Relaxation of the isothermal Euler-Poisson system to the Drift-Diffusion Equations

S. Junca and M. Rascle.
Laboratoire de Mathématiques, U.R.A. C.N.R.S. 168
Université de Nice, Parc Valrose B.P. 71, F06108 Nice Cédex 2

Abstract : We consider the one dimensional Euler-Poisson system, in the isothermal case, with a friction coefficient ε^{-1}. When $\varepsilon \to 0_+$, we show that the sequence of entropy-admissible weak solutions constructed in [PRV] converges to the solution to the drift-diffusion equations. We use the scaling introduced in [MN2], who proved a quite similar result in the isentropic case, using the theory of compensated compactness. On one hand this theory cannot be used in our case, on the other hand, exploiting the linear pressure law, we can give here a much simpler proof by only using the entropy inequality and de la Vallée-Poussin criterion of weak compactness in L^1.

1 Introduction

In this paper, we consider a classical fluid model of the transport of electric charges in semi-conductors, namely the Euler-Poisson system, written here in the one-dimensional case :

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) = 0, \tag{1.1}$$

$$\frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\left(nu^2 + p(n)\right) = -\frac{nu}{\varepsilon} + nE. \tag{1.2}$$

In this model, n is the density of electrons, u is the velocity, p is the pressure, and E is the “negative” electric field. Moreover, $1/\varepsilon$ is a positive friction term which describes the collisions between the electrons and the atoms of the crystal. Obviously, equation (1.1) describes the conservation of electrons, and equation (1.2) is the momentum equation, in which the nE term describes the acceleration of the electrons due to the electric field.

This electric field is assumed to be self-consistent:

$$\frac{\partial E}{\partial x} = n - N, \tag{1.3}$$

where N is the constant density of a fixed background charge, that we assume to be constant. For more details on the modeling, we refer e.g. to [MRS] and to the references therein.

For a fixed positive friction coefficient $1/\varepsilon$, Bo Zhang [Z] and Marcati-Natalini [MN1] have considered the *isentropic* case :

$$p(n) = \frac{1}{\gamma} n^\gamma, \qquad 1 < \gamma < \frac{5}{3}. \tag{1.4}$$

Using the theory of compensated compactness, they have proved the global existence of a weak entropy-admissible solution, namely a weak solution such that

$$\frac{\partial}{\partial t}\left(n\frac{u^2}{2} + \frac{1}{\gamma(\gamma-1)}n^\gamma\right) + \frac{\partial}{\partial x}\left(n\frac{u^3}{2} + \frac{1}{\gamma}n^\gamma u\right) + \frac{n(u)^2}{\varepsilon} - nuE \le 0.$$

On the other hand, still for a fixed $\varepsilon > 0$, Poupaud-Rascle-Vila [PRV] have considered the *isothermal* case

$$p(n) = n$$

and obtained the global existence of a weak entropy-admissible solution by using the Glimm scheme [G] and the very particular properties of the underlying nonlinear hyperbolic system, namely the so-called Nishida system [Ni]

$$\begin{cases} \dfrac{\partial n}{\partial t} + \dfrac{\partial}{\partial x}(nu) = 0, \\ \dfrac{\partial}{\partial t}(nu) + \dfrac{\partial}{\partial x}\left(nu^2 + n\right) = 0, \end{cases} \tag{1.5}$$

written here in Eulerian coordinates.

We are interested here in studying the limit of the full system (1.1), (1.2), (2.8) when $\varepsilon \to 0_+$. In the isentropic case, Marcati & Natalini, ([MN2]), have introduced a " parabolic scaling " $s := \varepsilon t$, $x := x$, and shown that in the new variables , the solution converges to the solution of the drift-diffusion system.

In this paper, we are going to show the same convergence result in the isothermal case $\gamma = 1$. We use the same scaling as in [MN2], but our proof is totally different, since the behavior of the Riemann invariants is entirely different near by the vacuum $n = 0$. In particular, we cannot use the method of compensated compactness as in [MN2]. In fact, our proof essentially uses two arguments. First, the pressure depends *linearly* on the density n. Therefore, there is no problem here to pass to the limit in this term ! The second crucial ingredient is the entropy inequality - with a suitable modification of the classical term $n \ln n$ to take into account the behavior of n at infinity - which classically implies the weak compactness in L^1, by de la Vallée-Poussin criterion. We also remark that, using energy estimates for higher order derivatives, a similar result of *strong* convergence when ($\varepsilon \to 0_+$) has been recently obtained in [CJZ1] for the same problem, starting with small smooth initial data. [In fact, the result is stated with a pressure law $p(n) = \frac{1}{\gamma}n^\gamma$, $\gamma > 1$, but the same method could probably cover the case $\gamma = 1$]. The same Authors have also obtained in [CJZ2] a similar result for a more complete hydrodynamic model, which involves an energy equation and a *fixed* positive heat conductivity.

The outline of the paper is as follows. After this introduction, we briefly state the problem in Section 2. Then in Section 3 we recall a few basic facts, before studying the entropy inequality in Section 4 and finally proving the convergence to the solution of the drift-diffusion system in Section 5.

