PERIODIC SOLUTIONS OF CONSERVATION LAWS CONSTRUCTED THROUGH GLIMM SCHEME

HERMANO FRID

ABSTRACT. In this note we present a periodic version of an adaptation of the Glimm scheme introduced by Nishida (1968) and further extended by Bakhvalov (1970) and DiPerna (1973). For special classes of 2×2 systems of conservation laws these adaptions of the Glimm scheme give global existence of solutions of the Cauchy problem with large initial data in $L^\infty \cap BV_{loc}(\mathbb{R})$, for Nishida-Bakhvalov's class, and in $L^\infty \cap BV(\mathbb{R})$, in the case of DiPerna's class. For periodic initial data, our periodic formulation establishes that the periodic solutions so constructed, $u(\cdot, t)$, are uniformly bounded in $L^\infty \cap BV([0, \ell])$, for all $t > 0$, where ℓ is the period. We then obtain the asymptotic decay of these solutions by applying a theorem of Chen-Frid in [2] combined with a compactness theorem of DiPerna in [5]. The question about the decay of Nishida's solution was proposed by Glimm-Lax [7] and remained open since then. The classes considered include systems motivated by isentropic gas dynamics.

1. INTRODUCTION

In [12], Nishida proved the global existence of weak solutions of the Cauchy problem for the system

$$v_t - u_x = 0,$$
$$u_t + (1/v)_x = 0,$$

constructed by an adaptation of Glimm's method, for any bounded initial data belonging to $BV_{loc}(\mathbb{R})$, assuming values in the region $v > \delta > 0$. According to Nishida's theorem, the L^∞ norm of these weak solutions may increase unboundedly with time. His result was later extended by Bakhvalov [1], who identified a class of 2×2 systems

$$U_t + F(U)_x = 0, \tag{1.1}$$

with $U = (u_1, u_2)$, $F(U) = (f_1(U), f_2(U))$, to which Nishida's reasonings could be applied. Here we will not a give a description of Bakhvalov's class but instead we will consider the main representative of this class:

$$\begin{cases} v_t - u_x = 0, \\ u_t + g(v)_x = 0, \end{cases} \tag{1.2}$$

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where g is a smooth function defined in $v>0$ satisfying

$$g'(v)<0, \quad g''(v)>0, \quad 3(g''(v))^2 \le 2g'(v)g'''(v), \quad \text{for } v>0. \tag{1.3}$$

For instance, $g(v)=1/v^\gamma$, with $0<\gamma\le 1$, or $g(v)=-v^\gamma$, with $0<\gamma<1$. It is not difficulty to show that if g_1 and g_2 satisfy (1.3) then $C_1g_1+C_2g_2$ satisfies (1.3), for any pair of nonnegative constants $(C_1,C_2)\ne(0,0)$.

Set $U=(v,u)$ and

$$z(U)\equiv u+\int^v\sqrt{-g'(v)}\,dv, \qquad w(U)\equiv u-\int^v\sqrt{-g'(v)}\,dv. \tag{1.4}$$

Consider the initial condition

$$U(x,0)=U_0(x), \qquad x\in\mathbb{R}, \tag{1.5}$$

with

$$0<\delta\le v_0(x)\le M, \qquad |u_0(x)|\le M, \tag{1.6}$$

for some positive constants δ, M. Suppose that

$$T.V.(U_0|[0,\ell))<\infty, \tag{1.7}$$

and

$$U_0(x+\ell)=U_0(x). \tag{1.8}$$

If $G(v)=-\int^v\sqrt{-g'(v)}\,dv \nrightarrow +\infty$ as $v\to 0$, we further assume that

$$w_0=\sup_{x\in\mathbb{R}} w(U_0(x))<z_0=\inf_{x\in\mathbb{R}} z(U_0(x)). \tag{1.9}$$

In this note we prove the following theorem concerning the problem (1.2)-(1.8).

Theorem 1.1. *There exists a global periodic entropy solution of* (1.2)-(1.9), *which belongs to* $L^\infty\cap BV_{loc}(\mathbb{R}\times\mathbb{R}_+)$. *This solution* $U(x,t)$ *also satisfies* $w(U(x,t))\le w_0$, $z(U(x,t))\ge z_0$. *Furthermore, it has the following decay property*

$$ess\lim_{t\to\infty}\int_0^\ell |U(x,t)-\bar U|\,dx=0, \tag{1.10}$$

where $\bar U=\frac{1}{\ell}\int_0^\ell U_0(x)\,dx$.

We recall that the problem of the decay of the large data solutions obtained by Nishida was proposed in [7] and remained open since then. On passing from Lagrangian to Eulerian coordinates, Theorem 1.1 gives a global periodic entropy solution in $L^\infty\cap BV_{loc}(\mathbb{R}\times\mathbb{R}_+)$ of the corresponding problem for the system

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ (\rho u)_t+\big(\rho u^2+p(\rho)\big)_x=0,\end{cases} \tag{1.11}$$

where $\rho=v^{-1}$ and $p(\rho)=g(1/\rho)$.

