A Note on the Riemann Problem

for General n×n Conservation Laws

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Abstract. We analize the structure of the general solution of the Riemann problem for a striclty hyperbolic system of conservation laws whose characteristic fields are neither genuinely non-linear nor linearly degenerate in the sense of Lax.

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1. Introduction and Preliminaries

Consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension

$$u_t + \big[F(u)\big]_x = 0, \tag{1.1}$$

where F is a smooth map taking values in $\mathbb{R}^n$ and defined on an open set $\Omega \subset \mathbb{R}^n$ containing the origin. Denote by $\lambda_1(u) < \cdots < \lambda_n(u)$ the eigenvalues of the Jacobian matrix $A(u) \doteq DF(u)$, and with $r_1(u), \ldots, r_n(u)$, a basis of eigenvectors. We suppose that in such a domain the characteristic fields are neither genuinely nonlinear nor linearly degenerate in the sense of Lax [LA], [S]. Instead, denoting by

$$r \bullet \phi\,(u) \doteq D\phi \cdot r(u) = \lim_{h\to 0} \frac{\phi\big(u + h\,r(u)\big) - \phi(u)}{h}.$$

the directional derivative of a function $\phi = \phi(u)$ in the direction of a vector field $r = r(u)$, throughout the paper we adopt the following assumption (see Figure 1):

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(A) There exist n smooth $n-1$-dimensional manifolds Ω_i^0, $i=1,\dots,n$, such that

$$\Omega_i^0 \doteq \{\, u\in\Omega;\ r_i\bullet\lambda_i(u)=0\,\}, \tag{1.2}$$

$$r_i\bullet\big(r_i\bullet\lambda_i\big)\,\big(u^0\big)\neq 0, \qquad \forall\ u^0\in\Omega_i^0. \tag{1.3}$$

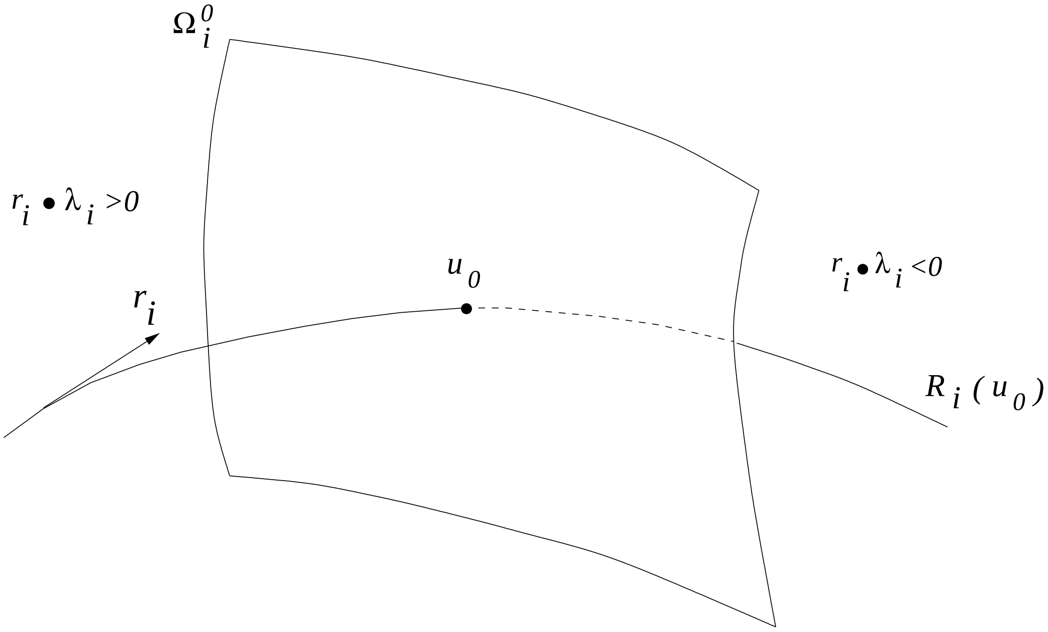

FIGURE 1

Systems of conservations laws which are neither genuinely non-linear nor linearly degenerate arise in several contexts, in particular in studying elastodynamic or rigid heat conductors at low temperature.

The purpose of this note is to analyze the structure of the general solution of the Riemann problem associated with (1.1), i.e. of the Cauchy problem (1.1) with arbitrary initial data

$$u(0,x)=\begin{cases} u^L & \text{if } x<0,\\ u^R & \text{if } x>0.\end{cases} \tag{1.4}$$

The Riemann Problem plays a key role in the theory of hyperbolic systems of conservation laws. In fact, the solution of the Riemann problem represents the basic building block towards the construction of solutions to the general Cauchy Problem for (1.1).

We recall that, for hyperbolic systems of conservation laws, to achieve uniqueness of solutions it is necessary to consider weak solutions that satisfy some additional admissibility condition, possibly motivated by physical considerations. In [LI1], [LI2], T.P. Liu proposed an admissibility criterion valid for $n\times n$ systems of conservation laws having the property that, for each characteristic speed λ_i, the directional derivative $r_i\bullet\lambda_i$ vanishes on a finite, disjoin union of $n-1$-dimensional manifolds. This criterion, in the case of a system where $r_i\bullet\lambda_i=0$ on a single hypersurface in the u-space, is equivalent to the stability condition introduced by Lax:

(L) A shock connecting the left state u^L and the right state u^R, travelling with speed s is an admissible discontinuity of the i-th family if

$$\lambda_i(u^L) \geq s \geq \lambda_i(u^R).$$

In this setting, an *entropy weak solution* of (1.1) will be a weak solution of (1.1) that has admissible discontinuities in the sense of Lax. The existence of a unique, self-similar, entropy weak solution of the Riemann problem (1.1), (1.4), was established by T.P. Liu in [LI1], [LI2]. Such solution consists of elementary waves that can be compressive shocks, centered rarefactions or *composed waves* made of one-side contact-discontinuities adjacent to rarefaction waves.

