Well-Posedness for

General 2×2 Systems of Conservation Laws

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Abstract. We consider the Cauchy problem for a strictly hyperbolic 2×2 system of conservation laws in one space dimension

$$u_t + [F(u)]_x = 0, \qquad\qquad u(0,x) = \bar{u}(x), \tag{1}$$

which is neither linearly degenerate nor genuinely non-linear. We make the following assumption on the characteristic fields. If $r_i(u)$, $i = 1,2$, denotes the i-th right eigenvector of $DF(u)$ and $\lambda_i(u)$ the corresponding eigenvalue, then the set $\{u : \nabla\lambda_i \cdot r_i(u) = 0\}$ is a smooth curve in the u-plane that is transversal to the vector field $r_i(u)$.

Systems of conservation laws that fulfill such assumptions arise in studying elastodynamics or rigid heat conductors at low temperature.

For such systems we prove the existence of a closed domain $\mathcal{D} \subset L^1$, containing all functions with sufficiently small total variation, and of a uniformly Lipschitz continuous semigroup $S : \mathcal{D} \times [0, +\infty) \to \mathcal{D}$ with the following properties. Each trajectory $t \mapsto S_t\bar{u}$ of S is a weak solution of (1). Viceversa, if a piecewise Lipschitz, entropic solution $u = u(t,x)$ of (1) exists for $t \in [0,T]$, then it coincides with the trajectory of S, i.e. $u(t,\cdot) = S_t\bar{u}$.

This result yields the uniqueness and continuous dependence of weak, entropy-admissible solutions of the Cauchy problem (1) with small initial data, for systems satysfying the above assumption.

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Contents

1. Introduction

Consider the Cauchy problem for a strictly hyperbolic 2×2 system of conservation laws in one space dimension

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

$$u(0,x) = \overline{u}(x). \tag{1.2}$$

Let the flux function $F:\Omega\mapsto\mathbb{R}^2$ be a smooth vector field defined on a neighborhood of the origin $\Omega\subset\mathbb{R}^2$. Denote by $\lambda_1(u)<\lambda_2(u)$ the eigenvalues of the Jacobian matrix $A(u)\doteq DF(u)$, and let $\big\{r_1(u),\, r_2(u)\big\}$ be a basis of right eigenvectors.

We assume that the system is neither genuinely non-linear nor linearly degenerate in the sense of Lax [LA], [S]. Instead, we require that for each characteristic speed $\lambda_i,\ i=1,2,$ the directional derivative of λ_i in the direction of the characteristic field r_i

$$r_i\bullet\lambda_i(u)\doteq\lim_{h\to 0}\frac{\lambda_i\big(u+h\,r_i(u)\big)-\lambda_i(u)}{h},$$

vanishes on a curve in Ω which is transversal to the field r_i. More precisely, we make the following assumption.

(A) For $i=1,2,$ there exists a smooth curve $\Gamma_i\subset\Omega$ such that

$$\Omega_i^0\doteq\{\,u\in\Omega;\ r_i\bullet\lambda_i(u)=0\,\}=\Gamma_i,\qquad 0\in\Gamma_i. \tag{1.3}$$

Moreover, there holds

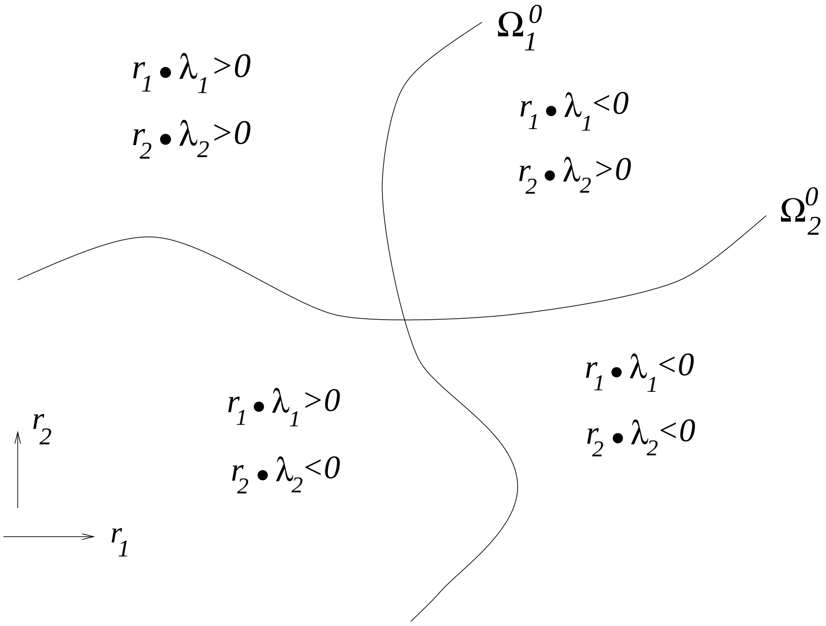

FIGURE 1.1

$$r_1 \bullet (r_1 \bullet \lambda_1)(u) < 0, \qquad \forall u \in \Gamma_1 \tag{1.4}$$

$$r_2 \bullet (r_2 \bullet \lambda_2)(u) > 0, \qquad \forall u \in \Gamma_2. \tag{1.5}$$

Systems of conservations laws which are neither genuinely non-linear nor linearly degenerate physically arise in several contexts, in particular in studying elastodynamic or rigid heat conductors at low temperature. In elasticity, genuine nonlinearity is typically precluded by the fact that the medium in question can sustain discontinuities in both compressive and expansive phases of the motion. In the generalized nonlinear Cattaneo model, proposed by T. Ruggeri, A. Muracchini, L. Seccia [RMS1], [RMS2] to describe the heat propagation in rigid heat conductors, the lack of genuine nonlinearity results in the presence of two kinds of shocks (*hot* and *cold* shock) both physically stable.

We recall that, for hyperbolic systems of conservation laws, to achieve uniqueness of solutions it is necessary to consider weak solutions that satisfy some additional admissibility condition, possibly motivated by physical considerations. In [LI1], [LI2], T.P. Liu proposed an admissibility criterion valid for $n \times n$ systems of conservation laws having the property that, for each characteristic speed λ_i, the directional derivative $r_i \bullet \lambda_i$ vanishes on a finite, disjoin union of n–1-dimensional manifolds. This criterion, in the case of a system where $r_i \bullet \lambda_i = 0$ on a single hypersurface in the u-space, is equivalent to the stability condition introduced by Lax:

(L) A shock connecting the left state u^L and the right state u^R, travelling with speed s is an admissible discontinuity of the i-th family if

$$\lambda_i(u^L) \geq s \geq \lambda_i(u^R).$$

Thus, for a system satisfying the assumption **(A)**, with an *entropy weak solution* of the Cauchy problem (1.1)-(1.2) we shall always mean a weak solution of (1.1)-(1.2) that has admissible discontinuities in the sense of Lax.

For such systems, T.P. Liu established in [LI1], [LI2] the existence of a unique, self-similar, entropy weak solution of the Riemann problem with arbitrary initial data

$$u(0,x) = \begin{cases} u^L & \text{if} \quad x < 0, \\ u^R & \text{if} \quad x > 0. \end{cases}$$

The existence of global weak solutions of the Cauchy problem (1.1)-(1.2) with small BV data was proved in [LI3] using the Glimm scheme. In this paper, following the approach introduced by A. Bressan, R.M. Colombo in [BC1], we construct a new algorithm, based on wave-front tracking, which yields a sequence of flow maps S_n whose trajectories are approximate solutions of (1.1) that depend on the initial conditions in a uniformly Lipschitz continuous way. Such a sequence converges, on a set $\mathcal{D}$ of integrable functions with small total variation, to a globally Lipschitz flow S whose trajectories are weak solutions to (1.1). More precisely, the following holds.

Theorem 1. *Let F be a smooth map from a neighborhood of the origin $\Omega \subset \mathbb{R}^2$ into $\mathbb{R}^2$. Assume that the system (1.1) is strictly hyperbolic and that the characteristic fields $r_i, i = 1, 2$, satisfy the basic assumption* **(A)**. *Then, there exist a closed domains $\mathcal{D} \subset \mathbb{L}^1(\mathbb{R};\mathbb{R}^2)$, constants $L,\ \overline{\delta} > 0$, and a continuous semigroup $S : [0,\infty[\times \mathcal{D} \mapsto \mathcal{D}$, with the properties:*

(i) Every function $\overline{u} \in \mathbb{L}^1$ with $Tot.Var.(\overline{u}) \leq \overline{\delta}$ lies in $\mathcal{D}$.

(ii) For all $\overline{u} \in \mathcal{D}$, one has $S_0\overline{u} = \overline{u}, \quad S_t S_s \overline{u} = S_{t+s}\overline{u}, \quad t,s \geq 0$.

(iii) For all $\overline{u}, \overline{v} \in \mathcal{D}$, $t,s \geq 0$, one has $\left\|S_t\overline{u} - S_s\overline{v}\right\|_{\mathbb{L}^1} \leq L\left(|t-s| + \left\|\overline{u} - \overline{v}\right\|_{\mathbb{L}^1}\right)$.

(iv) If $\overline{u} \in \mathcal{D}$ is piecewise constant, then for $t > 0$ sufficiently small the function $u(t,\cdot) \doteq S_t\overline{u}$ coincides with the solution of (1.1)-(1.2) obtained by piecing together the entropy-admissible, self-similar solutions of the corresponding Riemann problems.

(v) Each trajectory $t \mapsto u(t,\cdot) \doteq S_t\overline{u}$ yields a weak solution to the Cauchy problem (1.1)-(1.2).

Following [B3], we say that a map with the properties (i)-(iv) is a *Standard Riemann Semigroup* (SRS) generated by (1.1). The existence of such a map plays a key role in proving the uniqueness of entropy weak solutions for a given Cauchy-problem with small BV data. Indeed, with the same technique in [B3] one can show that a map S with the properties (i)-(iv) is necessarily unique (up to the domain). Besides, by similar arguments as in [BG], [BL], one can prove that any entropy weak solution to (1.1)-(1.2), satisfying some mild assumptions either on the oscillation or on the local boundedness of the total variation, must coincide with the trajectory of S. Results in this direction can be found in [BLP]. A further consequence of the existence of a SRS that is worth mentioning is the derivation of sharp error estimates on approximate solutions constructed by the Glimm scheme as in the case of genuinely non-linear systems (cfr. [BM2]).

The existence of a map S with the properties (i)-(v), under the assumption that the characteristic fields were genuinely non-linear or linearly degenerate, was first proved by A.Bressan, R.M. Colombo in [BC1], for systems of two equations, and by A.Bressan, G.Crasta, B.Piccoli in [BCP] and, more recently, by A.Bressan, T.P.Liu, T.Yang, in [BLY], for $n \times n$ systems. Our main theorem here, is a first step towards the extensions of these results to the more general class of not genuinely non-linear systems considered by T.P. Liu in [LI1], [LI2], [LI3].

As in [BC1], the main features of our wave-front tracking algorithm are:

1. Every Riemann Problem is solved by the same procedure. In particular, we never introduce any non-physical wave-front.

2. Centered rarefaction waves are cut along a fixed grid of step size ε and approximated by a corresponding piecewise constant rarefaction fan.

3. Shock waves of small amplitude coincide with rarefactions.

A Riemann Solver with the above properties, valid for genuinely non-linear (or linearly degenerate) systems, is obtained in [BC1] by constructing ε-approximate solutions in which

- shocks of strength $|\sigma| \geq 2\sqrt{\varepsilon}$ satisfy the exact Rankine-Hugoniot equations;
- shocks of strength $|\sigma| \leq \sqrt{\varepsilon}$ are rarefactions;
- shocks of strength $|\sigma| \in \left[\sqrt{\varepsilon},\, 2\sqrt{\varepsilon}\right]$ connect a left and right state lying on a smooth interpolation between the shock and the rarefaction curve.

Unfortunately, in the case of not genuinely non-linear systems, approximate solutions of this type fail to produce a Riemann Solver with the properties **1-3**, depending continuously on the initial data. This is the consequence of the different structure of the elementary waves contained in the general solution of the Riemann Problem. In fact, for such systems, the exact solution to the Riemann Problem may consist of composed waves made of one-sided contact discontinuities adjacent to rarefaction waves. Then, to guarantee the stability of the composite wave pattern of a piecewise constant function that approximates a wave of this type, we need to modify the Rankine-Hugoniot equations for shocks of any amplitude. This fact, in turn, forces us to introduce three different types of interpolations between the rarefaction and the approximate shock curves defined in connection with the modified Rankine-Hugoniot conditions.

A basic step in the proof of the global existence of approximate solutions generated by wave-front tracking consists in deriving a-priori bounds on the total variation. As customary, these a-priori estimates are obtained using a functional Q that measures the *potential interaction* of waves in the solution.

In particular, for not genuinely non-linear systems, T.P. Liu ([LI3]) introduced a functional Q in which the amount of potential interaction between waves of the same family is proportional to the product of the strength of the waves and of their angle. This choice is motivated by the fact that no interaction between two waves is expected when their angle is zero. On the other hand, this corresponds to consider an interaction functional that is fourth order w.r.t. the strength of the waves of the same family. In fact, the angle between two waves of strength $|\sigma'|$, $|\sigma''|$, belonging to the the same family, is proportional either to $|\sigma'\sigma''|$ or to $\max\{|\sigma'|^2, |\sigma''|^2\}$.

In our analysis, however, to establish the Lipschitz continuous dependence of the approximate solutions on the initial data, it is essential to have a disposal an interaction-potential that is second order w.r.t. the strength of all waves. For this reason we define here the amount of potential interaction between *any* couple of wave-fronts in the approximate solution as proportional to the product of their strengths, no matter if they belong to the same family or not. More precisely, for any given approximate solution u generated by wave-front tracking, with several wave-fronts, say located at $x_\alpha = x_\alpha(t)$, $\alpha = 1, \dots, N$, calling $|\sigma_i^\alpha|$ the strength of the wave at x_α belonging to the i-th family, we define the interaction potential $Q(t) \doteq Q(u(t))$ by setting

$$Q(t) = K_1 \Bigg[\sum_{x_\alpha < x_\beta} \big|\sigma_2^\alpha \cdot \sigma_1^\beta\big| + \sum_{i=1,2} \sum_{\sigma_i^\alpha \cdot \sigma_i^\beta < 0} \big|\sigma_i^\alpha \cdot \sigma_i^\beta\big| \Bigg] + \sum_{i=1,2} \sum_{\sigma_i^\alpha \cdot \sigma_i^\beta > 0} \big|\sigma_i^\alpha \cdot \sigma_i^\beta\big| \tag{1.6}$$

where K_1 is a suitable constant > 1. Notice that, since we are assuming by this definition that the amount of interaction-potential between two wave-fronts of the same family is non zero no matter which angle they make, one would expect that the global interaction-potential may increase after an interaction that produces a composed wave containing a piecewise constant rarefaction fan. However, interactions of this type always involve waves of different families or waves of the same family having an opposite sign. Therefore, observing that the total strength of i-rarefaction waves produced by such an interaction is dominated by the total strength of j-waves (or of i-waves of opposite sign w.r.t. the outgoing waves) involved in the interaction, one can always control the increase of Q due to the new rarefaction waves by assigning a sufficiently large weight K_1 to the amount of interaction-potential of the interacting waves.

The proof of the Lipschitz continuous dipendence on the initial data of the approximate solutions constructed by our algorithm, is based on the same technique developed in [BC1]. The basic idea consists in "differentiating" a family of approximate solutions w.r.t. a parameter which determines the locations of the jumps. More precisely, consider a piecewise constant approximate solution u having

$$u(0,x) = \sum_{\alpha=1}^{N_0} \overline{\omega}_\alpha \cdot \chi_{]\overline{x}_{\alpha-1},\, \overline{x}_\alpha[}(x), \tag{1.7}$$

as initial data. Here χ_I denotes the characteristic function of the interval I. Then one can construct a one-parameter family of initial data $u^\theta(0,\cdot)$ obtained from $u(0,\cdot)$ by shifting the positions of the jumps $\overline{x}_0, \overline{x}_1, \dots, \overline{x}_{N_0}$, at the constant rates $\overline{\xi}_0, \overline{\xi}_1, \dots, \overline{\xi}_{N_0}$, i.e. letting

$$u^\theta(0,x) = \sum_{\alpha=1}^{N_0} \overline{\omega}_\alpha \cdot \chi_{]x^\theta_{\alpha-1},\, x^\theta_\alpha[}(x), \qquad x^\theta_\alpha = \overline{x}_\alpha + \theta\, \overline{\xi}_\alpha, \qquad \theta \in [-a,\, a].$$

Here the states $\overline{\omega}_\alpha$ remain fixed for all values of the parameter θ. Let $u^\theta = u^\theta(t,x)$ be the corresponding approximate solution. If a is sufficiently small, at any time t where no interaction occurs, the functions $u^\theta(t,\cdot)$ are still obtained from $u(t,\cdot)$ by shifting the positions of its jumps:

$$u^\theta(t,x) = \sum_{\beta=1}^{N_t} \omega_\beta \cdot \chi_{]x^\theta_{\beta-1}(t),\, x^\theta_\beta[}(x), \qquad\qquad x^\theta_\beta = x_\beta + \theta\,\xi_\beta,$$

for some shift rates $\xi_0, \xi_1, \dots, \xi_{N_t}$. The $\mathbb{L}^1$-length of the path $\gamma_t : \theta \mapsto u^\theta(t,\cdot)$, is then computed by

$$\|\gamma_t\|_{\mathbb{L}^1} = \int_{-a}^{a} \sum_\beta \left|\Delta u^\theta(t,\, x^\theta_\beta)\right| \left|\xi_\beta\right|\, d\theta, \qquad\qquad \xi_\beta = \frac{dx^\theta_\beta}{d\theta}, \tag{1.8}$$

where $\Delta u^\theta = \omega_{\beta+1} - \omega_\beta$ is the jump of u^θ at x^θ_β.

Thus, to estimate the $\mathbb{L}^1$-distance between two nearby approximate solutions u^{-a}, u^a, one has to provide an a-priori bound on the integrand in (1.8). As soon as an estimate of this type is obtained, a standard argument yields the stability of the approximate solutions. Clearly the integrand in (1.8) can change (as a function of time) only at times where some interaction takes place between wave-fronts in u^θ. Therefore the key point is to study how the summation in (1.8) is changed across each interaction time. As one would expect, the major difference in this analysis, between the standard case of genuinely non-linear systems and the case of systems not genuinely non-linear, regards the behaviour of the summation in (1.8) at any time where two waves of the same family interact. We consider the following basic case.

Example 1.1. Let $u = u(t,x)$ be a piecewise constant solution, consisting of two shocks of the first family which interact at some time τ. Let $x'(t) < x''(t)$ be the locations of the shocks for $t < \tau$, and assume that the interaction produces an outgoing shock for each i-th family, at $x_i^+(t)$ (see fig 1.2). Call σ', σ'' the sizes of the two incoming waves, and σ_i^+ the size of the outgoing i-shock. Assume that σ', σ'' are both positive.