We also consider the class identified by DiPerna [4], with similar properties to that studied in [1]. We will not give a description of DiPerna's class but instead we will consider the main representative of this class:

$$\begin{cases}\rho_t+(\rho u)_x=0,\\ u_t+\big(\frac12u^2+\frac{\kappa^2}{\gamma-1}\rho^{\gamma-1}\big)_x=0, \quad \kappa>0,\ 1<\gamma<3,\end{cases} \tag{1.12}$$

which is motivated by the isentropic gas dynamics equations for a polytropic gas. Let $U=(\rho,u)$, $F(U)=\frac{1}{2}u^2+\frac{\kappa^2}{\gamma-1}\rho^{\gamma-1}$, $\rho>0$, and

$$z(U)=u-\frac{\kappa}{\beta}\rho^\beta,\qquad w(U)=u+\frac{\kappa}{\beta}\rho^\beta,\quad \beta=\frac{1}{2}(\gamma-1). \tag{1.13}$$

Denote $v=\rho^{-1}$ and consider the initial conditions (1.5)-(1.8) and

$$z_0=\sup_{x\in\mathbb{R}} z(U_0(x))<w_0=\inf_{x\in\mathbb{R}} w(U_0(x)). \tag{1.14}$$

In this note we also prove the following result concerning the system (1.12).

Theorem 1.2. *There exists a global entropy solution of* (1.12), (1.5)-(1.8), (1.14), *which belongs to* $L^\infty\cap BV_{loc}(\mathbb{R}\times\mathbb{R}_+)$ *and is periodic with period* ℓ. *This solution* $U(x,t)$ *also satisfies* $z(U(x,t))\le z_0$, $w(U(x,t))\ge w_0$. *Moreover, it accomplishes the decay property* (1.10).

In both Theorems 1.1 and 1.2, the global boundedness of the periodic solutions is obtained by means of a periodic version of Nishida's modification of the Glimm scheme. The decay property (1.10) is then a consequence of a combination of a theorem in [2] and a compactness theorem in [5]. The remaining of this paper is organized as follows. In section 2 we address the existence of uniformly bounded periodic solutions for systems (1.2) in Bakhvalov's class, proving the first part of Theorem 1.1. In section 3 we address the same question for the systems (1.12), proving the first part of Theorem 1.2. In section 4 we then show how the decay property (1.10) can be achieved in both cases.

2. Systems in the class of Nishida-Bakhvalov

We recall that a weak solution of (1.1), (1.5) is a locally integrable function defined over $\mathbb{R}\times[0,\infty)$ satisfying

$$\iint_{\mathbb{R}\times[0,\infty)} U\phi_t+F(U)\phi_x\,dx\,dt+\int_{\mathbb{R}} U_0(x)\phi(x,0)\,dx=0, \tag{2.1}$$

for any $\phi\in C_0^1(\mathbb{R}^2)$. Assume that (1.1) is endowed with a strictly convex entropy $\eta(U)$ with entropy flux $q(U)$, that is, η,q satisfy

$$\nabla\eta(U)\nabla F(U)=\nabla q(U). \tag{2.2}$$

We then say that $U(x,t)$ is an entropy solution to (1.1), (1.5) if it is a weak solution of this problem and satisfies

$$\iint_{\mathbb{R}\times[0,\infty)} \eta(U)\phi_t+q(U)\phi_x\,dx\,dt+\int_{\mathbb{R}} \eta(U_0(x))\phi(x,0)\,dx\ge 0, \tag{2.3}$$

for any $\phi\in C_0^1(\mathbb{R}^2)$, with $\phi\ge0$.

System (1.1) is strictly hyperbolic if $F'(U)$ has real and distinct eigenvalues $\lambda_1(U)<\lambda_2(U)$. It is genuinely nonlinear if the right eigenvectors of $F'(U)$, r_1 and r_2, corresponding to λ_1 and λ_2, respectively, satisfy $r_i\cdot\nabla\lambda_i\neq0$, for $i=1,2$. A pair of functions $z(U)$, $w(U)$ is a pair of Riemann invariants for (1.1) if $\nabla z\cdot r_2=\nabla w\cdot r_1=0$ everywhere.

Given a point U_0, the rarefaction curves by U_0 are constituted by the points U which can be connected to U_0 on the right by a rarefaction wave, that is, a solution of (1.1), with initial data

$$U_0(x) = \begin{cases} U_0, & x < 0 \\ U, & x > 0 \end{cases}, \tag{2.4}$$

continuous for $t > 0$ and depending only on x/t. For 2×2 systems there are 2 such curves defined by

$$\begin{aligned} &\frac{dU}{d\xi} = \tilde{r}_i(U), \quad \xi \geq \lambda_i(U_0), \quad i = 1, 2, \\ &U(\lambda_i(U_0)) = U_0, \end{aligned}$$

where $\tilde{r}_i$ are the right eigenvectors normalized by $\nabla \lambda_i \cdot \tilde{r}_i = 1$. In particular, the Riemann invariant z is constant along the rarefaction curves of the second characteristic family while w is constant along the ones of the first characteristic family.