For each characteristic family, one can define an *elementary curve* that characterizes the set of all left and right states u^L, u^R, which are connected by means of an elementary wave of that family. In this note, in particular, we shall derive some basic properties of the elementary curves associated to each characteristic family.

Here, we make some standard assumptions and introduce some basic notations. We choose right and left eigenvectors $r_i(u)$, $l_i(u)$ of $A(u)$ normalized so that

$$\big|r_i(u)\big| = 1, \tag{1.5}$$

$$\langle l_i(u),\, r_j(u)\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{1.6}$$

For any fixed $u^0 \in \Omega$, and $i = 1,\dots,n$, as usual we define the i-th rarefaction curve $s \mapsto R_i(u^0)(s)$ as the integral curve of r_i through u^0, parametrized by arc-length. We shall often use also the notation $\big(\exp sr_i\big)(u^0) \doteq R_i\big(u^0\big)(s)$. Moreover, by the implicit function theorem one can define the i-th shock curve $S_i(u^0)$ as the set of points u such that

$$\langle l_j(u,\, u^0),\ u - u^0\rangle = 0 \qquad \forall j \neq i, \tag{1.7}$$

where $l_j(u,\, u^0)$ is the j-th left eigenvector of the averaged matrix

$$A(u,\, u^0) = \int_0^1 A\big(u^0 + \xi(u^0 - u)\big)d\xi$$

normalized as in (1.6). We parametrize also the curve $s \mapsto S_i(u^0)(s)$ by arc-length.

It is well known [S] that the two curves $R_i(u^0)$, $S_i(u^0)$ have second order tangency at $s = 0$. Thus the following hold

$$\frac{d}{ds}S_i(u^0)(s)\Big|_{s=0} = r_i\big(u^0\big), \tag{1.8}$$

$$\frac{d^2}{ds^2}S_i(u^0)(s)\Big|_{s=0} = r_i \bullet r_i\big(u^0\big). \tag{1.9}$$

The speed of the shock wave connecting the left state u^L to the right state $S_i(u^L)(s) = u^R$ is denoted $\lambda_i^s\big(u^0, s\big)$ or $\lambda_i\big(u^L,\, u^R\big)$ and satisfies

$$\lambda_i^s\big(u^L,\, 0\big) = \lambda_i\big(u^L\big), \tag{1.10}$$

$$\frac{d}{ds}\lambda_i^s\big(u^L,\, s\big)\Big|_{s=0} = \frac{1}{2} r_i \bullet \lambda_i\big(u^L\big). \tag{1.11}$$

We shall focus our attention, in particular, on the Riemann problem with constant states u^L, u^R in (1.4) satisfying

$$\big[r_k \bullet \lambda_k\big(u^L\big)\big] \cdot \big[r_k \bullet \lambda_k\big(u^R\big)\big] < 0, \tag{1.12}$$

for some $k \in \{1, \dots, n\}$, i.e. on the case in which the left state u^L and the right state u^R lie on opposite sides w.r.t. the hypersurface Ω_k^0. In this case, by the analysis in [LI1], [LI3], the solution of the Riemann problem (1.1), (1.4) contains a *composed wave* of the k-th characteristic family that consists of

Case 1: a rarefaction wave followed by a left contact discontinuity both of the k-th family, if

$$r_k \bullet \big(r_k \bullet \lambda_k\big)\big(u^0\big) < 0, \qquad \forall\ u^0 \in \Omega_k^0, \tag{1.13}$$

i.e. if, for each $u^0 \in \Omega_k^0$, the function $s \mapsto \lambda_k(\exp sr_k)\big(u^0\big)$ attains its maximum value at $s = 0$;

Case 2: a right contact discontinuity followed by a rarefaction wave both of the k-th family, if

$$r_k \bullet \big(r_k \bullet \lambda_k\big)\big(u^0\big) > 0, \qquad \forall\ u^0 \in \Omega_0^k, \tag{1.14}$$

i.e. if, for each $u^0 \in \Omega_k^0$, the function $s \mapsto \lambda_k(\exp sr_k)\big(u^0\big)$ attains its minimum value at $s = 0$.

To take advantage of the symmetry of the system, we shall express the general solution of the Riemann problem in terms of two types of elementary curves:

1. If Case 1 holds, then we construct the *elementary curve of right states of the k-th family*, that consists of all right states that are connected with a given left state by an elementary wave of the k-th characteristic family.

2. If Case 2 holds, then we construct the *elementary curve of left states of the k-th family*, that consists of all left states that are connected with a given right state by an elementary wave of the k-th characteristic family.

In particular, this means that, if Case 1 holds, for any fixed $u^0 \in \Omega_k^0$ we define a *mixed curve of right states of the k-th family,* denoted $s \mapsto T_k\big(u^0\big)(s)$, having the property

M1. For every point $u^R \in T_k(u^0)$, there exists a unique point $u^L \in R_k(u^0)$ such that $u^R \in S_k(u^L)$ and $\lambda_k(u^L, u^R) = \lambda_k(u^L)$.