2 Statement of the problem

As we said in the Introduction, we consider the isothermal one-dimensional Euler-Poisson system, that we rewrite here

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nu) &= 0, && (2.6)\\ \frac{\partial}{\partial t}(nu) + \frac{\partial}{\partial x}\Big(nu^2 + p(n)\Big) &= -\frac{nu}{\varepsilon} + nE, && (2.7) \end{aligned}$$

with $p(n) = n$ and

$$\frac{\partial E}{\partial x} = n - N, \tag{2.8}$$

where N is a given non-negative constant. Moreover, we add the initial data

$$n(0,x) = n_0(x), u(0,x) = u_0(x), \qquad x \in \mathbb{R}. \tag{2.9}$$

These initial data, as well as the function N, are not necessary small, and can be discontinuous. However, in order to use the results of [PRV], we assume that

$$u_0 \text{ and } \ln n_0 \in BV(\mathbb{R}), \tag{2.10}$$

where $BV(\mathbb{R})$ is the space of functions with bounded variation:

$$VT(f) := \sup_{\substack{k \in \mathbb{N} \\ x_0 < x_1 < \cdots < x_k}} \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| < \infty.$$

Moreover we make the following assumption:

$$E(t, -\infty) = V_\infty^\varepsilon(t), \qquad t \geq 0, \tag{2.11}$$

with $E_\infty^\varepsilon(s) := V_\infty^\varepsilon(s/\varepsilon)$ satisfies

$$\begin{aligned} E_\infty^\varepsilon &\quad \text{is bounded in} \quad L^\infty(\mathbb{R}_s^+) \cap W_{loc}^{1,1}(\mathbb{R}_s^+), && (2.12)\\ E_\infty^\varepsilon &\quad \text{converges to} \quad E_\infty. && (2.13) \end{aligned}$$

For example, we can take E_∞^ε constant, as in [MN2].

We assume also that there exists a $L_0 > 0$ such that,

$$\text{for } |x| \geq L_0, \quad u_0(x) \equiv u_\infty(0), \qquad n_0(x) \equiv N \equiv N_\infty, \tag{2.14}$$

and

$$\int_{\mathbb{R}} (n_0(x) - N)dx = 0. \tag{2.15}$$

As we will see in Lemma 3.1, these assumptions imply that for any fixed positive ε and t the electric field and the velocity are compactly supported in x.

Let us now compare our assumptions with these considered in [MN2] and [PRV]. Here, as in [PRV], the density is assumed to be constant at infinity. The main reason for this assumption is that the existence result in [PRV] uses the Glimm scheme for the Nishida system (1.5). Now the special Glimm functional for this system involves $\ln n$, and therefore the density cannot be compactly supported, contrarily to the assumptions of [MN2]. On the other hand, we assume here that the background charge N is constant and we consider here the slightly restrictive case where the limits of $E(s,.)$ at $\pm\infty$ are the same. However this assumption is natural in view of (2.15), which expresses a global neutrality.

Let $(n^\varepsilon, u^\varepsilon, E^\varepsilon)$ be a solution to (2.6),(2.7),(2.8), (2.9),(2.11),(2.14). We first recall formal arguments which lead to the "parabolic" scaling introduced in [MN2]. If the problem is well posed, then in equation (2.7) - at least in regions where u^ε is smooth - all the terms are of order 1, except the damping term $-\dfrac{n^\varepsilon u^\varepsilon}{\varepsilon}$.

Therefore , we expect that

$$n^\varepsilon \frac{u^\varepsilon}{\varepsilon} = \mathrm{O}(1), \tag{2.16}$$

and in fact that $\dfrac{u^\varepsilon}{\varepsilon} = \mathrm{O}(1)$.

Therefore, in order to balance the term $\partial_x(n^\varepsilon u^\varepsilon) = \partial_x(0(\varepsilon))$ in equation (2.6), we need to introduce a slow time

$$s := \varepsilon t,$$

so that (2.6) can be rewritten as

$$\frac{\partial n^\varepsilon}{\partial \varepsilon t} + \frac{\partial}{\partial x}\left(n^\varepsilon \frac{u^\varepsilon}{\varepsilon}\right) = 0.$$

In variables s and x, the isothermal Euler-Poisson system reads

$$\begin{aligned}
\frac{\partial n^\varepsilon}{\partial s} + \frac{\partial}{\partial x}\left(\frac{n^\varepsilon u^\varepsilon}{\varepsilon}\right) &= 0, && (2.17)\\
\varepsilon \frac{\partial}{\partial s}(n^\varepsilon u^\varepsilon) + \frac{\partial}{\partial x}\left(n^\varepsilon (u^\varepsilon)^2 + n^\varepsilon\right) &= -\frac{n^\varepsilon u^\varepsilon}{\varepsilon} + n^\varepsilon E, && (2.18)\\
\frac{\partial E^\varepsilon}{\partial x} &= n^\varepsilon - N. && (2.19)
\end{aligned}$$

Let us now assume that, say in the distribution sense,

$$\begin{array}{lcl} n^\varepsilon & \to & n, \\ E^\varepsilon & \to & E. \end{array}$$