We also remember that the shock curves by U_0 consist of the points U satisfying the Rankine-Hugoniot relations

$$s(U - U_0) = F(U) - F(U_0), \tag{2.5}$$

where s is the shock speed. They give rise to solutions of the problem (1.1), (2.4) defined by

$$U(x,t) = \begin{cases} U_0, & x < st \\ U, & x > st \end{cases}.$$

A shock connecting U_r on the right of U_l is of the first kind if

$$\lambda_1(U_r) < s < \lambda_1(U_l), \quad s < \lambda_2(U_r),$$

and it is of the second kind if

$$\lambda_2(U_r) < s < \lambda_2(U_l), \quad s > \lambda_1(U_l).$$

In this section we consider the strictly hyperbolic genuinely nonlinear system (1.2) for which $U = (v, u)$ and $F(U) = (f_1(U), f_2(U)) \equiv (-u, g(v))$, $\lambda_1 = -\sqrt{-g'(v)}$, $\lambda_2 = \sqrt{-g'(v)}$, $r_1 = (1, \sqrt{-g'(v)})^\top$, $r_2 = (-1, \sqrt{-g'(v)})^\top$, and z, w given in (1.4) form a pair of Riemann invariants. System (1.2) is endowed with the following strictly convex entropy and associated entropy flux:

$$\eta(U) = \frac{u^2}{2} - \int^v g(v)\, dv, \qquad q(U) = ug(v). \tag{2.6}$$

Let $\mathcal{R}$ be the region of the v,u-plane given by

$$\mathcal{R} = \{U \in \mathbb{R}^2 \,:\, z(U) \geq z_0,\ w(U) \leq w_0\},$$

with z_0 and w_0 given in (1.9). Given a point $U_0 \in \mathcal{R}$, there are curves $R_i(\gamma; U_0)$, $i = 1, 2$, with $R_i(0; U_0) = U_0$, such that, for $\gamma \geq 0$, U_0 can be connected to $R_i(\gamma; U_0)$ on the right by an i-rarefaction wave, and for $\gamma < 0$, U_0 can be connected to $R_i(\gamma; U_0)$ on the right by an i-shock wave. We also define the curves $L_i(\gamma; U_0)$, $i = 1, 2$, satisfying $L_i(0; U_0) = U_0$, such that, for $\gamma \geq 0$, U_0 can be connected to $L_i(\gamma; U_0)$ on the left by an i-rarefaction wave, and for $\gamma < 0$, U_0 can be connected to $L_i(\gamma; U_0)$ on the left by an i-shock wave. Given points $U_1, U_2 \in \mathcal{R}$, the Riemann problem with left state U_1 and right state U_2, that is, the initial value problem for (1.2), with initial data

$$U_0(x) = \begin{cases} U_1, & x < 0 \\ U_2, & x > 0 \end{cases}, \tag{2.7}$$

is solved in the following way. We find the point $\iota(U_1, U_2) = R_1(\gamma_1; U_1) = L_2(\gamma_2; U_2)$, the intersection of $R_1(\gamma; U_1)$ with $L_2(\gamma; U_2)$. By the properties of the system (1.2), such a point $\iota(U_1, U_2)$ always exists, is unique, and belongs to R (see [12, 1]). Then the solution is obtained by connecting U_1 to $\iota(U_1, U_2) = R_1(U_1; \gamma_1)$ on the right by a 1-shock (if $\gamma_1 < 0$) or a 1-rarefaction wave (if $\gamma_1 \geq 0$) and then connecting $\iota(U_1, U_2)$ to U_2 by a 2-shock (if $\gamma_2 < 0$) or a 2-rarefaction (if $\gamma_2 \geq 0$). The properties of the system (1.2), described in [12, 1], also ensure that the solution of the Riemann problem (1.2) and (2.7) takes its values in the region R.

The construction of approximate solutions of (1.1), (1.5) proposed by Glimm is well known by now (see, *e.g.*, [6, 14, 8]). Let $R > 0$, U_* be a point in $\mathcal{R}$ and let

$$\mathcal{V} = \{U \in \mathcal{R} : |U - U_*| < 2R\}. \tag{2.8}$$

Assume the initial data $U_0(x)$ satisfies

$$\|U_0 - U_*\|_\infty < R, \qquad \mathrm{TV}\,(U_0|[0, \ell)) < \nu(R), \tag{2.9}$$

where $\mathrm{TV}\,(U_0|[0, \ell))$ denotes the total variation of U_0 over one period and $\nu(R)$ is to be suitably chosen small enough such that for any two points $U_1, U_2 \in B(U_*; R)$, the open ball centered in U_* with radius R, satisfying $|U_1 - U_2| < \nu(R)$, the solution of the Riemann problem (1.1), (2.7) takes its values in $\mathcal{V}$.

Given mesh widths $l = \Delta x$, $h = \Delta t$, satisfying the Courant-Friedrichs-Lewy condition

$$Ah \leq l, \qquad A = \sup\{|\nabla F(U)| : U \in \mathcal{V}\}, \tag{2.10}$$

we start by taking a piecewise constant approximation of the initial data, say, by setting

$$U^h(x, 0) = \frac{1}{2l} \int_{ml}^{(m+2)l} U_0(x)\, dx, \quad \text{for } ml < x < (m+2)l, \text{ and } m \in \mathbb{Z} \text{ is even.}$$

We define the approximate solution in the strip $\mathbb{R} \times [0, h)$ by solving Riemann problems for each of the discontinuities at the points $(ml, 0)$, m even; condition (2.10) guarantees that these Riemann solutions will not interact in this strip. To define the solution in the next strip $\mathbb{R} \times [h, 2h)$ we first choose a number $\theta_1 \in (-1, 1)$ and set

$$U^h(x, h) = U^h((m + 1 + \theta_1)l, h - 0), \qquad \text{if } ml < x < (m+2)l, \text{ and } m \text{ is odd.}$$

Then we proceed as in the first step solving Riemann problems. The process then continues in the same way for the successive strips $\mathbb{R} \times [kh, (k+1)h)$, $k = 2, 3, \ldots$, making use of a sampling sequence $\theta_k \in (-1, 1)$ to define U^h as a piecewise constant function at $t = kh$, and then solving Riemann problems. The method can be carried out as long as U^h assumes its values in $\mathcal{V}$.