Instead, if Case 2 holds, for any fixed $u^0 \in \Omega_k^0$ we define a *mixed curve of left states of the k-th family,* still denoted $s \mapsto T_k\big(u^0\big)(s)$, having the property

M2. For every point $u^L \in T_k(u^0)$, there exists a unique point $u^R \in R_k(u^0)$ such that $u^R \in S_k(u^L)$ and $\lambda_k(u^L, u^R) = \lambda_k(u^R)$.

Throughout, for a one-parameter map $s \mapsto \phi(s)$, $s \in \mathcal{I} \subset \mathbb{R}$, we shall often use a dot to denote the differentiation of ϕ w.r.t. s.

2. The Mixed Curves

In this section we want to study the properties of the mixed curves of a fixed k-th characteristic family, $k \in \{1, \dots, n\}$. For sake of simplicity, we assume that (1.13) holds, the other case being entirely similar because of the above definitions.

Given a point $u^0 \in \Omega_k^0$, i.e. such that $r_k \bullet \lambda_k(u^0) = 0$, the curve of states in a neighborhood of u^0 which can be connected *to the left* by a composed wave made of a rarefaction followed by a left contact discontinuity of the k-th characteristic family can be determined by solving the equation

$$F(u) - F\big((\exp\, s r_k)(u^0)\big) = \lambda_k\big((\exp\, s r_k)(u^0)\big) \cdot \big(u - (\exp\, s r_k)(u^0)\big) \tag{2.1}$$

in terms of $(s, u) \in \mathbb{R} \times \Omega$. Notice that (2.1) implies that the point $u = u^R$ lies on the shock curve trough $(\exp\, s r_k)(u^0) = u^L$ and that the speed of the shock wave connecting $(\exp\, s r_k)(u^0)$ with u is precisely $\lambda_k\big((\exp\, s r_k)(u^0)\big)$. Therefore the vector equation (2.1) is equivalent to the scalar equation

$$\lambda_k^s\big[(\exp\, s r_k)(u^0),\ t\big] = \lambda_k\big((\exp\, s r_k)(u^0)\big). \tag{2.2}$$

Indeed, (s, t) is a solution of (2.2) iff $\big(s,\ S_k\big((\exp\, s r_k)(u^0),\ t\big)\big)$ is a solution of (2.1). Clearly a trivial solution of (2.2) is $(s, t) = (s, 0)$. A non-trivial solution of (2.2) will be determined by first finding a curve of solutions of the equation

$$\frac{\partial}{\partial t}\lambda_k^s\big[(\exp\, s r_k)(u^0),\ t\big] = 0 \tag{2.3}$$

and then using such a curve to determine an approximate solution of (2.2) which in turn allows to determine the exact solution of (2.2) by applying the Newton-Kantorovich Theorem [De]. The main result of this note is the following

Theorem 2.1. *Let $k \in \{1, \ldots, n\}$ be such that (1.13) holds. Then for any $u^0 \in \Omega_k^0$, there exists a smooth curve $T \doteq T(u^0) = T_k(u^0) : s \mapsto T(u^0)(s) \in \Omega$, together with two smooth maps $\zeta \doteq \zeta(u^0) = \zeta_k(u^0) : s \mapsto \zeta(u^0)(s) \in \mathbb{R}$, $\mu \doteq \mu(u^0) = \mu_k(u^0) : s \mapsto \mu(u^0)(s) \in \mathbb{R}$, defined in a neighborhood of zero $\mathcal{J}$, such that*

$$F\big(T(s)\big) - F\Big((\exp\, \zeta(s)\, r_k)(u^0)\Big) = \lambda_k\Big((\exp\, \zeta(s)\, r_k)(u^0)\Big) \cdot \Big(T(s) - (\exp\, \zeta(s)\, r_k)(u^0)\Big), \tag{2.4}$$

$$\frac{d}{ds}T(s) = \frac{d}{dt}S_k\left((\exp\, \zeta(s)\, r_k)(u^0),\ t\right)\bigg|_{t=\mu(s)}, \tag{2.5}$$

for all $s \in \mathcal{J}$. Moreover, the map $s \mapsto \zeta(s) + \mu(s)$ is strictly increasing and, at $s = 0$, one has

$$T(0) = u^0, \tag{2.6}$$

$$\frac{d}{ds}T(s)\bigg|_{s=0} = r_k(u^0), \tag{2.7}$$

$$\frac{d^2}{ds^2}T(s)\bigg|_{s=0} = r_k \bullet r_k(u^0). \tag{2.8}$$

Remark 2.1. If we fix some $k \in \{1, \ldots, n\}$ for which (1.14) holds, then the statement of Theorem 2.1 continue to hold, except for the fact that the map $s \mapsto \zeta(s) + \mu(s)$ is *strictly decreasing*. Towards a proof of Theorem 2.1, we first establish two intermediate results.