Next, consider a family of perturbed solution u^θ, $\theta \in [0,\,1]$, obtained from u by shifting the position of x', x'' at the constant rates ξ', ξ''. The incoming shocks of u^θ thus occur on the lines $x = x'_\theta(t)$, $x = x''_\theta(t)$, with

$$x'_\theta(t) = x'(t) + \xi'\,\theta, \qquad\qquad x''_\theta(t) = x''(t) + \xi''\,\theta, \qquad\qquad \theta \in [0,\,1].$$

For any fixed $\theta \in [0,\,1]$, after the interaction is taken place (both in u and in u^θ) the perturbed solution u^θ will still contain wave-fronts of exactly the same strength as u. The locations $x^+_{1,\theta}$, $x^+_{2,\theta}$, of these wave-fronts, however, will be shifted by certain amounts, say ξ_1^+, ξ_2^+. An elementary calculation shows that the shift rates of the outgoing i-th shock is

$$\xi_i^+ = \frac{\partial x^+_{i,\theta}}{\partial\theta} = \frac{\xi''\big(\dot{x}' - \dot{x}_i^+\big) - \xi'\big(\dot{x}'' - \dot{x}_i^+\big)}{\dot{x}' - \dot{x}''}, \qquad\qquad i = 1,2, \tag{1.9}$$

where $\dot{x}$ denotes the speed of the wave-front at x. By the above arguments, the quantities

$$E^- \doteq \left|\sigma'\,\xi'\right| + \left|\sigma''\,\xi''\right|, \qquad\qquad E^+ \doteq \left|\sigma_1^+\,\xi_1^+\right| + \left|\sigma_2^+\,\xi_2^+\right|, \tag{1.10}$$

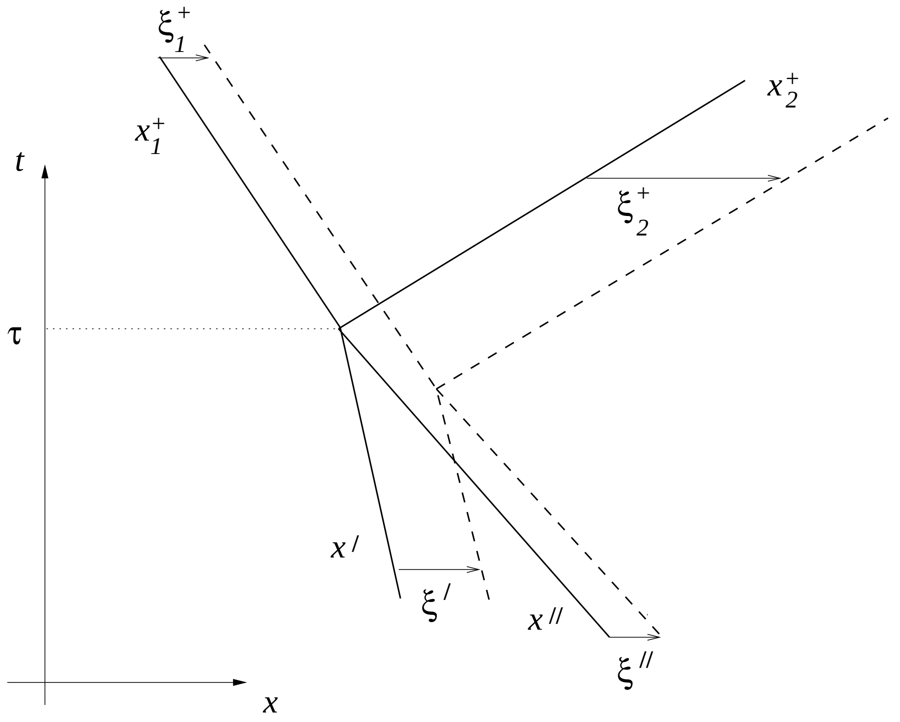

FIGURE 1.2

provide a measure of the $\mathbb{L}^1$-distance between the two solutions u, u^1 before and after the interaction time τ. Standard interaction estimates on the size and the speed of the outgoing waves imply that

$$\left|\sigma_2^+\right| + \left|\sigma_1^+ - (\sigma' + \sigma'')\right| = \mathcal{O}(1) \cdot \left|\sigma' \cdot \sigma''\right| \cdot \max\left\{\left|\sigma'\right|, \left|\sigma''\right|\right\}, \tag{1.11}$$

$$\left|\dot{x}_1^+ - \frac{\sigma'\xi' + \sigma''\xi''}{\sigma' + \sigma''}\right| = \mathcal{O}(1) \cdot \left|\sigma' \cdot \sigma''\right|, \tag{1.12}$$

where $\mathcal{O}(1)$ denotes a quantity uniformly bounded by a constant that depends only on the system (1.1). Observe that the shifts ξ_i^+ in (1.9) may be very large compared with ξ', ξ'', because of the small denominator. However, in the case the system (1.1) is genuinely non-linear, a supplementary bound holds

$$\left|\dot{x}' - \dot{x}''\right| \geq c\big(|\sigma'| + |\sigma''|\big) \qquad \text{for some} \quad c > 0. \tag{1.13}$$

Thus, from (1.11)-(1.13) we obtain the basic estimate

$$E^+ - E^- \leq \left|\sigma_1^+\xi_1^+ - \sigma'\xi' - \sigma''\xi''\right| + \left|\sigma_2^+\xi_2^+\right| = \mathcal{O}(1) \cdot \left|\sigma' \cdot \sigma''\right|\big(|\xi'| + |\xi''|\big). \tag{1.14}$$

This estimates indicates that, although the quantity in (1.8) may well increase in time, one can derive a uniform bound on a weighted sum of the form

$$\sum_\beta \left|\Delta u^\theta(t, x_\beta^\theta)\right|\left|\xi_\beta\right|W^\beta \tag{1.15}$$

provided that the weights W^β are suitably chosen, depending on the amount of waves in u which approach the wave located at x_β. Clearly this arguments fails for not genuinely non-linear systems since an estimate as in (1.13) no more holds.

For this reason, it is necessary to provide sharper interaction estimates than the ones in (1.11)-(1.12). Indeed, if the system (1.1) satisfies the assumption **(A)**, the following holds

$$\left|\sigma_2^+\right| + \left|\sigma_1^+ - (\sigma' + \sigma'')\right| = \mathcal{O}(1) \cdot \left|\sigma' \cdot \sigma''\right| \cdot \Big[\left|\sigma'\right| + \left|\sigma''\right|\Big] \cdot \Big[\left|\sigma'\right| + \left|\sigma''\right| + \left|\delta_1\right|\Big], \tag{1.16}$$

$$\left|\dot{x}_1^+ - \frac{\sigma'\xi' + \sigma''\xi''}{\sigma' + \sigma''}\right| = \mathcal{O}(1) \cdot \left|\sigma' \cdot \sigma''\right| \cdot \Big[\left|\sigma'\right| + \left|\sigma''\right| + \left|\delta_1\right|\Big], \tag{1.17}$$

$$\left|\dot{x}' - \dot{x}''\right| \geq c' \cdot \Big[\left|\sigma'\right| + \left|\sigma''\right|\Big] \cdot \Big[\left|\sigma'\right| + \left|\sigma''\right| + \left|\delta_1\right|\Big] \qquad \text{for some} \quad c' > 0, \tag{1.18}$$

where $\delta_1 \doteq \delta_1(u^L)$ denotes the distance between the state u^L on the left of the wave at x', and the curve Γ_1 defined in (1.3). Estimates of this type were first established in [LI3] in the case of interacting waves of the same family and of the same sign. A derivation of such estimate in the general case of interacting waves of the same family and of any sign can be found in [AM2]. Thanks to (1.16)-(1.18), one can then recover the basic estimate (1.14) also in the case of a system satisfying the assumption **(A)**.

Relying on (1.14), we can now derive a uniform a-priori bound on a weighted sum as in (1.15), choosing weights of the form

$$W_i^\alpha \doteq \left[1 + K_2\Big[\sum_{\substack{j\neq i \\ (x_\alpha - x_\beta)\cdot(i-j)<0}} \left|\sigma_j^\beta\right| + \sum_{\sigma_i^\beta\cdot\sigma_i^\alpha<0} \left|\sigma_i^\beta\right|\Big] + K_3 \sum_{\sigma_i^\beta\cdot\sigma_i^\alpha>0} \left|\sigma_i^\beta\right|\right] e^{K_4\,Q}, \tag{1.19}$$

where K_2, K_3, K_4, $(K_2 > K_3)$ are suitable positive constants, and Q is the interaction potential at (1.6). Here, with σ_i^β (σ_j^β) we denote the size of a wave-front of the i-th (j-th) characteristic family, and W_i^α is the weight assigned to an i-th wave located at x_α.

The paper is organized as follows. In section 2 we introduce the main notations and we define the elementary curves, associated to each characteristic field, used to define the exact solution of the Riemann Problem. Section 3 contains an outline of the proof of Theorem 1, given as a sequence of Propositions. The technical details involved in the proof of these Propositions are then worked out in Sections 6 to 9, relying on the basic interaction estimates collected in Section 5. The construction of the approximate elementary curves that determine the approximate solution of the Riemann Problem is carried out in Section 4.

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^2$ be an open neighborhood of the origin and $F : \Omega \to \mathbb{R}^2$ a smooth map. Denote with $\lambda_1(u) < \lambda_2(u)$ the eigenvalues of the Jacobian matrix $A(u) \doteq DF(u)$. Performing a linear change of coordinates in the (t, x)-plane, we can assume that

$$\lambda_1 < 0 < \lambda_2, \qquad 0 < \lambda^{\min} < |\lambda_i| < \lambda^{\max}, \tag{2.1}$$

for some constant $\lambda^{\min}$, $\lambda^{\max}$. The directional derivative of a function $\phi = \phi(u)$ in the direction of a vector field $r = r(u)$ is written

$$r \bullet \phi\,(u) \doteq D\phi \cdot r(u) = \lim_{h \to 0} \frac{\phi\big(u + h\, r(u)\big) - \phi(u)}{h}.$$

Choose right and left eigenvectors $r_i(u)$, $l_i(u)$, $i = 1, 2$, of $A(u)$ normalized so that

$$[r_i, r_j](u) \doteq r_j \bullet r_i(u) - r_i \bullet r_j(u) \equiv 0, \qquad i, j = 1, 2, \tag{2.2}$$

$$\langle l_i(u),\, r_j(u)\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{2.3}$$

Since we deal with a 2×2 system, it is convenient to work with a set of Riemann coordinates $v = (v_1, v_2)$ associated with a local diffeomorphism $v \mapsto u(v)$, defined on a neighborhood of zero $\mathcal{V}$. We shall assume that $u(0) = 0$, and that

$$\partial_i\, u(v) = r_i(u(v)) \qquad \forall\, v \in \mathcal{V}, \qquad i = 1, 2. \tag{2.4}$$

Since the curves Γ_1, Γ_2, at (1.3) are transversal to the characteristic fields r_1 and r_2, we can expresse them (in the Riemann coordinates) in the form

$$\gamma_1(v_2) = \big(\widetilde{\gamma}_1(v_2),\, v_2\big), \qquad \gamma_2(v_1) = \big(v_1,\, \widetilde{\gamma}_2(v_1)\big), \tag{2.5}$$

for some smooth functions $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$. It is not restrictive to assume that $\widetilde{\gamma}_1$, $\widetilde{\gamma}_2$ are defined, respectively, on the set $\{v_2 : v = (v_1, v_2) \in \mathcal{V}\}$, and $\{v_1 : v = (v_1, v_2) \in \mathcal{V}\}$. Consider the sets

$$\mathcal{V}_1^0 \doteq \big\{v = (v_1, v_2) \in \mathcal{V} : v = \gamma_1(v_2)\big\}, \qquad \mathcal{V}_2^0 \doteq \big\{v = (v_1, v_2) \in \mathcal{V} : v = \gamma_2(v_1)\big\}, \tag{2.6}$$

and define the maps $\delta_i : \mathcal{V} \to \mathbb{R}$, $\pi_i : \mathcal{V} \to \mathcal{V}_i^0$, $i = 1, 2$, by setting

$$\begin{aligned} \delta_1(v) &\doteq v_1 - \widetilde{\gamma}_1(v_2), & \delta_2(v) &\doteq v_2 - \widetilde{\gamma}_2(v_1), \\ \pi_1(v) &\doteq \big(\widetilde{\gamma}_1(v_2),\, v_2\big), & \pi_2(v) &\doteq \big(v_1,\, \widetilde{\gamma}_2(v_1)\big). \end{aligned} \tag{2.7}$$

In the Riemann coordinates, the rarefaction curves $R_1(v)$, $R_2(v)$, through a point $v = (v_1, v_2) \in \mathcal{V}$, are naturally parametrized by

$$R_1(v, \sigma) = (v_1 + \sigma,\, v_2), \qquad R_2(v, \sigma) = (v_1,\, v_2 + \sigma), \qquad \sigma \in \mathcal{J}, \tag{2.8}$$

with $\mathcal{J}$ a suitable neighborhood of zero. On the other hand, since shock and rarefaction curves have second order contact, the shock curves $S_1(v)$, $S_2(v)$, through v can be expressed in the form

$$S_1(v,\sigma) = \big(v_1+\sigma,\ v_2+\widetilde{S}_1(v,\sigma)\cdot\sigma^3\big), \qquad S_2(v,\sigma) = \big(v_1+\widetilde{S}_2(v,\sigma)\cdot\sigma^3,\ v_2+\sigma\big), \qquad \sigma\in\mathcal{J}, \tag{2.9}$$

where $\widetilde{S}_1$, $\widetilde{S}_2$ are suitable smooth functions. We shall often use the notation $\lambda_1(v) \doteq \lambda_1(u(v))$, $\lambda_2(v) \doteq \lambda_2(u(v))$, for the eigenvalues of $A(u(v))$ and let $\lambda_i(v^L, v^R)$ denote the eigenvalue of the averaged matrix

$$A\big(u(v^L), u(v^R)\big) = \int_0^1 A\big(u(v^L) + \xi(u(v^R) - u(v^L))\big)\, d\xi. \tag{2.10}$$

We shall also adopt the notation $\lambda_i^s(v,\sigma) \doteq \lambda_i\big(v,\ S_i(v,\sigma)\big)$ for the speed of the shock wave connecting the state v with the state $S_i(v,\sigma)$. Throughout the paper we shall often use the Landau symbol $\mathcal{O}(1)$ to denote quantities uniformly bounded by a constant which depends only on the system (1.1) and not on ε or on the particular solution considered.

We briefly review next the standard procedure adopted to construct the exact solution of a Riemann problem for a system with non genuinely nonlinear characteristic speeds (a more detailed discussion can be found in [LI1], [LI2], [AM1]).

2.1 - General solutions of the Riemann Problem.

The general entropy-admissible solution of a Riemann Problem $u^L = u(v^L)$, $u^R = u(v^R)$, for a system satisfying the assumption **(A)**, consists of rarefaction waves, compressive shocks and *composed waves* made of one side contact-discontinuity adjacent to a rarefaction wave. In particular, because of (1.4)-(1.5), a composed wave of the i-th family consists of

$i = 1$: a left contact-discontinuity on the right of a rarefaction wave. In this case u^L, u^R lie on opposite sides with respect to the curve Γ_1 at (1.3).

$i = 2$: a right contact-discontinuity on the left of a rarefaction wave. In this case u^L, u^R lie on opposite sides with respect to the curve Γ_2 at (1.3).

To take advantage of the symmetry of the system, we express the general solution of a Riemann Problem in terms of the following two curves (given in v-coordinates).

1. The *elementary curve of right states of the first family*, that consists of all right states that are connected with a given left state by a wave of the first characteristic family.
2. The *elementary curve of left states of the second family*, that consists of all left states that are connected with a given right state by a wave of the second characteristic family.

In particular, this means that, for any fixed $v^0 \in \mathcal{V}_1^0$, we define a *mixed curve of right states of the first family*, parametrized by $\sigma \mapsto T_1(v^0,\sigma)$ and denoted $T_1(v^0)$, having the property

M1. For every point $v^R \in T_1(v^0)$, there exists a unique point $v^L \in R_1(v^0)$ such that $v^R \in S_1(v^L)$ and $\lambda_1(v^L, v^R) = \lambda_1(v^L)$.

Instead, for any fixed $v^0 \in \mathcal{V}_2^0$, we define a *mixed curve of left states of the second family*, parametrized by $\sigma \mapsto T_2(v^0,\sigma)$ and denoted $T_2(v^0)$, having the property

M2. For every point $v^L \in T_2(v^0)$, there exists a unique point $v^R \in R_2(v^0)$ such that $v^R \in S_1(v^L)$ and $\lambda_2(v^L, v^R) = \lambda_2(v^R)$.

Since the mixed curves have second order contact with the rarefaction curves (see [AM1]), they admit a representation of the same form as the shock curves, i.e.

$$\begin{aligned} T_1(v^0,\sigma) &= \big(v_1^0+\sigma,\ v_2^0+\widetilde{T}_1(v^0,\sigma)\cdot\sigma^3\big),\\ T_2(v^0,\sigma) &= \big(v_1^0+\widetilde{T}_2(v^0,\sigma)\cdot\sigma^3,\ v_2^0+\sigma\big), \end{aligned} \qquad \sigma\in\mathcal{J}, \tag{2.11}$$

with $\widetilde{T}_1,\ \widetilde{T}_2,$ suitable smooth functions. We recall now some basic properties of the mixed curves that will be used throughout.

i) For any fixed $v^0 \in \mathcal{V}_i^0,\ i=1,2,$ there exists a smooth map $\sigma \mapsto \mu_i(v^0,\sigma)$ such that

$$\mu_i(v^0,0)=0, \tag{2.12}$$

$$\lambda_i^s\Big[R_i\big(v^0,\,\sigma-\mu_i(v^0,\sigma)\big),\ \mu_i(v^0,\sigma)\Big] = \lambda_i\Big(R_i\big(v^0,\,\sigma-\mu_i(v^0,\sigma)\big)\Big), \qquad \forall\,\sigma\in\mathcal{J}. \tag{2.13}$$

ii) For any fixed $v^0 \in \mathcal{V}_i^0,\ i=1,2,$ there holds

$$\begin{aligned} T_i(v^0,\sigma) &= S_i\Big(R_i\big(v^0,\,\sigma-\mu_i(v^0,\sigma)\big),\ \mu_i(v^0,\sigma)\Big),\\ \frac{d}{d\sigma}T_i(v^0,\sigma) &= \frac{d}{ds}S_i\Big(R_i\big(v^0,\,\sigma-\mu_i(v^0,\sigma)\big),\ s\Big)\Big|_{s=\mu_i(v^0,\sigma)}, \end{aligned} \qquad \forall\,\sigma\in\mathcal{J}. \tag{2.14}$$

iii) The map $\sigma \mapsto \zeta_i(v^0,\sigma) \doteq \sigma - \mu_i(v^0,\sigma)$ is strictly decreasing. Moreover, if we denote $s \mapsto \zeta_i^{-1}(v^0,s)$ its inverse map and set

$$\nu_i(v) \doteq \mu_i\big(\pi_i(v),\ \zeta_i^{-1}\big(\pi_i(v),\,\delta_i(v)\big)\big), \qquad \forall\,v\in\mathcal{V}, \tag{2.15}$$

then, there holds

$$T_i\big(\pi_i(v),\ \delta_i(v)+\nu_i(v)\big) = S_i\big(v,\ \nu_i(v)\big), \qquad \forall\,v\in\mathcal{V}. \tag{2.16}$$

The map $\nu_i : \mathcal{V}_i \mapsto \mathbb{R}$ has the following expansion

$$\nu_i(v) = -3\delta_i(v) + \mathcal{O}(1)\big|\delta_i(v)\big|^2 \qquad \forall\,v\in\mathcal{V}. \tag{2.17}$$

Next, for each point $v\in\mathcal{V}$, we define the *elementary curve of right states of the first characteristic family* $\Psi_1(v)$, and the *elementary curve of left states of the second characteristic family* $\Psi_2(v)$. Consider the sets

$$\mathcal{V}_i^+ \doteq \{v\in\mathcal{V}:\ r_i\bullet\lambda_i(v)>0\} \qquad \mathcal{V}_i^- \doteq \{v\in\mathcal{V}:\ r_i\bullet\lambda_i(v)<0\} \qquad i=1,2. \tag{2.18}$$

Then, define the maps $\Psi_i(v,\cdot):\mathcal{J}\to\mathcal{V}$, $i=1,2$, as follows:

$$\Psi_1(v,\sigma)=\begin{cases} S_1(v,\sigma) \qquad \forall\,\sigma & \text{if} \quad v\in\mathcal{V}_1^0,\\[2mm] \begin{cases} S_1(v,\sigma) & \text{if} \quad \sigma<0\\ R_1(v,\sigma) & \text{if} \quad 0\le\sigma\le-\delta_1(v)\\ T\Big(\pi_1(v),\ \sigma+\delta_1(v)\Big) & \text{if} \quad -\delta_1(v)<\sigma\le\nu_1(v)\\ S_1(v,\sigma) & \text{if} \quad \sigma>\nu_1(v)\end{cases} & \text{if} \quad v\in\mathcal{V}_1^+,\\[2mm] \begin{cases} S_1(v,\sigma) & \text{if} \quad \sigma>0\\ R_1(v,\sigma) & \text{if} \quad -\delta_1(v)\le\sigma\le 0\\ T\Big(\pi_1(v),\ \sigma+\delta_1(v)\Big) & \text{if} \quad \nu_1(v)\le\sigma<-\delta_1(v)\\ S_1(v,\sigma) & \text{if} \quad \sigma<\nu_1(v)\end{cases} & \text{if} \quad v\in\mathcal{V}_1^-,\end{cases} \tag{2.19}$$

$$\Psi_2(v,\sigma)=\begin{cases} S_2(v,\sigma) \qquad \forall\,\sigma & \text{if} \quad v\in\mathcal{V}_2^0,\\[2mm] \begin{cases} S_2(v,\sigma) & \text{if} \quad \sigma>0\\ R_2(v,\sigma) & \text{if} \quad -\delta_2(v)\le\sigma\le 0\\ T_2\Big(\pi_2(v),\ \sigma+\delta_2(v)\Big) & \text{if} \quad \nu_2(v)\le\sigma<-\delta_2(v)\\ S_2(v,\sigma) & \text{if} \quad \sigma<\nu_2(v)\end{cases} & \text{if} \quad v\in\mathcal{V}_2^+,\\[2mm] \begin{cases} S_2(v,\sigma) & \text{if} \quad \sigma<0\\ R_2(v,\sigma) & \text{if} \quad 0\le\sigma\le-\delta_2(v)\\ T_2\Big(\pi_2(v),\ \sigma+\delta_2(v)\Big) & \text{if} \quad -\delta_2(v)<\sigma\le\nu_2(v)\\ S_2(v,\sigma) & \text{if} \quad \sigma>\nu_2(v)\end{cases} & \text{if} \quad v\in\mathcal{V}_2^-.\end{cases} \tag{2.20}$$

By the implicit function theorem, given $v^L,\,v^R\in\mathcal{V}$, there exists a unique intermediate state $v^M\in\mathcal{V}$, and wave sizes $\sigma_1,\sigma_2\in\mathcal{J}$, such that

$$v^M=\Psi_1(v^L,\sigma_1),\qquad v^M=\Psi_2(v^R,\sigma_2). \tag{2.21}$$

Then, if we set

$$\overline{v}^1(x)\doteq\begin{cases} v^L & \text{if} \quad x<0,\\ v^M & \text{if} \quad x>0,\end{cases}\qquad\qquad \overline{v}^2(x)\doteq\begin{cases} v^M & \text{if} \quad x<0,\\ v^R & \text{if} \quad x>0,\end{cases} \tag{2.22}$$

the solution of the Riemann Problem with initial data given (in terms of the v-coordinates) by

$$\overline{v}(x)\doteq\begin{cases} v^L & \text{if} \quad x<0,\\ v^R & \text{if} \quad x>0,\end{cases}$$

is constructed by piecing together the solutions of the Riemann problem with initial data $\overline{v}^1$ on the quadrant where $x\le 0$, with the solution of the Riemann problem with initial data $\overline{v}^2$ on the quadrant where $x \geq 0$. The solution to each Riemann problem with initial data $\overline{v}^i$, $i = 1, 2$, is defined as follows.

