10), (1.11) in the sense of Definition 3.1, are equivalent to Kružkov entropy solutions of (3.3). Furthermore, from the stability estimate (2.15), we find

$$\begin{aligned}
\int_{\mathbb{R}} \left|s(\cdot,t) - \hat{s}(\cdot,t)\right| dx &\leq \int_{\mathbb{R}} \left|s(\cdot,t) - \hat{s}(\cdot,t)\right| \frac{1}{s(\cdot,t)}\, dx \\
&= \int_{\mathbb{R}} \left|\frac{1}{s(\cdot,\tau)} - \frac{1}{\hat{s}(\cdot,\tau)}\right| dy \\
&\leq \int_{\mathbb{R}} \left|\frac{1}{s_0} - \frac{1}{\hat{s}_0}\right| dy \\
&= \int_{\mathbb{R}} \frac{1}{s_0} \left|\hat{s}_0 - s_0\right| dx \\
&\leq \frac{1}{\alpha}\int_{\mathbb{R}} \left|\hat{s}_0 - s_0\right| dx
\end{aligned} \tag{3.5}$$

where s and $\hat{s}$ are entropy solutions of (3.3) with initial data s_0 and $\hat{s}_0$ respectively. It is also now straightforward to translate the stability estimate (2.27) into a corresponding estimate for the polymer system

$$\begin{aligned}
&\left(\left\|s(\cdot,t) - \hat{s}(\cdot,t)\right\|_1 + \left\|c(\cdot,t) - \hat{c}(\cdot,t)\right\|_1 + \left\|c_x(\cdot,t) - \hat{c}_x(\cdot,t)\right\|_1\right) \\
&\qquad \leq M\left(\left\|s_0 - \hat{s}_0\right\|_1 + \left\|c_0 - \hat{c}_0\right\|_1 + \left\|c_{0,x} - \hat{c}_{0,x}\right\|_1\right)
\end{aligned} \tag{3.6}$$

where (s,c) and $(\hat{s},\hat{c})$ are entropy solutions of (1.10), (1.11), with initial data (s_0,c_0) and $(\hat{s}_0,\hat{c}_0)$. Collecting the bounds obtained in this section, we have:

Theorem 3.1. *There exists a unique entropy solution, in the sense of Definition 3.1 to* (1.10), (1.11). *Furthermore two entropy solutions with the same initial c data, satisfies* (3.5), *and two entropy solutions with different s and c initial data satisfies* (3.6).

3.2. **Stability of entropy solutions for general c.** Let now

$$\beta = \frac{1}{s}, \quad q = \frac{1}{\mu(c)}, \quad \text{and} \quad h(\beta, q) = f\left(\frac{1}{\beta}, c(q)\right).$$

We wish to use front tracking to construct approximate solutions to

$$\begin{aligned} \beta_\tau + h(\beta, q)_y = 0, \qquad q_\tau = 0 \\ \beta(y, 0) = \beta_0(y), \end{aligned} \tag{3.7}$$

where β_0 is in the interval $[1, 1/\alpha]$ and $q > 0$. Below we have shown the function h as a function of β for two different r values. Using the solution of the Riemann problem in [4], it is now a matter

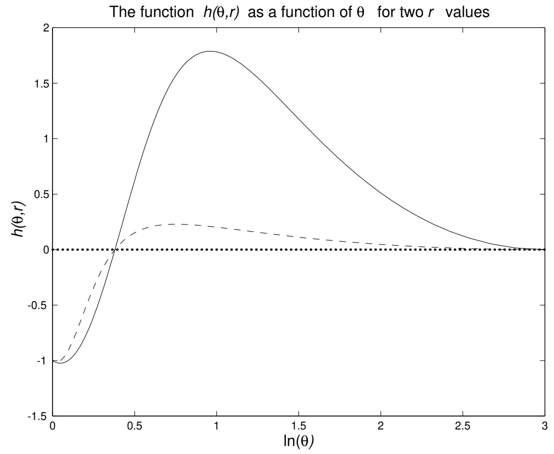


FIGURE 3. The function $h(\beta, q)$ as a function of $\ln(\beta)$.