Proposition 2.1. *For every $u^0 \in \Omega_k^0$, there exists a unique smooth map $\eta \doteq \eta\,(u^0) : s \mapsto \eta(u^0)(s) \in \mathbb{R}$ defined in a neighborhood of zero $\mathcal{J}$, such that*

$$\eta(0) = 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}\lambda_k^s\Big[(\exp\, s r_k)(u^0),\ t\Big]\bigg|_{t=\eta(s)} = 0 \qquad \forall s \in \mathcal{J}. \tag{2.10}$$

Moreover, one has

$$\dot{\eta}(0) = -\frac{3}{2}. \tag{2.11}$$

Proof. Since $u^0 \in \Omega_k^0$, by (1.11) we have

$$\frac{d}{dt}\lambda_k^s(u^0,\, t)\Big|_{t=0} = 0. \tag{2.12}$$

Then, by differentiating three times the Rankine-Hugoniot equation as in [LI3] p.8, one obtains

$$\begin{aligned}\Big(A(u^0) - \lambda_k(u^0)\Big)\left(\frac{d^3}{dt^3}S_k(u^0)(t)\Big|_{t=0} - \frac{d^2}{dt^2}r_k\big(S_k(u^0)(t)\big)\Big|_{t=0}\right) = \\ = \left(3\,\frac{d^2}{dt^2}\lambda_k^s(u^0,\, t)\Big|_{t=0} - \frac{d^2}{dt^2}\lambda_k\Big(S_k(u^0)(t)\Big)\Big|_{t=0}\right)r_k(u^0).\end{aligned} \tag{2.13}$$

But, the left hand side of (2.13) is proportional to $r_2\big(u^0\big)$ and hence we deduce that

$$\frac{d^2}{dt^2}\lambda_k^s(u^0,\, t)\Big|_{t=0} = \frac{1}{3}\,\frac{d^2}{dt^2}\lambda_k\Big(S_k(u^0)(t)\Big)\Big|_{t=0}. \tag{2.14}$$

Thus, since

$$\frac{d^2}{dt^2}\lambda_k\Big(S_k(u^0)(t)\Big)\Big|_{t=0} = r_k \bullet \big(r_k \bullet \lambda_k\big)\big(u^0\big), \tag{2.15}$$

from (2.14) and because of (1.11) it follows that

$$\frac{\partial^2}{\partial t^2}\lambda_k^s\big[(\exp\, s\,r_k)(u^0),\, t\big]\Big|_{(s,t)=(0,0)} = \frac{d^2}{dt^2}\lambda_k^s(u^0,\, t)\Big|_{t=0} \neq 0. \tag{2.16}$$

Therefore, by the implicit function theorem, (2.3) admits a unique smooth solution $\eta : s \mapsto \eta(s)$ in a neighborhood of zero and by (1.11), (2.14), we find

$$\dot{\eta}(0) = -\frac{\dfrac{\partial^2}{\partial s \partial t}\lambda_k^s\big[(\exp\, s\,r_k)(u^0),\, t\big]\Big|_{(s,t)=(0,0)}}{\dfrac{\partial^2}{\partial t^2}\lambda_k^s\big[(\exp\, s\,r_k)(u^0),\, t\big]\Big|_{(s,t)=(0,0)}} = -\frac{r_k \bullet \big(r_k \bullet \lambda_k\big)\big(u^0\big)}{2\,\dfrac{d^2}{dt^2}\lambda_k^s(u^0,\, t)\Big|_{t=0}} = -\frac{3}{2}.$$

□

Proposition 2.2. *For every $u^0 \in \Omega_k^0$, there exists a unique smooth map $\mu \doteq \mu(u^0) : s \mapsto \mu(u^0)(s) \in \mathbb{R}$ defined in a neighborhood of zero $\mathcal{J}$, such that $\mu(s) \neq 0$ whenever $s \neq 0$, and*

$$\mu(0) = 0, \tag{2.17}$$

$$\lambda_k\Big((\exp\, s\,r_k)\big(u^0\big)\Big) = \lambda_k^s\Big[(\exp\, s\,r_k)\big(u^0\big),\, \mu(s)\Big] \qquad \forall s \in \mathcal{J}. \tag{2.18}$$

Proof. For any fixed $u^0 \in \Omega_k^0$, and $s \in \mathbb{R}$, consider the map

$$t \mapsto G^s(t) \doteq \lambda_k\Big((\exp\, s\,r_k)\big(u^0\big)\Big) - \lambda_k^s\Big[(\exp\, s\,r_k)\big(u^0\big),\, t\Big]. \tag{2.19}$$

Then $\mu(s)$ is precisely the non trivial solution of

$$G^s(t) = 0. \tag{2.20}$$

Let η be the map defined by Proposition 2.1 and consider the map $s \mapsto \tilde{\mu}(s)$ defined by

$$\tilde{\mu}(s) = \eta(s) + \Big|2\lambda_k\big((\exp\, s\, r_k)(u^0)\big) - 2\lambda_k\big(S_k\big((\exp\, s\, r_k)(u^0),\, \eta(s)\big)\big)\Big|^{\frac{1}{2}} \left|\frac{d^2}{dt^2}\lambda_k^s\big[(\exp\, s\, r_k)(u^0),\, t\big]\Big|_{t=\eta(s)}\right|^{-\frac{1}{2}},$$

if $\quad s \leq 0$,

$$\tilde{\mu}(s) = \eta(s) - \Big|2\lambda_k\big((\exp\, s\, r_k)(u^0)\big) - 2\lambda_k\big(S_k\big((\exp\, s\, r_k)(u^0),\, \eta(s)\big)\big)\Big|^{\frac{1}{2}} \left|\frac{d^2}{dt^2}\lambda_k^s\big[(\exp\, s\, r_k)(u^0),\, t\big]\Big|_{t=\eta(s)}\right|^{-\frac{1}{2}},$$

if $\quad s > 0$.