Actually, for systems in Nishida-Bakhvalov's class, the eigenvalues satisfy the apriori bound

$$\sup\{|\lambda_i(U)| : U \in \mathcal{R}\} \leq \Lambda, \qquad i = 1, 2, \tag{2.11}$$

for some constant $\Lambda > 0$. So, the verification of the condition (2.10), and hence the possibility of carring out indefinitely the construction of the approximate solutions, would not require the uniform boudedness of the approximate solution, which in general does not hold. Nevertheless, our main concern here is to prove the uniform boundedness of the approximate solutions in the case of periodic initial data, so we will not take advantage of (2.11). On the other hand, this procedure will be necessary even for the construction of the approximate solutions in the case of systems in DiPerna's class, which will be considered in the next section, for which there is no analogue of (2.11).

From the construction described above we immediately have.

Lemma 2.1. *The approximate solution $U^h(\cdot,t)$ is periodic with period ℓ for any $t > 0$ in its interval of existence.*

For $U_1, U_2 \in \mathcal{R}$ we define

$$d(U_1,U_2) = \big(z(U_1) - z(\iota(U_1,U_2))\big)_+ + \big(w(\iota(U_1,U_2)) - w(U_2)\big)_+ \tag{2.12}$$

with z, w given in (1.4) and $\iota(U_1,U_2)$ being, as above, the median state in the solution of the Riemann problem (1.2), (2.7). As usual, for any real number y, we denote $(y)_+ = \max\{y, 0\}$. The special feature of Nishida-Bakhvalov's system (1.2) is sumarized in the following lemma whose proof is given in detail in [1].

Lemma 2.2 (Nishida [12], Bakhvalov [1])**.** *Let $U_1, U_2, U_3 \in \mathcal{R}$. Then*

$$d(U_1,U_3) \le d(U_1,U_2) + d(U_2,U_3). \tag{2.13}$$

Moreover, if U_2 is a value assumed by the solution of the Riemann problem with left state U_1 and right state U_3 then equality holds in (2.13).

For a function $U = (v,u) \in BV_{loc}(\mathbb{R},\mathbb{R}^2)$, we define

$$\mu([a,b]) = \sup \sum_{k=0}^{N-1} d(U(x_k),U(x_{k+1})), \qquad i = 1,2 \tag{2.14}$$

where the supremum is taken over all partitions $a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a,b]$. Lemma 2.2 implies that μ extends to a Radon measure over $\mathbb{R}$. Let $U^h = (v^h,u^h)$ be an approximate solution constructed by Glimm's method. The periodic version of the Glimm functional, as adapted by Nishida [12], is defined by

$$F_{per}(U^h(t)) = \mu^{h,t}([a,a+\ell)),$$

where $\mu^{h,t}$ is the measure obtained from $U^h(\cdot,t)$ by (2.14) and $a \in \mathbb{R}$ is arbitrary.

A trivial but crucial property of F_{per} is stated in the following lemma.

Lemma 2.3. *F_{per} does not depend on a,* i.e., *$\mu^{h,t}([a,a+\ell)) = \mu^{h,t}([0,\ell))$, for any $a \in \mathbb{R}$.*

Proof. Immediate from the definition of F_{per} and Lemma 2.1. □

We take $l = \ell/(2N)$, for some positive integer N and hold fixed h/l satisfying (2.10). To prove Theorem 1.1 we need the following decisive proposition.

Proposition 2.1. *$F_{per}(U^h(t))$ is non-increasing in t.*

Proof. Since $F_{per}(U^h(t))$ is constant for $t \in (kh,(k+1)h)$, $k \in \mathbb{N}$, because of the periodicity (Lemma 2.1) and independence with respect to a (Lemma 2.3), the proof of the proposition reduces to showing that

$$F_{per}(U^h(kh+0)) \le F_{per}(U^h(kh-0)). \tag{2.15}$$

Let $m_0 \le m \le m_0 + 2N$, with $m, m_0 \in \{m' \in \mathbb{Z} \mid m' + k \text{ is even}\}$, and set $a = (m_0 + \theta_k + 1)l$, $a_1 = (m + \theta_k + 1)l$, $a_2 = (m + \theta_k + 3)l$, and then $a + \ell = (m_0 + \theta_k + 2N + 1)l$. Denote $U(x) = U^h(x, kh-0)$ and let $\widetilde{U}(x)$ be equal to $U(x)$ if $x \notin [a_1, a_2)$ and equal to $U^h(x, kh+0)$ if $x \in [a_1, a_2)$. A trivial induction reduces the proof of (2.15) to proving that

$$F_{per}(\widetilde{U}) \le F_{per}(U).$$

But Lemma 2.2 implies directly the above inequality. Indeed, if $U_1 = U^h(a_1, kh-0)$, $U_2 = U^h((m+2)l, kh-0)$ and $U_3 = U^h(a_2, kh-0)$, we have

$$\mu^{h,kh-0}([a_1,a_2)) = d(U_1,U_2) + d(U_2,U_3) \ge d(U_1,U_3) = \mu^{h,kh+0}([a_1,a_2)).$$

We then arrive at the conclusion that $F(U^h(t))$ is decreasing in time as desired. □

As a direct consequence of Proposition 2.1 we have the following result.