Then

$$\varepsilon \frac{\partial}{\partial s}(n^\varepsilon u^\varepsilon) \to 0,$$

and

$$\frac{\partial}{\partial x}\left(n^\varepsilon (u^\varepsilon)^2\right) \to 0,$$

since $u^\varepsilon \to 0$. If we can pass to the limit in the product $n^\varepsilon E^\varepsilon$ in (2.18), then the limit current satisfies

$$\lim_{\varepsilon \to 0} \frac{n^\varepsilon u^\varepsilon}{\varepsilon} = nE - \frac{\partial n}{\partial x}.$$

In view of (2.17), we obtain the drift-diffusion equations

$$\frac{\partial n}{\partial s} + \frac{\partial}{\partial x}\left(nE - \frac{\partial n}{\partial x}\right) = 0, \tag{2.20}$$

$$\frac{\partial E}{\partial x} = n - N, \tag{2.21}$$

together with the initial data

$$n(0,x) = n_0(x), \quad x \in \mathbb{R}, \tag{2.22}$$

$$E(s,-\infty) = E_\infty(s), \quad s \in \mathbb{R}^+. \tag{2.23}$$

As in [MN2], the aim of this paper is to give a rigorous justification of the above asymptotic analysis. However, as in [PRV], we consider here the isothermal case. As we said in the Introduction, the compensated compactness method used in [MN2] does not work in our case, in particular since the Riemann invariants of the hyperbolic system (2.6), (2.7) are now

$$u \pm \ln n$$

and therefore are unbounded near by the vacuum $n = 0$. Our main result is

Theorem 2.1
Under the assumptions (2.9),(2.10),(2.11),(2.12),(2.13),(2.14), for any positive ε let $(n^\varepsilon, u^\varepsilon, E^\varepsilon)$ be an entropy-admissible solution to (2.6),(2.7),(2.8), constructed in [PRV], an let (n,E) be the unique solution to (2.20), (2.21), (2.22), (2.23). Then the whole sequence $\left(n^\varepsilon, n^\varepsilon \frac{u^\varepsilon}{\varepsilon}, E^\varepsilon\right)$ converges in the following way :

$$\begin{array}{llcll} (i) & n^\varepsilon - N_\infty & \to & n - N_\infty & \textit{in } L^1_{loc}(\mathbb{R}^+_s \times \mathbb{R}_x) \textit{ weak}, \\ (ii) & E^\varepsilon & \to & E & \textit{in } L^p_{loc}(\mathbb{R}^+_s \times \mathbb{R}_x) \textit{ strong}, 1 \le p < \infty, \\ (iii) & \dfrac{n^\varepsilon u^\varepsilon}{\varepsilon} & \to & nE - \dfrac{\partial n}{\partial x} & \textit{in } M^1_{loc}(\mathbb{R}^+_s \times \mathbb{R}_x) \textit{ weak-star}. \end{array}$$

In the above Theorem, L^1_{loc} and M^1_{loc} respectively denote the space of locally integrable functions and the space of (locally bounded) measures on $\mathbb{R}^+_s \times \mathbb{R}_x$.

Let us point out that there is no natural $L^1([0,S] \times \mathbb{R}_x)$ estimate of $(n^\varepsilon - N_\infty)$. Therefore the above convergence results can only be *local.* However, we can show that the electric field $(E^\varepsilon)_{\varepsilon>0}$ is uniformly bounded in $L^\infty(\mathbb{R}^+_s \times \mathbb{R}_x)$.

If we don't use the assumption(2.13), we have only this Theorem for a subsequence.

3 Basic facts

In this Section, ε is a fixed positive number. We consider an entropy-admissible solution to (2.17), (2.18), (2.19) constructed in [PRV]. This solution $(n^\varepsilon, u^\varepsilon, E^\varepsilon)$ is constructed with a splitting, via the Glimm scheme applied to the hyperbolic system (1.5) which is nothing but the famous Nishida system, see [Ni]. We will essentially use the following properties of this solution, that for convenience we rewrite here in the new variables (s,x) :

Proposition 3.1 ([PRV])
For any fixed positive ε, we have
i)

$$(n^\varepsilon, u^\varepsilon, E^\varepsilon)(s,x) \in L^\infty_{loc}(\mathbb{R}^+_s, L^\infty(\mathbb{R}_x)) \cap C^0(\mathbb{R}^+_s, L^1_{loc}(\mathbb{R}_x)), \tag{3.24}$$

ii) the solution is compactly supported in x : $\exists\ L(s,\varepsilon); L(0,\varepsilon) := L_0,\ L(.,\varepsilon)$ is an decreasing function, an for $s \geq 0$ and $|x| > L(s,\varepsilon)$

$$\frac{\partial}{\partial x}(n^\varepsilon, u^\varepsilon, E^\varepsilon)(s,x) = (0,0,0), \tag{3.25}$$

iii) the solution is entropy-admissible :

$$\begin{aligned}&\frac{\partial}{\partial s}\left(n^\varepsilon \frac{(u^\varepsilon)^2}{2} + n^\varepsilon \ln n^\varepsilon\right) \\ &+\frac{1}{\varepsilon}\frac{\partial}{\partial x}\left(n^\varepsilon \frac{(u^\varepsilon)^3}{2} + n^\varepsilon u^\varepsilon \ln n^\varepsilon + n^\varepsilon u^\varepsilon\right) + n^\varepsilon \left(\frac{u^\varepsilon}{\varepsilon}\right)^2 - n^\varepsilon \frac{u^\varepsilon}{\varepsilon} E^\varepsilon \quad \leq 0.\end{aligned} \tag{3.26}$$