We will show that $\tilde{\mu}(s)$ is an approximate solution of (2.20) in the sense that

$$G^s(\tilde{\mu}(s)) = o(s^2), \qquad \frac{d}{dt}G^s(t)\Big|_{t=\tilde{\mu}(s)} = c\, s + o(s), \tag{2.21}$$

where $o(s), o(s^2)$, are the usual Landau symbols and $c \neq 0$ a suitable constant. This is sufficient to establish the Proposition which then follows by applying the Newton-Kantorovich Theorem [De]. To prove (2.21) first observe that, since η satisfies (2.10) and hence

$$\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ \eta(s)\right] = \lambda_k\left(S_k\Big((\exp\, s\, r_k)(u^0),\ \eta(s)\Big)\right),$$

it follows that

$$\lambda_k\left((\exp\, s\, r_k)(u^0)\right) = \lambda_k\Big(S_k\Big((\exp\, s\, r_k)(u^0)\Big)\left[\eta(s)\right]\Big) + \frac{1}{2}\frac{d^2}{dt^2}\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ t\right]\Big|_{t=\eta(s)} (\eta(s))^2 + o(\eta^2(s)). \tag{2.22}$$

Therefore, by using (2.11) and observing that by construction $s \mapsto \frac{d^2}{dt^2}\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\, t\right]\Big|_{t=\eta(s)}$ is bounded away from zero in a neighborhood of zero, we deduce that

$$\tilde{\mu}(s) = 2\eta(s) + o(\eta(s)) = -3s + o(s), \tag{2.23}$$

and that

$$\begin{aligned} G^s(\tilde{\mu}(s)) &= \lambda_k\Big(S_k\Big((\exp\, s\, r_k)(u^0)\Big)\left[\eta(s)\right]\Big) + \frac{1}{2}\frac{d^2}{dt^2}\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ t\right]\Big|_{t=\eta(s)} (\tilde{\mu}(s) - \eta(s))^2 - \\ &\quad - \lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ \tilde{\mu}(s)\right] = \\ &= o((\tilde{\mu}(s) - \eta(s))^2) = o(s^2), \end{aligned} \tag{2.24}$$

$$\begin{aligned} \frac{d}{dt}G^s(t)\Big|_{t=\tilde{\mu}(s)} &= -\frac{d^2}{dt^2}\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ t\right]\Big|_{t=\eta(s)} (\tilde{\mu}(s) - \eta(s)) + o((\tilde{\mu}(s) - \eta(s)) = \\ &= \frac{3}{2}\frac{d^2}{dt^2}\lambda_k^s\left[(\exp\, s\, r_k)(u^0),\ t\right]\Big|_{t=\eta(s)} s + o(s). \end{aligned} \tag{2.25}$$

□

Proof of Theorem 2.1 Let μ be the map defined by Proposition 2.2 and ζ the identity. Then, by (2.17), (2.18), the curve

$$T(s) \doteq S_k\Big((\exp\ \zeta(s)\, r_k)(u^0)\Big)\,[\mu(s)] \tag{2.26}$$

satisfies (2.4), (2.6). Regarding (2.5), (2.7) and (2.8), first riparametrize by its arclength the curve defined by (2.26) and use μ, ζ, T, to denote the corresponding riparametrized maps as well. Then observe that since $T(s)$ belongs to the shock curve through $\omega(s) \doteq (\exp\ \zeta(s)\, r_k)\big(u^0\big)$, differentiating twice the Rankine-Hugoniot condition and setting

$$\dot\lambda_k(s) \doteq \frac{d}{dt}\ \lambda_k^s\big(\omega(s),\ t\big)\Big|_{t=\mu(s)}, \qquad \dot S_k(s) \doteq \frac{d}{dt} S_k\big(\omega(s)\big)(t)\Big|_{t=\mu(s)}, \qquad \dot T(s) \doteq \frac{d}{ds}T(s),$$

$$\ddot\lambda_k(s) \doteq \frac{d^2}{dt^2}\ \lambda_k^s\big(\omega(s),\ t\big)\Big|_{t=\mu(s)}, \qquad \ddot S_k(s) \doteq \frac{d^2}{dt^2} S_k\big(\omega(s)\big)(t)\Big|_{t=\mu(s)}, \qquad \ddot T(s) \doteq \frac{d^2}{ds^2}T(s),$$

we have

$$\big(DF\,(T(s)) - \lambda_k\,(\omega(s))\,\big)\cdot \dot S_k(s) = \dot\lambda_k(s)\big(T(s)-\omega(s)\big) \tag{2.27}$$

and

$$\big(DF\big(T(s)\big) - \lambda_k(\omega(s))\big)\cdot \ddot S_k(s) = \ddot\lambda_k(s)\big(T(s)-\omega(s)\big) + 2\dot\lambda_k(s)\dot S_k(s) - D^2F\big(T(s)\big)\cdot \dot S_k(s)\otimes \dot S_k(s). \tag{2.28}$$

On the other hand, differentiating twice (2.4) w.r.t s, one finds

$$\begin{aligned}\big(DF\,(T(s)) - \lambda_k\,(\omega(s))\,\big)\cdot \dot T(s) &= \dot\zeta(s)\big(DF\,(\omega(s)) - \lambda_k\,(\omega(s))\,\big)\cdot r_k(\omega(s)) + \\ &\quad + \dot\zeta(s)\ r_k \bullet \lambda_k\big(\omega(s)\big)\big(T(s)-\omega(s)\big) \\ &= \dot\zeta(s)\ r_k \bullet \lambda_k\big(\omega(s)\big)\big(T(s)-\omega(s)\big),\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\big(DF\big(T(s)\big) - \lambda_k(\omega(s))\big)\cdot \ddot T(s) &= (\dot\zeta(s))^2\ r_k \bullet \big(r_k\bullet\lambda_k\big)\big(\omega(s)\big)\big(T(s)-\omega(s)\big) + 2\dot\zeta(s)\ r_k\bullet\lambda_k\big(\omega(s)\big)\ \dot T(s) - \\ &\quad - D^2F\big(T(s)\big)\cdot \dot T(s)\otimes \dot T(s) - (\dot\zeta(s))^2\ r_k\bullet\lambda_k\big(\omega(s)\big)\ r_k\big(\omega(s)\big).\end{aligned} \tag{2.30}$$