Proposition 2.2. *$TV(U^h(t)|[0,\ell))$ is uniformly bounded for $t > 0$.*

Proof. The properties of system (1.2) described in [1] imply that the measure μ is concentrated at the shocks. This is due to the fact that the Riemann invariants z, w decrease across shocks of both characteristic families and increase across rarefaction waves of the corresponding family, being constant across rarefaction waves of the complementary family. It is also a consequence of the properties of system (1.2) described in [1] the fact that across shocks of the first family one has $|[w]| \le |[z]|$ and for shocks of the second family one has $|[z]| \le |[w]|$, where as usual the square brackets denote the jump of the enclosed function across the discontinuity. Taking the latter into account and the periodicity we have that

$$\begin{aligned}
\mathrm{TV}\,(U^h(t)|[0,\ell)) &\le C_1\big(\mathrm{TV}\,\{z(U^h(t)|[0,\ell)\} + \mathrm{TV}\,\{w(U^h(t))|[0,\ell)\}\big)\\
&\le 2C_1\big(\mathrm{DV}\,\{z(U^h(t)|[0,\ell)\} + \mathrm{DV}\,\{w(U^h(t)|[0,\ell)\}\big)\\
&\le 4C_1 F(U^h(t)) \le 4C_1 F(U^h(0)) \le 4C_1C_2 \mathrm{TV}\,(U^h(0)|[0,\ell))\\
&\le 4C_1C_2\mathrm{TV}\,(U_0|[0,\ell)),
\end{aligned}$$

for certain positive constants C_1, C_2 depending only on F and R, where DV denotes the decreasing variation. So, we have the uniform boundedness of the variation over one period. □

We then impose the following bound on the total variation over one period of the initial data

$$4C_1C_2\mathrm{TV}\,(U_0|[0,\ell)) \equiv r_0 < r < R, \tag{2.16}$$

for some $0 < r < R$. Let

$$\bar{U} = \frac{1}{\ell}\int_0^\ell U_0(x)\,dx.$$

Our goal now is to prove the following proposition:

Proposition 2.3. *There exist a sequence of approximate solutions constructed as above which converge almost everywhere to an entropy solution of* (1.2), (1.5), *such that, given any* $t > 0$, *the approximate solutions* U^h *are defined up to certain time* $\bar{t} > t$, *provided* h *is smaller than certain* $h_0(\bar{t}) > 0$, *and satisfy*

$$\|U^h(t) - \bar{U}\|_\infty < R, \qquad 0 \le t < \bar{t}. \tag{2.17}$$

In order to prove Proposition 2.3 we need a standard result which is a consequence of the uniform bound of the total variation over one period.

Lemma 2.4. *The approximate solutions satisfy*

$$\int_0^\ell |U^h(x,t_1) - U^h(x,t_2)|\,dx \le C(|t_1 - t_2| + h), \tag{2.18}$$

for t_1, t_2 *belonging to the interval of existence of* U^h, *where* C *depends only on* C_1, C_2 *and* $TV(U_0|[0,\ell))$.

Proof. Follows as the one for the analogous result in [6] (see also [14, 8]). In this case we should also use the periodicity in x of U^h. □

The following lemma is an important step in the proof of Proposition 2.3.

Lemma 2.5. *Let* $0 < T < (R-r)\ell/2C$, *with* r, R *as in* (2.16) *and* C *given in* (2.18). *Assume that for* h *sufficiently small a sequence of approximate solutions* U^h *is defined up to certain* $\bar{t} > NT$, *for some* $0 \le N \in \mathbb{Z}$ *satisfying both*

$$\|U^h(t) - \bar{U}\|_\infty < R, \qquad 0 \le t \le NT,$$
$$\Big|\frac{1}{\ell}\int_0^\ell U^h(x,NT)\,dx - \bar{U}\Big| < \varepsilon < r - r_0.$$

Then, if $h < T$, *the approximate solutions may be extended up to a certain time* $\bar{\bar{t}} > (N+1)T$ *satisfying*

$$\|U^h(t) - \bar{U}\|_\infty < R, \qquad 0 \le t \le (N+1)T. \tag{2.19}$$

In particular, from (2.8)-(2.9) *it follows that the values of* $U^h(t)$ *lie in* $\mathcal{V}$, *for* $0 \le t \le (N+1)T$.

Proof. For $NT < t < \bar{t}$, applying Lemma 2.4 we get

$$\begin{aligned}\|U^h(t) - \bar{U}\|_\infty &\le \|U^h(t) - \frac{1}{\ell}\int_0^\ell U^h(x,t)\,dx\|_\infty \\ &\quad + \Big|\frac{1}{\ell}\int_0^\ell U^h(x,t)\,dx - \frac{1}{\ell}\int_0^\ell U^h(x,NT)\,dx\Big| + \Big|\frac{1}{\ell}\int_0^\ell U^h(x,NT)\,dx - \bar{U}\Big| \\ &\le \mathrm{TV}\,\{U^h(t)|[0,\ell)\} + \frac{C}{\ell}(t - NT + h) + \varepsilon \\ &\le r_0 + \varepsilon + \frac{C}{\ell}(t - NT + h).\end{aligned}$$

So, if T and h satisfy the required inequalities, it is possible to prolong U^h to an interval $[0,\bar{\bar{t}})$, with $\bar{\bar{t}} > (N+1)T$ verifying the estimate (2.19).