Here, in view of the above assumptions on the data, we have

Lemma 3.1
From assumption (2.15), for all $\varepsilon > 0$ and $s \geq 0$, $n^\varepsilon - N_\infty, u^\varepsilon - u^\varepsilon_\infty, E^\varepsilon - E^\varepsilon_\infty$ are compactly supported in x, with

$$u^\varepsilon_\infty(s) := \frac{1}{\varepsilon}\int_0^s \exp\left(-\frac{s-\tau}{\varepsilon^2}\right) E^\varepsilon_\infty(\tau)d\tau + u_\infty(0)\exp\left(-\frac{s}{\varepsilon^2}\right).$$

Proof : For any $S>0$, let us introduce $W=W^+\cup W^-$ where $W^\pm=\{(s,x)/\ s\in[0,S] \text{ and } \pm x>L(s/\varepsilon,\varepsilon)\}$. By (2.17), (2.18) we clearly have

$$\frac{\partial n^\varepsilon}{\partial s}=0; \qquad n^\varepsilon(0,x)=N_\infty, \qquad |x|>L_0, \tag{3.27}$$

$$\frac{\partial u^\varepsilon}{\partial s}=-\frac{u^\varepsilon}{\varepsilon^2}+\frac{E}{\varepsilon}; \qquad u^\varepsilon(0,x)=u_\infty(0), \qquad |x|>L_0, \tag{3.28}$$

$$\frac{\partial E^\varepsilon}{\partial x}=n-N; \quad E^\varepsilon(s,-\infty)=E^\varepsilon_\infty(s), \quad s\geq 0. \tag{3.29}$$

From (3.27), we have $n^\varepsilon=N_\infty$ on W, therefore $E^\varepsilon=E^\varepsilon_\infty$ on W^-. Now, by (2.15) and (2.17),

$$\int_{\mathbb{R}}(n^\varepsilon(s,x)-N)dx=\int_{\mathbb{R}}(n_0(x)-N)dx=0,$$

which implies $E^\varepsilon=E^\varepsilon_\infty$ on W^+ and therefore

$$\frac{\partial u^\varepsilon}{\partial s}=-\frac{u^\varepsilon}{\varepsilon^2}+\frac{E^\varepsilon_\infty}{\varepsilon}, \quad |x|>L(s/\varepsilon,\varepsilon) \quad \textit{and} \quad u^\varepsilon(0,x)=u_\infty(0), \quad |x|>L_0. \tag{3.30}$$

□

In the next Section, we are going to show the entropy inequality, which will be the crucial argument to show the convergence when $\varepsilon\to 0$.

4 An entropy inequality

Obviously, the natural entropy

$$\eta_0 \;=\; n\frac{u^2}{2}+n\ln n, \tag{4.31}$$

used in [PRV] is not a non-negative function! Moreover, the involved functions are not integrable on $\mathbb{R}$, due to the behavior of the data when $|x|\to+\infty$. Therefore, in order to overcome this classical difficulty, we first modify η_0 into

$$\eta_1 \;:=\; n\frac{u^2}{2}+\varphi(n-N_\infty), \tag{4.32}$$

where the $n\ln n$ term in (4.31) is now contained in the first term in the right-hand side of (4.32):

$$\varphi(y) \;:=\; (y+N_\infty)\ln(y+N_\infty)-[N_\infty\ln N_\infty+(1+\ln N_\infty)y]\geq 0. \tag{4.33}$$

Clearly $\varphi:]-N_\infty,+\infty[\to\mathbb{R}^+$ is convex and super-linear at infinity:

$$\lim_{y\to+\infty}\frac{\varphi(y)}{y}=+\infty. \tag{4.34}$$

Moreover, it is easy to check that

$$\forall y>-N_\infty\ ,\ \varphi(|y|)\leq\varphi(y). \tag{4.35}$$

Using the entropy inequality (3.26), adding and subtracting the linear part of φ, and using (2.17), we have

$$\frac{\partial \eta_1}{\partial s} + \frac{\partial q_1}{\partial x} + r_1 \leq 0, \tag{4.36}$$

with

$$\begin{aligned} q_1 &:= \frac{1}{\varepsilon}\left(n^\varepsilon \frac{(u^\varepsilon)^3}{2} + n^\varepsilon u^\varepsilon \ln\left(\frac{n^\varepsilon}{N_\infty}\right)\right), \\ r_1 &:= n^\varepsilon \frac{(u^\varepsilon)^2}{\varepsilon^2} - n^\varepsilon \frac{u^\varepsilon}{\varepsilon} E^\varepsilon. \end{aligned}$$