By comparing (2.27) and (2.29), we derive

$$\big(DF\,(T(s)) - \lambda_k\,(\omega(s))\,\big)\cdot\left(\dot T(s) - \frac{\dot\zeta(s)\ r_k\bullet\lambda_k\big(\omega(s)\big)}{\dot\lambda_k(s)}\dot S_k(s)\right) = 0 \qquad \forall\, s. \tag{2.31}$$

Since $\lambda_k(\omega(s)) > \lambda_k(T(s))$ for all $s \neq 0$, and since $|\dot S_k(s)| = |\dot T(s)| \equiv 1$, from (2.31) we recover (2.5) in the case $s \neq 0$. By continuity, (2.5) still holds if $s = 0$ and hence, recalling that $\dot S_k(0) = r_k\big(u\big)$, we derive (2.7). Finally, using (2.7) and evaluating (2.28), (2.30) at $s = 0$, we obtain

$$\Big(DF\big(u^0\big) - \lambda_k\big(u^0\big)\Big)\cdot\big(\ddot T(0) - \ddot S_k(0)\big) = 0$$

which, recalling that $\ddot S_k(0) = Dr_k\cdot r_k\big(u^0\big)$, yields (2.8). To conclude the proof we only need to observe that, differentiating (2.26) at $s = 0$ one finds $\dot T(0) = \big(\dot\mu(0) + \dot\zeta(0)\big)r_k\big(u^0\big)$, which by (2.7) implies $\dot\mu(0) + \dot\zeta(0) = 1$. Hence the map $s \mapsto \zeta(s) + \mu(s)$ is strictly increasing in a neighborhood of zero. □

Remark 2.2. Since by Theorem 2.1 the map $s \mapsto \zeta_k(u^0)(s) + \mu_k(u^0)(s),\ u^0 \in \Omega_k^0$, is strictly increasing, we can riparametrize the map

$$s \mapsto T_k(u^0)(s) = S_k\Big(R_k(u^0)\big[\zeta_k(u^0)(s)\big]\Big)\big[\mu_k(u^0)(s)\big] \tag{2.32}$$

as

$$\sigma \mapsto T_k(u^0)\big(\varsigma_k(u^0)^{-1}(\sigma)\big), \qquad \varsigma_k(u^0) \doteq \zeta_k(u^0) + \mu_k(u^0). \tag{2.33}$$

Thus, using still $\mu_k,\ \zeta_k,\ T_k$ to denote the riparametrized maps, we can write the mixed curve in the form

$$T_k(u^0)(\sigma) = S_k\Big(R_k(u^0)\big[\sigma - \mu_k(u^0)(\sigma)\big]\Big)\big[\mu_k(u^0)(\sigma)\big]. \tag{2.34}$$

Notice that, in the new parametrization one has $\zeta_k(u^0)(\sigma) = \sigma - \mu_k(u^0)(\sigma)$. On the other hand, by construction, there holds $d/d\sigma\ \zeta_k(u^0)(\sigma) < 0$ for all σ, which, in turn, implies

$$\frac{d}{d\sigma}\mu_k(u^0)(\sigma) > 1 \qquad \forall\ \sigma. \tag{2.35}$$

3. The Elementary Curves

Through each point $u \in \Omega$, we construct now the n elementary curves, one for each characteristic family, that are used to define the general entropy-admissible solution of the Riemann problem (1.1), (1.4). As observed in the introduction, we shall construct, alternatively, the *elementary curve of right states*, or the *elementary curve of left states* of the k-th characteristic family, depending on which inequality holds between (1.13) and (1.14).

First we need some more notations. For each $i = 1, \ldots, n$, we introduce two maps

i) $\delta_i : \Omega \to \mathbb{R}$, implicitly defined by

$$R_i(u)\big(-\delta_i(u)\big) \in \Omega_i^0; \tag{3.1}$$

ii) $\pi_i : \Omega \to \Omega_i^0$, defined by

$$\pi_i(u) \doteq R_i(u)\big(-\delta_i(u)\big). \tag{3.2}$$

Recalling the definition (1.2), one clearly has

$$r_i \bullet \lambda_i\big(\pi_i(u)\big) = 0 \qquad \forall\ u \in \Omega. \tag{3.3}$$

By construction, the maps $\delta_i, \pi_i,\ i = 1, \ldots, n$ are smooth. From (2.35) it follows that, for any $i = 1, \ldots, n$, the map $\sigma \mapsto \zeta_i(u^0)(s) \doteq s - \mu_i(u^0)(s)$ is strictly decreasing. Denote $\sigma \mapsto \zeta_i^{-1}(u^0)(\sigma)$ the inverse map of $s \mapsto \zeta_i\big(u^0\big)(s)$, and set

$$\nu_i(u) \doteq \mu_i\big(\pi_i(u)\big)\Big[\zeta_i^{-1}\big(\pi_i(u)\big)\big[\delta_i(u)\big]\Big], \qquad u \in \Omega. \tag{3.4}$$