□

We are now ready to prove Proposition 2.3.

Proof of the Proposition 2.3. By Lemma 2.5 with $N = 0$ all U^h are defined in the strip $\mathcal{S}_T = \mathbb{R} \times [0, T]$. Applying Glimm's consistence proof, we obtain a set

$$\Theta_1 \subset \Theta = \Pi_{k=1}^{\infty}(-1, 1),$$

with $meas(\Theta_1) = 1$ such that any convergent subsequence of $\{U^h\}$, constructed with a sampling sequence belonging to Θ_1, converges to a weak solution of (1.2)-(1.5), defined in the strip $\mathcal{S}_T$. So, we take $\theta^{(1)} = \{\theta_k^{(1)}\}_{k=1}^{\infty} \in \Theta_1$ and a subsequence $\{h_s\}$, $s \in \mathbb{N}$, corresponding to a convergent subsequence $\{U^{h_s}\}$. Let U be the limit of this subsequence, which is defined in the same strip. Since U is a weak solution of (1.2)-(1.5) in $\mathcal{S}_T$, and is clearly periodic with period ℓ, we must have

$$\frac{1}{\ell}\int_0^{\ell} U(x, T)\, dx = \bar{U}, \tag{2.20}$$

which is immediate from (1.2) and the periodicity. Let $\varepsilon > 0$ satisfy $\varepsilon < r - r_0$, with r, r_0 given by (2.16). By (2.20), given $\theta^{(1)} \in \Theta_1$, there exists an $h_0 = h_0(\varepsilon, \theta^{(1)}) < T$, such that, for all $h_s < h_0$

$$|\frac{1}{\ell}\int_0^{\ell} U^{h_s}(x, T)\, dx - \bar{U}| < \varepsilon.$$

But then, again by Lemma 2.5, with this $\theta^{(1)} \in \Theta_1$ fixed, to be used in the strip $[0, T]$, we may extend the approximate solutions U^{h_s}, with $h_s < h_0$, to the strip $\mathcal{S}_{2T} = \mathbb{R} \times [0, 2T]$, applying Glimm's scheme, using any random sequence in the strip $(T, 2T]$. Now, using again, Glimm's consistence proof, we obtain a Θ_2, with $meas(\Theta_2) = 1$, such that any convergent subsequence of $\{U^{h_s}\}$, extended to the strip $(T, 2T]$ by means of a sampling sequence belonging to Θ_2, converges to a weak solution of (1.2)-(1.5), defined in the strip $\mathcal{S}_{2T}$. We then fix $\theta^{(2)} \in \Theta_2$. Then, there exists an $h_1 = h_1(\varepsilon, \theta^{(2)}) < h_0$, such that, for all $h_s < h_1$,

$$|\frac{1}{\ell}\int_0^{\ell} U^{h_s}(x, 2T)\, dx - \bar{U}| < \varepsilon.$$

Again we can apply Lemma 2.5, and for the fixed $\theta^{(1)} \in \Theta_1$, which is to be used in the strip $[0, T]$, and $\theta^{(2)} \in \Theta_2$, which is to be used in the strip $(T, 2T]$, the approximate solutions U^{h_s}, with $h_s < h_1$, may be extended using Glimm's scheme to the strip $\mathcal{S}_{3T} = \mathbb{R} \times [0, 3T]$, using any random sequence in the strip $(2T, 3T]$. Apply Glimm's consistence proof to obtain a Θ_3, with $meas(\Theta_3) = 1$, such that any convergent subsequence of $\{U^{h_s}\}$, extended to the strip $(2T, 3T]$ by means of a sampling sequence belonging to Θ_3, converges to a weak solution of (1.2)-(1.5), defined in the strip $\mathcal{S}_{3T}$, etc.. It is clear that this process may be continued indefinitely. A subsequence satisfying the statement of Proposition 2.3 is then obtained by a standard diagonal process. The fact that the global weak solution so constructed is an entropy solution follows from a well known theorem of Lax [10]. □

The results in this section clearly imply the existence and uniform boundedness of a periodic entropy solution of (1.2), (1.5) proving Theorem 1.1, except for the last assertion concerning the decay property (1.10). The latter will follow from the results in Section 4.

3. Systems in DiPerna's class

In this section we consider the strictly hyperbolic genuine nonlinear systems in DiPerna's class given in (1.12). For this system we have $U = (\rho, u)^\top$, $F(U) = (u\rho, \frac{1}{2}u^2 + \frac{\kappa}{\gamma-1}\rho^{\gamma-1})^\top$, the eigenvalues are $\lambda_1 = u - \kappa\rho^\beta$, $\lambda_2 = u + \kappa\rho^\beta$, where $\beta = (\gamma-1)/2$, with corresponding eigenvectors $r_1 = (-\rho, \kappa\rho^\beta)^\top$, $r_2 = (\rho, \kappa\rho^\beta)^\top$, respectively. The Riemann invariants are given by (1.13). System (1.12) is endowed with the following strictly convex entropy and associated entropy flux

$$\eta(U) = \frac{\kappa^2}{(\gamma-1)(\gamma-2)}\rho^{\gamma-1} + \frac{u^2}{2},\ q(U) = \frac{\kappa^2}{(\gamma-2)}\rho^{\gamma-1}u + \frac{u^3}{3}.$$

The analysis here is quite similar to the one in the last section for the systems in Nishida-Bakhvalov's class, so we will just give an outline paying more attention to the points where it differs from the latter.