Now, the above term $-n^\varepsilon \dfrac{u^\varepsilon}{\varepsilon} E^\varepsilon$ is dissipative. Indeed

Lemma 4.2
For all $\varepsilon > 0$, $E^\varepsilon \in W^{1,1}_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x)$, and in particular

$$\frac{\partial (E^\varepsilon - E^\varepsilon_\infty)}{\partial s} = -n^\varepsilon \frac{u^\varepsilon}{\varepsilon} + N_\infty \frac{u^\varepsilon_\infty}{\varepsilon}. \tag{4.37}$$

Therefore, by the chain rule formula

$$\left(-n^\varepsilon \frac{u^\varepsilon}{\varepsilon} + N_\infty \frac{u^\varepsilon_\infty}{\varepsilon}\right)(E^\varepsilon - E^\varepsilon_\infty) = \frac{\partial}{\partial s}\left(\frac{(E^\varepsilon - E^\varepsilon_\infty)^2}{2}\right). \tag{4.38}$$

Proof : Deriving (2.19) with respect to s and using (2.17), we obtain

$$\frac{\partial^2 E^\varepsilon}{\partial s \partial x} = \frac{\partial n^\varepsilon}{\partial t} = -\frac{\partial}{\partial x}\left(n^\varepsilon \frac{u^\varepsilon}{\varepsilon}\right).$$

Deriving with respect to x the boundary condition $E^\varepsilon(s, -\infty) = E^\varepsilon_\infty(s)$, we obtain

$$\frac{\partial (E^\varepsilon - E^\varepsilon_\infty)}{\partial s} = -n^\varepsilon \frac{u^\varepsilon}{\varepsilon} + N_\infty \frac{u^\varepsilon_\infty}{\varepsilon}.$$

Combining with (2.19) and Proposition 3.1, we obtain the $W^{1,1}_{loc}(\mathbb{R}^+_t \times \mathbb{R}_x)$ regularity for the electric field E^ε. $\square$
We can now obtain a more useful entropy inequality. By (2.18), (2.19) and (3.30) we have

$$\partial_s(n^\varepsilon u^\varepsilon u^\varepsilon_\infty) = \left(-\frac{1}{\varepsilon}\partial_x\left(n^\varepsilon (u^\varepsilon)^2 + n^\varepsilon\right) - \frac{n^\varepsilon u^\varepsilon}{\varepsilon^2} + n^\varepsilon \frac{E^\varepsilon}{\varepsilon}\right) u^\varepsilon_\infty + n^\varepsilon u^\varepsilon \left(-\frac{u^\varepsilon_\infty}{\varepsilon^2} + \frac{E^\varepsilon_\infty}{\varepsilon}\right),$$

and, by (2.17) and (3.30),

$$\partial_s\left(n^\varepsilon \frac{(u^\varepsilon_\infty)^2}{2}\right) = -\frac{1}{\varepsilon}\partial_x(n^\varepsilon u^\varepsilon)\frac{(u^\varepsilon_\infty)^2}{2} + n^\varepsilon u^\varepsilon_\infty \left(-\frac{u^\varepsilon_\infty}{\varepsilon^2} + \frac{E^\varepsilon_\infty}{\varepsilon}\right).$$

Subtracting and adding $\partial_s \left(n^\varepsilon u^\varepsilon u^\varepsilon_\infty - n^\varepsilon \frac{(u^\varepsilon_\infty)^2}{2} \right)$ in inequality (4.36) we have

$$\frac{\partial \eta_2}{\partial s} + \frac{\partial q_2}{\partial x} + r_2 \leq 0, \tag{4.39}$$

with

$$\begin{aligned}
\eta_2 &:= n^\varepsilon \frac{(u^\varepsilon - u^\varepsilon_\infty)^2}{2} + \varphi(n^\varepsilon - N_\infty), \\
q_2 &:= q_1 - \frac{1}{\varepsilon}\left(n^\varepsilon (u^\varepsilon)^2 + n^\varepsilon \right) u^\varepsilon_\infty - \frac{1}{\varepsilon} n^\varepsilon u^\varepsilon \left(\frac{(u^\varepsilon_\infty)^2}{2} \right), \\
r_2 &:= r_1 + \left(-\frac{n^\varepsilon u^\varepsilon}{\varepsilon^2} + n^\varepsilon \frac{E^\varepsilon_\infty}{\varepsilon} \right) u^\varepsilon_\infty + n^\varepsilon (u^\varepsilon - u^\varepsilon_\infty) \left(-\frac{u^\varepsilon_\infty}{\varepsilon^2} + \frac{E^\varepsilon_\infty}{\varepsilon} \right).
\end{aligned}$$