1 - Waves. Initial data $\overline{v}^1$. There are 3 possibilities:

1) $v^L \in \mathcal{V}_1^0$: the solution consists of a shock of the first characteristic family travelling with speed $\lambda_1^s(v^L, \sigma_1)$.

2) $v^L \in \mathcal{V}_1^+$: the solution consists of

a) $0 \leq \sigma_1 \leq -\delta_1(v^L)$: a rarefaction wave of the first characteristic family, with characteristic speeds ranging over the interval

$$\left[\lambda_1(v^L),\, \lambda_1(v^M)\right];$$

b) $\sigma_1 < 0$ or $\sigma_1 \geq \nu_1(v^L)$: a shock of the first characteristic family travelling with speed $\lambda_1^s(v^L, \sigma_1)$;

c) $-\delta_1(v^L) < \sigma_1 < \nu_1(v^L)$; a composed wave of the first characteristic family made of a rarefaction wave of size

$$\widetilde{\sigma}_1 \doteq \sigma_1 - \mu_1\big(\pi_1(v^L),\ \sigma_1 + \delta_1(v^L)\big),$$

with characteristic speeds ranging over the interval

$$\left[\lambda_1(v^L),\, \lambda_1(\widetilde{v}^1)\right], \qquad \widetilde{v}^1 \doteq R_1\big(v^L, \widetilde{\sigma}_1\big),$$

followed by a left contact discontinuity $\widehat{\sigma}_1 \doteq \sigma_1 - \widetilde{\sigma}_1$, travelling with speed

$$\lambda_1^s\left[\widetilde{v}^1, \widehat{\sigma}_1\right] = \lambda_1(\widetilde{v}^1).$$

3) $v^L \in \mathcal{V}_1^-$: the solution consists of

a) $-\delta_1(v^L) \leq \sigma_1 \leq 0$: a rarefaction wave as in 2-*a)*;

b) $\sigma_1 > 0$ or $\sigma_1 \leq \nu_1(v^L)$: a shock as in 2-*b)*;

c) $\nu_1(v^L) < \sigma_1 < -\delta_1(v^L)$: a composed wave as in 2-*c)*, made of a rarefaction wave followed by a left contact discontinuity.

2 - Waves. Initial data $\overline{v}^2$. There are 3 possibilities:

1) $v^R \in \mathcal{V}_2^0$: the solution consists of a shock of the second characteristic family travelling with speed $\lambda_2^s\big(v^R,\,\sigma_2\big)$.

2) $v^R \in \mathcal{V}_2^+$: the solution consists of

   a) $-\delta_2(v^R) \leq \sigma_2 \leq 0$; a rarefaction wave of the second characteristic family, with characteristic speeds ranging over the interval

$$\big[\lambda_2(v^M),\,\lambda_2(v^R)\big];$$

   b) $\sigma_2 > 0$ or $\sigma_2 \leq \nu_2(v^R)$: a shock of the second characteristic family travelling with speed $\lambda_2^s\big(v^R,\,\sigma_2\big)$;

   c) $\nu_2(v^R) < \sigma_2 < -\delta_2(v^R)$: a composed wave of the second characteristic family made of a right contact discontinuity of size

$$\widehat{\sigma}_2 \doteq \mu_2\big(\pi_2(v^R),\ \sigma_2 + \delta_2(v^R)\big),$$

   travelling with speed

$$\lambda_2^s\big[\widetilde{v}^2,\,\widehat{\sigma}_2\big] = \lambda_2(\widetilde{v}^2), \qquad \widetilde{v}^2 \doteq R_2\big(v^R,\,\sigma_2 - \widehat{\sigma}_2\big),$$

   followed by a rarefaction wave of size $\widetilde{\sigma}_2 \doteq \sigma_2 - \widehat{\sigma}_2$, with characteristic speeds ranging over the interval

$$\big[\lambda_2(\widetilde{v}^2),\,\lambda_2(v^R)\big].$$

3) $v^R \in \mathcal{V}_2^-$: the solution consists of

   a) $0 \leq \sigma_2 \leq -\delta_2(v^R)$: a rarefaction wave as in 2-*a)*;

   b) $\sigma_2 < 0$ or $\sigma_2 \geq \nu_2(v^R)$: a shock as in 2-*b)*;

   c) $-\delta_2(v^R) < \sigma_2 < \nu_2(v^R)$: a composed wave as in 2-*c)*, made of a right contact discontinuity followed by a rarefaction wave.

3. Outline of the Proof

We collect here the basic Steps in the proof of Theorem 1. Technical details will be worked out in the remaining sections.

Since we are dealing with a 2×2 system, it will be convenient to work always with a set of Riemann coordinates $v = (v_1, v_2)$. Throughout we shall often adopt the notation

$$\lfloor v \rfloor_1 \doteq \left\lfloor \frac{v_1}{\varepsilon} \right\rfloor, \qquad \lfloor v \rfloor_2 \doteq \left\lfloor \frac{v_2}{\varepsilon} \right\rfloor, \qquad v = (v_1, v_2), \qquad \varepsilon > 0,$$

where $\lfloor \alpha \rfloor$ denotes the integer part of α.

In the initial part of the proof, we fix $\varepsilon > 0$ and introduce a refined version of the wave-front tracking algorithm adopted in [BC1] which yields piecewise constant ε-approximate solutions, well defined for all $t > 0$, for a system satisfying the assumption **(A)**. These are piecewise constant functions with jumps located along finitely many segments in the (t, x)-plane, and satisfying a set of approximate Rankine-Hugoniot conditions. To this purpose, for $v \in \mathcal{V} \subset \mathbb{R}^2$, and σ in a suitable neighborhood of zero $\mathcal{I} \subset \mathbb{R}$, we shall construct some maps

$$\begin{array}{ll} (v,\sigma) \mapsto \Psi_1^\varepsilon(v,\, \sigma), & (v,\sigma) \mapsto \Psi_2^\varepsilon(v,\, \sigma), \\ (v,\sigma) \mapsto \lambda_1^{\psi,\varepsilon}(v,\, \sigma), & (v,\sigma) \mapsto \lambda_2^{\psi,\varepsilon}(v,\, \sigma), \\ (v,\sigma) \mapsto \mu_1^\varepsilon(\pi_1(v),\, \sigma), & (v,\sigma) \mapsto \mu_2^\varepsilon(\pi_2(v),\, \sigma), \\ v \mapsto \nu_1^\varepsilon(v), & v \mapsto \nu_2^\varepsilon(v), \end{array} \qquad v = (v_1, v_2) \in \mathcal{V}, \quad \sigma \in \mathcal{I}, \tag{3.1}$$

that provide a careful approximation of the functions $\Psi_i,\, \lambda_i^s,\, \mu_i,\, \nu_i,\ i = 1, 2,$ involved in the definition of the exact solution of a Riemann problem and, at the same time, yield an ε-Approximate Riemann Solver whose main features are:

- Every Riemann Problem is solved with the same procedure and the solution depends continuously on the Riemann data v^L, v^R.
- Centered rarefaction waves are approximated by means of a piecewise constant rarefaction fan, inserting intermediate states on a fixed grid of step size ε.
- Shock waves of strength $|\sigma| < 2\sqrt{\varepsilon}$ connect a left and right state lying on the same rarefaction curve.

The detailed definition of such maps is given in Section 4. Here, we collect only their basic properties.

P1. As $\varepsilon \to 0+$, for $i = 1, 2$, there holds

$$\Psi_i^\varepsilon \to \Psi_i, \qquad \lambda_i^{\psi,\varepsilon} \to \lambda_i^s, \qquad \mu_i^\varepsilon \to \mu_i, \qquad \nu_i^\varepsilon \to \nu_i,$$

the convergence being uniform on $\mathcal{V}$ and $\mathcal{I}$. The maps Ψ_i^ε, $\lambda_i^{\psi,\varepsilon}$, μ_i^ε, ν_i^ε, admit left and right partial derivatives with respect to v_1, v_2, σ, and are piecewise continuously differentiable on their domain. Moreover, Ψ_i^ε, $\lambda_i^{\psi,\varepsilon}$, are continuous in any point.

P2.

$$\begin{aligned} \Psi_i^\varepsilon(v,\,\sigma) &= R_i(v,\sigma), \\ \lambda_i^{\psi,\varepsilon}(v,\sigma) &= \lambda_i^{r,\varepsilon}(v,\sigma), \end{aligned} \qquad \text{if} \qquad |\sigma| \le 2\sqrt{\varepsilon}, \qquad i=1,2, \tag{3.2}$$

where $\lambda_i^{r,\varepsilon}$ denotes the averaged speed defined, for the first characteristic family, as

$$\begin{aligned} \lambda_1^{r,\varepsilon}(v,\sigma) &\doteq \sum_{\ell=\lfloor v\rfloor_1}^{\lfloor R_1(v,\sigma)\rfloor_1} \left[\frac{\mathrm{meas}\Big([\ell\,\varepsilon,\,(\ell+1)\,\varepsilon] \cap \big[v_1, v_1+\sigma\big]\Big)}{\sigma} \cdot \widehat{\lambda}_1^\ell(v) \right] \qquad \sigma \ge 0, \\ \widehat{\lambda}_1^\ell(v) &\doteq \sup_{0\le s\le\varepsilon} \lambda_1\big((\ell\,\varepsilon + s,\, v_2)\big), \end{aligned} \tag{3.3}$$

and, for the second characteristic family, as

$$\begin{aligned} \lambda_2^{r,\varepsilon}(v,\sigma) &\doteq \sum_{\ell=\lfloor v\rfloor_2}^{\lfloor R_2(v,\sigma)\rfloor_2} \left[\frac{\mathrm{meas}\Big([\ell\,\varepsilon,\,(\ell+1)\,\varepsilon] \cap \big[v_2, v_2+\sigma\big]\Big)}{\sigma} \cdot \widehat{\lambda}_2^\ell(v) \right] \qquad \sigma \ge 0, \\ \widehat{\lambda}_2^\ell(v) &\doteq \inf_{0\le s\le\varepsilon} \lambda_2\big((v_1,\, \ell\,\varepsilon + s)\big). \end{aligned} \tag{3.4}$$

The definitions of the averaged speed $\lambda_1^{r,\varepsilon}$, $\lambda_2^{r,\varepsilon}$, in the case $\sigma < 0$ are entirely similar.

P3. For any fixed $v \in \mathcal{V}_1^+$ and $\sigma \in [-\delta_1(v),\, \nu_i^\varepsilon(v)]$, setting

$$\widetilde{\sigma} \doteq \sigma - \mu_1^\varepsilon(\pi_1(v),\, \sigma + \delta_1(v)), \qquad \widehat{\sigma} \doteq \sigma - \widetilde{\sigma}, \tag{3.5}$$

the equation

$$\Psi_1^\varepsilon(v,\,\sigma) - \Psi_1^\varepsilon\Big[R_1(v,\sigma_1),\ \sigma_2\Big] = 0 \tag{3.6}$$

admits:

i) A unique solution $(\sigma_1,\,\sigma_2) = (\widetilde{\sigma},\,\widehat{\sigma})$, if $\widehat{\sigma} > \nu_1^\varepsilon\big(R_1(v,\widetilde{\sigma})\big)$;

ii) A set of solutions

$$\Big\{ (\sigma_1,\sigma_2) \ :\ \widetilde{\sigma} \le \sigma_1 \le \widetilde{\sigma} + \varepsilon, \quad \sigma_2 = \sigma - \sigma_1 \Big\}, \tag{3.7}$$

if $\widehat{\sigma} = \nu_1^\varepsilon\big(R_1(v,\widetilde{\sigma})\big)$.

Moreover, one has $v_1 + \widetilde{\sigma} \in \mathbb{Z}\,\varepsilon$ and, in the case $\widehat{\sigma} = \nu_1^\varepsilon\big(R_1(v,\widetilde{\sigma})\big)$, there holds

$$\lambda_1^{\lfloor R_1(v,\widetilde{\sigma})\rfloor_1}(v) = \lambda_1^{\psi,\varepsilon}\big[R_1(v,\widetilde{\sigma}),\ \widehat{\sigma}\big] = \lambda_1^{\psi,\varepsilon}\big[R_1(v,\sigma_1),\ \sigma_2\big], \tag{3.8}$$

for all (σ_1,σ_2) in (3.7).

Similar relations are verified in the case $v \in \mathcal{V}_1^-$ and $v \in \mathcal{V}_2^\pm$.

P4. For any fixed $v^L, v^R \in \mathcal{V}$, the equation

$$\Psi_1^\varepsilon(v^L, \sigma_1) - \Psi_2^\varepsilon(v^R, \sigma_2) = 0 \tag{3.9}$$

admits a unique solution in $\mathcal{I} \times \mathcal{I}$. Moreover, one can find some constant $C > 0$ independent of ε such that, for all $v^L, v^R \in \mathcal{V}$, if σ_1, σ_2 are the corresponding values determined by (3.9), then one has

$$\frac{1}{C}\left|v^R - v^L\right| \le |\sigma_1| + |\sigma_2| \le C\left|v^R - v^L\right|. \tag{3.10}$$

Remark 3.1 Fix some state $v^L \in \mathcal{V}_1^-$, and let $\alpha \mapsto \widetilde{F}_1(\alpha)$ be a scalar, continuous, piecewise linear function such that

$$\widetilde{F}_1'(\alpha) = \widehat{\lambda}_1^\ell(v^L) \qquad \text{if} \qquad \alpha \in]\ell\,\varepsilon,\ (\ell+1)\,\varepsilon[,$$

where $\widehat{\lambda}_1^\ell(v^L)$ is the discrete characteristic speed defined at (3.3) (see fig. 3.1). Consider the function

$$\omega(t,x) = \begin{cases} v^L & \text{if} \quad x < \lambda_1^{\psi,\varepsilon}\big(v^L, \sigma\big)\cdot t, \\ v^R & \text{if} \quad \lambda_1^{\psi,\varepsilon}\big(v^L, \sigma\big)\cdot t < x \le 0, \end{cases}$$

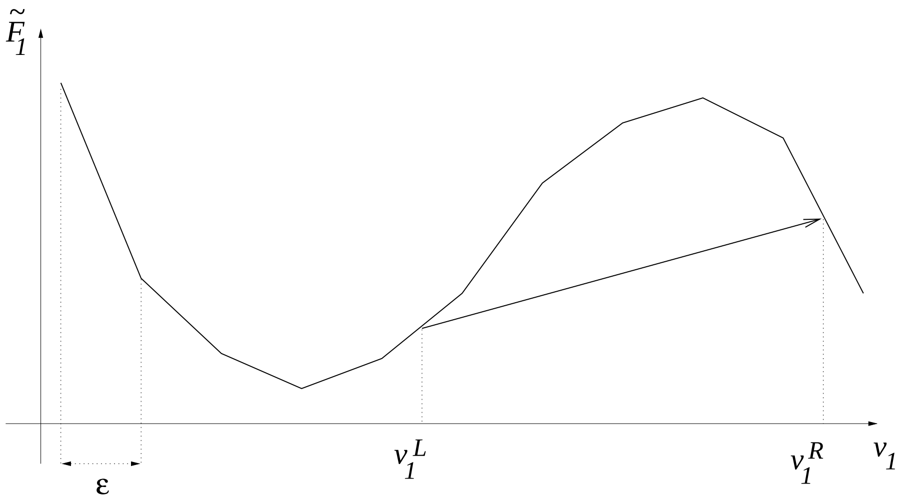

FIGURE 3.1

where $v^R = \Psi_1^\varepsilon(v, \sigma)$. Then, because of the property **P2**, as long as $0 \le \sigma \le 2\sqrt{\varepsilon}$, the first component $\omega_1(t,x)$ of ω provides a weak, entropy-admissible solution to the scalar conservation law

$$(v_1)_t + \big[\widetilde{F}_1(v_1)\big]_x = 0. \tag{3.11}$$

The second component is constant $\omega_2(t,x) = v_2^L$. This property motivates the definition (3.3) of averaged characteristic speed for a small shock. Of course, the same property holds for all shock waves σ that connect a left and right state lying on the same curve Ψ_i^ε, $i = 1, 2$, and have strength $|\sigma| \le 2\sqrt{\varepsilon}$.