of routine to verify that there exist smooth mappings $\theta(\beta, q)$ and $r(q)$, such that the solution of the Riemann problem for (3.7) is given by figure 2. Using this Riemann solution, we can repeat the arguments in section 2 to show that there exists a weak solution for any initial data β_0, and r such that $F(\theta(\beta_0, r), q(r))$ is bounded. In particular, this includes the case where β_0 and r are of bounded variation. This means that also the polymer system (1.10), (1.11), has a weak solution for s_0 and c_0 in $BV \cap L^1$.

We can also mimic the definition of entropy solutions in the case where $c_0(x)$ is discontinuous. Let $(s^\varepsilon, c^\varepsilon)$ be entropy solutions, in the sense of Definition 3.1, to the initial value problem

$$\begin{aligned} s^\varepsilon_t + f\left(s^\varepsilon, c^\varepsilon\right)_x = 0, \qquad \left(s^\varepsilon c^\varepsilon\right)_t + \left(c^\varepsilon f\left(s^\varepsilon, c^\varepsilon\right)\right)_x = 0 \\ s^\varepsilon(x, 0) = s_0(x), \qquad c^\varepsilon(x, 0) = c_0 * \omega_\varepsilon(x) \end{aligned} \tag{3.8}$$

where ω_ε is a standard mollifier. By using the arguments in the previous section, we find that $\{(s^\varepsilon, c^\varepsilon)\}$ converges in L^1_{loc} to a unique limit as $\varepsilon \to 0$, and that this limit is a weak solution to (1.10), (1.11). Furthermore, this unique limit can be constructed by using front tracking on the corresponding problem (3.7). We define this unique limit as an entropy solution to (1.10), (1.11). Hence we have shown

Theorem 3.2. *Let $f(s, c)$ be as in section 1.1, with $f(s, c) = 0$ for all $s \leq \alpha$ and all $c \in [0, 1]$. Assume that s_0 and c_0 are in $BV \cap L^1$ and take values in $[\alpha, 1]$ and $[0, 1]$ respectively. Then there exists a unique entropy solution to the initial value problem*

$$\begin{aligned} s_t + f(s, c)_x = 0, \qquad (sc)_t + (cf(s, c))_x = 0 \\ s(x, 0) = s_0(x), \qquad c(x, 0) = c_0(x). \end{aligned}$$

If $\hat{s}$ is another entropy solution with initial value $\hat{s}_0$, but with the same initial c value, then

$$\|s(\cdot,t)-\hat{s}(\cdot,t)\|_1 \leq \frac{1}{\alpha}\|s_0-\hat{s}_0\|_1 .$$

Remark. The above uniqueness results enable us to give an independent proof of the equivalence between weak (entropy) solutions of (1.10), (1.11) and (3.3). To prove such a result we can do the following: First show that front tracking converge to the unique entropy solutions of (1.10), (1.11) and (3.3) respectively. Then we can use the fact that the front tracking approximations are weak solutions to approximate problems, and the fact that the Rankine-Hugoniot condition for the front tracking approximations to (1.10), (1.11) and (3.3) are equivalent. This means that under the coordinate transformation $x \mapsto y$, where

$$\frac{\partial y}{\partial x} = s^\delta, \quad \frac{\partial y}{\partial t} = -f^\delta\left(s^\delta, c^\delta\right),$$

where s^δ, c^δ and f^δ are the front tracking approximations and the approximate flux function respectively, front tracking approximations to (1.10), (1.11) are mapped to front tracking approximations to (3.3). Since front tracking converges to unique entropy solutions, these must also be equivalent.

Remark. The front tracking approximation can also yield an existence result without the restriction $f(s,c)=0$ for $s \leq \alpha$. This follows by using the functional F and the arguments in [21] or [5].

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