We can rewrite r_2

$$\begin{aligned}
r_2 &= n^\varepsilon \frac{(u^\varepsilon)^2}{\varepsilon^2} - 2 n^\varepsilon u^\varepsilon \frac{u^\varepsilon_\infty}{\varepsilon^2} + n^\varepsilon \frac{(u^\varepsilon_\infty)^2}{\varepsilon^2} \\
&\quad - n^\varepsilon \frac{u^\varepsilon}{\varepsilon} E^\varepsilon + n^\varepsilon \frac{u^\varepsilon_\infty}{\varepsilon} E^\varepsilon + n^\varepsilon \frac{u^\varepsilon}{\varepsilon} E^\varepsilon_\infty - n^\varepsilon \frac{u^\varepsilon_\infty}{\varepsilon} E^\varepsilon_\infty \\
&= n^\varepsilon \frac{(u^\varepsilon - u^\varepsilon_\infty)^2}{\varepsilon^2} \\
&\quad + \left(-n^\varepsilon \frac{u^\varepsilon}{\varepsilon} + N_\infty \frac{u^\varepsilon_\infty}{\varepsilon} \right) (E^\varepsilon - E^\varepsilon_\infty) + (n^\varepsilon - N_\infty) \frac{u^\varepsilon_\infty}{\varepsilon} (E^\varepsilon - E^\varepsilon_\infty).
\end{aligned}$$

Using (4.38), (2.19), and the chain rule formula we obtain

$$r_2 = n^\varepsilon \frac{(u^\varepsilon - u^\varepsilon_\infty)^2}{\varepsilon^2} + \partial_s \left(\frac{(E^\varepsilon - E^\varepsilon_\infty)^2}{2} \right) + \partial_x \left(\frac{(E^\varepsilon - E^\varepsilon_\infty)^2}{2} \frac{u^\varepsilon_\infty}{\varepsilon} \right).$$

We obtain the final entropy:

$$\eta^\varepsilon := n^\varepsilon \frac{(u^\varepsilon - u^\varepsilon_\infty)^2}{2} + \varphi(n^\varepsilon - N_\infty) + \frac{(E^\varepsilon - E^\varepsilon_\infty)^2}{2},$$

which correspond to

$$\begin{aligned}
q^\varepsilon &:= \frac{1}{\varepsilon} \Bigg(n^\varepsilon \frac{(u^\varepsilon)^3}{2} + n^\varepsilon u^\varepsilon \ln \left(\frac{n^\varepsilon}{N_\infty} \right) \\
&\qquad + \left(\frac{(E^\varepsilon - E^\varepsilon_\infty)^2}{2} + n^\varepsilon u^\varepsilon \frac{u^\varepsilon_\infty}{2} - n^\varepsilon (u^\varepsilon)^2 - n^\varepsilon \right) u^\varepsilon_\infty \Bigg), \\
r^\varepsilon &:= n^\varepsilon \frac{(u^\varepsilon - u^\varepsilon_\infty)^2}{\varepsilon^2}.
\end{aligned}$$

Proposition 4.2 (Entropy inequality)
i) With the above notations, for all $\varepsilon > 0$, we have:

$$\frac{\partial \eta^\varepsilon}{\partial s} + \frac{\partial q^\varepsilon}{\partial x} + r^\varepsilon \leq 0. \tag{4.40}$$

ii) Therefore, there exists a constant C, independent of ε, such that for all $S>0$

$$\int_{\mathbb{R}}\left(n^\varepsilon\frac{(u^\varepsilon-u^\varepsilon_\infty)^2}{2}+\varphi(|n^\varepsilon-N_\infty|)+\frac{(E^\varepsilon-E^\varepsilon_\infty)^2}{2}\right)(S,x)dx$$
$$+\int_0^S\int_{\mathbb{R}}n^\varepsilon\frac{(u^\varepsilon-u^\varepsilon_\infty)^2}{\varepsilon^2}(s,x)dxds \quad \leq C. \tag{4.41}$$

iii) Consequently, we have

$$n^\varepsilon(u^\varepsilon-u^\varepsilon_\infty)^2 \;=O(\varepsilon^2) \quad \textit{in } L^1(\mathbb{R}^+_s\times\mathbb{R}_x), \tag{4.42}$$
$$E^\varepsilon-E^\varepsilon_\infty \;=O(1) \quad \textit{in } L^\infty(\mathbb{R}^+_s,L^2(\mathbb{R}_x)), \tag{4.43}$$
$$\varphi(|n^\varepsilon-N_\infty|) \;=O(1) \quad \textit{in } L^\infty(\mathbb{R}^+_s,L^1(\mathbb{R}_x)). \tag{4.44}$$

Proof : We have already proved (4.40). Let $L:=L(S/\varepsilon,\varepsilon)$, integrating (4.40) on $\{s\}\times[-L,L]$, with $0\leq s\leq S$, we have $\displaystyle\int_{-L}^{L}\partial_x q^\varepsilon(s,x)dx=[q^\varepsilon]^L_{-L}=0$. Therefore, integrating (4.40) on $[0,S]\times[-L,L]$, and using (4.35) we obtain (4.41). □
We are now ready to justify the convergence to the drift-diffusion equations when $\varepsilon\to 0$.

5 Proof of Theorem 1

We start with estimate (4.44). Unfortunately, since we have no control on the size of the support of $n^\varepsilon-N_\infty$ and since the function $\varphi(y)$ is quadratic near by $y=0$, (4.44) only implies a control in L^1_{loc} of $(n^\varepsilon-N_\infty)$.