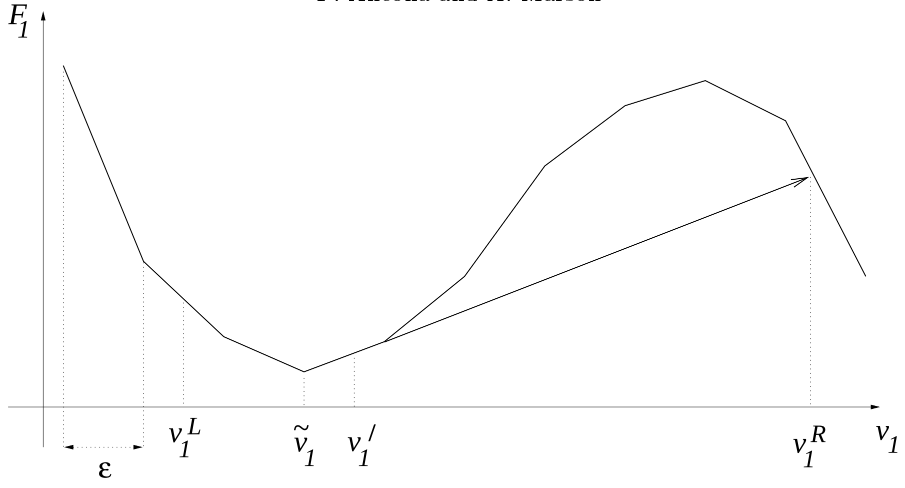

FIGURE 3.2

Remark 3.2 Fix $v^L \in \mathcal{V}_1^+$, $\sigma \in [-\delta_1(v^L),\, 2\sqrt{\varepsilon}]$, and assume that there exists some $\widetilde\sigma \in [0,\,\sigma]$ such that

$$v_1^L + \widetilde\sigma \in \mathbb{Z}\,\varepsilon, \qquad\qquad \widehat\lambda_1^{\lfloor R_1(v^L,\widetilde\sigma)\rfloor_1}(v^L) = \lambda_1^{r,\varepsilon}\big[R_1(v^L,\widetilde\sigma),\ \sigma - \widetilde\sigma\big]. \tag{3.12}$$

Let $\sigma' \in [\widetilde\sigma,\, \widetilde\sigma + \varepsilon]$, and set

$$\widehat\sigma \doteq \sigma - \widetilde\sigma, \qquad \widehat\sigma' \doteq \sigma - \sigma', \qquad \widetilde v \doteq R_1\big(v^L,\, \widetilde\sigma\big), \qquad v' \doteq R_1\big(v^L,\, \sigma'\big), \qquad v^R \doteq \Psi_1^\varepsilon(v,\sigma).$$

Notice that, from the properties of $\lambda_1^{r,\varepsilon}$ and by (3.12), it follows

$$\widehat\lambda_1^{\lfloor\widetilde v\rfloor_1}(v^L) = \lambda_1^{r,\varepsilon}\big(\widetilde v,\, \widehat\sigma\big) = \lambda_1^{r,\varepsilon}\big(v',\, \widehat\sigma'\big). \tag{3.13}$$

By **P2**, this relation can be rewritten as

$$\widehat\lambda_1^{\lfloor\widetilde v\rfloor_1}(v^L) = \lambda_1^{\psi,\varepsilon}\big(\widetilde v,\, \widehat\sigma\big) = \lambda_1^{\psi,\varepsilon}\big(v',\, \widehat\sigma'\big), \tag{3.13}$$

which is the analogous of (3.8) in **P3**. Then, for any given $\xi > 0$, consider the function

$$\omega^{\sigma',\xi}(t,x) = \begin{cases} v^L & \text{if} \quad x < \widehat\lambda_1^{\lfloor v^L\rfloor_1}(v^L)\cdot t, \\ \big(\ell\varepsilon, v_2^L\big) & \text{if} \quad \widehat\lambda_1^{\ell-1}(v^L)\cdot t < x < \widehat\lambda_1^{\ell}(v^L)\big)\cdot t, \qquad \ell = \lfloor v^L\rfloor_1 + 1, \dots, \lfloor\widetilde v\rfloor_1, \\ v' & \text{if} \quad \widehat\lambda_1^{\lfloor\widetilde v\rfloor_1}(v^L)\cdot t < x < \xi + \widehat\lambda_1^{\lfloor\widetilde v\rfloor_1}(v^L)\cdot t, \\ v^R & \text{if} \quad \xi + \lambda_1^{\psi,\varepsilon}\big(v',\, \widehat\sigma'\big)\cdot t < x \le 0, \end{cases}$$

that provides a piecewise constant approximation of a composed wave made of a centered rarefaction followed by a shock (see figures 3.2-3.3). As in the previous Remark, one can easily check that the first component of $\omega^{\sigma',\xi}(t,x)$ is a weak, entropy-admissible solutions of the scalar conservation law (3.11), while the second component is constant $\omega_2^{\sigma',\xi}(t,x) = v_2^L$. Moreover, thanks to (3.13), we find that the $\mathbb{L}^1$-limit

$$\omega(t,\cdot) \doteq \lim_{\xi\to 0+} \omega^{\sigma',\xi}(t,\cdot) \qquad\qquad t \ge 0,$$

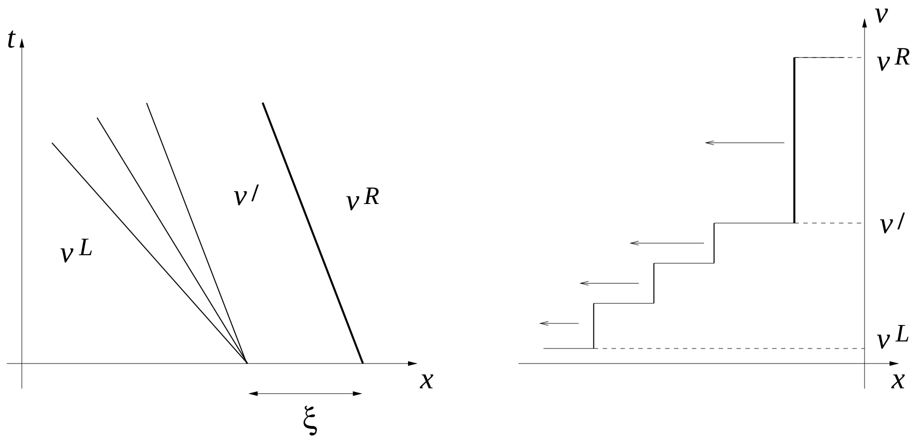

FIGURE 3.3

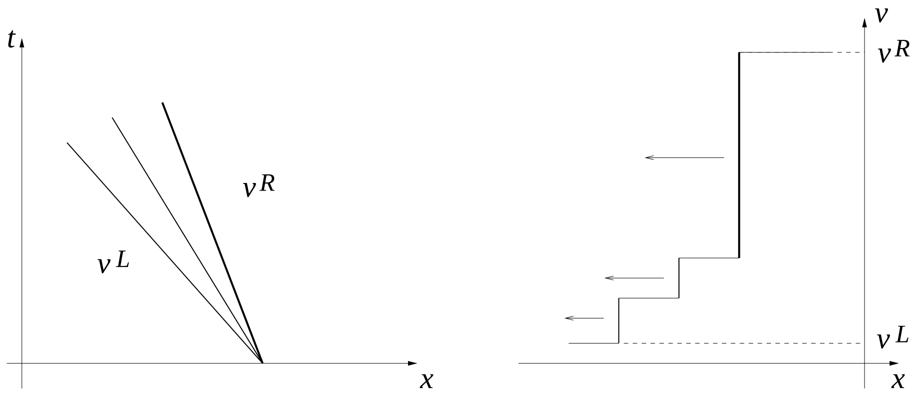

FIGURE 3.4

is

$$\omega(t,x) = \begin{cases} v^L & \text{if } \ x < \widehat{\lambda}_1^{\lfloor v^L \rfloor_1}(v^L)\cdot t, \\ \big(\ell\varepsilon, v_2^L\big) & \text{if } \ \widehat{\lambda}_1^{\ell-1}(v^L)\cdot t < x < \widehat{\lambda}_1^{\ell}(v^L)\big)\cdot t, \qquad \ell = \lfloor v^L\rfloor_1 + 1, \dots, \lfloor \widetilde{v} \rfloor_1, \\ v^R & \text{if } \ \lambda_1^{\psi,\varepsilon}\big(\widetilde{v},\, \widehat{\sigma}\big)\cdot t < x \le 0. \end{cases}$$

and the first component of $\omega(t,x)$ is a weak, entropy-admissible solution of (3.11) as well (see figures 3.2-3.4). Clearly, the same property holds in the case we consider a piecewise constant function that approximates a composed waves σ, of strength $|\sigma| \le 2\sqrt{\varepsilon}$, belonging to the second characteristic family. This observation shows the crucial importance of property **P3** to guarantee the stability and the continuous dependence on the initial data of approximate solutions constructed for a system satisfying the assumption **(A)**.

Definition 3.1. Let (v^L, v^R) be a jump located at $x = x(t)$ and traveling with speed $\dot{x}$.

1. We say that (v^L, v^R) satisfies the Ψ^ε-*Rankine-Hugoniot condition* $\big(\Psi^\varepsilon\text{-RH}\big)$ *of the first family* if one of the following two cases occurs:

i)

$$v^R = R_1(v^L,\sigma), \qquad \dot{x} = \widehat{\lambda}_1^{\lfloor v^L \rfloor_1}(v^L), \tag{3.14}$$

for some $v^L \in \mathcal{V}_1^+$ and $\sigma \in \big[0,\ (\lfloor v^L \rfloor_1 + 1)\varepsilon - v_1^L\big]$, or for some $v^L \in \mathcal{V}_1^-$ and $\sigma \in \big[\lfloor v^L \rfloor_1 \varepsilon - v_1^L,\ 0\big]$ ($\widehat{\lambda}_1^{\lfloor v^L \rfloor_1}(v^L)$ is the discrete characteristic speed defined as in (3.3)).

ii)

$$v^R = \Psi_1^\varepsilon(v^L,\sigma), \qquad \dot{x} = \lambda_1^{\psi,\varepsilon}(v^L,\sigma), \tag{3.15}$$

for some $v^L \in \mathcal{V}_1^+$ and $\sigma \in \big\{s\ :\ s \leq 0\big\} \cup \big\{s\ :\ s \geq \nu_1^\varepsilon\big(v^L\big)\big\}$, or for some $v^L \in \mathcal{V}_1^-$ and $\sigma \in \big\{s\ :\ s \geq 0\big\} \cup \big\{s\ :\ s \leq \nu_1^\varepsilon\big(v^L\big)\big\}$.

2. We say that (v^L, v^R) satisfies the Ψ^ε-*Rankine-Hugoniot condition* $\big(\Psi^\varepsilon$-RH$\big)$ *of the second family* if one of the following two cases occurs:

i)

$$v^L = R_2(v^R,\sigma), \qquad \dot{x} = \widehat{\lambda}_2^{\lfloor v^R \rfloor_2}(v^R), \tag{3.16}$$

for some $v^R \in \mathcal{V}_2^+$ and $\sigma \in \big[0,\ (\lfloor v^R \rfloor_2 + 1)\varepsilon - v_2^R\big]$, or for some $v^R \in \mathcal{V}_2^-$ and $\sigma \in \big[0,\ \lfloor v^R \rfloor_2 \varepsilon - v_2^R\big]$ ($\widehat{\lambda}_2^{\lfloor v^R \rfloor_2}(v^R)$ is the discrete characteristic speed defined as in (3.4)).

ii)

$$v^L = \Psi_2^\varepsilon(v^R,\sigma), \qquad \dot{x} = \lambda_2^{\psi,\varepsilon}(v^R,\sigma), \tag{3.17}$$

for some $v^R \in \mathcal{V}_2^-$ and $\sigma \in \big\{s\ :\ s \leq 0\big\} \cup \big\{s\ :\ s \geq \nu_2^\varepsilon\big(v^R\big)\big\}$, or for some $v^R \in \mathcal{V}_2^+$ and $\sigma \in \big\{s\ :\ s \geq 0\big\} \cup \big\{s\ :\ s \leq \nu_2^\varepsilon\big(v^R\big)\big\}$.

Definition 3.2. We say that a piecewise constant function $v = v(t,x)$ is an ε-*approximate solution* if all of its jumps satisfy the Ψ^ε-RH conditions. Any line $x = x(t)$ across which v has a jump satisfying the Ψ^ε-RH conditions of the i-th family is an ε-*admissible wave-front* of the i-th family.

Remark 3.3 By introducing the modified Rankine Hugoniot conditions of Definition 3.1, we impose that each admissible jump of the ε-approximate solution correspond to one of the following two types of waves:

- A shock wave whose left and right states are connected by an “approximate” shock-curve, that travels with an “approximate” shock speed. The errors in the size and in the speed are small and go to zero as $\varepsilon \to 0$.
- A rarefaction wave-front of size $\leq \varepsilon$ that approximates a piece of a rarefaction fan and travels with the discrete characteristic speed defined at (3.3)-(3.4).

Notice that, because of the property **P2**, whenever a shock has strength less than $2\sqrt{\varepsilon}$, its left and right states are connected by a rarefaction curve (instead of an approximate-shock curve) and the shock travels with the averaged characteristic speed defined at (3.3)-(3.4).

The modified Rankine Hugoniot conditions and the above definition of approximate solution determine a new way of approximately solving a Riemann problem. More precisely, consider the initial data given (in terms of the v-coordinates) by

$$\overline{v}(x) \doteq \begin{cases} v^L & \text{if} \quad x < 0, \\ v^R & \text{if} \quad x > 0. \end{cases} \tag{3.18}$$

We seek a self-similar, piecewise constant function $\omega = \omega(t, x)$ that satisfies the Ψ^ε-RH conditions along each discontinuity line and attains the value (3.18) at time $t = 0$. The solution to this problem is provided by the following

ε-Approximate Riemann solver: Thanks to the property **P4**, given v^L, $v^R \in \mathcal{V}$, there exists a unique intermediate state $v^M \in \mathcal{V}$, and wave sizes $\sigma_1, \sigma_2 \in \mathcal{J}$, such that

$$v^M = \Psi_1^\varepsilon\big(v^L\big)(\sigma_1), \qquad v^M = \Psi_2^\varepsilon\big(v^R\big)(\sigma_2). \tag{3.19}$$

Then, if we set

$$\overline{v}^1(x) \doteq \begin{cases} v^L & \text{if} \quad x < 0, \\ v^M & \text{if} \quad x > 0, \end{cases} \qquad \overline{v}^2(x) \doteq \begin{cases} v^M & \text{if} \quad x < 0, \\ v^R & \text{if} \quad x > 0, \end{cases}$$

the ε-approximate solution of the Riemann problem with initial data (3.18) consists of the ε-approximate solution of the Riemann problem with initial data $\overline{v}^1$ on the quadrant where $x \leq 0$, and of the ε-approximate solution of the Riemann problem with initial data $\overline{v}^2$ on the quadrant where $x \geq 0$. The ε-approximate solution to each Riemann problem with initial data $\overline{v}^i$, $i = 1, 2$, is constructed as follows.

1 - Waves. Initial data $\overline{v}^1$. There are 3 possibilities:

1) $v^L \in \mathcal{V}_1^0$: the solution consists of a shock of the first characteristic family travelling with speed $\lambda_1^{\psi,\varepsilon}\big(v^L, \sigma_1\big)$, i.e. (see fig. 3.5)

$$\omega(t, x) = \begin{cases} v^L & \text{if} \quad x < \lambda_1^{\psi,\varepsilon}\big(v^L, \sigma_1\big) \cdot t, \\ v^M & \text{if} \quad \lambda_1^{\psi,\varepsilon}\big(v^L, \sigma_1\big) \cdot t < x \leq 0. \end{cases} \tag{3.20}$$

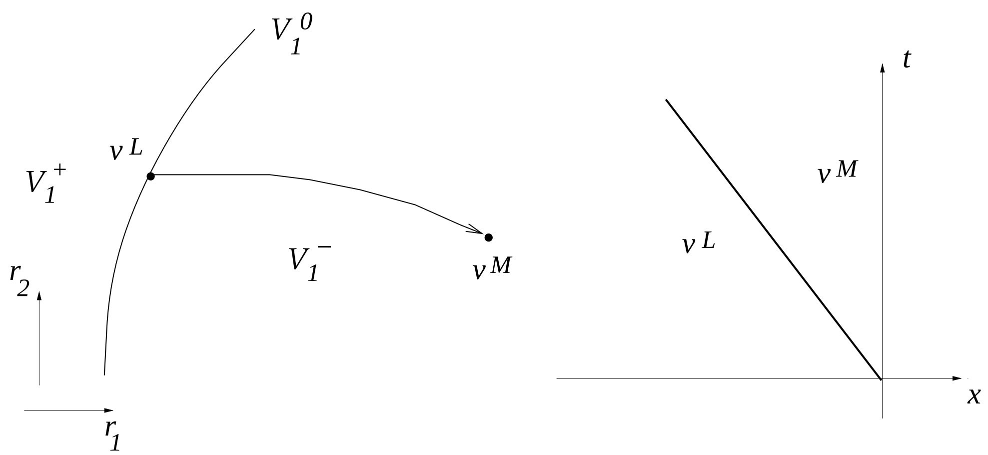

FIGURE 3.5

2) $v^L \in \mathcal{V}_1^+$: the solution consists of

a) $0 \le \sigma_1 \le -\delta_1(v^L)$: a piecewise constant rarefaction fan of the first characteristic family, i.e. (see fi.g 3.6)

$$\omega(t,x) = \begin{cases} v^L & \text{if} \quad x < \widehat{\lambda}_1^{\lfloor v^L \rfloor_1}(v^L)\cdot t, \\ (\ell\,\varepsilon,\, v_2^L) & \text{if} \quad \widehat{\lambda}_1^{\ell-1}(v^L)\cdot t < x < \widehat{\lambda}_1^{\ell}(v^L)\cdot t, \qquad \ell = \lfloor v^L\rfloor_1 + 1, \dots, N_m, \\ v^M & \text{if} \quad \widehat{\lambda}_1^{N_m}(v^L)\cdot t < x \le 0, \end{cases} \tag{3.21}$$

where

$$N_m = \begin{cases} \lfloor v^M \rfloor_1 & \text{if} \quad \lfloor v^M\rfloor_1\,\varepsilon \ne v_1^M, \\ \lfloor v^M \rfloor_1 - 1 & \text{otherwise}, \end{cases}$$

while $\widehat{\lambda}_1^{\ell}(v^L)$ are defined as in (3.3);

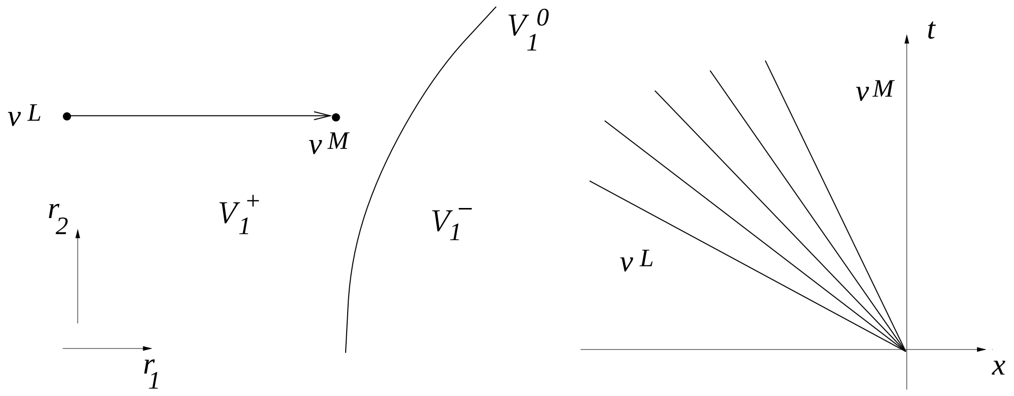

FIGURE 3.6

b) $\sigma_1 < 0$ or $\sigma_1 \ge \nu_1^{\varepsilon}(v^L)$: a shock of the first characteristic family as in (3.20), travelling with speed $\lambda_1^{\psi,\varepsilon}(v^L,\, \sigma_1)$;

c) $-\delta_1(v^L) < \sigma_1 < \nu_1^{\varepsilon}(v^L)$: a composed wave of the first characteristic family, made of a piecewise constant rarefaction fan as in (3.21), of size

$$\widetilde{\sigma}_1 \doteq \sigma_1 - \mu_1^{\varepsilon}\big(\pi_1(v^L),\; \sigma_1 + \delta_1(v^L)\big),$$

followed by a shock $\widehat{\sigma}_1 \doteq \sigma_1 - \widetilde{\sigma}_1$, travelling with speed

$$\lambda_1^{\psi,\varepsilon}(\widetilde{v}^1,\, \widehat{\sigma}_1) \le \widehat{\lambda}_1^{\lfloor \widetilde{v}^1\rfloor_1}(v^L), \qquad \widetilde{v}^1 \doteq R_1(v^L,\, \widetilde{\sigma}_1). \tag{3.22}$$

It follows that the solution is

$$\omega(t,x) = \begin{cases} v^L & \text{if} \quad x < \widehat{\lambda}_1^{\lfloor v^L \rfloor_1}(v^L)\cdot t, \\ (\ell\varepsilon, v_2^L) & \text{if} \quad \widehat{\lambda}_1^{\ell-1}(v^L)\cdot t < x < \widehat{\lambda}_1^{\ell}(v^L))\cdot t, \qquad \ell = \lfloor v^L\rfloor_1 + 1, \dots, \lfloor \widetilde{v}^1\rfloor_1 - 1, \\ \widetilde{v}^1 & \text{if} \quad \widehat{\lambda}_1^{\lfloor \widetilde{v}^1\rfloor_1 - 1}(v^L)\cdot t < x < \lambda_1^{\psi,\varepsilon}(\widetilde{v}^1,\, \widehat{\sigma}_1)\cdot t, \\ v^M & \text{if} \quad \lambda_1^{\psi,\varepsilon}(\widetilde{v}^1,\, \widehat{\sigma}_1)\cdot t < x \le 0, \end{cases} \tag{3.23}$$

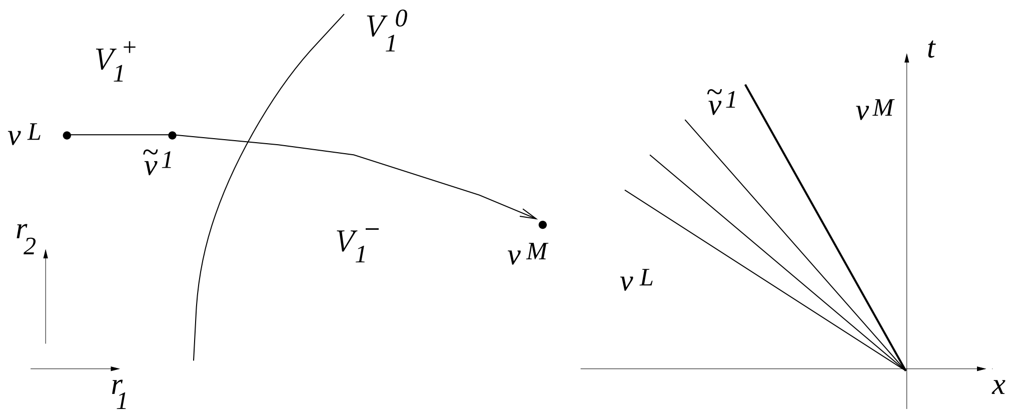

FIGURE 3.7

with $\widehat{\lambda}_1^{\ell}(v^L)$ defined as in (3.3) (see fig. 3.7);

3) $v^L \in \mathcal{V}_1^-$: the solution consists of

a) $-\delta_1(v^L) \leq \sigma_1 \leq 0$: a piecewise constant rarefaction fan as in 2-*a)*;

b) $\sigma_1 > 0$ or $\sigma_1 \leq \nu_1^{\varepsilon}(v^L)$: a shock as in 2-*b)*;

c) $\nu_1^{\varepsilon}\big(v^L\big) < \sigma_1 V - \delta_1\big(v^L\big)$: a composed wave as in 2-*c)*, made of a piecewise constant rarefaction fan followed by a shock.