Notice that, since

$$\zeta_i^{-1}\big(\pi_i(u)\big)\big[\delta_i(u)\big] = \delta_i(u) + \nu_i(u)$$

one has

$$R_i\big(\pi_i(u)\big)\Big[\zeta_i\big(\pi_i(u)\big)\big[\delta_i(u) + \nu_i(u)\big]\Big] = u.$$

Thus, from (2.34) we deduce

$$T_i\big(\pi_i(u)\big)\Big[\delta_i(u)+\nu_i(u)\Big] = S_i(u)\Big[\mu_i\big(\pi_i(u)\big)\big[\delta_i(u)+\nu_i(u)\big]\Big] = S_i(v)\big[\nu_i(u)\big]. \tag{3.5}$$

Call

$$\begin{aligned} \Omega_i^+ &\doteq \{u\in\Omega:\ r_i\bullet\lambda_i(u)>0\} \\ \Omega_i^- &\doteq \{u\in\Omega:\ r_i\bullet\lambda_i(u)<0\} \end{aligned} \qquad i=1,\dots,n. \tag{3.6}$$

Fix $u\in\Omega$, $k\in\{1,\dots,n\}$, and let $\mathcal{J}$ be the neighborhood of zero at Theorem 2.1. Then, define the elementary curve of the k-th characteristic family through u, $\Psi_k(u):\mathcal{J}\to\Omega$, as follows:

1. If (1.13) holds, then

$$\Psi_k(u)(s)=\begin{cases} S_k(u)(s) & \forall\ s \qquad\qquad \text{if}\quad u\in\Omega_k^0, \\[2mm] \begin{cases} S_k(u)(s) & \text{if}\ \ s<0 \\ R_k(u)(s) & \text{if}\ \ 0\le s\le -\delta_k(u) \\ T_k\big(\pi_k(u)\big)\big[s+\delta_k(u)\big] & \text{if}\ \ -\delta_k(u)<s\le\nu_k(u) \\ S_k(u,\,s) & \text{if}\ \ s>\nu_k(u) \end{cases} & \text{if}\quad u\in\Omega_k^+, \\[2mm] \begin{cases} S_k(u)(s) & \text{if}\ \ s>0 \\ R_k(u,\,s) & \text{if}\ \ -\delta_k(u)\le s\le 0 \\ T_k\big(\pi_k(u)\big)\big[s+\delta_k(u)\big] & \text{if}\ \ \nu_k(u)\le s<-\delta_k(u) \\ S_k(u)(s) & \text{if}\ \ s<\nu_k(u) \end{cases} & \text{if}\quad u\in\Omega_k^-. \end{cases} \tag{3.7}$$

2. If (1.14) holds, then

$$\Psi_k(u)(s)=\begin{cases} S_k(u)(s) & \forall\ s \qquad\qquad \text{if}\quad u\in\Omega_k^0, \\[2mm] \begin{cases} S_k(u)(s) & \text{if}\ \ s>0 \\ R_k(u)(s) & \text{if}\ \ -\delta_k(u)\le s\le 0 \\ T_k\big(\pi_k(u)\big)\big[s+\delta_k(u)\big] & \text{if}\ \ \nu_k(u)\le s<-\delta_k(u) \\ S_k(u,\,s) & \text{if}\ \ s<\nu_k(u) \end{cases} & \text{if}\quad u\in\Omega_k^+, \\[2mm] \begin{cases} S_k(u)(s) & \text{if}\ \ s<0 \\ R_k(u)(s) & \text{if}\ \ 0\le s\le -\delta_k(u) \\ T_k\big(\pi_k(u)\big)\big[s+\delta_k(u)\big] & \text{if}\ \ -\delta_k(u)<s\le\nu_k(u) \\ S_k(u)(s) & \text{if}\ \ s>\nu_k(u) \end{cases} & \text{if}\quad u\in\Omega_k^-. \end{cases} \tag{3.8}$$

Notice that (1.8), (2.5), (2.7) and (3.5) together, imply that, for any fixed $u\in\Omega$, $k\in\{1,\dots,n\}$, the map $s\mapsto\Psi_k(u)(s)$ is everywhere continuously differentiable. Thus, by the implicit function theorem, given u^L, $u^R\in\Omega$, we can uniquely determine intermediate states $u^L=\omega_0,\omega_1,\dots,\omega_n=u^R$ and wave sizes $\sigma_1,\dots,\sigma_n\in\mathcal{J}$, such that there holds

$$\begin{aligned} \omega_k &= \Psi_k(\omega_{k-1})(\sigma_k) \qquad \text{if} \qquad (1.13)\quad \text{occurs}, \\ \omega_{k-1} &= \Psi_k(\omega_k)(\sigma_k) \qquad \text{if} \qquad (1.14)\quad \text{occurs}. \end{aligned} \tag{3.9}$$

Then the solution to the Riemann Problem with initial data (1.4) can be constructed by piecing together the solutions to the Riemann problems with initial data

$$\overline{u}_k(x)=u_k(0,\,x)=\begin{cases} \omega_{k-1} & \text{if}\ x<0, \\ \omega_k & \text{if}\ x<0, \end{cases} \qquad k=1,\dots,n. \tag{3.10}$$

The solution to each Riemann problem with initial data $\overline{u}_k,\ k=1,\dots,n$, is defined as follows.

1. Assume that (1.13) holds. There are 3 possibilities:

1) $\omega_{k-1}\in\Omega_k^0$: the solution consists of a compressive shock of the k-th characteristic family travelling with speed $\lambda_k^s\big(\omega_{k-1},\,\sigma_k\big)$.