For all $X>0$, $\exists\, C_1(X)$ such that :

$$\|n^\varepsilon\|_{L^\infty(\mathbb{R}^+_s,L^1(]-X,X[)}\leq C_1(X). \tag{5.45}$$

From now on, let S be a positive number, and let us first consider the bounded domain

$$Q_X:=]0,S[\times]-X,X[.$$

Lemma 5.3
There exists a constant $C_2=C_2(S,X)$ independent of ε, such that:

$$\int\int_{Q_X}n^\varepsilon\frac{|u^\varepsilon_\infty|}{\varepsilon}dsdx+\int\int_{Q_X}n^\varepsilon\frac{|u^\varepsilon-u^\varepsilon_\infty|}{\varepsilon}dsdx\leq C_2.$$

Proof : For the first term we use the lemma 3.1 and (5.45). By Cauchy-Schwarz inequality,

$$\begin{aligned}\int\int_{Q_X}n^\varepsilon|u^\varepsilon-u^\varepsilon_\infty|dxds &= \int\int_{Q_X}\sqrt{(n^\varepsilon)}\sqrt{(n^\varepsilon)}|u^\varepsilon-u^\varepsilon_\infty|dxds\\ &\leq \left(\int\int_{Q_X}n^\varepsilon dxds\right)^{\frac12}\left(\int\int_{Q_X}n^\varepsilon(u^\varepsilon-u^\varepsilon_\infty)^2dxds\right)^{\frac12},\end{aligned}$$

and we conclude by (4.42) and (5.45). □

Lemma 5.4
The sequence (E^ε) is bounded in the space

$$W^{1,1}(Q_X) \quad \cap \quad L^\infty(\mathbb{R}_s^+, W^{1,1}(]-X,X[)) \quad \cap \quad L^\infty(\mathbb{R}_s^+ \times \mathbb{R}_x).$$

Proof : By (4.43) and Cauchy-Schwarz inequality, we obtain the $L^\infty(\mathbb{R}_s^+, L^1(]-X,X[))$ estimates. Now, using (2.19) and (5.45), we see that $\dfrac{\partial E^\varepsilon}{\partial x}$ is bounded in $L^\infty(\mathbb{R}_s^+, L^1(]-X,X[))$. Therefore (E^ε) is bounded in $L^\infty(\mathbb{R}_s^+, W^{1,1}(]-X,X[))$. Then, by lemma 4.2 and lemma 5.3 we show that (E^ε) is bounded in $W^{1,1}(Q_X)$.

Finally, we remark that these bounds only depend on the measure of the domain Q_X. Now, since

$$\sup_{k\in\mathbb{Z}} \|E^\varepsilon\|_{L^\infty(\mathbb{R}_s^+, W^{1,1}(]k,k+1[))} < \infty,$$

we obtain the bound $L^\infty(\mathbb{R}_s^+ \times \mathbb{R}_x)$. $\square$

We can now justify the formal calculations of section 2 :

Proposition 5.3
At least for a subsequence, we have

$$\begin{array}{llccll}
(i) & \dfrac{\partial}{\partial x}\Big(n^\varepsilon(u^\varepsilon)^2\Big) & \rightharpoonup & 0 & in & \mathcal{D}'(Q_X), \\
(ii) & \varepsilon\dfrac{\partial}{\partial s}(n^\varepsilon u^\varepsilon) & \rightharpoonup & 0 & in & \mathcal{D}'(Q_X), \\
(iii) & n^\varepsilon - N_\infty & \rightharpoonup & n - N_\infty & in & L^1(Q_X) \ weak\ , \\
(iv) & E^\varepsilon & \to & E & in & L^p(Q_X) \ strong,\ 1\le p<\infty, \\
(v) & n^\varepsilon E^\varepsilon & \rightharpoonup & nE & in & \mathcal{D}'(Q_X), \\
(vi) & \dfrac{n^\varepsilon u^\varepsilon}{\varepsilon} & \rightharpoonup & nE + \dfrac{\partial n}{\partial x} & in & M^1(Q_X) \ weak\text{-}star\ ,
\end{array}$$

where (n,E) is a solution to (2.20), (2.21), (2.22) , (2.23), on Q_X, and $\mathcal{D}'(Q_X)$ is the space of distributions on Q_X.

Proof : Writing $n^\varepsilon(u^\varepsilon)^2 = n^\varepsilon(u^\varepsilon - u^\varepsilon_\infty)^2 + n^\varepsilon(u^\varepsilon_\infty)^2 - 2n^\varepsilon(u^\varepsilon - u^\varepsilon_\infty)u^\varepsilon_\infty$, we first obtain (i) as a consequence of (4.42) and Lemma 5.3. We have (ii) in view of Lemma 5.3. Now Q_X is bounded and φ is super-linear at infinity. Therefore by (4.44) and de la Vallée-Poussin criterion, the sequence $(n^\varepsilon - N_\infty)$ is weakly compact in $L^1(Q_X)$, which implies (iii), at least for a subsequence, still denoted by $(n^\varepsilon - N_\infty)$.