2 - Waves. Initial data $\overline{v}^2$. There are 3 possibilities:

1) $v^R \in \mathcal{V}_2^0$: the solution consists of a shock of the second characteristic family travelling with speed $\lambda_2^{\psi,\varepsilon}\big(v^R, \sigma_2\big)$, i.e. (see fig. 3.8)

$$\omega(t,x) = \begin{cases} v^M & \text{if} \quad 0 \leq x < \lambda_2^{\psi,\varepsilon}\big(v^R,\, \sigma_2\big) \cdot t, \\ v^R & \text{if} \quad x > \lambda_2^{\psi,\varepsilon}\big(v^R,\, \sigma_2\big) \cdot t. \end{cases} \tag{3.24}$$

2) $v^R \in \mathcal{V}_2^+$: the solution consists of

a) $-\delta_2\big(v^R\big) \leq \sigma_2 \leq 0$: a piecewise constant rarefaction fan of the second characteristic family, i.e. (see fig. 3.9)

$$\omega(t,\, x) = \begin{cases} v^M & \text{if} \quad 0 \leq x < \widehat{\lambda}_2^{\lfloor v^M \rfloor_2}(v^R) \cdot t, \\ \big(v_2^R,\, \ell\varepsilon\big) & \text{if} \quad \widehat{\lambda}_2^{\ell-1}(v^R) \cdot t < x < \widehat{\lambda}_2^{\ell}(v^R) \cdot t, \qquad \ell = \lfloor v^M \rfloor_2 + 1, \ldots, N_r, \\ v^R & \text{if} \quad x > \widehat{\lambda}_2^{N_r}(v^R) \cdot t, \end{cases} \tag{3.25}$$

where

$$N_r = \begin{cases} \lfloor v^R \rfloor_1 & \text{if} \qquad \lfloor v^R \rfloor_1\, \varepsilon \neq v_1^R, \\ \lfloor v^R \rfloor_1 - 1 & \text{otherwise,} \end{cases}$$

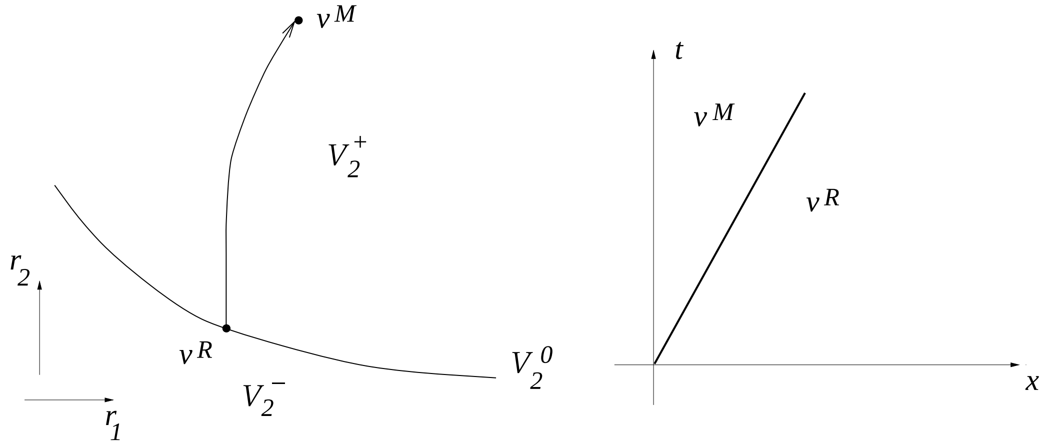

FIGURE 3.8

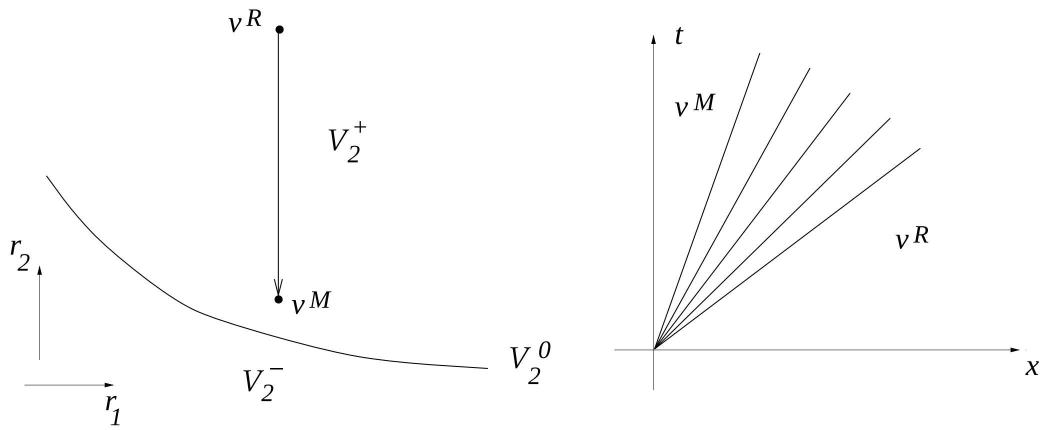

FIGURE 3.9

while $\widehat{\lambda}_2^\ell(v^L)$ are defined as in (3.4);

b) $\sigma_2 > 0$ or $\sigma_2 \le \nu_2^\varepsilon\big(v^R\big)$: a shock of the second characteristic family as in (3.24), travelling with speed $\lambda_2^{\psi,\varepsilon}\big(v^R,\, \sigma_2\big)$;

c) $\nu_2^\varepsilon\big(v^R\big) < \sigma_2 < -\delta_2\big(v^R\big)$: a composed wave of the second characteristic family, made of a shock of size

$$\widehat{\sigma}_2 \doteq \mu_2^\varepsilon\big(\pi_2(v^R),\ \sigma_2 + \delta_2(v^R)\big),$$

travelling with speed

$$\lambda_2^{\psi,\varepsilon}\big(\widetilde{v}^2,\, \widehat{\sigma}_2\big) \ge \widehat{\lambda}_2^{\lfloor \widetilde{v}^2 \rfloor_2 - 1}(v^R), \qquad \widetilde{v}^2 \doteq R_2\big(v^R,\ \sigma_2 - \widehat{\sigma}_2\big), \tag{3.26}$$

followed by a piecewise constant rarefaction fan of size $\widetilde{\sigma}_2 \doteq \sigma_2 - \widehat{\sigma}_2$. Then, the solution is

$$\omega(t,x) = \begin{cases} v^M & \text{if} \quad 0 \le x < \lambda_2^{\psi,\varepsilon}\big(\widetilde{v}^2,\, \widehat{\sigma}_2\big) \cdot t, \\ \widetilde{v}^2 & \text{if} \quad \lambda_2^{\psi,\varepsilon}\big(\widetilde{v}^2,\, \widehat{\sigma}_2\big) \cdot t < x < \widehat{\lambda}_2^{\lfloor \widetilde{v}^2 \rfloor_2}(v^R) \cdot t, \\ \big(v_1^R, \ell\varepsilon\big) & \text{if} \quad \widehat{\lambda}_2^{\ell-1}(v^R) \cdot t < x < \widehat{\lambda}_2^{\ell}(v^R) \cdot t, \qquad \ell = \lfloor \widetilde{v}^2 \rfloor_2 + 1, \dots, \lfloor v^R \rfloor_2, \\ v^R & \text{if} \quad x > \widehat{\lambda}_2^{\lfloor v^R \rfloor_2}(v^R) \cdot t, \end{cases} \tag{3.27}$$

with $\widehat{\lambda}_2^{\ell}(v^R)$ defined as in (3.4) (see fig. 3.10);

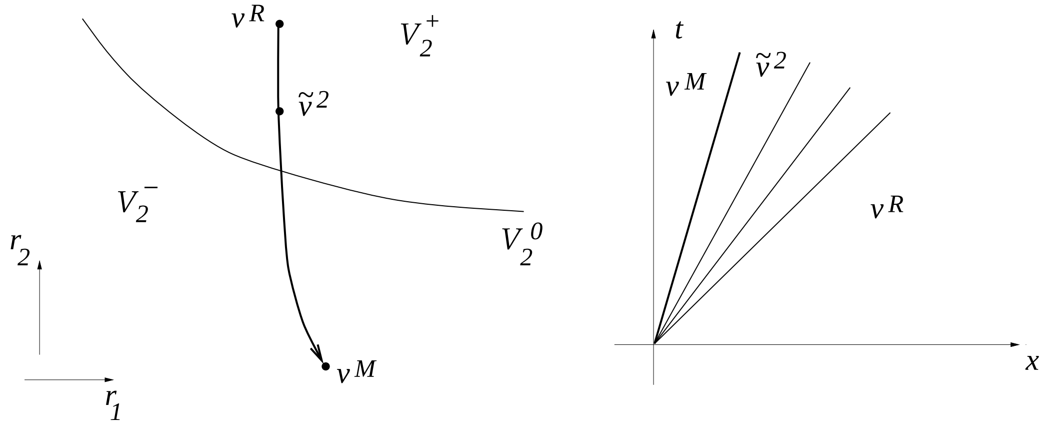

FIGURE 3.10

3) $v^R \in \mathcal{V}_2^-$: the solution consists of

a) $0 \le \sigma_2 \le -\delta_2\big(v^R\big)$: a piecewise constant rarefaction fan as in 2-*a)*;

b) $\sigma_2 < 0$ or $\sigma_2 \ge \nu_2\big(v^R\big)$: a shock as in 2-*b)*;

c) $-\delta_2\big(v^R\big) \le \sigma_2 < \nu_2^{\xi}\big(v^R\big)$: a composed wave as in 2-*c)*, made of a shock followed by a piecewise constant rarefaction fan.

At this stage, for every fixed $\varepsilon > 0$, we have defined an algorithm for approximately solving the Riemann problem within the class of piecewise constant functions. Consider now a piecewise constant initial data $\overline{v}$ with bounded support and small total variation. An ε-approximate solution of the corresponding Cauchy problem can then be constructed (in the v-coordinates) by applying the above Riemann Solver at each point where two or more wave-fronts interact. More precisely, we first solve the Riemann problems at time $t = 0$ determined by the jumps in $\overline{v}$ using the ε-approximate Riemann Solver. The ε-approximate solution $v(t,x)$ is then prolonged until a time t_1 is reached, when the first set of interactions take place. Since $v(t_1, \cdot)$ is still a piecewise constant function, the corresponding Rieman problems can be solved applying again the above algorithm.

This determines the solution v up to the time t_2 where the second set of interactions take place, etc...

Our goal is to prove that the ε-approximate solution generated by this algorithm are well defined for all $t \geq 0$. The core of the argument consists in showing that, as time increases,

1. the total variation of $v(t,\cdot)$ remains uniformly bounded;
2. the number of wave-fronts in v remains finite.

Regarding the first point, to derive a-priori estimates on the total variation, we shall proceed using the standard technique introduced by Glimm [G].

Let $v : \mathbb{R} \to \mathbb{R}^2$ be a piecewise constant function with bounded support. Call $x_1 < \cdots < x_n$ the points where v suffers a jump. Assume that at each point x_α the corresponding Riemann problem with data $\omega_\alpha^L = v(x_\alpha-)$, $\omega_\alpha^R = v(x_\alpha+)$ is solved by our algorithm with a single ε-admissible wave-front. Call σ_i^α the size of this wave, where the index $i \in \{1,2\}$ refers to its family, i.e.

$$\begin{aligned} \Psi_1^\varepsilon\Big(\omega_\alpha^L,\, \sigma_1^\alpha\Big) &= \omega_\alpha^R, \qquad \text{if} \qquad i = 1,\\ \Psi_2^\varepsilon\Big(\omega_\alpha^R,\, \sigma_2^\alpha\Big) &= \omega_\alpha^L, \qquad \text{if} \qquad i = 2. \end{aligned} \tag{3.28}$$

Then, we define the *total strength* of waves in v as

$$V^\varepsilon(v) \doteq \sum_{\alpha=1}^{N} \big|\sigma_i^\alpha\big|, \tag{3.29}$$

while the *interaction potential* is

$$Q^\varepsilon(v) \doteq K_1\Bigg[\sum_{x_\alpha < x_\beta} \big|\sigma_2^\alpha \cdot \sigma_1^\beta\big| + \sum_{i=1,2}\ \sum_{\sigma_i^\alpha\cdot\sigma_i^\beta<0} \big|\sigma_i^\alpha \cdot \sigma_i^\beta\big|\Bigg] + \sum_{i=1,2}\ \sum_{\sigma_i^\alpha\cdot\sigma_i^\beta>0} \big|\sigma_i^\alpha \cdot \sigma_i^\beta\big|, \tag{3.30}$$

where $K_1 > 0$ denotes a suitable constant whose precise value will be specified later.

Proposition 3.1. *There exist positive constants K_1, δ_1, independent of ε, for which, in the above setting, the following holds. Let $\overline{v}$ be a piecewise constant initial data with bounded support such that*

$$V^\varepsilon\big(\overline{v}\big) + Q^\varepsilon\big(\overline{v}\big) < \delta, \tag*{$(3.31)_\delta$}$$

with $\delta \leq \delta_1$, and let $v = v(t,\,x)$ be the corresponding ε-approximate solution constructed on some interval $[0,T[$. Then, by construction $v(t,\cdot)$ is a piecewise constant function, and

$$V^\varepsilon\big(v(t,\,\cdot)\big) + Q^\varepsilon\big(v(t,\,\cdot)\big) \leq (1+4\,\varepsilon)\Big[V^\varepsilon\big(\overline{v}\big) + Q^\varepsilon\big(\overline{v}\big)\Big], \qquad \forall\, 0 \leq t < T. \tag{3.32}$$

Now, fix $\delta_1 > 0$ according with Proposition 3.1 and, for any $\delta \leq \delta_1$, define the domain

$$\mathcal{D}^{\varepsilon,\delta} \doteq \Big\{v \in \mathbb{L}^1\big(\mathbb{R};\,\mathbb{R}^2\big)\ :\ v \text{ is piecewise constant},\, V^\varepsilon\big(\overline{v}\big) + Q^\varepsilon\big(\overline{v}\big) < \delta\Big\}\,. \tag{3.33}$$

Then, taking $\varepsilon < 1/4$, if v is an ε-approximate solution with initial data $v(0,\cdot) \in \mathcal{D}^{\varepsilon,\delta}$, $\delta \le \delta_1$, and if v can be constructed on some initial interval $[0,T[$, from (3.32) we derive the uniform bound

$$V^{\varepsilon}\big(v(t,\,\cdot)\big) + Q^{\varepsilon}\big(v(t,\,\cdot)\big) \le 2\,\delta, \qquad \forall\ 0 < t < T. \tag{3.34}$$

In particular, this means that the total variation of $v(t,\cdot)$ remains uniformly bounded for all $t \in [0,T]$. Then, thanks to the Property **P4**, every time an interaction occurs the corresponding Riemann Problem can be uniquely solved applying our algorithm. Thus, in order to prove that v can actually be prolonged beyond time $t = T$, we only need to show that the total number of wave-fronts and of interaction points remains finite. We first use the estimates on the decrease of the interaction potential to derive an a-priori bound on the total number of wave-fronts with strength $|\sigma| \ge 2\sqrt{\varepsilon}$.

Proposition 3.2. *There exists $\delta_2 \in]0,\,\delta_1]$, independent of ε, for which the following holds. Let $\overline{v} \in \mathcal{D}^{\varepsilon,\delta_2}$ and let $v = v(t,\,x)$ be the corresponding ε-approximate solution constructed on some interval $[0,T[$. Then, all of the shocks of v with strength $|\sigma| > 2\sqrt{\varepsilon}$ are located along a finite number of polygonal lines.*

From Proposition 3.2 and by property **P2** it follows that, if $v(0,\cdot) \in \mathcal{D}^{\varepsilon,\delta_2}$ and we construct the corresponding ε-solution v on some interval $[0,T[$, then outside a finite number of polygonal lines v behaves as the solution of a system with coinciding shock and rarefaction curves. This, in particular, implies that, whenever two small waves interact,

1) the outgoing waves have the same size of the incoming ones;
2) the possible generation of new wave-fronts can occur only when are involved in the interaction wave-fronts of different family or wave-fronts of the same family having sizes of opposite sign. Thanks to these two key properties we can establish

Proposition 3.3. *Let v be as in Proposition 3.2. Then the set of all points where two or more wave-fronts interact has no limit point in the $(t,\,x)$-plane.*

Then, using Propositions 3.1-3.3, we derive

Proposition 3.4. *There exists $\delta_3 \in]0,\,\delta_2]$, such that, for any $0 < \varepsilon < 1/4$ and every initial data $v(0,\cdot) = \overline{v} \in \mathcal{D}^{\varepsilon,\delta_3}$, the corresponding ε-approximate solution $v = v(t,\,x)$ is well defined for all $t > 0$ and contains a finite number of wave-fronts and interaction points in the $(t,\,x)$-plane. Moreover*

$$V^{\varepsilon}\big(v(t,\cdot)\big) + Q^{\varepsilon}\big(v(t,\cdot)\big) < 2\,\delta_3, \qquad \forall\ t \ge 0. \tag{3.35}$$

To denote the unique, globally defined, ε-approximate solution $v(t,\cdot)$ with initial data $\overline{v} \in \mathcal{D}^{\varepsilon,\delta_2}$, we shall often use the semigroup notation

$$S^{\varepsilon}_t\overline{v} \doteq v(t,\cdot). \tag{3.36}$$

The next part of the proof of Theorem 1 consists in showing that the map

$$S^{\varepsilon} : [0,\infty[\,\times\, \mathcal{D}^{\varepsilon,\delta_3} \mapsto \mathcal{D}^{\varepsilon,\,2\delta_3}, \qquad (t,\,\overline{v}) \to S_t^{\varepsilon}\overline{v}, \tag{3.37}$$

is globally Lipschitz in the $\mathbb{L}^1$ norm, with a Lipschitz constant independent of ε. To estimate the distance between two approximate solutions v, w, the basic idea is, as in [BC1], to construct a one-parameter family of solutions v^{θ} connecting v, w, and then study how the length of the path $\gamma_t : \theta \to v^{\theta}(t,\cdot)$ varies in time. More precisely, given two piecewise constant initial data $\overline{v}$, $\overline{w} \in \mathbb{L}^1$, with small total variation, we shall consider continuous paths $\gamma : \theta \to \overline{v}^{\theta}$ that connect $\overline{v}$ with $\overline{w}$ by merely shifting the position of the jumps. Then, by studying the rates at which the jumps in the corresponding solution $\gamma_t(\theta) \doteq S_t^{\varepsilon}\overline{v}^{\theta}$ are shifted, for any fixed time $t > 0$, we will derive an a-priori estimate on the norm of a generalized tangent vector. In turn, this provides a bound on the length of γ_t and hence on the distance between $v(t,\cdot) \doteq S_t^{\varepsilon}\overline{v}$ and $w(t,\cdot) \doteq S_t^{\varepsilon}\overline{w}$.