2) $\omega_{k-1}\in\Omega_k^+$: the solution consists of

a) $0\le\sigma_k\le-\delta_k(\omega_{k-1})$: a rarefaction wave of the k-th characteristic family, with characteristic speeds ranging over the interval

$$\big[\lambda_k(\omega_{k-1}),\ \lambda_k(\omega_k)\big];$$

b) $\sigma_k<0$ or $\sigma_k\ge\nu_k(\omega_{k-1})$: a compressive shock of the k-th characteristic family travelling with speed $\lambda_k^s\big(\omega_{k-1},\,\sigma_k\big)$;

c) $-\delta_k(\omega_{k-1})<\sigma_k<\nu_k(\omega_{k-1})$: a composed wave of the k-th characteristic family made of a a rarefaction wave of size

$$\widetilde{\sigma}_k\doteq\sigma_k-\mu_k\big(\pi_k(\omega_{k-1}),\ \sigma_k+\delta_k(\omega_{k-1})\big),$$

with characteristic speeds ranging over the interval

$$\big[\lambda_k(\omega_{k-1}),\,\lambda_k(\widetilde{\omega}_{k-1})\big],\qquad\qquad \widetilde{\omega}_{k-1}\doteq R_k\big(\omega_{k-1},\,\widetilde{\sigma}_k\big),$$

followed by a left-contact discontinuity $\widehat{\sigma}_k\doteq\sigma_k-\widetilde{\sigma}_k$, travelling with speed

$$\lambda_k^s\big[\widetilde{\omega}_{k-1},\,\widehat{\sigma}_k\big]=\lambda_k(\widetilde{\omega}_{k-1}).$$

3) $\omega_{k-1}\in\Omega_k^-$: the solution consists of

a) $-\delta_k(\omega_{k-1})\le\sigma_k\le 0$: a rarefaction wave as in 2-*a)*;

b) $\sigma_k>0$ or $\sigma_k<\nu_i(\omega_{k-1})$: a compressive shock as in 2-*b)*;

c) $\nu_k(\omega_{k-1})<\sigma_k<-\delta_k(\omega_{k-1})$: a composed wave as in 2-*c)*, made of a rarefaction wave followed by a left-contact discontinuity.

2. Assume that (1.14) holds. There are 3 possibilities:

1) $\omega_k\in\Omega_k^0$: the solution consists of a compressive shock of the k-th characteristic family travelling with speed $\lambda_k^s\big(\omega_k,\,\sigma_k\big)$.

2) $\omega_k\in\Omega_k^+$: the solution consists of

a) $-\delta_k(\omega_k)\le\sigma_k\le 0$: a rarefaction wave of the k-th characteristic family, with characteristic speeds ranging over the interval

$$\big[\lambda_k(\omega_{k-1}),\ \lambda_k(\omega_k)\big];$$

b) $\sigma_k>0$ or $\sigma_k<\nu_k(\omega_k)$: a compressive shock of the k-th characteristic family travelling with speed $\lambda_k^s\big(\omega_k,\,\sigma_k\big)$;

c) $\nu_k(\omega_k)<\sigma_k<-\delta_k(\omega_k)$: a composed wave of the k-th characteristic family made of a right-contact discontinuity of size

$$\widehat{\sigma}_k\doteq\mu_k\big(\pi_k(\omega_k),\ \sigma_k+\delta_k(\omega_k)\big),$$

travelling with speed

$$\lambda_k^s\big[\widetilde{\omega}_k,\,\widehat{\sigma}_k\big]=\lambda_k(\widetilde{\omega}_k),\qquad\qquad \widetilde{\omega}_k\doteq R_k\big(\omega_k,\,\sigma_k-\widehat{\sigma}_k\big),$$

followed by a rarefaction wave of size $\widetilde{\sigma}_k\doteq\sigma_k-\widehat{\sigma}_k$, with characteristic speeds ranging over the interval

$$\big[\lambda_k(\widetilde{\omega}_k),\,\lambda_k(\omega_k)\big].$$

3) $\omega_k\in\Omega_k^-$: the solution consists of

a) $0\le\sigma_k\le-\delta_k(\omega_k)$: a rarefaction wave as in 2-*a)*;

b) $\sigma_k<0$ or $\sigma_k\ge\nu_k(\omega_k)$: a compressive shock as in 2-*b)*;

c) $-\delta_k(\omega_k)<\sigma_k<\nu_k(\omega_k)$: a composed wave as in 2-*c)*, made of a right-contact discontinuity followed by a rarefaction wave.

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References

[Da] C.M. Dafermos, Regularity and large time behaviour of solutions of a conservation law without convexity, *Proc. Royal Soc. Edinburgh.* , **99 A** (1985), pp. 201-239.

[De] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[LA] P.D. Lax, Hyperbolic Systems of Conservation Laws II, *Comm. on Pure and Applied Math.*, **10** (1957), pp. 537-566.

[LI1] T.P. Liu, The Riemann problem for general 2×2 conservation laws, *Trans. Amer. Math. Soc.*, **199** (1974), pp. 89-112.

[LI2] T.P. Liu, The Riemann problem for general systems of conservation laws, *J. Differential Equations*, **18** (1975), pp. 218-234.

[LI3] T.P. Liu, Admissible solutions of hyperbolic conservation laws, *Memoir Amer. Math. Soc.*, **30** (1981), no. 240.

[S] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Second Edition, Springer-Verlag, 1994.