Let $\bar{\mathcal{R}}$ be the region of the ρ, u-plane given by

$$\bar{\mathcal{R}} = \{U \in \mathbb{R}^2\ :\ z(U) \le z_0,\ w(U) \ge w_0\},$$

with z_0 and w_0 given in (1.14). Again, given any two points $U_1, U_2 \in \bar{\mathcal{R}}$, the Riemann problem (1.12),(2.7) has a unique solution whose values remain in $\bar{\mathcal{R}}$. The construction of approximate solutions through the Glimm scheme is described in the above section, and Lemma 2.1 holds in general. Denote

$$\sigma = w + z = 2u, \qquad \zeta = w - z = \frac{2\kappa}{\beta}\rho^\beta.$$

The properties described in [4] imply that σ decreases from left to right across shocks of both characteristic families and increases across rarefaction waves of both families. For $U_1, U_2 \in \bar{\mathcal{R}}$ we define

$$\bar{d}(U_1, U_2) = \big(\sigma(U_1) - \sigma(\iota(U_1, U_2))\big)_+ + \big(\sigma(\iota(U_1, U_2)) - \sigma(U_2)\big)_+ \tag{3.1}$$

with $\iota(U_1, U_2)$ as in the last section. For $\bar{d}$ we have the following analogue of Lemma 2.2, whose detailed proof is given in [4].

Lemma 3.1 (DiPerna [4])**.** *Let $U_1, U_2, U_3 \in \bar{\mathcal{R}}$. Then*

$$\bar{d}(U_1, U_3) \le \bar{d}(U_1, U_2) + \bar{d}(U_2, U_3). \tag{3.2}$$

Moreover, if U_2 is a value assumed by the solution of the Riemann problem with left state U_1 and right state U_3 then equality holds in (3.2)*.*

Again, for a function $U = (v, u) \in BV_{loc}(\mathbb{R}, \mathbb{R}^2)$, we define

$$\mu([a,b]) = \sup \sum_{k=0}^{N-1} \bar{d}(U(x_k), U(x_{k+1})), \qquad i = 1, 2 \tag{3.3}$$

with supremum taken over all partitions $a = x_0 < x_1 < \cdots < x_N = b$ of the interval $[a,b]$. Lemma 3.1 implies that μ extends to a Radon measure over $\mathbb{R}$. Let $U^h = (\rho^h, u^h)$ be an approximate solution constructed by Glimm's method. Again we define

$$F_{per}(U^h(t)) = \mu^{h,t}([a, a+\ell)),$$

where $\mu^{h,t}$ is obtained from $U^h(\cdot, t)$ by (3.3) and $a \in \mathbb{R}$ is arbitrary. Clearly, Lemma 2.3 holds also here. Similarly, Proposition 2.1 remains true with identical proof.

Now, in order to obtain the analogue of Proposition 2.2 we need the following result.

Lemma 3.2. *Let (σ,ζ) be on an arbitrary shock curve with initial point (σ_0,ζ_0). Then*

$$|\zeta-\zeta_0| \le \frac{1}{\sqrt{\beta}}|\sigma-\sigma_0|. \tag{3.4}$$

Proof. Following [4], we notice that the Rankine-Hugoniot relations in σ,ζ variables yield

$$\begin{aligned}\left(\frac{\sigma-\sigma_0}{\zeta-\zeta_0}\right)^2 &= \beta\left(\frac{\zeta+\zeta_0}{\zeta^{1/\beta}+\zeta_0^{1/\beta}}\right)\left(\frac{\zeta^{1/\beta}-\zeta_0^{1/\beta}}{\zeta-\zeta_0}\right)\\ &= \beta\frac{(\epsilon+1)}{(\epsilon-1)}\frac{(\epsilon^{1/\beta}-1)}{(\epsilon^{1/\beta}+1)} \equiv W(\epsilon),\end{aligned}$$

where $\epsilon=\eta/\eta_0$. Now, as stated in [4], since $0<\beta<1$, we have $\frac{dW}{d\epsilon}>0$, for $0<\epsilon<1$, and $\frac{dW}{d\epsilon}<0$, for $1<\epsilon<\infty$. Since $W(\epsilon)\to\beta$ as $\epsilon\to 0$ and $W(\epsilon)\to\beta$ when $\epsilon\to+\infty$, we deduce that $W(\epsilon)\ge\beta$, and hence we get (3.4). □

The proof of Proposition 2.2 in the present context now follows as in the last section, the main estimate becoming

$$\begin{aligned}\mathrm{TV}\,(U^h(t)|[0,\ell)) &\le C_1\big(\mathrm{TV}\,\{z(U^h(t)|[0,\ell)\} + \mathrm{TV}\,\{w(U^h(t))|[0,\ell)\}\big)\\ &= 2C_1\big(\mathrm{DV}\,\{z(U^h(t)|[0,\ell)\} + \mathrm{DV}\,\{w(U^h(t))|[0,\ell)\}\big)\\ &\le 2C_1\sum_{\text{across shocks}}\{|[\sigma]|+|[\zeta]|\}\\ &\le 2C_1(1+\frac{1}{\sqrt{\beta}})\big(\mathrm{DV}\,\{\sigma(U^h(t)|[0,\ell)\}\big)\\ &= 2C_1(1+\frac{1}{\sqrt{\beta}})F(U^h(t)) \le 2C_1(1+\frac{1}{\sqrt{\beta}})F(U^h(0))\\ &\le 2(1+\frac{1}{\sqrt{\beta}})C_1C_2\mathrm{TV}\,(U^h(0)|[0,\ell))\\ &\le 2(1+\frac{1}{\sqrt{\beta}})C_1C_2\mathrm{TV}\,(U_0|[0,\ell)),\end{aligned}$$