In particular, our paths will be *pseudopolygonals*, i.e. finite concatenation of *elementary paths* of the form

$$\gamma(\theta) = \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{]x^{\theta}_{\alpha-1},\, x^{\theta}_{\alpha}[}, \qquad x^{\theta}_{\alpha} = \overline{x}_\alpha + \xi_\alpha\theta, \qquad \theta \in]a,\,b[, \tag{3.38}$$

with $x^{\theta}_{\alpha-1} < x^{\theta}_{\alpha}$ for all $\theta \in]a,b[$ and $\alpha = 1,\dots,N$. Here $\chi_{_I}$ is the characteristic function of the interval I, $\omega_1,\dots,\omega_N \in \mathbb{R}$ are constant states and ξ_α is the shift rates of the jump at x_α. Using entirely similar arguments to those developed in [BC1], one can show that a basic property of our algorithm is that it preserves pseudopolygonals. In fact, if we consider a pseudopolygonal $\gamma_0 : \theta \mapsto \overline{v}^{\theta}$ with $\overline{v}^{\theta} \in \mathcal{D}^{\varepsilon,\delta_3}$ for all $\theta \in [0,\,1]$, then, for all $t > 0$, the path

$$\gamma_t : \theta \mapsto v^{\theta}(t,\cdot) \doteq S_t^{\varepsilon}\overline{v}^{\theta} \tag{3.39}$$

is also a pseudopolygonal. Moreover, there are finitely many parameters values $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$ such that the wave-front configuration of the solution v^{θ} on $[0,t]$ remains the same as θ ranges on each of the open interval $J_h =]\theta_{h-1},\,\theta_h[$. Since, by definition, the $\mathbb{L}^1$-length of a continuous path $\gamma : [a,b] \to \mathbb{L}^1(\mathbb{R})$ is

$$\|\gamma\|_{\mathbb{L}^1} \doteq \sup\Big\{ \sum_{i=1}^{N} \big\|\gamma(\theta_i) - \gamma(\theta_{i-1})\big\|_{\mathbb{L}^1} \;:\; N \geq 1,\; a = \theta_0 < \theta_1 < \cdots < \theta_N = b \Big\}, \tag{3.40}$$

the length of the pseudopolygonal γ_t at (3.39) will be just the sum of the lengths of the elementary paths obtained by restricting γ_t to each subinterval J_h. Then, to estimate the length of γ_τ for any fixed $\tau > 0$, it will be sufficient to consider the case where $\gamma_0 :]a,\,b[\to \mathbb{L}^1$ is an elementary path and the wave-front configurations of the corresponding solutions $v^{\theta}(t,x) \doteq S_t^{\varepsilon}\gamma_0(\theta)$, on the strip $[0,\tau] \times \mathbb{R}$, are the same for all $\theta \in]a,\,b[$.

If $v^{\theta}(t,\cdot)$ is the piecewise constant function

$$v^{\theta}(t,\,x) = \sum_{\alpha=1}^{N(t)} \omega_\alpha \cdot \chi_{]x^{\theta}_{\alpha-1}(t),\, x^{\theta}_{\alpha}(t)[}(x), \qquad x^{\theta}_{\alpha}(t) = x_\alpha(t) + \xi_\alpha\theta, \tag{3.41}$$

and we let $\Delta v^\theta(x_\alpha) \doteq \omega_{\alpha+1} - \omega_\alpha$ denote the jump of $v^\theta(t,\cdot)$ at x_α, the length of $\gamma_t : \theta \mapsto v^\theta(t,\cdot)$ is measured by

$$\|\gamma_t\|_{\mathbb{L}^1} = \int_a^b \sum_\alpha \left|\Delta v^\theta(x_\alpha)\right| \left|\frac{dx_\alpha^\theta}{d\theta}\right| d\theta. \tag{3.42}$$

Thus, in order to relate the length of γ_τ with the length of the path γ_0 of initial conditions, we will study how the integrand in (3.42)

$$\sum_\alpha \left|\Delta v^\theta(x_\alpha)\right| \left|\frac{dx_\alpha^\theta}{d\theta}\right| = \sum_\alpha \left|\omega_{\alpha+1} - \omega_\alpha\right| \left|\xi_\alpha\right|, \tag{3.43}$$

varies in time, for any fixed $\theta \in]a,b[$. Clearly, this sum can change only at times $\bar{t}$ where some interaction takes place between wave-fronts of v^θ. In general, an interaction occuring at time $\bar{t}$ may well produce an increase of the sum at (3.43). However, by carefully studying the relation between $\|\gamma_{\bar{t}+}\|_{\mathbb{L}^1}$ and $\|\gamma_{\bar{t}-}\|_{\mathbb{L}^1}$, one can show that, as in the case of systems satisfying the standard hypothesis of genuine non-linearity or linearly degeneracy, also for systems satisfying our basic assumption **(A)** it is possible to control such increase of $\|\gamma_t\|_{\mathbb{L}^1}$ in terms of an interaction potential. Hence, as in [BC1], we shall define a weighted length $\|\cdot\|_\varepsilon$ equivalent to the standard $\mathbb{L}^1$-length and then prove that, for every elementary path $\gamma_t : \theta \mapsto v^\theta(t,\cdot)$ of ε-approximate solutions, there holds $\|\gamma_t\|_\varepsilon \le L\|\gamma_0\|_\varepsilon$, for some constant L independent of ε.

Let γ be the elementary path at (3.38) and assume that the Riemann problen with left data ω_α and right data $\omega_{\alpha+1}$ is solved by our algorithm with a single ε-admissible wave-front σ_i^α of the i-th family, $i \in \{1,2\}$, i.e. that (3.38) holds with ω_α^L replaced by ω_α and ω_α^R replaced by $\omega_{\alpha+1}$. We define the *weighted length* of γ by setting

$$\|\gamma\|_\varepsilon \doteq \sum_\alpha (b-a)\left|\sigma_i^\alpha\, \xi_\alpha\right| \cdot W_i^\alpha, \tag{3.44}$$

$$W_i^\alpha \doteq \left[1 + K_2\left[\sum_{\substack{j\neq i\\ (\overline{x}_\alpha - \overline{x}_\beta)\cdot(i-j)<0}} \left|\sigma_j^\beta\right| + \sum_{\sigma_i^\beta\cdot\sigma_i^\alpha<0} \left|\sigma_i^\beta\right|\right] + K_3 \sum_{\sigma_i^\beta\cdot\sigma_i^\alpha>0} \left|\sigma_i^\beta\right|\right] e^{K_4 Q^\varepsilon(\gamma)}, \tag{3.45}$$

where K_2, K_3, K_4, $(K_2 > K_3)$ are positive constants to be determined later and $Q^\varepsilon(\gamma) \doteq Q^\varepsilon\big(\gamma(\theta)\big)$ for any $\theta \in]a,b[$. Whenever γ is a pseudopolygonal, its *weighted length* $\|\gamma\|_\varepsilon$ is defined as the sum of the weighted lengths of its elementary paths. Notice that, because of (3.10), there exists some constant $C_0 > 0$ independent of ε, such that

$$C_0^{-1}\|\gamma\|_{\mathbb{L}^1} \le \|\gamma\|_\varepsilon \le C_0\|\gamma\|_{\mathbb{L}^1}. \tag{3.46}$$

Proposition 3.5. *In the above setting, there exist positive constants K_2, K_3, K_4, L_1, and $\delta_4 \le \delta_3$, independent of ε, such that the following holds. Let $\gamma_0 : \theta \mapsto \bar{v}^\theta \in \mathcal{D}^{\varepsilon,\delta_4}$, $\theta \in]a,b[$ be an elementary path as well as $\gamma_\tau : \theta \mapsto v^\theta(\tau,\cdot) \doteq S_\tau^\varepsilon \bar{v}^\theta$, for some $\tau > 0$. Assume that on the strip $[0,\tau] \times \mathbb{R}$ the wave-front configuration of the ε-solution v^θ remains constant for all θ. Then there holds*

$$\|\gamma_t\|_\varepsilon \le L_1\|\gamma_0\|_\varepsilon, \qquad \forall\ 0 \le t \le \tau. \tag{3.47}$$

Proposition 3.6. *There exist constants* $\delta^* \in]0,\, \delta_4]$, $L_1 > 0$, *independent on* ε, *such that, setting*

$$\mathcal{D}^\varepsilon \doteq \Big\{ v \in \mathbb{L}^1\left(\mathbb{R};\, \mathbb{R}^2\right) \ :\ v \textit{ is piecewise constant, } V^\varepsilon\left(\overline{v}\right) + Q^\varepsilon\left(\overline{v}\right) < \delta^* \Big\},$$

the following holds

(i)
$$S^\varepsilon \overline{v} \in \mathcal{D}^{\varepsilon, 2\delta^*}, \qquad \forall\, \overline{v} \in \mathcal{D}^\varepsilon, \tag{3.48}$$

(ii)
$$\left\| S^\varepsilon_t \overline{v} - S^\varepsilon_t \overline{w} \right\|_{\mathbb{L}^1} \leq L_2 \|\overline{v} - \overline{w}\|_{\mathbb{L}^1}, \qquad \forall\, t \geq 0, \quad \overline{v}, \overline{w} \in \mathcal{D}^\varepsilon. \tag{3.49}$$

Relying on Proposition 3.6, we can now construct a Standard Riemann Semigroup generated by (1.1) as a suitable limit of the flows S^ε. Consider the sequence of maps $S^{\varepsilon_n} : \mathcal{D}^{\varepsilon_n} \mapsto \mathcal{D}^{\varepsilon_n, 2\delta^*}$, with $\varepsilon_n = 2^{-n}$, and define the map S on the domain

$$\begin{aligned} \mathcal{D} \doteq \Big\{ \overline{v} \in \mathbb{L}^1\left(\mathbb{R};\, \mathbb{R}^2\right) \ :\ \exists\ \left\{\overline{v}_n\right\}_{n \in \mathbb{N}}, \quad & \overline{v}_n \in \mathcal{D}^{\varepsilon_n}\ \forall\, n, \\ & \lim_n \overline{v}_n = \overline{v}, \quad \limsup_n \Big[V^{\varepsilon_n}\left(\overline{v}_n\right) + Q^{\varepsilon_n}\left(\overline{v}_n\right) \Big] \leq \delta^*/2 \Big\}, \end{aligned} \tag{3.50}$$

by setting

$$S_t \overline{v} \doteq \lim_{n \to +\infty} S^{\varepsilon_n}_t \overline{v}_n, \tag{3.51}$$

where $\left\{\overline{v}_n\right\}_{n \in \mathbb{N}}$ is any sequence with $\overline{v}_n \in \mathcal{D}^{\varepsilon_n}$ converging to $\overline{v}$ in $\mathbb{L}^1$, and such that

$$\limsup_n \left[V^{\varepsilon_n}\left(\overline{v}_n\right) + Q^{\varepsilon_n}\left(\overline{v}_n\right) \right] \leq \delta^*/2.$$

Notice that, thanks to (3.32), the domain $\mathcal{D}$ at (3.51) is positively invariant with respect to S. Then, we conclude by proving

Proposition 3.7. *The semigroup* S *in (3.51) is well defined and satisfies the conditions (i)-(v) stated in Theorem 1.*

4. The Algorithm

In this section, for any fixed $\varepsilon > 0$, we first construct two suitable approximations $F^{1,\varepsilon}$, $F^{2,\varepsilon}$, of the flux function F, which, in turn, determine a corresponding definition of *ε-approximate shock curve* S^ε, for each characteristic family. Next, we modify the ε-approximate shock curve S^ε by defining a curve $S^{\psi,\varepsilon}$ that coincides with the rarefaction curve for small amplitudes, and with S^ε for large ones. A suitable interpolation between the two curves is used for intermediate values. Finally, we give the definition of the *ε-approximate elementary curves* Ψ_1^ε, Ψ_2^ε, that are used to construct the ε-approximate solution of a Riemann problem.

Throughout the following we shall often adopt the notation

$$\begin{aligned} &\lfloor v \rfloor_1 \doteq \left\lfloor \frac{v_1}{\varepsilon} \right\rfloor, && \lfloor v \rfloor_2 \doteq \left\lfloor \frac{v_2}{\varepsilon} \right\rfloor, \\ &\lfloor\lfloor v \rfloor\rfloor_1 \doteq \big(\lfloor v \rfloor_1\, \varepsilon,\, v_2\big), && \lfloor\lfloor v \rfloor\rfloor_2 \doteq \big(v_1,\, \lfloor v \rfloor_2\, \varepsilon\big), \end{aligned} \qquad v = (v_1, v_2) \in \mathcal{V}. \tag{4.1}$$

where $\lfloor a \rfloor$ represents the integer part of $a \in \mathbb{R}$. As usual, the Landau sympol $\mathcal{O}(1)$ always denote a quantity whose absolute value satisfies a uniform bound, depending only on the system (1.1) and not on ε or on the functions of the variable v appearing in the sequel. We also recall the notation

$$\begin{aligned} &\delta_1(v) \doteq v_1 - \widetilde{\gamma}_1(v_2), && \delta_2(v) \doteq v_2 - \widetilde{\gamma}_2(v_1), \\ &\pi_1(v) \doteq \big(\widetilde{\gamma}_1(v_2),\, v_2\big), && \pi_2(v) \doteq \big(v_1,\, \widetilde{\gamma}_2(v_1)\big). \end{aligned}$$

introduced in Section 2.

4.1 - Approximate flux functions.

Here we introduce two ε-approximations of the flux function $u \mapsto F(u)$ defined as follows. Given $u \doteq u(v)$, $v = (v_1, v_2) \in \mathcal{V}$, assume that

$$h_i \varepsilon \leq v_i < \big(h_i + 1\big)\varepsilon, \qquad i = 1, 2, \tag{4.2}$$

for some integers h_i. Then define:

1.

$$F^{1,\varepsilon}\big(u(v)\big) \doteq F\big(u(\pi_1(v))\big) + \operatorname{sgn}\big(\delta_1(v)\big) \left[\sum_{\ell=\min\{0,h_1\}}^{\max\{0,h_1\}} \widehat{\lambda}_1^\ell(v) \Big[u(w_1^{\ell+1}) - u(w_1^\ell) \Big] \right], \tag{4.3}$$

where

$$\begin{aligned} w_1^{\min\{0,h_1\}} &\doteq \begin{cases} v & \text{if } \; h_1\, \varepsilon \leq \widetilde{\gamma}_1(v_2), \\ \pi_1(v) & \text{otherwise}, \end{cases} \\ w_1^\ell &\doteq \big(\ell\, \varepsilon,\, v_2\big), \qquad \text{if } \; \ell \neq \min\{0, h_1\},\ \max\{0, h_1\} + 1, \\ w_1^{\max\{0,h_1\}+1} &\doteq \begin{cases} \pi_1(v) & \text{if } \; h_1\, \varepsilon \leq \widetilde{\gamma}_1(v_2), \\ v & \text{otherwise}, \end{cases} \end{aligned} \tag{4.4}$$

and $\widehat{\lambda}_1^\ell(v)$ are defined as in (3.3).

2.

$$F^{2,\varepsilon}\big(u(v)\big) \doteq F\big(u(\pi_2(v))\big) + \mathrm{sgn}\,\big(\delta_2(v)\big)\left[\sum_{\ell=\min\{0,h_2\}}^{\max\{0,h_2\}} \widehat{\lambda}_2^{\ell}(v)\Big[u(w_2^{\ell+1}) - u(w_2^{\ell})\Big]\right] \tag{4.5}$$

where

$$w_2^{\min\{0,h_2\}} \doteq \begin{cases} v & \text{if} \quad h_2\,\varepsilon \le \widetilde{\gamma}_2(v_1), \\ \pi_2(v) & \text{otherwise,} \end{cases}$$

$$w_2^{\ell} \doteq \big(v_1,\, \ell\,\varepsilon\big), \qquad \text{if} \quad \ell \ne \min\{0,h_2\},\ \max\{0,h_2\}+1, \tag{4.6}$$

$$w_2^{\max\{0,h_2\}+1} \doteq \begin{cases} \pi_2(v) & \text{if} \quad h_2\,\varepsilon \le \widetilde{\gamma}_2(v_1), \\ v & \text{otherwise,} \end{cases}$$

and $\widehat{\lambda}_2^{\ell}(v)$ are defined as in (3.4).

Notice that the maps $u \mapsto F^{l,\varepsilon}(u)$, $l = 1, 2$, defined by (4.2)-(4.6), are continuous on Ω, continuously differentiable on

$$\Omega^{\varepsilon} \doteq \{\, u = u(v_1, v_2) \in \Omega \;:\; v_i \notin \mathbb{Z}\varepsilon, \quad i = 1, 2\,\}, \tag{4.7}$$

and, at any point $u \in \Omega \setminus \Omega^{\varepsilon}$, they admit left and right directional derivatives along the characteristic fields $r_i(u)$. The derivatives of $u \mapsto F^{l,\varepsilon}(u)$ along the fields r_i are written

$$DF^{l,\varepsilon}(u) \cdot r_i(u) \doteq \lim_{s\to 0} \frac{F^{l,\varepsilon}\left(u + s\; r_i(u)\right) - F^{l,\varepsilon}(u)}{s}. \tag{4.8}$$

The right and left directional derivatives of $F^{l,\varepsilon}$ along r_i are denoted $DF^{l,\varepsilon+}(u) \cdot r_i(u)$, and $DF^{l,\varepsilon-}(u) \cdot r_i(u)$, respectively. We call $\lambda_i^{l,\varepsilon}(u)$, $\lambda_i^{l,\varepsilon+}(u)$, and $\lambda_i^{l,\varepsilon-}(u)$, the eigenvalues of $A^{l,\varepsilon}(u) \doteq DF^{l,\varepsilon}(u)$, $A^{l,\varepsilon+}(u) \doteq DF^{l,\varepsilon+}(u)$, and $A^{l,\varepsilon-}(u) \doteq DF^{l,\varepsilon-}(u)$. Similar notations are used for the corresponding left and right eigenvectors. The right eigenvectors $r_i^{l,\varepsilon}$, $r_i^{l,\varepsilon\pm}$ are chosen so that

$$\langle l_i(u),\, r_i^{l,\varepsilon}(u)\rangle = \langle l_i(u),\, r_i^{l,\varepsilon\pm}(u)\rangle = 1, \qquad i, l = 1, 2, \tag{4.9}$$

and the left eigenvector $l_i^{l,\varepsilon}$, $l_i^{l,\varepsilon\pm}$ are normalized as in (2.3) by

$$\langle l_i^{l,\varepsilon}(u),\, r_j^{l,\varepsilon}(u)\rangle = \langle l_i^{l,\varepsilon\pm}(u),\, r_j^{l,\varepsilon\pm}(u)\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \ne j \end{cases} \qquad i, j, l = 1, 2. \tag{4.10}$$

Remark 4.1. A straight computation shows that, at any point $u = u(v) \in \Omega^{\varepsilon}$, the derivatives of $F^{l,\varepsilon}$ take the form

$$\begin{aligned} DF^{1,\varepsilon}(u) \cdot r_1(u) &= \widehat{\lambda}_1^{\lfloor v \rfloor_1}(v)\; r_1(u), \\ DF^{2,\varepsilon}(u) \cdot r_2(u) &= \widehat{\lambda}_2^{\lfloor v \rfloor_2}(v)\; r_2(u). \end{aligned} \tag{4.11}$$

Similar formulas hold for the left and right directional derivatives of $F^{l,\varepsilon}$ along the characteristic fields r_i at any point $u \in \Omega \setminus \Omega^{\varepsilon}$. In particular, at any point $u = u(v_1, v_2)$ with $v_i \in \mathbb{Z}\,\varepsilon$, $i = 1, 2$, one has

$$\begin{aligned} DF^{i,\varepsilon+}(u) \cdot r_i(u) &= \widehat{\lambda}_i^{\lfloor v \rfloor_i}(v)\; r_i(u), \\ DF^{i,\varepsilon-}(u) \cdot r_i(u) &= \widehat{\lambda}_i^{\lfloor v \rfloor_i - 1}(v)\; r_i(u). \end{aligned} \tag{4.12}$$

Condition (4.9) and the expression (4.11) of $DF^{l,\varepsilon}$ imply that, at any point $u = u(v) \in \Omega$, one has

$$\begin{aligned} \lambda_i^{i,\varepsilon+}(u) &= \widehat{\lambda}_i^{\lfloor v \rfloor_i}(v), \\ \lambda_i^{i,\varepsilon-}(u) &= \begin{cases} \widehat{\lambda}_i^{\lfloor v \rfloor_i}(v) & \text{if} \quad v_i \notin \mathbb{Z}\,\varepsilon, \\ \widehat{\lambda}_i^{\lfloor v \rfloor_i - 1}(v) & \text{otherwise}, \end{cases} \\ r_i^{i,\varepsilon}(u) &= r_i^{i,\varepsilon\pm}(u) = r_i(u). \end{aligned} \tag{4.13}$$

The derivatives of $F^{l,\varepsilon}$, $l = 1, 2$, converge uniformly to the corresponding derivatives of F. Namely, one has

Lemma 4.1. *For* $i, l = 1, 2$, $\varepsilon > 0$, *and any* $u = u(v) \in \Omega^\varepsilon$, *there holds*

$$\left| DF^{l,\varepsilon}(u) \cdot r_i(u) - DF(u) \cdot r_i(u) \right| = \mathcal{O}(1) \left| \delta_l(v) \right| \varepsilon. \tag{4.14}$$

A proof of Lemma 4.1 can be found in [AM2]. Uniform estimates as (4.14) hold for the right and left directional derivatives of $u \mapsto F^{l,\varepsilon}(u)$ along the characteristic fields $r_i(u)$, at any point $u \in \Omega \setminus \Omega^\varepsilon$.