Now, $W^{1,1}(Q_X)$ is compactly embedded into $L^1(Q_X)$. By Lemma 5.4, we can extract a new subsequence (E^ε) which satisfies (iv).

Passing to the limit in (2.18) we obtain

$$\frac{\partial E}{\partial x} = n - N. \tag{5.46}$$

Therefore, $E \in L^\infty(\mathbb{R}_s^+, W^{1,1}(]-X,X[))$. Now

$$n^\varepsilon E^\varepsilon = (n^\varepsilon - N)E^\varepsilon + NE^\varepsilon.$$

By (iv) and Lemma 5.4, using (2.19) and the chain rule formula, we obtain

$$\begin{aligned} n^\varepsilon E^\varepsilon &= \frac{\partial}{\partial x}\left(\frac{(E^\varepsilon)^2}{2}\right) + NE^\varepsilon \rightharpoonup \frac{\partial}{\partial x}\left(\frac{E^2}{2}\right) + NE \\ &= \frac{\partial E}{\partial x}E + NE = (n-N)E + NE = nE. \end{aligned}$$

Combining the above results with Lemma 5.3 and Lemma 5.4, we can now pass to the limit in (2.18) to obtain (vi).

Finally, using this last relation and passing to the limit in (2.17) and (2.19), we obtain the system of drift-diffusion equations on Q_X, for all S and X. □

By the diagonal process, it is easy to show that for all S, (n, E) is *a* solution of (2.20), (2.21) , (2.22) , (2.23), on the whole strip $Q_\infty = (0, S) \times \mathbb{R}$. The boundary condition (2.23) is a consequence of assumption (2.13).

Let be $E^\# := E - E_\infty$, the Drift-Diffusion equations is equivalent to :

$$\begin{aligned} \frac{\partial E^\#}{\partial s} &= -NE^\# - E_\infty \frac{\partial E^\#}{\partial x} - \frac{\partial}{\partial x}\left(\frac{(E^\#)^2}{2}\right) + \frac{\partial^2 E^\#}{\partial x^2}, \\ E^\#(0,x) &= \int_{-\infty}^{x} (n_0(y) - N)dy. \end{aligned}$$

Since this solution is *unique*, we conclude that the whole sequence $(n^\varepsilon, E^\varepsilon)$ converges. Obviously, these convergences are only local. Therefore the formal calculations of section 2 are rigorously justified.

References

[B] H. Brezis, *Analyse fonctionnelle,* Masson, Paris, (1983).

[DCL] X. Ding, G.-Q. Chen & P. Luo, *Convergence of the fractional step Lax-Friedrichs scheme an Godounov scheme for isentropic gas dynamics,* Comm. Math. Phys., 121, (1989), p. 63-84.

[CJZ1] G.-Q. Chen, J. W. Jerome & Bo Zhang, *Particle hydrodynamic moment models in biology and microelectronics: singular relaxation limits,* preprint, (1996).

[CJZ2] G.-Q. Chen, J. W. Jerome & Bo Zhang, *Existence and the singular relaxation limit for the inviscid hydrodynamic energy model,* preprint, (1996).

[DiP] R. DiPerna, *Convergence of approximate solutions of conservation laws,* Arch. Rat. Mech. Anal., 82, (1983), p. 27-70.

[ET] I. Ekeland & R. Temam, *Analyse convexe et problèmes variationnels,* Dunod, Gauthier-Villars, (1974).

[G] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations,* Com.Pure.Appl.Math., 18, (1965), p. 698-715.

[J] S. Junca, *Optique géométrique non linéaire, chocs forts, relaxation,* Thesis, (III), Univ. Nice-Sophia Antipolis, (1995).

[Ni] T. Nishida, *Global solutions for an initial boundary value problem of a quasi-linear hyperbolic system.* Japan Acad., 44, (1968), p. 642-646.

[MN1] P. Marcati & R. Natalini, *Weak Solutions to a hydrodynamic model for semi-conductors: the Cauchy problem,* to appear Proc. Roy. Edinburgh.

[MN2] P. Marcati & R. Natalini, *Weak solutions to a hydrodynamic model for semi-conductors and relaxation to the Drift-Diffusion equation.* Arch. Rat. Mech. Anal., 129, (1995), p. 129-145.

[MRS] P. A. Markowich, C. Ringhofer & C. Schmeiser, *Semiconductors equations,* Springer, Wien-New York, (1989).

[PRV] F. Poupaud, M. Rascle & J. P. Vila, *Global Solutions to the Isothermal Euler-Poisson System with Arbitrarily Large Data* , J.D.E., 123, (1995), p. 93-121.

[T] L. Tartar, *Compensated compactness and applications to partial differential equations,* Research notes in mathematics, nonlinear analysis and mechanics: Heriot-Watt Symposium, ed. R. J. Knops, New York, Pitman Press, 4, (1979), p. 136-212.

[Z] B. Zhang, *Convergence of the Godunov scheme for a simplified one dimensional hydrodynamic model for semiconductor devices,* preprint, Dept. Math. Purdue Univ., (1992).