for certain positive constants C_1, C_2 depending only on F and R. Here, we have used the fact that the Riemann invariants increase from left to right across rarefaction waves, so that the decreasing variation is concentrated on the shocks. We then impose the following bound on the total variation over one period of the initial data

$$2(1+\frac{1}{\sqrt{\beta}})C_1C_2\mathrm{TV}\,(U_0|[0,\ell)) \equiv r_0 < r < R, \tag{3.5}$$

for some $0<r<R$.

From Proposition 2.2 we again obtain Lemma 2.4 and then Lemma 2.5. Finally, the analogue of Proposition 2.3 in the present context now follows exactly as in the last section for systems in Nishida-Bakhvalov's class. In this way we have proved the existence and uniform boundedness of a periodic entropy solution of (1.12), (1.2), as asserted in Theorem 1.2. To conclude the proof of Theorem 1.2 it remains to prove the decay property (1.10) which will follow from the results in Section 4.

4. Decay of Periodic Solutions

In this section we show how the decay property of the periodic solutions obtained in Sections 1 and 2, as asserted in Theorems 1.1 and 1.2, can be achieved. In sum, it will be a consequence of their uniform boundedness and the fact that the systems considered are strictly hyperbolic and genuinely nonlinear, by applying the results in [2] and [5]. As we will see these decay results show a remarkable interplay between Glimm's [6] and DiPerna's [5] framework, which meet each other through the link provided by the following theorem of [2].

Theorem 4.1 (Chen-Frid [2]). *Assume that* (1.1) *is endowed with a strictly convex entropy, and let* $U \in L^\infty(\mathbb{R}\times\mathbb{R}_+)$ *be a periodic entropy solution of* (1.1)-(1.5)*, with period, say, one. Denote* $U^T(x,t) = U(Tx,Tt)$*. If the sequence* U^T*,* $T\to\infty$*, is compact in* $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$ *then one has*

$$ess \lim_{t\to\infty} \int_0^1 |U(x,t)-\bar U|\,dx = 0,$$

where $\bar U = \int_0^1 U_0(x)\,dx$.

We also recall here the following theorem of DiPerna [5].

Theorem 4.2 (DiPerna [5]). *Assume* (1.1) *is a strictly hyperbolic* 2×2 *system with both characteristic families genuinely nonlinear. Let* U^T*,* $T\in I$*, for some index set* I*, be a family of entropy solutions of initial value problems for* (1.1)*, which is uniformly bounded in* $L^\infty(\mathbb{R}\times\mathbb{R}_+)$*. Then* U^T *is compact in* $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$*.*

The proof of (1.10) for the periodic solutions constructed in the Sections 2 and 3, for systems (1.2) and (1.12), now follows easily.

Proof of (1.10). Let $U(x,t)$ be one such periodic solution. From the results in Sections 2 and 3 it follows that $\|U\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)} = M < \infty$. Since $\|U^T\|_{L^\infty(\mathbb{R}\times\mathbb{R}_+)} = M$, the sequence is uniformly bounded in $L^\infty(\mathbb{R}\times\mathbb{R}_+)$. Now, since both systems are strictly hyperbolic and genuinely nonlinear, by Theorem 4.2 the sequence U^T is compact in $L^1_{loc}(\mathbb{R}\times\mathbb{R}_+)$. But, these systems are also endowed with strictly convex entropy and so Theorem 4.1 directly implies (1.10), which is what we desired to prove. □

Final Remarks

Remark 4.1. Using Glimm's method, Glimm and Lax [7] prove an existence result for a large class of 2×2 systems, including the p-systems of the isentropic gas dynamics, for small L^∞ initial data and also the decay in L^∞, $O(1/t)$, of the periodic solutions so constructed. A careful analysis of the asymptotics and the mechanism generating the asymptotic patterns for solutions having the decay property proved in [7] is provided in [3]. Although the classes of systems considered here are more restrict, our results apply to large initial data.

Remark 4.2. Since its publication, Theorem 4.2 has hardly found application because the assumption of uniform boundedness of the sequence is in general too difficult of being verified for genuinely nonlinear systems, except in the special case where the rarefaction curves are straight lines (*e.g.*,[15, 13, 11, 2]). Actually, as far as the author knows, toghether with this special case the decay results proved here constitute so far the only applications of Theorem 4.2 which do not assume apriori the uniform boundedness.

Remark 4.3. The special feature of the systems considered here which allowed a simple periodic version of the adapted Glimm scheme is that they permit the use of a linear (*i.e.*, first-order) Glimm functional. The original Glimm functional contains a quadratic part which is the most difficult obstacle for a periodic version of the general result in [6]. It is immediate to verify that the methods used here are also applicable to the easier case of $n \times n$ systems with n independent Riemann invariants and coinciding shock and rarefaction curves.

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Instituto de Matemática, Universidade Federal do Rio de Janeiro, C. Postal 68530, CEP 21945-970, Rio de Janeiro, RJ, Brazil
E-mail address: hermano@im.ufrj.br