Remark 4.2. From Lemma 4.1 and by Remark 4.1 it follows that, for $l, i = 1, 2$, $u = u(v) \in \Omega$, and $\varepsilon > 0$, there holds

$$\begin{aligned} \left| \lambda_i^{l,\varepsilon\pm}(u) - \lambda_i(u) \right| &= \mathcal{O}(1) \left| \delta_l(v) \right| \varepsilon, \\ \left| r_i^{l,\varepsilon\pm}(u) - r_i(u) \right| &= \mathcal{O}(1) \left| \delta_l(v) \right| \varepsilon, \\ \left| l_i^{l,\varepsilon\pm}(u) - l_i(u) \right| &= \mathcal{O}(1) \left| \delta_l(v) \right| \varepsilon, \end{aligned} \tag{4.15}$$

and

$$\left| \lambda_i^{i,\varepsilon\pm}\big(u\big(R_j(v,\sigma)\big)\big) - \lambda_i^{i,\varepsilon\pm}(u) \right| = \mathcal{O}(1) |\sigma|, \qquad j \neq i. \tag{4.16}$$

4.2 - Approximate shock-curves.

Given two points $v, v' \in \mathcal{V}$, and setting $u \doteq u(v)$, $u' \doteq u(v')$, for any $i, l = 1, 2$, we call $\lambda_i^{l,\varepsilon}(v, v')$ the eigenvalue of the averaged matrix

$$A^{l,\varepsilon}(u, u') \doteq \int_0^1 DF^{l,\varepsilon}\left(u' + \xi(u' - u)\right)\, d\xi \tag{4.17}$$

ordered so that

$$\lambda_1^{l,\varepsilon}(v, v') < 0 < \lambda_2^{l,\varepsilon}(v, v') \tag{4.18}$$

and $l_i^{l,\varepsilon}(v, v')$, $r_i^{l,\varepsilon}(v, v')$, respectively, the corresponding left and right eigenvectors of $A^{l,\varepsilon}$ normalized by

$$\langle l_i(v, v'), r_i^{l,\varepsilon}(v, v') \rangle = 1 \qquad\qquad \langle l_i^{l,\varepsilon}(v, v'), r_j^{l,\varepsilon}(v, v') \rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}. \tag{4.19}$$

Definition 4.1. Let u^L, u^R be two points in Ω. We say that (u^L, u^R) satisfies the ε-*Rankine-Hugoniot condition* (ε-RH) of the i-th family if, for some scalar λ, there holds

$$F^{i,\varepsilon}(u^R) - F^{i,\varepsilon}(u^L) = \lambda\ (u^R - u^L) \qquad\qquad (-1)^i\,\lambda > 0. \tag{4.20$_i$}$$

Notice that the solutions of $(4.20)_i$ are precisely the non-trivial solutions of the scalar equation

$$\langle l_j^{i,\varepsilon}(u^L, u^R),\, u^R - u^L\rangle = 0, \qquad j \neq i, \tag{4.21$_i$}$$

for $i = 1, 2$. Moreover, the map $u^R \mapsto l_j^{i,\varepsilon}(u^L, u^R)$ has the same regularity of the map $u^R \mapsto F^{i,\varepsilon}(u^L, u^R)$, i.e. it's continuous on Ω, continuously differentiable on Ω^ε and, at any point in $\Omega \setminus \Omega^\varepsilon$, admits left and right directional derivatives along the vector fields $r_j(u^R)$. Therefore, by applying the implicit function theorem, for any $i = 1, 2$, one can show the existence of a continuous, piecewise smooth curve of solutions of $(4.21)_i$ that will be denoted as the ε-*approximate shock curves* through u^L of the i-th characteristic family. Such curves can be parametrized by

$$S_1^\varepsilon(v,\sigma) = \big(v_1 + \sigma,\, v_2 + \widetilde{S}_1^\varepsilon(v,\sigma)\big), \qquad S_2^\varepsilon(v,\sigma) = \big(v_1 + \widetilde{S}_2^\varepsilon(v,\sigma),\, v_2 + \sigma\big), \qquad \sigma \in \mathcal{J}, \tag{4.22}$$

for some piecewise smooth maps $\widetilde{S}_1^\varepsilon$, $\widetilde{S}_2^\varepsilon$. Call $\lambda_i^{s,\varepsilon}(v,s) \doteq \lambda_i^{i,\varepsilon}(v,\, S_i^\varepsilon(v,s))$ the speed of the ε-approximate shock wave connecting the left state $u(v)$ to the right state $u(S_i^\varepsilon(v,s))$.

Remark 4.3. Let $v \in \mathcal{V}$ be given. Then, from Definition 4.1 and by Remark 4.1 it follows that

$$\begin{aligned}
&S_1^\varepsilon(v,\sigma) = (v_1 + \sigma, v_2) \qquad && \forall \quad \lfloor v\rfloor_1 \varepsilon - v_1 \leq \sigma \leq (\lfloor v\rfloor_1 + 1)\varepsilon - v_1, \qquad && \text{if} \quad v_1 \notin \mathbb{Z}\varepsilon,\\
&S_1^\varepsilon(v,\sigma) = (v_1 + \sigma, v_2) \qquad && \forall \quad -\varepsilon \leq \sigma \leq \varepsilon, \qquad && \text{if} \quad v_1 \in \mathbb{Z}\varepsilon.
\end{aligned} \tag{4.23}$$

Moreover, if $v \in \mathcal{V}_1^+$ and

$$\exists\ \sigma' > 0 \quad : \quad \lambda_1^{s,\varepsilon}(v,\sigma') = \widehat{\lambda}_1^{\lfloor v\rfloor_1}(v), \tag{4.24}$$

then there holds

$$\left.\begin{aligned}
S_1^\varepsilon\big(R_1(v,-\sigma''),\, \sigma' + \sigma''\big) &= S_1^\varepsilon(v,\sigma')\\
\lambda_1^{s,\varepsilon}\big(R_1(v,-\sigma''),\, \sigma' + \sigma''\big) &= \lambda_1^{s,\varepsilon}(v,\sigma')
\end{aligned}\right. \qquad \forall\ \lfloor v\rfloor_1\,\varepsilon - v_1 \leq \sigma'' \leq (\lfloor v\rfloor_1 + 1)\,\varepsilon - v_1. \tag{4.25}$$

Similar statements hold for $v \in \mathcal{V}_1^-$ and for $v \in \mathcal{V}_2^+$. Notice that (4.25) is the analogous of (3.8) in **P3**.

Remark 4.4. From Lemma 4.1 it follows that, for $i = 1, 2,\ v = (v_1, v_2) \in \mathcal{V},\ \sigma \in \mathcal{J},$ and $\varepsilon > 0$, there holds

$$\begin{aligned}
\big|S_i^\varepsilon(v,\sigma) - S_i(v,\sigma)\big| &= \mathcal{O}(1)\ \max\big\{\big|\delta_i(v)\big|,\, |\sigma|\big\}\cdot\varepsilon,\\
\big|\lambda_i^{s,\varepsilon}(v,\sigma) - \lambda_i^s(v,\sigma)\big| &= \mathcal{O}(1)\ \max\big\{\big|\delta_i(v)\big|,\, |\sigma|\big\}\cdot\varepsilon.
\end{aligned} \tag{4.26}$$

Indeed, using the ε-RH conditions, one can derive sharper estimates of the convergence of the maps S_i^ε, λ_i^ε, and of their derivatives. A proof is given in [AM2].

4.3 - Interpolations.

For any given $v \in \mathcal{V}$ and $i = 1, 2$, we construct here two maps $\sigma \to S_i^{\psi,\varepsilon}(v,\sigma)$, $\sigma \to \lambda_i^{\psi,\varepsilon}(v,\sigma)$, having the basic properties **P2-P3** required by our algorithm for a "correct" definition of the approximate solution of a Riemann problem. Such maps are obtained as suitable interpolations between, respectively, the functions S_i^ε, $\lambda_i^{s,\varepsilon}$, and R_i, $\lambda_i^{r,\varepsilon}$. This is done in four steps.

Step 1. Consider a smooth function $\psi : \mathbb{R} \to [0,1]$ such that

$$\begin{cases} \psi(\sigma) = 0 & \text{if} \quad \sigma \le 0, \\ \psi(\sigma) = 1 & \text{if} \quad \sigma \ge 1, \\ \psi'(\sigma) \in [0,1] & \text{if} \quad \sigma \in [0,1], \end{cases}$$

and, given $v \in \mathcal{V}$, let $\varphi_i(v,\cdot) : \mathbb{R}^+ \to [0,1]$ be the map defined by

$$\varphi_i(v,\sigma) \doteq \psi\left(\frac{\sigma - 3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right) \qquad \sigma \ge 0. \tag{4.27}$$

Then, for $i = 1, 2$, $v \in \mathcal{V}$, and $\sigma \ge 0$, define the interpolated curves

$$S_i^{\varphi,\varepsilon}(v,\sigma) = \begin{cases} \varphi_i\big(v,\,\sigma\big)\cdot S_i^\varepsilon(v,\sigma) + \big[1 - \varphi_i\big(v,\,\sigma\big)\big]\cdot R_i(v,\sigma) & \text{if} \quad 0 \le \delta_i(v), \\[2ex] \varphi_i\big(v,\,\sigma + \delta_i(v)\big)\cdot S_i^\varepsilon(v,\sigma) + \\ \qquad + \big[1 - \varphi_i\big(v,\,\sigma + \delta_i(v)\big)\big]\cdot R_i(v,\sigma) & \text{if} \quad -\sqrt{\varepsilon} \le \delta_i(v) \le 0, \\[2ex] \varphi_i\big(v,\,\sigma - \sqrt{\varepsilon}\big)\cdot S_i^\varepsilon(v,\sigma) + \\ \qquad + \big[1 - \varphi_i\big(v,\,\sigma - \sqrt{\varepsilon}\big)\big]\cdot R_i(v,\sigma) & \text{if} \quad \delta_i(v) \le -\sqrt{\varepsilon}, \end{cases} \tag{4.28}$$

together with the interpolated speeds

$$\lambda_i^{\varphi,\varepsilon}(v,\sigma) = \begin{cases} \varphi_i\big(v,\,\sigma\big)\cdot \lambda_i^{s,\varepsilon}(v,\sigma) + \big[1 - \varphi_i\big(v,\,\sigma\big)\big]\cdot \lambda_i^{r,\varepsilon}(v,\sigma) & \text{if} \quad 0 \le \delta_i(v), \\[2ex] \varphi_i\big(v,\,\sigma + \delta_i(v)\big)\cdot \lambda_i^{s,\varepsilon}(v,\sigma) + \\ \qquad + \big[1 - \varphi_i\big(v,\,\sigma + \delta_i(v)\big)\big]\cdot \lambda_i^{r,\varepsilon}(v,\sigma) & \text{if} \quad -\sqrt{\varepsilon} \le \delta_i(v) \le 0, \\[2ex] \varphi_i\big(v,\,\sigma - \sqrt{\varepsilon}\big)\cdot \lambda_i^{s,\varepsilon}(v,\sigma) + \\ \qquad + \big[1 - \varphi_i\big(v,\,\sigma - \sqrt{\varepsilon}\big)\big]\cdot \lambda_i^{r,\varepsilon}(v,\sigma) & \text{if} \quad \delta_i(v) \le -\sqrt{\varepsilon}, \end{cases} \tag{4.29}$$

where $\lambda_i^{r,\varepsilon}$ denotes the averaged characteristic speed defined at (3.3)-(3.4).

In the case $\sigma \le 0$, the definitions of $S_i^{\varphi,\varepsilon}(v,\sigma)$, $\lambda_i^{\varphi,\varepsilon}(v,\sigma)$, are entirely similar. Notice that the curve defined by (4.28) coincides with the rarefaction curve through v whenever $0 \le \sigma \le 2\sqrt{\varepsilon}$, and with the ε-approximate shock curve through v whenever $\sigma \ge 5\sqrt{\varepsilon}$.

Step 2. For any fixed $v^0 \in \mathcal{V}_i^0$, we define now two sequences of numbers $\big(\sigma_i^{r,\ell}(v^0)\big)_{\ell\in\mathbb{Z}}$, $\big(\sigma_i^{s,\ell}(v^0)\big)_{\ell\in\mathbb{Z}}$, by setting

$$\sigma_i^{r,\ell}(v^0) \doteq \begin{cases} 0 & \text{if} \quad \ell = 0, \\ \inf\Big\{ t > \varepsilon \ : \ \lambda_i^{r,\varepsilon}\Big[R_i\big(\lfloor\lfloor v^0 \rfloor\rfloor_i, -\ell\,\varepsilon\big), \ \ell\,\varepsilon + t\Big] = \widehat{\lambda}_i^{\lfloor v^0 \rfloor_i - \ell}(v^0)\Big\} & \text{if} \quad 0 < \ell, \end{cases}$$

$$\sigma_i^{s,\ell}(v^0) \doteq \begin{cases} 0 & \text{if} \quad \ell = 0, \\ \inf\Big\{ t > \varepsilon \ : \ \lambda_i^{s,\varepsilon}\Big[R_i\big(\lfloor\lfloor v^0 \rfloor\rfloor_i, -\ell\,\varepsilon\big), \ \ell\,\varepsilon + t\Big] = \widehat{\lambda}_i^{\lfloor v^0 \rfloor_i - \ell}(v^0)\Big\} & \text{if} \quad 0 < \ell. \end{cases} \tag{4.30}$$

Similar definitions are given for $\sigma_i^{r,\ell}(v^0)$, $\sigma_i^{s,\ell}(v^0)$, in the case $\ell < 0$. Using the properties of the maps $\lambda_i^{s,\varepsilon}$, $\lambda_i^{r,\varepsilon}$, one can show (see [AM2]) that there exists some positive constant c, independent of ε and v^0, such that

$$c{\cdot}\varepsilon \leq \big|\sigma_i^{r,\ell+1}(v^0) - \sigma_i^{r,\ell}(v^0))\big|, \qquad c{\cdot}\varepsilon \leq \big|\sigma_i^{s,\ell+1}(v^0) - \sigma_i^{s,\ell}(v^0))\big|, \qquad \forall\, v^0 \in \mathcal{V}_i^0, \quad \ell \in \mathbb{Z}, \quad \varepsilon > 0. \tag{4.31}$$

Next, for every $v \in \mathcal{V}$ such that $\delta_i(v) \leq 0$, we define the map

$$\lambda_i^{\widetilde{r},\varepsilon}\big(v, \sigma\big) \doteq \lambda_i^{r,\varepsilon}\Big[v, \ \sigma + \sigma_i^{r,\ell(v)}(\pi_i(v)) - \sigma_i^{s,\ell(v)}(\pi_i(v))\Big], \qquad \sigma \geq 0, \tag{4.32}$$

with $\ell(v) \doteq \lfloor(\pi_i(v))\rfloor_i - \lfloor v \rfloor_i$. Denote $\lambda_i^{\widetilde{\varphi},\varepsilon}\big(v,\sigma\big)$ the function defined as in (4.29) with $\lambda_i^{r,\varepsilon}$ replaced by $\lambda_i^{\widetilde{r},\varepsilon}$. Of course, similar definitions are given in the case $\delta_i(v) \geq 0$ and $\sigma < 0$.

Step 3. For every $v \in \mathcal{V}$ with $\delta_i(v) \leq 0$, define the map

$$\lambda_i^{\widetilde{\psi},\varepsilon}(v,\sigma) \doteq \psi\left(\frac{2\delta_i(v) + 3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\lambda_i^{\varphi,\varepsilon}(v,\sigma) + \left[1 - \psi\left(\frac{2\delta_i(v) + 3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\right]\lambda_i^{\widetilde{\varphi},\varepsilon}(v,\sigma) \qquad \sigma \geq 0. \tag{4.33}$$

Then, for any fixed $v^0 \in \mathcal{V}_i^0$, define the sequence $\big(\sigma_i^\ell(v^0)\big)_{\ell\in\mathbb{Z}}$, by setting

$$\sigma_i^{\ell}(v^0) \doteq \begin{cases} 0 & \text{if} \quad \ell = 0, \\ \inf\Big\{ t > \varepsilon \ : \ \lambda_i^{\widetilde{\psi},\varepsilon}\Big[R_i\big(\lfloor\lfloor v^0 \rfloor\rfloor_i, -\ell\,\varepsilon\big), \ \ell\,\varepsilon + t\Big] = \widehat{\lambda}_i^{\lfloor v^0 \rfloor_i - \ell}(v^0)\Big\} & \text{if} \quad 0 < \ell. \end{cases} \tag{4.34}$$

The same type of estimates in (4.31) hold for $\big(\sigma_i^\ell(v^0)\big)_{\ell\in\mathbb{Z}}$. Notice that, by (4.33), the two maps $\lambda_i^{\widetilde{\varphi},\varepsilon}(v,\sigma)$, $\lambda_i^{\widetilde{\psi},\varepsilon}(v,\sigma)$, coincide for all $\sigma \geq 0$ whenever $\delta_i(v) \leq -(3/2)\sqrt{\varepsilon}$. Thus, comparing the definitions (4.30) and (4.34), it follows that

$$\sigma_i^\ell(v^0) = \sigma_i^{s,\ell}(v^0) \qquad \forall\, \ell \geq \frac{3}{2\sqrt{\varepsilon}}.$$

Entirely similar definitions are given in the case $\delta_i(v) \geq 0$ and $\sigma < 0$.

Step 4. For any $v \in \mathcal{V}$ with $\delta_i(v) \leq 0$, in connection with the sequence $\big(\sigma_i^\ell(\pi_i(v))\big)_\ell$ defined at (4.34), consider the map $\widehat{\varphi}_i(v,\cdot\,):\mathbb{R}\to[0,1]$ defined by

$$\widehat{\varphi}_i(v,\sigma)\doteq\psi\left(\frac{\sigma-\sigma_i^{\ell(v)}(\pi_i(v))+v_i-\lfloor\pi_i(v)\rfloor_i\,\varepsilon}{\sigma_i^{\ell(v)+1}(\pi_i(v))-\sigma_i^{\ell(v)}(\pi_i(v))}\right),\qquad \ell(v)\doteq\lfloor\pi_i(v)\rfloor_i-\lfloor v\rfloor_i. \tag{4.35}$$

Then, define the interpolated curve

$$S_i^{\widehat{\varphi},\varepsilon}(v,\sigma)\doteq\widehat{\varphi}_i(v,\sigma)\cdot S_i^{\varphi,\varepsilon}\big(R_i(v,-\varepsilon),\ \sigma+\varepsilon\big)+\big(1-\widehat{\varphi}_i(v,\sigma)\big)\cdot S_i^{\varphi,\varepsilon}(v,\sigma),\qquad \sigma\geq 0, \tag{4.36}$$

and the interpolated speed

$$\lambda_i^{\widehat{\varphi},\varepsilon}(v,\sigma)\doteq\widehat{\varphi}_i(v,\sigma)\cdot\lambda_i^{\widetilde{\varphi},\varepsilon}\Big[R_i(v,-\varepsilon),\ \sigma+\varepsilon\Big]+\big(1-\widehat{\varphi}_i(v,\sigma)\big)\cdot\lambda_i^{\widetilde{\varphi},\varepsilon}(v,\sigma),\qquad \sigma\geq 0. \tag{4.37}$$

One can easily check that these maps satisfy a relation of the same type as (3.8) in **P3**. However, the map $S_i^{\widehat{\varphi},\varepsilon}(v,\sigma)$ never coincides with the approximate shock curve $S_i^{\varepsilon}(v,\sigma)$ even for large σ, while the map $\lambda_i^{\widehat{\varphi},\varepsilon}(v,\sigma)$ assumes different values from $\lambda_i^{r,\varepsilon}(v,\sigma)$ even for small σ. For this reason, we perform a last interpolation by setting, for all $\sigma\geq 0$,

$$\begin{aligned}S_i^{\psi,\varepsilon}(v,\sigma)\doteq{}&\psi\left(\frac{\sigma+[\delta_i(v)]^--4\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)S_i^{\varepsilon}(v,\sigma)\ +\\&+\left[1-\psi\left(\frac{\sigma+[\delta_i(v)]^--4\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\right]\cdot\left[\psi\left(\frac{2\delta_i(v)+3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)S_i^{\varphi,\varepsilon}(v,\sigma)+\right.\\&\left.+\left[1-\psi\left(\frac{2\delta_i(v)+3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\right]S_i^{\widehat{\varphi},\varepsilon}(v,\sigma)\right],\end{aligned} \tag{4.38}$$

$$\begin{aligned}\lambda_i^{\psi,\varepsilon}(v,\sigma)\doteq{}&\psi\left(\frac{\sigma+[\delta_i(v)]^--4\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\lambda_i^{s,\varepsilon}(v,\sigma)\ +\\&+\left[1-\psi\left(\frac{\sigma+[\delta_i(v)]^--4\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\right]\cdot\left[\psi\left(\frac{2\delta_i(v)+3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\lambda_i^{\varphi,\varepsilon}(v,\sigma)+\right.\\&\left.+\left[1-\psi\left(\frac{2\delta_i(v)+3\sqrt{\varepsilon}}{\sqrt{\varepsilon}}\right)\right]\lambda_i^{\widehat{\varphi},\varepsilon}(v,\sigma)\right],\end{aligned} \tag{4.39}$$

where

$$[\delta_i(v)]^-\doteq\min\{0,\ \delta_i(v)\}.$$

The maps $S_i^{\psi,\varepsilon}(v,\sigma)$, $\lambda_i^{\psi,\varepsilon}(v,\sigma)$, are called, respectively, the *modified approximate shock curve* and the *modified approximate shock speed of the i-th family.*

From the above definitions it follows that, for $\sigma\geq 0$, there holds

$$S_i^{\psi,\varepsilon}(v,\sigma)=\begin{cases}R_i(v,\sigma) & \text{whenever}\quad \sigma\leq 2\sqrt{\varepsilon},\\ S_i^{\varphi,\varepsilon}(v,\sigma) & \text{whenever}\quad \delta_i(v)\geq-\sqrt{\varepsilon}\quad\text{and}\quad \sigma+[\delta_i(v)]^-\leq 4\sqrt{\varepsilon},\\ S_i^{\widehat{\varphi},\varepsilon}(v,\sigma) & \text{whenever}\quad \delta_i(v)\leq-(3/2)\sqrt{\varepsilon}\quad\text{and}\quad \sigma+[\delta_i(v)]^-\leq 4\sqrt{\varepsilon},\\ S_i^{\varepsilon}(v,\sigma) & \text{whenever}\quad \sigma+[\delta_i(v)]^-\geq 5\sqrt{\varepsilon},\end{cases} \tag{4.40}$$

and analogous relations are verified by the map $\lambda_i^{\psi,\varepsilon}(v,\sigma)$. Similar definitions are given for the maps $S_i^{\psi,\varepsilon}(v,\sigma)$, $\lambda_i^{\psi,\varepsilon}(v,\sigma)$, in the case $\sigma \leq 0$. In the following Lemma, for a function depending on the variables $v = (v_1, v_2)$ and σ, we use a dot to denote the differention w.r.t. σ, and ∂_i to denote the differentiation w.r.t. v_i.

Lemma 4.2. *For $i = 1,2,\ \varepsilon > 0,\ v \in \mathcal{V}$, the following estimates holds*

$$\begin{aligned}
S_i^{\psi,\varepsilon}(v,\sigma) &= S_i(v,\sigma) + \mathcal{O}(1)\,|\sigma| \max\big\{\big|\delta_i(v)\big|,\,|\sigma|\big\}\cdot\varepsilon,\\
\dot S_i^{\psi,\varepsilon}(v,\sigma) &= \dot S_i + \mathcal{O}(1)\,\max\big\{\big|\delta_i(v)\big|,\,|\sigma|\big\}\cdot\varepsilon,\\
\partial_j\, S_i^{\psi,\varepsilon}(v,\sigma) &= \partial_j\, S_i + \mathcal{O}(1)\,\max\big\{\big|\delta_i(v)\big|,\,|\sigma|\big\}\cdot\varepsilon, \qquad j=1,2,\\
\dot S_i^{\psi,\varepsilon}(v,\sigma) &= \partial_i\, S_i^{\psi,\varepsilon}(v,\sigma) + \mathcal{O}(1)\,|\sigma|\cdot\Big[\max\big\{\big|\delta_i(v)\big|,\,|\sigma|\big\}\cdot\varepsilon + |\sigma|\cdot\big|\sigma-\nu_i(v)\big|\Big],
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\lambda_i^{\psi,\varepsilon}(v,\sigma) &= \lambda_i^{s}\big(v,\sigma) + \mathcal{O}(1)\,\max\big\{\big|\delta_i(v)\big|,\,|\sigma|\big\}\cdot\varepsilon,\\
\dot\lambda_i^{\psi,\varepsilon}(v,\sigma) &= \dot\lambda_i^{s}(v,\sigma) + \mathcal{O}(1)\,\frac{\max\big\{|\delta_i(v)|,|\sigma|\big\}\cdot\varepsilon}{|\sigma|},\\
\partial_j\,\lambda_i^{\psi,\varepsilon}(v,\sigma) &= \partial_j\,\lambda_i^{s}(v,\sigma) + \mathcal{O}(1)\,\frac{\max\big\{|\delta_i(v)|,|\sigma|\big\}\cdot\varepsilon}{|\sigma|}, \qquad j=1,2,\\
\dot\lambda_i^{\psi,\varepsilon}(v,\sigma) &= \partial_i\,\lambda_i^{\psi,\varepsilon}(v,\sigma) + \mathcal{O}(1)\,\Big[\frac{\max\big\{|\delta_i(v)|,|\sigma|\big\}\cdot\varepsilon}{|\sigma|} + \big|\sigma-\nu_i(v)\big|\Big].
\end{aligned} \tag{4.42}$$

A proof of Lemma 4.2 and the derivation of further estimates on the second derivatives of $S_i^{\psi,\varepsilon}$, $\lambda_i^{\psi,\varepsilon}$, can be found in [AM2].

4.4 - Approximate mixed-curves.

We define now an approximation of the mixed-curve of the i-th family through a given point $v^0 \in \mathcal{V}_i^0$ as follows. In connection with the sequence $\big(\sigma_i^\ell\big)_\ell \doteq \big(\sigma_i^\ell(v^0)\big)_\ell$ introduced in (4.34) consider the map $\sigma \mapsto \mu_i^\varepsilon(v^0,\sigma)$ defined by

$$\mu_i^\varepsilon(v^0,\sigma) \doteq \ell\,\varepsilon + \sigma \qquad \text{if} \qquad \begin{cases} \sigma_i^\ell \leq \sigma < \sigma_i^{\ell+1}, & 0 \leq \ell,\\ \sigma_i^{\ell-1} < \sigma \leq \sigma_i^\ell, & \ell < 0. \end{cases} \tag{4.43}$$

Then, the curve $\sigma \mapsto T_i^\varepsilon(v^0,\sigma)$ defined by

$$T_i^\varepsilon(v^0,\sigma) \doteq S_i^{\psi,\varepsilon}\Big(R_i\big(\lfloor\lfloor v^0 \rfloor\rfloor_i,\, \sigma - \mu_i^\varepsilon(v^0,\sigma)\big),\ \mu_i^\varepsilon(v^0,\sigma)\Big) \tag{4.44}$$

is called the *ε-approximate mixed curve of the i-th family through v^0*. Such a curve converges with its derivative to the corresponding exact mixed curve T_i. Indeed, the following holds

Lemma 4.3. *For all $v^0 \in \mathcal{V}_i^0$, $\sigma \in \mathcal{J}$, and $\varepsilon > 0$ there holds*

$$\left|\mu_i^\varepsilon(v^0,\sigma) - \mu_i(v^0,\sigma)\right| = \mathcal{O}(1)\,\varepsilon, \tag{4.45}$$

$$\left|T_i^\varepsilon(v^0,\sigma) - T_i(v^0,\sigma)\right| = \mathcal{O}(1)\,\varepsilon, \tag{4.46}$$

$$\left|\frac{d}{d\sigma}T_i^\varepsilon(v^0,\sigma) - \frac{d}{d\sigma}T_i(v^0,\sigma)\right| = \mathcal{O}(1)\,\varepsilon. \tag{4.47}$$

Moreover, the second derivatives of T_i^ε w.r.t. σ are uniformly bounded.

A proof of Lemma 4.3 is given in [AM2]. Consider now the map $\nu_1^\varepsilon : \mathcal{V} \to \mathbb{R}$, defined by setting

$$\begin{aligned}
&\nu_1^\varepsilon(v) \doteq \big(\pi_1(v)\big)_1 - v_1 && \text{if} \quad v \in \mathcal{V}_1^+, \quad \lfloor \pi_1(v) \rfloor_1 = \lfloor v \rfloor_1,\\
&\nu_1^\varepsilon(v) \doteq \mu_1^\varepsilon\big(\pi_1(v),\ \sigma_1^{\ell(v)}(\pi_1(v))\big) + \big(\lfloor v \rfloor_1\,\varepsilon - v_1\big) && \text{if} \quad v \in \mathcal{V}_1^+, \quad \lfloor \pi_1(v) \rfloor_1 > \lfloor v \rfloor_1,
\end{aligned} \tag{4.48}$$

where $\ell(v) \doteq \lfloor \pi_1(v) \rfloor_1 - \lfloor v \rfloor_1$. For $v \in \mathcal{V}$, $\varepsilon > 0$, there holds

$$\nu_1^\varepsilon(v) = \nu_1(v) + \mathcal{O}(1)\,|\varepsilon|. \tag{4.49}$$

See [AM2] also for a proof of (4.49). From (4.49), recalling (2.19), it follows

$$\nu_1^\varepsilon(v) = -3\delta_1(v) + \mathcal{O}(1)\,\max\big\{|\varepsilon|,\, |\delta_1(v)|^2\big\}, \qquad v \in \mathcal{V}. \tag{4.50}$$

Notice that, using ν_1^ε, one can express the map $\widehat{\varphi}_1$ defined in (4.36) as

$$\widehat{\varphi}_1(v,\sigma) = \psi\left(\frac{\sigma - \nu_1^\varepsilon(v)}{\nu_1^\varepsilon\big(R_1(v,-\varepsilon)\big) - \nu_1^\varepsilon(v) - \varepsilon}\right), \qquad \delta_1(v) \le 0.$$

Then, observing that by (4.29) one has

$$\lambda_1^{\varphi,\varepsilon}\big(v,\sigma\big) = \lambda_1^{\widetilde{\varphi},\varepsilon}\big(v,\sigma\big) = \lambda_1^{\widetilde{\psi},\varepsilon}\big(v,\sigma\big) = \lambda_1^{s,\varepsilon}\big(v,\sigma\big), \qquad \text{if} \quad \sigma \ge 5\sqrt{\varepsilon},$$

and comparing the definition (4.34) of $\big(\sigma_1^\ell(v^0)\big)_\ell$ with the definitions (4.37) and (4.39), we deduce

$$\lambda_1^{\psi,\varepsilon}\big(v,\ \nu_1^\varepsilon(v)\big) = \lambda_1^{\widetilde{\psi},\varepsilon}\big(v,\ \nu_1^\varepsilon(v)\big) = \lambda_1^{1,\varepsilon+}(v) \qquad \forall\, v \in \mathcal{V}_1^+. \tag{4.51}$$

Moreover, by (4.40) and (4.50) it follows that

$$\begin{aligned}
&S_1^{\psi,\varepsilon}(v,\sigma) = S_1^\varepsilon(v,\sigma) \\
&\lambda_1^{\psi,\varepsilon}(v,\sigma) = \lambda_1^{s,\varepsilon}(v,\sigma)
\end{aligned}
\qquad \text{whenever} \qquad
\begin{array}{c}
\delta_1(v) \ge 0, \quad \sigma \ge 5\sqrt{\varepsilon}, \\
\text{or} \\
\delta_1(v) < 0, \quad \sigma \ge \max\big\{\nu_1^\varepsilon(v),\, 9\sqrt{\varepsilon}\big\}.
\end{array} \tag{4.52}$$

Similar definitions and relations hold in the case where $v \in \mathcal{V}_1^-$ or $v \in \mathcal{V}_2^\pm$.

Remark 4.5. The maps $S_1^{\psi,\varepsilon}(v,\sigma)$, $\lambda_1^{\psi,\varepsilon}(v,\sigma)$, $\nu_1^\varepsilon(v)$, satisfy the property **P3** stated in Section 3. In particular, if $v=(v_1,v_2)\in\mathcal{V}_1^+$ and

$$\exists\,\sigma'>0 \quad : \quad \lambda_1^{\psi,\varepsilon}(v,\sigma')=\widehat{\lambda}_1^{\lfloor v\rfloor_1}(v),$$

then it follows that $\sigma'=\nu_1^\varepsilon(v)$ and, for all $\sigma''\in\big[v_1-\lfloor v\rfloor_1\,\varepsilon,\ v_1-(\lfloor v\rfloor_1+1)\varepsilon\big]$, there holds

$$S_1^{\psi,\varepsilon}\Big(R_1(v,\sigma''),\ \nu_1^\varepsilon(v)-\sigma''\Big)=S_1^{\psi,\varepsilon}\big(v,\ \nu_i^\varepsilon(v)\big),$$

$$\lambda_1^{\psi,\varepsilon}\Big(R_1(v,\sigma''),\ \nu_1^\varepsilon(v)-\sigma''\Big)=\lambda_1^{\psi,\varepsilon}\big(v,\ \nu_i^\varepsilon(v)\big).$$

Similar relations hold for the map $\lambda_1^{\psi,\varepsilon}(v,\cdot)$ in the case $v\in\mathcal{V}_1^-$, and for the map $\lambda_2^{\psi,\varepsilon}(v,\cdot)$.

4.5 - Approximate elementary curves.

We define here the ε-*approximate elementary curves* associated to each characteristic field that are used to construct the ε-approximate solution of a Riemann problem.

Definition 4.2. For $i=1,2$, $v\in\mathcal{V}$, $\varepsilon>0$, define the maps $\Psi_i^\varepsilon(v):\mathcal{J}\to\mathcal{V}$, as follows:

$$\Psi_1^\varepsilon(v,\sigma)=\begin{cases} S_1^{\varphi,\varepsilon}(v,\sigma) \quad \forall\,\sigma & \text{if}\quad v\in\mathcal{V}_1^0,\\[2mm] \begin{cases} S_1^{\varphi,\varepsilon}(v,\sigma) & \text{if}\ \ \sigma<0\\ R_1(v,\sigma) & \text{if}\ \ 0\le\sigma<-\delta_1(v)\\ T_1^\varepsilon\Big(\pi_1(v),\ \sigma+\delta_1(v)\Big) & \text{if}\ \ -\delta_1(v)\le\sigma<\nu_1^\varepsilon(v)\\ S_1^{\psi,\varepsilon}\big(v,\sigma\big) & \text{if}\ \ \sigma\ge\nu_1^\varepsilon(v)\end{cases} & \text{if}\quad v\in\mathcal{V}_1^+,\\[2mm] \begin{cases} S_1^{\varphi,\varepsilon}(v,\sigma) & \text{if}\ \ \sigma>0\\ R_1(v,\sigma) & \text{if}\ \ -\delta_1(v)<\sigma\le0\\ T_1^\varepsilon\Big(\pi_1(v),\ \sigma+\delta_1(v)\Big) & \text{if}\ \ \nu_1^\varepsilon(v)<\sigma\le-\delta_1(v)\\ S_1^{\psi,\varepsilon}\big(v,\sigma\big) & \text{if}\ \ \sigma\le\nu_1^\varepsilon(v)\end{cases} & \text{if}\quad v\in\mathcal{V}_1^-,\end{cases}\tag{4.53}$$

$$\Psi_2^\varepsilon(v,\sigma)=\begin{cases} S_2^{\varphi,\varepsilon}(v,\sigma) \quad \forall\,\sigma & \text{if}\quad v\in\mathcal{V}_2^0,\\[2mm] \begin{cases} S_2^{\varphi,\varepsilon}(v,\sigma) & \text{if}\ \ \sigma>0\\ R_2(v,\sigma) & \text{if}\ \ -\delta_2(v)\le\sigma\le0\\ T_2^\varepsilon\Big(\pi_2(v),\ \sigma+\delta_2(v)\Big) & \text{if}\ \ \nu_2^\varepsilon(v)\le\sigma<-\delta_2(v)\\ S_2^{\psi,\varepsilon}\big(v,\sigma\big) & \text{if}\ \ \sigma<\nu_2^\varepsilon(v)\end{cases} & \text{if}\quad v\in\mathcal{V}_2^+,\\[2mm] \begin{cases} S_2^{\varphi,\varepsilon}(v,\sigma) & \text{if}\ \ \sigma<0\\ R_2(v,\sigma) & \text{if}\ \ -\delta_2(v)<\sigma\le0\\ T_2^\varepsilon\Big(\pi_2(v),\ \sigma+\delta_2(v)\Big) & \text{if}\ \ -\delta_2(v)<\sigma\le\nu_2^\varepsilon(v)\\ S_2^{\psi,\varepsilon}\big(v,\sigma\big) & \text{if}\ \ \sigma>\nu_2^\varepsilon(v)\end{cases} & \text{if}\quad v\in\mathcal{V}_2^-.\end{cases}\tag{4.54}$$

Remark 4.6. For each $v \in \mathcal{V}$, the curve $\sigma \mapsto \Psi_1^\varepsilon(v, \sigma)$ $(\sigma \mapsto \Psi_2^\varepsilon(v, \sigma))$ describes all the right (left) states that are connected to v by 1-waves (2-waves) of an ε-approximate solution. Notice that, by construction, the maps $\sigma \mapsto \Psi_i^\varepsilon(v, \sigma)$, $i = 1, 2$, are everywhere continuous and piecewise twice continuously differentiable. Moreover, left and right first derivatives are uniformly bounded with respect to σ and ε. These facts, together with Lemma 4.2 and Lemma 4.3, imply that the property **P1** stated in Section 3 is satisfied.

Remark 4.7. Let $\varepsilon > 0$ be given. For v^L, $v^R \in \mathcal{V}$ and (σ_1, σ_2) in a neighborhood of $0 \in \mathbb{R}^2$, consider the map

$$\Psi^\varepsilon\big(v^L, v^R;\, \sigma_1, \sigma_2\big) \doteq \Psi_1^\varepsilon\big(v^L, \sigma_1\big) - \Psi_2^\varepsilon\big(v^R, \sigma_2\big). \tag{4.55}$$

Then, because of the regularity of Ψ^ε and since

$$\left.\frac{\partial \Psi^\varepsilon\big(v^L, v^R;\, \sigma_1, \sigma_2\big)}{\partial \sigma_1}\right|_{\sigma_1, \sigma_2 = 0} = e_1,$$
$$\left.\frac{\partial \Psi^\varepsilon\big(v^L, v^R;\, \sigma_1, \sigma_2\big)}{\partial \sigma_2}\right|_{\sigma_1, \sigma_2 = 0} = -e_2,$$

by using the implicit function theorem one can show that there exists a neighborhood of $0 \in \mathbb{R}^2$, still denoted $\mathcal{V}$, such that for every $v^L, v^R \in \mathcal{V}$ we can *uniquely determine* σ_1, σ_2 for which

$$\Psi^\varepsilon\big(v^L, v^R;\, \sigma_1, \sigma_2\big) = 0 \tag{4.56}$$

holds. Notice that, thanks to the uniform bound on the derivatives of Ψ_i^ε, we can choose such neighborhood with size independent of ε. Moreover, we can find some constant C independent of ε such that, for all $v^L, v^R \in \mathcal{V}$, if σ_1, σ_2 are the corresponding values determined by (4.56), then one has

$$\frac{1}{C}\big|v^R - v^L\big| \le |\sigma_1| + |\sigma_2| \le C\,\big|v^R - v^L\big|. \tag{4.57}$$

This shows that the ε-approximate elementary curves above defined satsify also the property **P4** stated in Section 3.

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